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<https://hdl.handle.net/2134/374>.

Extending Two Fixpoint Theorems of Langley and Zheng

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Abstract

We extend two theorems on fixpoints of $f(z)$ by Langley and Zheng [1] to the consideration of points where $f(z) = Q(z)$ for some rational function Q such that $Q(\infty) = \infty$. In addition, we extend the class of functions f from transcendental entire functions to meromorphic functions with relatively few poles.

Mathematics Subject Classification (1991): 30D30, 30D35

Keywords: value distribution, meromorphic functions, Nevanlinna theory

1 Introduction

Let \mathcal{B} denote the class of functions f meromorphic in the plane, for which the set of finite singular values of the inverse function f^{-1} is bounded, that is, the class of all functions f whose set of finite asymptotic and critical values is bounded. This class \mathcal{B} has been considered extensively in iteration theory, see [2], [3], [1].

In [1], Langley and Zheng prove the following two fixpoint theorems, for certain types of functions in the class \mathcal{B} .

Theorem 1.1 (Langley and Zheng, [1]) *Let $0 < \alpha < 1$. There is a positive constant c , depending only on α , such that if f is a transcendental entire function in the class \mathcal{B} , then there are infinitely many fixpoints z satisfying*

$$f(z) = z, \quad |f'(z)| > c \log M(\alpha|z|, f). \quad (1)$$

Theorem 1.2 (Langley and Zheng, [1]) *Let f be a meromorphic function in the class \mathcal{B} , with order $\infty \geq \rho(f) > \mu > 0$. Then f has infinitely many fixpoints z with*

$$f(z) = z, \quad |f'(z)| > |z|^{\mu/2}.$$

Given a transcendental function f and a rational function Q , we define a Q -point of f to be any solution z_0 of the equation $f(z) = Q(z)$. We then extend these two fixpoint theorems to Q -point theorems. In particular, we extend them to the case where $f(z)$ is equal to a rational function $Q(z)$ such that $Q(\infty) = \infty$, that is, where Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞ .

In addition, we further extend Theorem 1.1 to transcendental meromorphic functions f such that $\delta(\infty, f) > 0$, that is, meromorphic functions with relatively few poles. We state the extended theorems as follows.

Theorem 1.3 *Let $0 < \alpha < 1$. There is a positive constant c , depending only on α , such that if f is a transcendental meromorphic function in the class \mathcal{B} with $\delta(\infty, f) > 0$, and Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞ , then there are infinitely many points z satisfying*

$$f(z) = Q(z), \quad |f'(z)| > c\delta(\infty, f) \frac{|Q(z)|}{|z|} T(\alpha|z|, f). \quad (2)$$

Theorem 1.4 *Let f be a meromorphic function in the class \mathcal{B} , with order $\infty \geq \rho(f) > \mu > 0$. Let Q be a rational function with a pole of multiplicity $p \geq 1$ at ∞ . Then f has infinitely many points z with*

$$f(z) = Q(z), \quad |f'(z)| > |z|^{\mu/2+p-1}.$$

In § 2 we prove a lemma that is used in the proof of Theorem 1.3 and Theorem 1.4. In § 3 and § 4 we prove Theorem 1.3 and Theorem 1.4 respectively.

2 A useful lemma

The following lemma is used in the proof of Theorem 1.3 and Theorem 1.4. We note that it is an extension of [1, Lemma 1], and we will provide the proof for completeness.

Lemma 2.1 *Suppose that f is a transcendental meromorphic function in the class \mathcal{B} . Let Q be a rational function such that $Q(\infty) = \infty$. Define a function G by*

$$G(z) = f(z)/Q(z), \quad G'(z)/G(z) = f'(z)/f(z) - Q'(z)/Q(z). \quad (3)$$

Suppose that δ is a positive constant. Then there exists a positive constant ϵ such that the following is true. If $|z_1|$ is large and $|G(z_1) - 1| < \frac{1}{4}\epsilon$, then z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$, such that C is contained in $B(z_1, \delta|z_1|)$. Define a function H by

$$H(z) = \frac{2}{\epsilon}(G(z) - G(z_1)). \quad (4)$$

Then C is mapped conformally onto $B(0, 1)$ by H . Furthermore, $|Q(z)G'(z)|$ is large on C , and given any $z_0 \in C$ such that $|G(z_0) - G(z_1)| < \frac{1}{4}\epsilon$, we have that

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_0)| \leq \frac{27}{4}|G'(z_1)|. \quad (5)$$

The following two results are used in the proof of Lemma 2.1.

Lemma 2.2 (Eremenko, Lyubich and Bergweiler, [2], [4], [3], [5]) *Suppose that f is a transcendental meromorphic function in the class \mathcal{B} . Then there are positive constants R , S and c such that*

$$|zf'(z)/f(z)| \geq c \log^+ |f(z)/R|$$

for $|z| > S$. Here, R and S depend on f , and c does not.

Theorem 2.3 (Koebe distortion theorem) *Let f be an analytic univalent function on $B(0, 1)$. Then for $0 < r < 1$,*

$$\frac{1-r}{(1+r)^3} |f'(0)| \leq \max_{|z| \leq r} |f'(z)| \leq \frac{1+r}{(1-r)^3} |f'(0)|.$$

Proof of Lemma 2.1 First we choose a large positive R_1 and a small positive constant ϵ such that $|G(z) - 1| > \epsilon$ on $|z| = R_1$. Suppose that $|z| > R_1$ and that $|G(z) - 1| < \epsilon$. Then we have that $|G(z)| > \frac{1}{2}$, and so by (3), $|f(z)| > \frac{1}{2}|Q(z)|$. Then by Lemma 2.2, there is a positive constant R such that

$$|zf'(z)/f(z)| \geq c \log^+ |f(z)/R| > c \log |Q(z)|, \quad (6)$$

where c denotes a positive constant which does not depend on R_1 or ϵ , but not necessarily the same constant at each occurrence.

For such z , we have by (3) that

$$|zG'(z)/G(z)| = |zf'(z)/f(z) - zQ'(z)/Q(z)| \geq |zf'(z)/f(z)| - |zQ'(z)/Q(z)|. \quad (7)$$

We recall that $|G(z) - 1| < \epsilon$ and so $|G(z)|$ is close to 1, and since $Q(\infty) = \infty$ and z is large, we have that $|zQ'(z)/Q(z)| = O(1)$ as $z \rightarrow \infty$. Then by (6) and (7) we have that

$$|zG'(z)| > R_2 = c \log |Q(R_1)|. \quad (8)$$

Then since $Q(\infty) = \infty$ and z is large, we have that $|Q(z)| \geq c|z|$ and so,

$$|Q(z)G'(z)| > R_3 = cR_2. \quad (9)$$

Suppose now that δ is a positive constant and that z_1 is as in the statement of the lemma, with $|z_1| > 2R_1$. That is, $|G(z_1) - 1| < \frac{1}{4}\epsilon$, and z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$. Let H be defined as in the statement of the lemma, that is, $H(z) = \frac{2}{\epsilon}(G(z) - G(z_1))$. Then $H(z_1) = 0$ and $|H'(z_1)| = \frac{2}{\epsilon}|G'(z_1)| > 2R_2/\epsilon|z_1|$ by (8), and so $H'(z_1) \neq 0$. Then H is a conformal mapping in a neighbourhood of z_1 .

Next define

$$h(w) = \sum_{k=0}^{\infty} c_k w^k, \quad c_0 = z_1, \quad (10)$$

to be that branch of the inverse function H^{-1} which maps 0 to z_1 , and let r_1 be the radius of convergence of this series. Then, by a standard compactness argument, there is some w^* with $w^* = r_1 e^{i\theta^*}$, for some real θ^* , such that h has no analytic continuation to a neighbourhood of w^* . Thus the image of the path $w = t e^{i\theta^*}$, $0 \leq t < r_1$, under h must, as $t \rightarrow r_1$, either tend to ∞ or to a multiple point z^* of H , with $H(z^*) = w^*$.

Set $r_2 = \min\{r_1, 1\}$. Let γ be the path $\gamma(t) = t e^{i\theta^*}$, $0 \leq t < r_2$. For $|z| = R_1$, we have that

$$|H(z)| = \frac{2}{\epsilon}|G(z) - G(z_1)| \geq \frac{2}{\epsilon}(|G(z) - 1| - |G(z_1) - 1|) > \frac{2}{\epsilon}(\epsilon - \frac{1}{4}\epsilon) = \frac{3}{2},$$

and since $h(0) = z_1$ we must have that the image path $h(\gamma)$ lies in $|z| > R_1$. Now $H(h(w)) = w$, which by differentiation gives $(H(h(w)))' = H'(z)h'(w) = 1$ where $z = h(w)$. Then,

$$h(w)/h'(w) = zH'(z) = \frac{2}{\epsilon}zG'(z) \quad (11)$$

and this is large on γ by (8). Then for w on γ we have by (8) and (11) that

$$\begin{aligned} \log |h(w)/z_1| &\leq \int_0^{|w|} |h'(se^{i\theta^*})/h(se^{i\theta^*})| ds \\ &\leq \frac{\epsilon}{2} \frac{|w|}{\inf_{z \in h(\gamma)} |zG'(z)|} \leq \frac{\epsilon}{2R_2}. \end{aligned}$$

Then if ϵ was chosen small enough, the path $h(\gamma)$ lies in $B(z_1, \delta|z_1|)$, and replacing θ^* by any $\theta \in [0, 2\pi]$ we see that $h(B(0, r_2)) \subseteq B(z_1, \delta|z_1|)$.

On the path $h(\gamma)$ we have

$$|G(z) - 1| \leq |G(z) - G(z_1)| + |G(z_1) - 1| < \frac{1}{2}\epsilon + \frac{1}{4}\epsilon = \frac{3}{4}\epsilon,$$

and, using (8), we have that $|H'(z)| = \frac{2}{\epsilon}|G'(z)| \geq 2R_2/\epsilon|z|$. In particular, we have that $h(\gamma)$ is bounded and does not tend to a critical point of H . Since $r_2 = \min\{r_1, 1\}$, we must then have that $r_1 \geq 1$. Thus, C is contained in $B(z_1, \delta|z_1|)$ and is mapped conformally onto $B(0, 1)$ by H . Furthermore, by (9), $|Q(z)G'(z)|$ is large on C .

Next, let $z_0 \in C$ such that $|G(z_0) - G(z_1)| < \frac{1}{4}\epsilon$. Then $|H(z_0)| = \frac{2}{\epsilon}|G(z_0) - G(z_1)| < \frac{1}{2}$. Then since h is univalent on $B(0, 1)$, there exists some $w_0 \in B(0, 1)$, such that $h(w_0) = z_0$. Further, $|w_0| < \frac{1}{2}$, and so by the Koebe distortion theorem we have,

$$\frac{4}{27}|h'(0)| \leq |h'(w_0)| \leq 12|h'(0)|. \quad (12)$$

Then since $H'(z)h'(w) = 1$ we have $h'(w) = 1/H'(z)$ for $w \in B(0, 1)$, and in particular, $h'(0) = 1/H'(z_1)$ and $h'(w_0) = 1/H'(z_0)$. Also, by (4), $H'(z) = \frac{2}{\epsilon}G'(z)$, and so by (12) we have

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_0)| \leq \frac{27}{4}|G'(z_1)|.$$

3 Proof of Theorem 1.3

We need the following theorem.

Theorem 3.1 (Toppila and Winkler, [6]) *Let f be a transcendental meromorphic function of order λ such that $\delta(\infty, f) > 0$. Then*

$$\limsup_{z \rightarrow \infty, z \in E(f)} \frac{|z|f^\#(z)}{T(|z|, f)} \geq A_0\delta(\infty, f)(1 + \lambda)$$

where $E(f) = \{z : |f(z)| = 1\}$, A_0 is a positive absolute constant and $f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative.

We now prove Theorem 1.3.

Proof of Theorem 1.3 Let $0 < \alpha < 1$ and let R be a large positive constant. Let δ and ϵ be as in Lemma 2.1, with $\frac{1}{1+\delta} > \alpha$. Define functions G and f_1 by

$$G(z) = \frac{f(z)}{Q(z)} = 1 + \frac{\epsilon f_1(z)}{16}. \quad (13)$$

Then $f_1(z) = \frac{16}{\epsilon}(\frac{f(z)}{Q(z)} - 1) = c_1\frac{f(z)}{Q(z)} - c_2$ where c_j denotes a positive constant not depending on R . Then since f is a transcendental meromorphic function and since Q is a rational function, we have that f_1 is a transcendental meromorphic function.

Further, we have that $\delta(\infty, f_1) > 0$ since $f(z) = (c_3 f_1(z) + c_4)Q(z)$ and,

$$\begin{aligned}
0 < \delta(\infty, f) &= \delta(\infty, (c_3 f_1 + c_4)Q) \\
&= \liminf_{r \rightarrow \infty} \frac{m(r, (c_3 f_1 + c_4)Q)}{T(r, (c_3 f_1 + c_4)Q)} \\
&\leq \liminf_{r \rightarrow \infty} \frac{m(r, c_3 f_1 + c_4) + m(r, Q) + O(1)}{T(r, c_3 f_1 + c_4) - T(r, Q) - O(1)} \\
&= \liminf_{r \rightarrow \infty} \frac{m(r, f_1) + O(\log r) + O(1)}{T(r, f_1) - O(\log r) - O(1)} \\
&= \delta(\infty, f_1),
\end{aligned}$$

since Q is a rational function and f_1 is a transcendental function function.

Let λ be the order of f_1 , and let $E(f_1) = \{z : |f_1(z)| = 1\}$. Then by Theorem 3.1, we have that

$$\limsup_{z \rightarrow \infty, z \in E(f_1)} \frac{|z| f_1^\sharp(z)}{T(|z|, f_1)} \geq A_0 \delta(\infty, f_1) (1 + \lambda),$$

where A_0 is a positive absolute constant. We note that for $z \in E(f_1)$ we have that $f_1^\sharp(z) = \frac{|f_1'(z)|}{1 + |f_1(z)|^2} = \frac{|f_1'(z)|}{2}$.

Then we have

$$\limsup_{z \rightarrow \infty, z \in E(f_1)} \frac{|z| |f_1'(z)|}{T(|z|, f_1)} > d_1 = A_0 \delta(\infty, f_1),$$

and so there exists a sequence (ζ_n) in $E(f_1)$, $\zeta_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$\frac{|\zeta_n| |f_1'(\zeta_n)|}{T(|\zeta_n|, f_1)} > d_1.$$

Thus we can choose z_1 in $E(f_1)$ arbitrarily large, and in particular such that $|z_1| > R$, with

$$|z_1| |f_1'(z_1)| > d_1 T(|z_1|, f_1). \tag{14}$$

Next, since $f(z) = (c_3 f_1(z) + c_4)Q(z)$ we have that

$$\begin{aligned}
T(|z_1|, f) &= T(|z_1|, (c_3 f_1 + c_4)Q) \\
&\leq T(|z_1|, c_3 f_1 + c_4) + T(|z_1|, Q) \\
&\leq T(|z_1|, f_1) + O(\log |z_1|),
\end{aligned}$$

and so, if R is large enough,

$$T(|z_1|, f_1) \geq T(|z_1|, f) - O(\log |z_1|) > \frac{1}{2} T(|z_1|, f), \tag{15}$$

since f is a transcendental function. Then since $|G'(z)| = \frac{\epsilon}{16} |f_1'(z)|$, we have by (14) and (15) that

$$|G'(z_1)| > \frac{\epsilon}{16} \frac{d_1}{|z_1|} T(|z_1|, f_1) > \frac{d_2}{|z_1|} T(|z_1|, f), \tag{16}$$

where $d_2 = \frac{\epsilon}{32} d_1$.

We recall that $G(z) = 1 + \frac{\epsilon f_1(z)}{16}$, and thus since $z_1 \in E(f_1)$, we have that $|G(z_1) - 1| = \frac{\epsilon}{16} < \frac{\epsilon}{4}$. Then by Lemma 2.1, z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$ such that C is contained in $B(z_1, \delta|z_1|)$. Also, by Lemma 2.1, C is mapped conformally onto $B(0, 1)$ by the function $H(z) = \frac{2}{\epsilon}(G(z) -$

$G(z_1)$). Then we can choose a point z_2 in C such that $G(z_2) = 1$ and $|G(z_2) - G(z_1)| < \frac{1}{4}\epsilon$. Furthermore, we have by (5) that,

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_2)| \leq \frac{27}{4}|G'(z_1)|,$$

and so by (16) we have

$$|G'(z_2)| > \frac{d_3}{|z_1|}T(|z_1|, f), \quad (17)$$

where $d_3 = \frac{d_2}{12}$. Then, since we have by (3) that,

$$z_2 \frac{f'(z_2)}{f(z_2)} = z_2 \frac{G'(z_2)}{G(z_2)} + z_2 \frac{Q'(z_2)}{Q(z_2)},$$

and since $f(z_2) = Q(z_2)$ and $G(z_2) = 1$, we have that

$$\left| z_2 \frac{f'(z_2)}{Q(z_2)} \right| \geq |z_2 G'(z_2)| - \left| z_2 \frac{Q'(z_2)}{Q(z_2)} \right|. \quad (18)$$

Now since Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞ , we have that $\left| z \frac{Q'(z)}{Q(z)} \right| = O(1)$ as $z \rightarrow \infty$. Since $z_2 \in B(z_1, \delta|z_1|)$, we have that $|z_1| \leq |z_1 - z_2| + |z_2| \leq \delta|z_1| + |z_2|$, which gives that $\left| \frac{z_2}{z_1} \right| \geq 1 - \delta$. Then by (17) and (18), we have that

$$\begin{aligned} \left| z_2 \frac{f'(z_2)}{Q(z_2)} \right| &\geq |z_2 G'(z_2)| - O(1) \\ &\geq \frac{1}{2} \left| \frac{z_2}{z_1} \right| d_3 T(|z_1|, f) \\ &\geq \frac{1 - \delta}{2} d_3 T(|z_1|, f) \\ &\geq \frac{1 - \delta}{2} d_3 T(\alpha|z_2|, f). \end{aligned}$$

Then since $d_3 = \frac{\epsilon}{384} A_0 \delta(\infty, f)$ and A_0 is an absolute constant, we have that

$$|f'(z_2)| > c\delta(\infty, f) \frac{|Q(z_2)|}{|z_2|} T(\alpha|z_2|, f).$$

where c is a positive constant depending only on α .

4 Proof of Theorem 1.4

We need the following theorem.

Theorem 4.1 (Langley and Zheng, [1]) *Let f be a transcendental meromorphic function in the class \mathcal{B} . If Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞ , then*

$$m(r, \frac{1}{f - Q}) = O(\log r T(r, f)) \quad (19)$$

as $r \rightarrow \infty$ outside a set of finite measure.

We note that the Nevanlinna counting function $N(r, \frac{1}{f-z})$ counts the poles of $\frac{1}{f-z}$, that is, the fixpoints of f . In [1] the following implication of Theorem 4.1 is noted, namely, that $N(r, \frac{1}{f-z})$ cannot satisfy $N(r, \frac{1}{f-z}) = o(T(r, f))$ as $r \rightarrow \infty$. We extend this to where Q is a rational function such that $Q(\infty) = \infty$, and state the result as a corollary. We provide a proof for completeness.

Corollary 4.2 *Let f and Q be as in the statement of Theorem 1.4. Then the Nevanlinna counting function $N(r, \frac{1}{f-Q})$, of points where $f(z) = Q(z)$, cannot satisfy $N(r, \frac{1}{f-Q}) = o(T(r, f))$ as $r \rightarrow \infty$. Also, for $\sigma > 0$, we may choose arbitrarily large r such that there are at least $2r^\sigma$ Q -points z_j , such that $f(z_j) = Q(z_j)$, in the annulus $\frac{1}{2}r \leq |z| \leq r$.*

Proof Since Q is a rational function we have that

$$T(r, f - Q) \leq T(r, f) + T(r, Q) + O(1) = T(r, f) + O(\log r) + O(1),$$

and that,

$$T(r, f) = T(r, f - Q + Q) \leq T(r, f - Q) + T(r, Q) + O(1) = T(r, f - Q) + O(\log r) + O(1).$$

Thus

$$|T(r, f) - T(r, f - Q)| \leq O(\log r) + O(1), \quad (20)$$

and since, $T(r, \frac{1}{f-Q}) = T(r, f - Q) + O(1)$ we have that

$$|T(r, f) - T(r, \frac{1}{f-Q}) - O(1)| \leq O(\log r) + O(1). \quad (21)$$

Then since $T(r, \frac{1}{f-Q}) = N(r, \frac{1}{f-Q}) + m(r, \frac{1}{f-Q})$ we have that

$$|T(r, f) - T(r, \frac{1}{f-Q}) - O(1)| \geq |T(r, f) - N(r, \frac{1}{f-Q})| - |m(r, \frac{1}{f-Q}) + O(1)|.$$

We may assume by (19) that $m(r, \frac{1}{f-Q}) = O(\log r T(r, f))$ as $r \rightarrow \infty$, with $r \notin E$, for some set E of finite measure. Then by (21) and Theorem 4.1 we have

$$\begin{aligned} |T(r, f) - N(r, \frac{1}{f-Q})| &\leq O(\log r) + O(1) + |m(r, \frac{1}{f-Q}) + O(1)| \\ &= O(\log r) + O(1) + O(\log r T(r, f)), \quad r \notin E. \end{aligned}$$

Then since f is a transcendental function, we have that $N(r, \frac{1}{f-Q})$ cannot satisfy $N(r, \frac{1}{f-Q}) = o(T(r, f))$ as $r \rightarrow \infty$.

Next we show that all large z_j such that $f(z_j) = Q(z_j)$ are simple zeros of $f - Q$. Let z_0 be large and suppose $f(z_0) = Q(z_0)$. Then since f and Q satisfy the hypotheses of Lemma 2.1 and Lemma 2.2, we have by (6) that

$$|z_0 f'(z_0)/f(z_0)| > R_1 = c \log |Q(z_0)|, \quad (22)$$

where c is a positive constant. Also, since Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞ , we have that $|\frac{Q'(z_0)}{Q(z_0)}| \leq \frac{p}{|z_0|}(1 + o(1))$. Then, by (22) and since $f(z_0) = Q(z_0)$, we have that

$$|f'(z_0)| > \frac{R_1}{|z_0|} |Q(z_0)| \geq \frac{p}{|z_0|} (1 + o(1)) |Q(z_0)| \geq |Q'(z_0)|,$$

since we can choose z_0 so large that p/R_1 is very small. Then this gives that $f'(z_0) - Q'(z_0) \neq 0$ and so z_0 is a simple zero of $f - Q$.

Suppose now that $r_0 \geq 0$ is such that for $r \geq r_0$, there are less than $2r^\sigma$ points z_j , such that $f(z_j) = Q(z_j)$, in $\frac{r}{2} \leq |z| \leq r$. Then for $m \in \mathbb{N}$ we have that

$$n \left(2^m r_0, \frac{1}{f-Q} \right) < n \left(r_0, \frac{1}{f-Q} \right) + 2 \left[(2^m r_0)^\sigma + (2^{m-1} r_0)^\sigma + \dots + (2r_0)^\sigma \right]. \quad (23)$$

Let r be large, with, in particular, $r > r_0$. Then there exists $m \in \mathbb{N}$ such that $2^{m-1}r_0 \leq r < 2^m r_0$. Then by (23) we have

$$\begin{aligned} n\left(r, \frac{1}{f-Q}\right) &\leq n\left(2^m r_0, \frac{1}{f-Q}\right) \\ &< n\left(r_0, \frac{1}{f-Q}\right) + 2[(2^m r_0)^\sigma + (2^{m-1} r_0)^\sigma + \dots + (2r_0)^\sigma] \\ &= c_1 + 2r_0^\sigma \left[\frac{2^\sigma(2^{\sigma m} - 1)}{2^\sigma - 1} \right] \\ &\leq c_1 + c_2 r^\sigma, \end{aligned}$$

where $c_1 = n(r_0, \frac{1}{f-Q})$ and c_2 is a positive constant. Then we have $N(r, \frac{1}{f-Q}) = O(r^\sigma)$.

Next, by (19), we may choose $r \leq r_1 \leq 2r$ such that $m(r_1, \frac{1}{f-Q}) = O(\log r_1 T(r_1, f))$. Then

$$\begin{aligned} T(r_1, \frac{1}{f-Q}) &= N(r_1, \frac{1}{f-Q}) + m(r_1, \frac{1}{f-Q}) \\ &= O(r_1^\sigma) + O(\log r_1 T(r_1, f)), \end{aligned}$$

and so,

$$T(r, \frac{1}{f-Q}) \leq T(r_1, \frac{1}{f-Q}) = O(r^\sigma).$$

Therefore we have that $T(r, f) = O(r^\sigma)$, and thus,

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ T(r, f)}{\log r} \leq \sigma.$$

This is a contradiction since $\rho(f) > \sigma$. Therefore, we may choose arbitrarily large r such that there are at least $2r^\sigma$ points z_j such that $f(z_j) = Q(z_j)$ in the annulus $\frac{1}{2}r \leq |z| \leq r$.

We now prove Theorem 1.4.

Proof of Theorem 1.4 Let f and Q be as in the statement of Theorem 1.4. Choose σ with $\rho(f) > \sigma > \mu$. Since $\rho(f) > 0$, we have that f is a transcendental function, and so by Corollary 4.2, we can choose arbitrarily large r such that f has at least $2r^\sigma$ points z_j such that $f(z_j) = Q(z_j)$ in the annulus $\frac{1}{2}r \leq |z| \leq r$.

Suppose δ is a small positive constant. Let $G(z) = f(z)/Q(z)$ be as defined in Lemma 2.1. Then for each z_j we have that $G(z_j) = 1$. Then by Lemma 2.1, we have that to each z_j there corresponds a component C_j of the set $\{z : |G(z) - 1| < \frac{1}{2}\epsilon\}$, and that each C_j is contained in $B(z_j, \delta|z_j|)$. Then choosing δ small enough gives that each C_j lies in the annulus $\frac{1}{4}r \leq |z| \leq 2r$. These C_j are disjoint simple islands since, by Lemma 2.1, they are mapped conformally onto $B(0, 1)$ by the function $H(z) = \frac{2}{\epsilon}(G(z) - G(z_j))$.

Let $h_j : B(0, 1) \rightarrow C_j$ be the inverse function of H . Then h_j is a univalent function on $B(0, 1)$ and by the Koebe distortion theorem we have that for $0 < r_0 < 1$,

$$c_1 |h'_j(0)| \leq \max_{|w| \leq r_0} |h'_j(w)| \leq c_2 |h'_j(0)|,$$

where c_1 and c_2 are positive constants, depending only on r_0 .

In particular, choose $r_0 = \frac{1}{2}$ and let $\hat{C}_j = h_j(B(0, \frac{1}{2}))$. Then $|h'_j(w)| \geq c_1 |h'_j(0)|$ for $|w| \leq \frac{1}{2}$. Then since $h'_j(w) = 1/H'(z)$ where $z = h_j(w)$, and in particular since $h'_j(0) = 1/H'(z_j) = \epsilon/2G'(z_j)$ we have that

$$|h'_j(w)| \geq c/|G'(z_j)|, \quad \text{for } |w| \leq \frac{1}{2}, \quad (24)$$

where, from here on, c denotes a positive constant, not necessarily the same at each occurrence, but not depending on r or σ .

Next, since the area of the annulus $\frac{1}{4}r \leq |z| \leq 2r$ is cr^2 , and since there are at least $2r^\sigma$ disjoint components \hat{C}_j in the annulus, we have that at least r^σ of these \hat{C}_j have area at most $cr^{2-\sigma}$. Then, for these z_j , we have

$$z_j \frac{f'(z_j)}{f(z_j)} = z_j \frac{G'(z_j)}{G(z_j)} + z_j \frac{Q'(z_j)}{Q(z_j)},$$

and since $f(z_j) = Q(z_j)$ and $G(z_j) = 1$, we have that

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq |z_j G'(z_j)| - \left| z_j \frac{Q'(z_j)}{Q(z_j)} \right|. \quad (25)$$

Now since Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞ , we have that $\left| z \frac{Q'(z)}{Q(z)} \right| = O(1)$ as $z \rightarrow \infty$. Therefore, (25) gives that

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq |z_j G'(z_j)| - O(1). \quad (26)$$

Now by [7, p.4] and (24) we have that

$$\text{area of } \hat{C}_j = \int \int_{B(0, \frac{1}{2})} |h'(w)|^2 du dv \geq c/|G'(z_j)|^2.$$

And so, since the area of \hat{C}_j is at most $cr^{2-\sigma}$ we have that $1/|G'(z_j)|^2 \leq cr^{2-\sigma}$, which gives $|G'(z_j)| \geq cr^{\sigma/2-1}$. Then by (26) and since $r \geq |z_j|$, and z_j is large, and $p \geq 1$, we have

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq cr^{\sigma/2} - O(1) \geq c|z_j|^{\sigma/2}.$$

Then since $|Q(z_j)| > c|z_j|^p$, we have that

$$|f'(z_j)| > c|z_j|^{\sigma/2+p-1} \geq |z_j|^{\mu/2+p-1}.$$

ACKNOWLEDGEMENTS: This research has been done as part of a PhD thesis at the University of Nottingham. The author gratefully acknowledges the advice and support of Prof J. K. Langley who suggested these results and provided useful ideas and comments. The author also acknowledges the referee(s) for their valuable contributions to this paper.

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Eingegangen am 12. November 2004