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THE STABLE MANIFOLD THEOREM FOR SEMI-LINEAR STOCHASTIC EVOLUTION EQUATIONS AND STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

I: THE STOCHASTIC SEMIFLOW

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Abstract. The main objective of this work is to characterize the pathwise local structure of solutions of semilinear stochastic evolution equations (see's) and stochastic partial differential equations (spde's) near stationary solutions. Such characterization is realized through the long-term behavior of the solution field near stationary points. The analysis falls in two parts I, II. In Part I (this paper), we establish a general existence and compactness theorem for C^k-cocycles of semilinear see's and spde's. Our results cover a large class of semilinear see's as well as certain semilinear spde's with non-Lipschitz terms such as stochastic reaction diffusion equations and the stochastic Burgers equation with additive infinite-dimensional noise. We adopt a notion of stationarity employed in previous work of one of the authors with M. Scheutzow ([M-S.2], cf. [E-K-M-S]). In Part II of this work ([M-Z-Z]), we establish a *local stable manifold theorem* for non-linear see's and spde's.

1. Introduction.

The construction of local stable and unstable manifolds near hyperbolic equilibria is a fundamental problem in deterministic and stochastic dynamical systems. The significance of these invariant manifolds consists in a characterization of the local behavior of the dynamical system in terms of long-time asymptotics of its trajectories near a stationary point. In recent years, it has been established that local stable/unstable manifolds exist for finite-dimensional stochastic ordinary differential equations (sode's) ([M-S.2]) and

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stochastic systems with finite memory (sfde's)([M-S.1]). On the other hand, existence of such manifolds for stochastic evolution equations (see's) and stochastic partial differential equations (spde's) has been an open problem since the early nineties ([F-S], [B-F], [B-F.1]).

The main objective of the present work is to establish the existence of local stable and unstable manifolds near stationary solutions of semilinear stochastic evolution equations (see's) and stochastic partial differential equations (spde's). Our approach consists in the following two major undertakings:

- A construction of a sufficiently Fréchet differentiable cocycle for mild/weak trajectories of the see or the spde. This is achieved in the see case by a combination of a chaos-type expansion and suitable lifting techniques, and for spde's by using stochastic variational representations and methods from deterministic pde's. Part I of this work is devoted to detailing the construction of the cocycle.
- The application of classical non-linear ergodic theory techniques developed by Oseledec [O] and Ruelle [Ru.2] in order to study the local structure of the above cocycle in a neighborhood of a hyperbolic stationary point. Stationary points correspond to stationary solutions and hyperbolicity is characterized via the Lyapunov spectrum of the linearized cocycle along the stationary trajectory. This is the subject of Part II of this work ([M-Z-Z]).

In [F-S], Flandoli and Schaumloeffl established the existence of a random evolution operator and its Lyapunov spectrum for a linear stochastic heat equation with finite-dimensional noise, on a bounded Euclidean domain. For linear see's with finite-dimensional noise, a stochastic semi-flow (i.e. random evolution operator) was obtained in [B-F]. Subsequent work on the dynamics of non-linear spde's has focused mainly on the question of existence of continuous semiflows and the existence and uniqueness of invariant measures.

The problem of existence of semiflows for see's and spde's is a non-trivial one, mainly due to the well-established fact that finite-dimensional methods for constructing (even

continuous) stochastic flows break down in the infinite-dimensional setting of spde's and see's. In particular, Kolmogorov's continuity theorem fails for random fields parametrized by infinite-dimensional Hilbert spaces (cf. [Mo.1], pp. 144-149, [Sk], [Mo.2], [F.1], [F.2]], [D-Z.1], pp. 246-248).

As indicated above, the existence of a smooth semiflow is a necessary tool for constructing the stable and unstable manifolds near a hyperbolic stationary random point, ala work of Ruelle ([Ru.1], [Ru.2]). In this article, we show the existence of smooth perfect cocycles for mild solutions of semilinear see's in Hilbert space (Theorem 2.6). Our construction employs a "chaos-type" representation coupled with lifting and variational techniques using the linear terms of the see (Theorems 2.1, 2.2, 2.4). This technique bypasses the need for Kolmogorov's continuity theorem and appears to be new. Applications to specific classes of spde's are given. In particular, we obtain smooth stochastic semiflows for semilinear spde's driven by cylindrical Brownian motion with a covariance Hilbert space K (Theorem 3.5). In these applications, it turns out that in addition to smoothness of the non-linear terms, one requires some level of dissipativity or Lipschitz continuity in order to guarantee the existence of smooth globally defined semiflows. Specific examples of spde's include semilinear parabolic spde's with Lipschitz nonlinearities (Theorem 3.5), stochastic reaction diffusion equations (Theorems 4.1, 4.2) and stochastic Burgers equations with additive infinite-dimensional noise (Theorem 4.3).

We begin by formulating the ideas of a *stochastic semiflow* and a *cocycle* which are central to the analysis in this work.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $\bar{\mathcal{F}}$ the P-completion of $\mathcal{F}$, and let $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions ([Pr]).

Denote $\Delta := \{(s,t) \in \mathbf{R}^2 : 0 \leq s \leq t\}$, and $\mathbf{R}^+ := [0, \infty)$. For a topological space E, let $\mathcal{B}(E)$ denote its Borel σ-algebra.

Let k be a positive integer and $0 < \epsilon \leq 1$. If E, N are real Banach spaces, we will denote by $L^{(k)}(E, N)$ the Banach space of all k-multilinear maps $A : E^k \to N$ with the uniform norm $\|A\| := \sup\{|A(v_1, v_2, \cdots, v_k)| : v_i \in E, |v_i| \leq 1, i = 1, \cdots, k\}$. Suppose $U \subseteq E$ is an open set. A map $f : U \to N$ is said to be *of class* $C^{k,\epsilon}$ if it is C^k and if $D^{(k)}f : U \to L^{(k)}(E, N)$ is ϵ-Hölder continuous on bounded sets in U. A $C^{k,\epsilon}$ map $f : U \to N$ is said to be *of class* $C_b^{k,\epsilon}$ if all its derivatives $D^{(j)}f, 1 \leq j \leq k$, are globally bounded on U, and $D^{(k)}f$ is ϵ-Hölder continuous on U. A mapping $\tilde{f} : [0, T] \times U \to N$ is *of class* $C^{k,\epsilon}$ *in the second variable uniformly with respect to the first* if for each $t \in [0, T]$, $\tilde{f}(t, \cdot)$ is $C^{k,\epsilon}$ on U, for every bounded set $U_0 \subseteq U$ the spatial partial derivatives $D^{(j)}\tilde{f}(t, x), j = 1, \cdots, k$, are uniformly bounded in $(t, x) \in [0, T] \times U_0$ and the corresponding ϵ-Hölder constant of $D^{(k)}\tilde{f}(t, \cdot)|U_0$ is uniformly bounded in $t \in [0, T]$.

The following definitions are crucial to the developments in this article.

Definition 1.1.

Let E be a Banach space, k a non-negative integer and $\epsilon \in (0, 1]$. A *stochastic* $C^{k,\epsilon}$ *semiflow* on E is a random field $V : \Delta \times E \times \Omega \to E$ satisfying the following properties:

(i) V is $(\mathcal{B}(\Delta) \otimes \mathcal{B}(E) \otimes \mathcal{F}, \mathcal{B}(E))$-measurable.

(ii) For each $\omega \in \Omega$, the map $\Delta \times E \ni (s, t, x) \mapsto V(s, t, x, \omega) \in E$ is continuous.

(iii) For fixed $(s, t, \omega) \in \Delta \times \Omega$, the map $E \ni x \mapsto X(s, t, x, \omega) \in E$ is $C^{k,\epsilon}$.

(iv) If $0 \leq r \leq s \leq t$, $\omega \in \Omega$ and $x \in E$, then

$$V(r, t, x, \omega) = V(s, t, V(r, s, x, \omega), \omega).$$

(v) For all $(s, x, \omega) \in \mathbf{R}^+ \times E \times \Omega$, one has $V(s, s, x, \omega) = x$.

Definition 1.2.

Let $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ be a P-preserving semigroup on the probability space $(\Omega, \mathcal{F}, P)$, E a Banach space, k a non-negative integer and $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (U, θ) on E is a $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(E) \otimes \mathcal{F}, \mathcal{B}(E))$-measurable random field $U : \mathbf{R}^+ \times E \times \Omega \to E$ with the following properties:

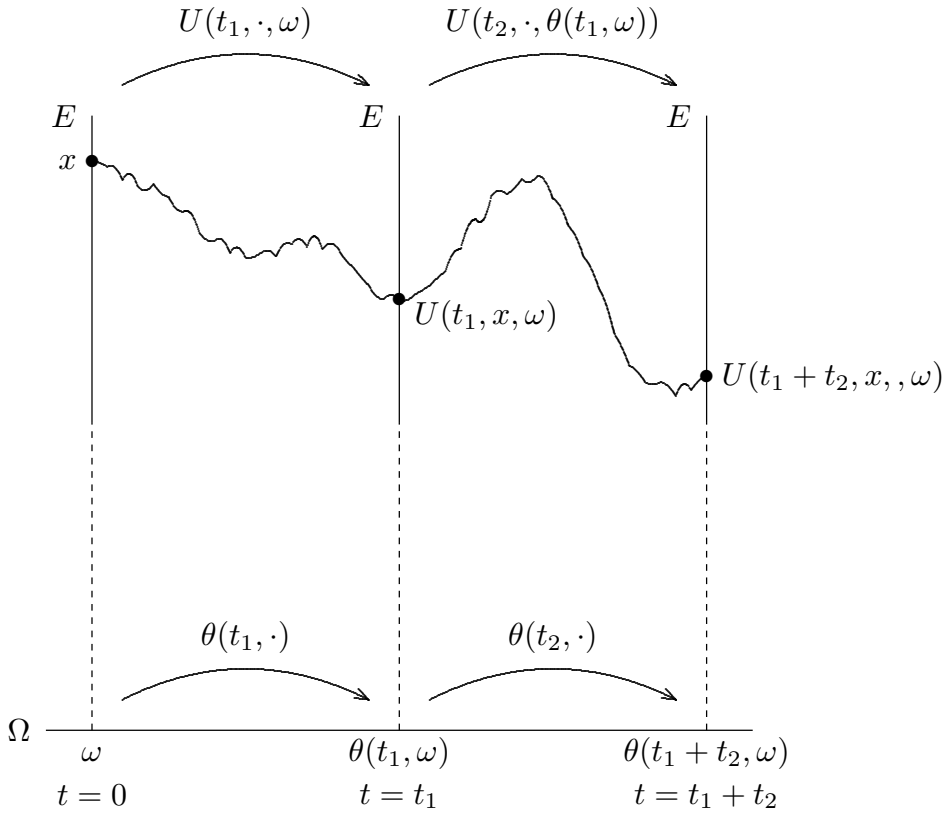
(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t, x) \mapsto U(t, x, \omega) \in E$ is continuous; and for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto U(t, x, \omega) \in E$ is $C^{k,\epsilon}$.

(ii) $U(t + s, \cdot, \omega) = U(t, \cdot, \theta(s, \omega)) \circ U(s, \cdot, \omega)$ for all $s, t \in \mathbf{R}^+$ and all $\omega \in \Omega$.

(iii) $U(0, x, \omega) = x$ for all $x \in E, \omega \in \Omega$.

Note that a cocycle (U, θ) corresponds to a one-parameter semigroup on $E \times \Omega$. The following figure illustrates the cocycle property. The vertical solid lines represent random copies of E sampled according to the probability measure P.

The Cocycle Property



The main objective of this article is to show that under sufficient regularity conditions on the coefficients, a large class of semilinear see's and spde's admits a $C^{k,\epsilon}$ semiflow $V : \Delta \times H \times \Omega \to H$ for a suitably chosen state space H and which satisfies $V(t_0, t, x, \cdot) = u(t, x)$ for all $x \in H$ and $t \geq t_0$, a.s., where u is the solution of the see/spde with initial function $x \in H$ at $t = t_0$. In the autonomous case, we show further that the semiflow V generates a cocycle (U, θ) on H, in the sense of Definition 1.2 above. The cocycle and its Fréchet derivative are compact in all cases.

2. Flows and cocycles of semilinear stochastic evolution equations.

In this section, we will establish the existence and regularity of semiflows generated by mild solutions of semilinear stochastic evolution equations. We will begin with the linear case. In fact, the linear cocycle will be used to represent the mild solution of the

semilinear stochastic evolution equation via a variational formula which transforms the semilinear stochastic evolution equation to a random integral equation (Theorem 2.5). The latter equation plays a key role in establishing the regularity of the stochastic flow of the semilinear see (Theorem 2.6).

One should note at this point the fact that Kolmogorov's continuity theorem fails for random fields parametrized by infinite-dimensional spaces. As a simple example, consider the random field $I : L^2([0,1],\mathbf{R}) \to L^2(\Omega,\mathbf{R})$ defined by the Wiener integral

$$I(x) := \int_0^1 x(t)\, dW(t), \quad x \in L^2([0,1],\mathbf{R}).$$

The above random field has no continuous measurable selection $L^2([0,1],\mathbf{R}) \times \Omega \to \mathbf{R}$ ([Mo.1], pp. 144-148).

(a) Linear stochastic evolution equations.

We will first prove the existence of semiflows associated with mild solutions of linear stochastic evolution equations of the form:

$$\left.\begin{aligned} du(t,x,\cdot) =& - Au(t,x,\cdot)dt + Bu(t,x,\cdot)\, dW(t), \quad t>0 \\ u(0,x,\omega) =& x. \end{aligned}\right\} \tag{2.1}$$

In the above equation $A : D(A) \subset H \to H$ is a closed linear operator on a separable real Hilbert space H. Assume that A has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding positive eigenvalues $\{\mu_n, n \geq 1\}$; i.e., $Ae_n = \mu_n e_n,\ n \geq 1$. Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H, t \geq 0$. Let E be a separable Hilbert space and $W(t), t \geq 0$, be a E-valued Brownian motion defined on the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and with a separable covariance Hilbert space K. Here $K \subset E$ is a Hilbert-Schmidt embedding. Indeed, Ω is the space of all continuous paths $\omega : \mathbf{R} \to E$ such that $\omega(0) = 0$ with the compact open topology, $\mathcal{F}$ is its Borel σ-field, $\mathcal{F}_t$ is the sub-σ-field generated by

all evaluations $\Omega \ni \omega \mapsto \omega(u) \in E, u \leq t$, and P is Wiener measure on Ω. The Brownian motion is given by

$$W(t,\omega) := \omega(t), \quad \omega \in \Omega,\, t \in \mathbf{R},$$

and may be represented by

$$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbf{R},$$

where $\{f_k : k \geq 1\}$ is a complete orthonormal basis of K, and $W^k, k \geq 1$, are standard independent one-dimensional Wiener processes ([D-Z], Chapter 4).

Denote by $L_2(K,H) \subset L(K,H)$ the Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, given the norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|^2\right]^{1/2},$$

where $|\cdot|$ is the norm on H. Suppose $B : H \to L_2(K,H)$ is a bounded linear operator. The stochastic integral in (2.1) is defined in the following sense ([D-Z], Chapter 4):

Let $F : [0,a] \times \Omega \to L_2(K,H)$ be $\mathcal{B}([0,a] \otimes \mathcal{F}, \mathcal{B}(L_2(K,H)))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted and such that $\int_0^a E\|F(t)\|^2_{L_2(K,H)}\, dt < \infty$. Define

$$\int_0^a F(t)\, dW(t) := \sum_{k=1}^{\infty} \int_0^a F(t)(f_k)\, dW^k(t)$$

where the H-valued stochastic integrals on the right hand side are with respect to the one-dimensional Wiener processes $W^k,\ k \geq 1$. Note that the above series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E\left|\int_0^a F(t)(f_k)\, dW^k(t)\right|^2 = \int_0^a E\|F(t)\|^2_{L_2(K,H)}\, dt < \infty.$$

Throughout the rest of the article, we will denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ the standard P-preserving ergodic Wiener shift on Ω:

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}.$$

Hence (W, θ) is a *helix*:

$$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega)), \quad t_1, t_2 \in \mathbf{R},\, \omega \in \Omega.$$

As usual, we let $L(H)$ be the Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$. Denote by $L_2(H) \subset L(H)$ the Hilbert space of all Hilbert-Schmidt operators $S : H \to H$. It is easy to see that if $S \in L_2(H)$ and $T \in L(H)$, then $\|S\|_{L(H)} \leq \|S\|_2$, $T \circ H$ (and $H \circ T$) $\in L_2(H)$ and $\|T \circ H\|_{L_2(H)} \leq \|T\|_{L(H)} \|S\|_{L_2(H)}$.

A *mild solution* of (2.1) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H$, $x \in H$, satisfying the following stochastic integral equation:

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot)\, dW(s), \quad t \geq 0. \tag{2.2}$$

The next lemma describes a canonical lifting of the strongly continuous semigroup $T_t : H \to H$, $t \geq 0$, to a strongly continuous semigroup of bounded linear operators $\tilde{T}_t : L_2(K, H) \to L_2(K, H), t \geq 0$.

Lemma 2.1.

Define the family of maps $\tilde{T}_t : L_2(K, H) \to L_2(K, H)$, $t \geq 0$, *by*

$$\tilde{T}_t(C) := T_t \circ C, \quad C \in L_2(K, H),\, t \geq 0.$$

Then the following is true:

(i) $\tilde{T}_t$, $t \geq 0$, *is a strongly continuous semigroup of bounded linear operators on* $L_2(K, H)$*; and* $\|\tilde{T}_t\|_{L(L_2(K,H))} = \|T_t\|_{L(H)}$ *for all* $t \geq 0$.

(ii) If $-\tilde{A} : \mathcal{D}(\tilde{A}) \subset L_2(K, H) \to L_2(K, H)$ *is the infinitesimal generator of* $\tilde{T}_t$, $t \geq 0$, *then*

$$\mathcal{D}(\tilde{A}) = \{C : C \in L_2(K, H),\, C(K) \subseteq \mathcal{D}(A), A \circ C \in L_2(K, H)\}$$

and

$$\tilde{A}(C) = A \circ C$$

for all $C \in \mathcal{D}(\tilde{A})$.

(iii) $\tilde{T}_t, t \geq 0$, *is a contraction semigroup if* $T_t, t \geq 0$, *is.*

Proof.

Observe that each $\tilde{T}_t : L_2(K,H) \to L_2(K,H), t \geq 0$, is a bounded linear map of $L_2(K,H)$ into itself. Indeed, it is easy to see that

$$\|\tilde{T}_t(C)\|_{L_2(K,H)} \leq \|T_t\|_{L(H)} \|C\|_{L_2(K,H)}, \quad C \in L_2(K,H),\, t \geq 0; \tag{2.3}$$

and hence $\|\tilde{T}_t\|_{L(L_2(K,H))} \leq \|T_t\|_{L(H)}$ for all $t \geq 0$. This implies assertion (iii). The reverse inequality

$$\|T_t\|_{L(H)} \leq \|\tilde{T}_t\|_{L(L_2(K,H))}, \quad t \geq 0,$$

is not hard to check. Hence the last assertion in (i) holds.

We next verify the semi-group property of $\tilde{T}_t, t \geq 0$. Let $t_1, t_2 \geq 0$, $C \in L_2(K,H)$. Then

$$(\tilde{T}_{t_2} \circ \tilde{T}_{t_1})(C) = T_{t_2} \circ (T_{t_1} \circ C) = T_{t_1+t_2} \circ C = \tilde{T}_{t_1+t_2}(C).$$

Note also that $\tilde{T}_0 = I_{L(L_2(K,H))}$, the identity map $L_2(K,H) \to L_2(K,H)$. Therefore, $\tilde{T}_t$, $t \geq 0$, is a semigroup on $L_2(K,H)$. To prove the strong continuity of $\tilde{T}_t$, $t \geq 0$, we will show that

$$\lim_{t \to 0+} \tilde{T}_t(C) = C \tag{2.4}$$

for each $C \in L_2(K,H)$. To prove the above relation, let $C \in L_2(K,H)$ and recall that $\{f_k : k \geq 1\}$ is a complete orthonormal basis of K. From the strong continuity of $T_t, t \geq 0$, it follows that

$$\lim_{t \to 0+} |T_t(C(f_k)) - C(f_k)|_H^2 = 0 \tag{2.5}$$

for each integer $k \geq 1$. Furthermore,

$$|T_t(C(f_k)) - C(f_k)|_H^2 \leq 2[\sup_{0\leq t\leq a} \|T_t\|_{L(H)}^2 + 1]|C(f_k)|_H^2, \quad k \geq 1. \tag{2.6}$$

Since C is Hilbert-Schmidt, (2.6) implies that the series $\sum_{k=1}^{\infty} |T_t(C(f_k)) - C(f_k)|_H^2$ converges uniformly w.r.t. t. Therefore, from (2.5), (2.6) and dominated convergence, it follows that

$$\begin{aligned}\lim_{t\to 0+} \|\tilde{T}_t(C) - C\|_{L_2(K,H)}^2 &= \lim_{t\to 0+} \sum_{k=1}^{\infty} |T_t(C(f_k)) - C(f_k)|_H^2 \\ &= \sum_{k=1}^{\infty} \lim_{t\to 0+} |T_t(C(f_k)) - C(f_k)|_H^2 = 0. \end{aligned} \tag{2.7}$$

Therefore, (2.4) holds and $\tilde{T}_t, t \geq 0$, is strongly continuous.

We next prove assertion (ii) of the lemma. Let $-\tilde{A} : \mathcal{D}(\tilde{A}) \subset L_2(K,H) \to L_2(K,H)$ be the infinitesimal generator of $\tilde{T}_t, t \geq 0$. We begin with a proof of the inclusion

$$\{C : C \in L_2(K,H),\, C(K) \subseteq \mathcal{D}(A), A \circ C \in L_2(K,H)\} \subseteq \mathcal{D}(\tilde{A}). \tag{2.8}$$

Let $C \in L_2(K,H)$ be such that $C(K) \subseteq \mathcal{D}(A)$ and $A \circ C \in L_2(K,H)$. We will show that

$$\lim_{t\to 0+} \frac{\tilde{T}_t(C) - C}{t} = A \circ C \tag{2.9}$$

in $L_2(K,H)$. To prove (2.9), note first that

$$\begin{aligned}\sup_{0\leq t\leq a} \frac{1}{t}|T_t(C(f_k)) - C(f_k)|_H &= \sup_{0\leq t\leq a} \frac{1}{t}\left|\int_0^t T_s(A(C(f_k)))\, ds\right|_H \\ &\leq \sup_{0\leq t\leq a} \|T_s\|_{L(H)}|A(C(f_k))|_H \end{aligned} \tag{2.10}$$

because $C(f_k) \in \mathcal{D}(A)$ for every $k \geq 1$. Since

$$\|A \circ C\|_{L_2(K,H)} = \sum_{k=1}^{\infty} |A(C(f_k))|_H^2 < \infty, \tag{2.11}$$

it follows from (2.10), (2.11) and dominated convergence that

$$\begin{aligned}\limsup_{t\to 0+}\left\|\frac{\tilde{T}_t(C)-C}{t}-A\circ C\right\|^2_{L_2(K,H)} &= \limsup_{t\to 0+}\sum_{k=1}^{\infty}\left|\frac{T_t(C(f_k))-C(f_k)}{t}-A(C(f_k))\right|^2_H \\ &\le \sum_{k=1}^{\infty}\limsup_{t\to 0+}\left|\frac{T_t(C(f_k))-C(f_k)}{t}-A(C(f_k))\right|^2_H \\ &=0. \end{aligned} \tag{2.12}$$

This proves (2.9). In particular, $C\in\mathcal{D}(\tilde{A})$ and $\tilde{A}(C)=A\circ C$.

It remains to prove the inclusion

$$\mathcal{D}(\tilde{A})\subseteq\{C: C\in L_2(K,H),\, C(K)\subseteq\mathcal{D}(A), A\circ C\in L_2(K,H)\}. \tag{2.13}$$

Suppose $C\in\mathcal{D}(\tilde{A})$. We will show that $C(K)\subseteq\mathcal{D}(A)$, $A\circ C\in L_2(K,H)$ and $\tilde{A}(C)=A\circ C$. Since

$$\lim_{t\to 0+}\left\|\frac{\tilde{T}_t(C)-C}{t}-\tilde{A}(C)\right\|^2_{L_2(K,H)} = \lim_{t\to 0+}\sum_{k=1}^{\infty}\left|\frac{T_t(C(f_k))-C(f_k)}{t}-\tilde{A}(C)(f_k)\right|^2_H = 0. \tag{2.14}$$

we have that

$$\lim_{t\to 0+}\left|\frac{T_t(C(f_k))-C(f_k)}{t}-\tilde{A}(C)(f_k)\right|^2_H = 0 \tag{2.15}$$

for every $k\ge 1$. Therefore, $C(f_k)\in\mathcal{D}(A)$ and $\tilde{A}(C)(f_k)=A(C(f_k))$ for each $k\ge 1$. Now pick any $f\in K$ and write

$$f^n := \sum_{k=1}^{n} < f, f_k > f_k, \quad n\ge 1.$$

Then $C(f^n)=\sum_{k=1}^{n} < f, f_k > C(f_k)\in\mathcal{D}(A)$, $n\ge 1$, and $C(f)=\lim_{n\to\infty}C(f^n)$ in H. Now since $\tilde{A}(C)\in L_2(K,H)\subseteq L(K,H)$, it follows that

$$\begin{aligned}\tilde{A}(C)(f) &= \lim_{n\to\infty}\tilde{A}(C)(f^n) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} < f, f_k > \tilde{A}(C)(f_k) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} < f, f_k > A(C(f_k)) \\ &= \lim_{n\to\infty} A(C(f^n)).\end{aligned}$$

Since A is a closed operator, the above relation implies that $C(f) \in \mathcal{D}(A)$ and $A(C(f)) = \tilde{A}(C)(f)$. As $\tilde{A}(C) \in L_2(K, H)$, and $f \in K$ is arbitrary, it follows that $C(K) \subseteq \mathcal{D}(A)$, $A \circ C \in L_2(K, H)$ and $\tilde{A}(C) = A \circ C$. This proves (2.13) and completes the proof of the lemma. $\square$

Our main results in this section give regular versions $u : \mathbf{R}^+ \times H \times \Omega \to H$ of mild solutions of (2.1) such that $u(t, \cdot, \omega) \in L(H)$ for all $(t, \omega) \in \mathbf{R}^+ \times \Omega$ (Theorems 2.1, 2.2). These regular versions are shown to be $L(H)$-valued cocycles with respect to the Brownian shift θ (Theorem 2.4). In order to formulate these regularity results, we will require the following lemma:

Lemma 2.2.

Let $B : H \to L_2(K, H)$ be continuous linear, and $v : \mathbf{R}^+ \times \Omega \to L_2(H)$ be a $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted process such that $\int_0^a E\|v(t)\|^2_{L_2(H)}\, dt < \infty$ for each $a > 0$. Then the random field $\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$, $x \in H$, $t \geq 0$, admits a jointly measurable version which will be denoted by $\int_0^t T_{t-s} B v(s)\, dW(s)$ (by abuse of notation) and has the following properties:

(i) $\left[\int_0^t T_{t-s} B v(s)\, dW(s)\right](x) = \int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$ *for all $x \in H, t \geq 0$, a.s..*

(ii) For a.a. $\omega \in \Omega$ and each $t \geq 0$, the map

$$H \ni x \mapsto \left[(\omega) \int_0^t T_{t-s} B v(s)\, dW(s)\right](x) \in H$$

is Hilbert-Schmidt.

Proof.

To prove the lemma, we will define $\int_0^t T_{t-s} B v(s)\, dW(s)$ as an Itô stochastic integral with values in the Hilbert space $L_2(H)$ in the sense of [D-Z], Chapter 4). To do this, we will introduce the following notation.

For any $V \in L_2(H)$ and $B \in L(H, L_2(K,H)$, define the linear map $B \star V : K \to L_2(H)$ by

$$(B \star V)(f)(x) := B(V(x))(f), \quad f \in K,\, x \in H. \tag{2.16}$$

Then $B \star V \in L_2(K, L_2(H))$ because of the following computation

$$\begin{aligned}
\|B \star V\|^2_{L_2(K,L_2(H))} &= \sum_{k=1}^{\infty} \|(B \star V)(f_k)\|^2_{L_2(H)} \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |(B \star V)(f_k)(e_n)|^2_H \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |B(V(e_n))(f_k)|^2 \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |B(V(e_n))(f_k)|^2 \\
&= \sum_{n=1}^{\infty} \|B(V(e_n))\|^2_{L_2(K,H)} \\
&\le \|B\|^2_{L(H,L_2(K,H))}\|V\|^2_{L_2(H)} < \infty.
\end{aligned}$$

Now let $v : \mathbf{R}^+ \times \Omega \to L_2(H)$ be as in the lemma. Denote by $\tilde{\tilde{T}}_t : L_2(K, L_2(H)) \to L_2(K, L_2(H))$ the induced lifting of $T_t : H \to H,\ t \ge 0$, via Lemma 2.1; i.e.

$$\tilde{\tilde{T}}_t(C)(f) := T_t \circ C(f), \quad C \in L_2(K, L_2(H)),\ f \in K.$$

Fix $t \in [0, a]$. Then the process $[0, t] \ni s \mapsto \tilde{\tilde{T}}_{t-s}(B \star v(s)) \in L_2(K, L_2(H))$ is $(\mathcal{F}_s)_{0\le s\le t}$-adapted and square-integrable, viz.

$$\begin{aligned}
&E\int_0^t \|\tilde{\tilde{T}}_{t-s}(B\star v(s))\|^2_{L_2(K,L_2(H))}\, ds \\
&\qquad \le \|B\|^2_{L(H,L_2(K,H))} \sup_{0\le u\le t} \|T_u\|^2_{L(H)} \int_0^t E\|v(s)\|^2_{L_2(H)}\, ds < \infty.
\end{aligned}$$

In view of this, the $L_2(H)$-valued Itô stochastic integral $\int_0^t \tilde{\tilde{T}}_{t-s}(B \star v(s))\, dW(s)$ is well-defined ([D-Z], Chapter 4). For simplicity of notation, we will denote this stochastic integral by

$$\int_0^t T_{t-s}Bv(s)\, dW(s) := \int_0^t \tilde{\tilde{T}}_{t-s}(B \star v(s))\, dW(s). \tag{2.17}$$

This gives the required version of the the random field $\int_0^t \tilde{T}_{t-s}(\{[B\circ v(s)](x)\})\,dW(s),\ x\in H$, $t\geq 0$, because

$$\begin{aligned}\left[\int_0^t T_{t-s}Bv(s)\,dW(s)\right](x) &:= \left[\int_0^t \tilde{\tilde{T}}_{t-s}(B\star v(s))\,dW(s)\right](x)\\ &=\sum_{k=1}^{\infty}\int_0^t\left[\tilde{\tilde{T}}_{t-s}(B\star v(s))(f_k)\right](x)\,dW^k(s)\\ &=\sum_{k=1}^{\infty}\int_0^t T_{t-s}\{B\star v(s))(f_k)(x)\}\,dW^k(s)\\ &=\sum_{k=1}^{\infty}\int_0^t T_{t-s}\{B(v(s)(x))(f_k)\}\,dW^k(s)\\ &=\sum_{k=1}^{\infty}\int_0^t \tilde{T}_{t-s}\{[B\circ v(s)](x)\}(f_k)\,dW^k(s)\\ &=\int_0^t \tilde{T}_{t-s}\{[B\circ v(s)](x)\}\,dW(s)\end{aligned}$$

for all $x\in H$ and $t\geq 0$ a.s.. In the above computation, we have used the fact that for fixed $x\in H$, the Itô stochastic integral commutes with the continuous linear evaluation map $L_2(H)\ni T\mapsto T(x)\in H$. $\square$

Theorem 2.1.

Assume that for some $\alpha\in(0,1)$, $A^{-\alpha}$ is trace-class, i.e., $\sum_{n=1}^{\infty}\mu_n^{-\alpha}<\infty$. Then the mild solution of the linear stochastic evolution equation (2.1) has a $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{B}(H)\otimes\mathcal{F},\mathcal{B}(H))$-measurable version $u:\mathbf{R}^+\times H\times\Omega\to H$ with the following properties:

(i) For each $x\in H$, the process $u(\cdot,x,\cdot):\mathbf{R}^+\times\Omega\to H$ is $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{F},\mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted and satisfies the stochastic integral equation (2.2).

(ii) For almost all $\omega\in\Omega$, the map $[0,\infty)\times H\ni(t,x)\to u(t,x,\omega)\in H$ is jointly continuous. Furthermore, for any fixed $a\in\mathbf{R}^+$,

$$E\sup_{0\leq t\leq a}\|u(t,\cdot,\cdot)\|_{L(H)}^{2p}<\infty,$$

whenever $p \in (1, \alpha^{-1}]$.

(iii) For each $t > 0$ *and almost all* $\omega \in \Omega$, $u(t, \cdot, \omega) : H \to H$ *is a Hilbert-Schmidt operator with the following representation:*

$$u(t,\cdot,\cdot) = T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n}\, dW(s_n) \cdots dW(s_2)\, dW(s_1). \tag{2.18}$$

In the above equation, the iterated Itô stochastic integrals are interpreted in the sense of Lemma 2.2, and the convergence of the series holds in the Hilbert space $L_2(H)$ *of Hilbert-Schmidt operators on* H.

(iv) For almost all $\omega \in \Omega$, *the path* $[0, \infty) \ni t \mapsto u(t, \cdot, \omega) - T_t \in L_2(H)$ *is continuous. In particular, the path* $(0, \infty) \ni t \mapsto u(t, \cdot, \omega) \in L_2(H)$ *is continuous for a.a.* $\omega \in \Omega$. *Furthermore, the process* $u : (0, \infty) \times \Omega \to L_2(H)$ *is* $(\mathcal{F}_t)_{t\geq 0}$*-adapted and* $(\mathcal{B}((0, \infty)) \otimes \mathcal{F}, \mathcal{B}(L_2(H)))$*-measurable.*

Proof.

Under the hypotheses on A, it is well known that the see (2.1) has a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted mild solution u satisfying the integral equation (2.2) in H. Moreover, (2.2) and a simple application of the Itô isometry together with Gronwall's lemma implies that

$$\sup_{\substack{0\leq t\leq a\\ |x|\leq 1}} E[|u(t,x,\cdot)|^2] < \infty$$

for each $a \in (0, \infty)$.

Applying (2.2) recursively, we obtain by induction

$$\begin{aligned} &u(t,x,\cdot)\\ =&T_t x + \sum_{k=1}^{n} \Big[\int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{k-1}} T_{s_{k-1}-s_k} B T_{s_k}\, dW(s_k) \cdots dW(s_2)\, dW(s_1) \Big] x\\ &+ \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_n} T_{s_n-s_{n+1}} B u(s_{n+1},x,\cdot)\, dW(s_{n+1}) \cdots dW(s_2)\, dW(s_1). \end{aligned}$$

Set $C_t := \sup_{0\le s\le t} ||T_sB||^2_{L_2(K,H)}$ for each $t>0$. Therefore,

$$
\begin{aligned}
&E[|\int_0^t T_{t-s_1}B\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_n} T_{s_n-s_{n+1}}Bu(s_{n+1},x,\cdot)dW(s_{n+1})\cdots dW(s_2)\,dW(s_1)|^2]\\
&=\int_0^t ds_1E[||T_{t-s_1}B\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_n} T_{s_n-s_{n+1}}Bu(s_{n+1},x,\cdot)dW(s_{n+1})\cdots dW(s_2)||^2_{L_2(K,H)}]\\
&\le C_t\int_0^t ds_1E[|\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_n} T_{s_n-s_{n+1}}Bu(s_{n+1},x,\cdot)dW(s_{n+1})\cdots dW(s_2)|^2]\\
&\le\cdots..\\
&\le C_t^n\int_0^t ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_n} E[|u(s_{n+1},x,\cdot)|^2]ds_{n+1}\le C_t^nM\frac{t^n}{n!}\to 0.
\end{aligned}
$$

This gives the following series representation of $u(t,x,\cdot)$:

$$
\begin{aligned}
u(t,x,\cdot)=T_tx+\sum_{n=1}^{\infty}\Big[\int_0^t T_{t-s_1}B\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_{n-1}}\\
T_{s_{n-1}-s_n}BT_{s_n}\,dW(s_n)\cdots dW(s_2)\,dW(s_1)\Big]x
\end{aligned}
\tag{2.19}
$$

for each $x\in H$. The above series of iterated Itô stochastic integrals converges in $L^2(\Omega,H)$ uniformly in compacta in t and for x in bounded sets in H.

Using the fact that A^{-1} is trace class, we will show further that the series expansion (2.18) actually holds in the Hilbert space $L^2(\Omega,L_2(H))$. To see this, first observe that T_t and all the terms in the series on the right hand side of (2.18) are Hilbert-Schmidt for any fixed $t>0$. We use the comparison test to conclude that the series on the right hand side of (2.18) converges (absolutely) in $L^2(\Omega,L_2(H))$. Fix $a>0$. Then by successive applications of the Itô isometry (in $L_2(H)$), one gets

$$
\begin{aligned}
&E\|\int_0^t T_{t-s_1}B\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_n} T_{s_n-s_{n+1}}BT_{s_{n+1}}dW(s_{n+1})\cdots dW(s_2)\,dW(s_1)\|^2_{L_2(H)}\\
&\le 4\int_0^a ds_1E[\|T_{t-s_1}B\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_n}\\
&\qquad\qquad T_{s_n-s_{n+1}}BT_{s_{n+1}}dW(s_{n+1})\cdots dW(s_2)\|^2_{L_2(K,L_2(H))}]\\
&\le C'_a\int_0^a ds_1E[\|\int_0^{s_1} T_{s_1-s_2}B\cdots\int_0^{s_n} T_{s_n-s_{n+1}}BT_{s_{n+1}}dW(s_{n+1})\cdots dW(s_2)\|^2_{L_2(H)}]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \cdots \\
&\leq C_a'^n \int_0^a ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_n} E[\|BT_{s_{n+1}}\|^2_{L_2(K,L_2(H))}] ds_{n+1} \\
&\leq C_a'^n \frac{a^n}{n!} \int_0^a \|T_s\|^2_{L_2(H)}\, ds = C_a'^n \frac{a^n}{n!} \int_0^a \sum_{k=1}^{\infty} e^{-2\mu_k s}\, ds \\
&\leq C_a'^n \frac{a^n}{n!} \sum_{k=1}^{\infty} \frac{1}{2\mu_k},
\end{aligned}
\tag{2.20}
$$

for each integer $n \geq 1$, where $C_a' := 4C_a$. This implies that the expansion (2.18) converges in $L^2(\Omega, L_2(H))$ for each $t > 0$. Hence assertion (iii) of the theorem holds.

We next prove assertion (iv). Consider the series in (2.18) and let $\Phi^n(t) \in L_2(H)$ be its general term, viz.

$$
\Phi^n(t) := \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n}\, dW(s_n) \cdots dW(s_2)\, dW(s_1),
$$

for $t \geq 0, n \geq 1$. Note the relations

$$
\left.
\begin{aligned}
\Phi^n(t) &= \int_0^t T_{t-s_1} B \Phi^{n-1}(s_1)\, dW(s_1), \quad n \geq 2, \\
\Phi^1(t) &= \int_0^t T_{t-s_1} B T_{s_1}\, dW(s_1),
\end{aligned}
\right\}
\tag{2.21}
$$

for $t \geq 0$.

First, we show by induction that for each $n \geq 1$, the process $\Phi^n : [0,\infty) \times \Omega \to L_2(H)$ has a version with a.a. sample paths continuous on $[0,\infty)$. In view of (2.21), this will follow from Proposition (7.3) ([D-Z], p. 184) provided we show that

$$
\int_0^a E\|\Phi^{n-1}(t)\|^{2p}_{L_2(H)}\, dt < \infty \tag{2.22}
$$

for all integers $n > 1$ and $p \in (1, \alpha^{-1}]$. For later use, we will actually prove the stronger estimate

$$
E \sup_{0 \leq s \leq t} \|\Phi^n(s)\|^{2p}_{L_2(H)} \leq K_1 \frac{(K_2 t)^{n-1}}{(n-1)!}, \quad t \in [0,a], \tag{2.22$'$}
$$

for all integers $n \geq 1$, and $p \in (1, \alpha^{-1}]$, where K_1, K_2 are positive constants depending only on p and a. We use induction on n to establish (2.22′). To check (2.22′) for $n = 1$, choose $p \in (1, \alpha^{-1}]$, and consider the following easy estimates:

$$\begin{aligned}\left\{\int_0^a \|T_s\|_{L_2(H)}^{2p}\, ds\right\}^{1/p} &= \left\{\int_0^a \left[\sum_{k=1}^{\infty} e^{-2\mu_k s}\right]^p ds\right\}^{1/p} \\ &\leq \sum_{k=1}^{\infty}\left\{\int_0^a e^{-2\mu_k ps}\, ds\right\}^{1/p} \\ &\leq \frac{1}{(2p)^{1/p}} \sum_{k=1}^{\infty} \mu_k^{-1/p} \\ &\leq \frac{1}{(2p)^{1/p}} \sum_{k=1}^{\infty} \mu_k^{-\alpha} < \infty.\end{aligned}$$

Now use the second equality in (2.21) and Proposition (7.3) ([D-Z], p. 184) to get the following estimate:

$$E \sup_{0\leq s\leq t} \|\Phi^1(s)\|_{L_2(H)}^{2p} \leq C_1 \int_0^t \|T_{s_1}\|_{L_2(H)}^{2p}\, ds_1 \leq \frac{C_1}{2p}\left[\sum_{k=1}^{\infty} \mu_k^{-\alpha}\right]^p \tag{2.23}$$

for all $t \in [0, a]$ and for $p \in (1, \alpha^{-1}]$. The constant C_1 does not depend on $t \in [0, a]$. Since $A^{-\alpha}$ is trace-class, the above inequality implies that (2.22′) holds with $K_1 := \frac{C_1}{2p}\left[\sum_{k=1}^{\infty} \mu_k^{-\alpha}\right]^p$, for all $n \geq 1$, and any $p \in (1, \alpha^{-1}]$. Now suppose that (2.22′) holds for some integer $n \geq 1$ and all $p \in (1, \alpha^{-1}]$. Then the first equality in (2.21) and Proposition (7.3) ([D-Z], p. 184) imply that there is a positive constant $K_2 := K_2(p, a)$ such that

$$\begin{aligned}E \sup_{0\leq s\leq t} \|\Phi^{n+1}(s)\|_{L_2(H)}^{2p} &\leq K_2 \int_0^t E\|\Phi^n(s_1)\|_{L_2(H)}^{2p}\, ds_1 \\ &\leq K_2 \int_0^t K_1 \frac{(K_2 s_1)^{n-1}}{(n-1)!}\, ds_1 = K_1 \frac{(K_2 t)^n}{n!},\end{aligned} \tag{2.24}$$

for all $t \in [0, a]$ and $p \in (1, \alpha^{-1}]$. Therefore by induction, (2.22′) (and hence (2.22)) holds for all integers $n \geq 1$ and any $p \in (1, \alpha^{-1}]$.

From the first equality in (2.21), (2.22) and Proposition 7.3 ([D-Z], p. 184), it follows that each $\Phi^n : [0,\infty) \times \Omega \to L_2(H)$ has a version with a.a. sample paths continuous on $[0,\infty)$. From the estimate (2.22′), it is easy to see that the series $\sum_{n=1}^{\infty} \Phi^n$ converges absolutely in $L^{2p}(\Omega, C([0,a], L_2(H)))$ for each $a > 0$ and $p \in (1, \alpha^{-1}]$. This gives a continuous modification for the sum $\sum_{n=1}^{\infty} \Phi^n$ of the series in (2.18). Hence the $L_2(H)$-valued process

$$u(t,\cdot,\cdot) - T_t = \sum_{n=1}^{\infty} \Phi^n(t), \quad t \geq 0,$$

has a version with almost all sample-paths continuous on $[0,\infty)$. This proves the first assertion in (iv). To prove the second assertion in (iv), it suffices to show that the mapping $(0,\infty) \ni t \mapsto T_t \in L_2(H)$ is locally Lipschitz. To see this, let $0 < t_1 < t_2 \leq a < \infty$. Then

$$\begin{aligned} \|T_{t_2} - T_{t_1}\|^2_{L_2(H)} &\leq \sum_{k=1}^{\infty} [e^{-\mu_k t_2} - e^{-\mu_k t_1}]^2 \\ &\leq (t_2 - t_1)^2 \sum_{k=1}^{\infty} \mu_k^2 e^{-\mu_k t_1} \\ &\leq \frac{3}{4t_1}(t_2 - t_1)^2 \sum_{k=1}^{\infty} \mu_k^{-1}. \end{aligned}$$

Since A^{-1} is trace-class, the above inequality implies that the mapping $(0,\infty) \ni t \mapsto T_t \in L_2(H)$ is locally Lipschitz. The second assertion in (iv) now follows immediately from this and the first assertion.

The measurability assertions in (iv) follow directly from the relation

$$u(t,\cdot,\cdot) = T_t + \sum_{n=1}^{\infty} \Phi^n(t), \quad t > 0,$$

and the fact that, as $L_2(H)$-valued Itô stochastic integrals, the processes $\Phi^n : (0,\infty) \times \Omega \to L_2(H)$, $n \geq 1$, are $(\mathcal{F}_t)_{t>0}$-adapted and $(\mathcal{B}((0,\infty)) \otimes \mathcal{F}, \mathcal{B}(L_2(H)))$-measurable.

The evaluation map

$$L_2(H) \times H \to H$$

$$(S, x) \mapsto S(x)$$

is continuous bilinear. Therefore the first assertion in (iv) implies that the map $[0,T]\times H \ni (t,x) \to u(t,x,\omega) - T_t(x) \in H$ is jointly continuous for almost all $\omega \in \Omega$. Since $[0,T]\times H \ni (t,x) \to T_t(x) \in H$ is jointly continuous (by strong continuity of the semigroup $T_t, t \geq 0$), the first assertion in (ii) follows.

Finally, it remains to prove the estimate in (ii). In view of (2.22′), the series in (2.18) converges absolutely in $L^{2p}(\Omega, C([0,a], L_2(H)))$, $p \in (1, \alpha^{-1}]$. Therefore,

$$\begin{aligned}\Big\{E \sup_{0\leq t\leq a} \|u(t,\cdot,\cdot)\|^{2p}_{L(H)}\Big\}^{1/(2p)} &\leq \sup_{0\leq t\leq a} \|T_t\|_{L(H)} + \sum_{n=1}^{\infty} \Big\{E \sup_{0\leq t\leq a} \|\Phi^n(t)\|^{2p}_{L_2(H)}\Big\}^{1/(2p)} \\ &\leq \sup_{0\leq t\leq a} \|T_t\|_{L(H)} + K_1^{1/(2p)} \sum_{k=1}^{\infty} \left\{ \frac{(K_2 a)^{n-1}}{(n-1)!} \right\}^{1/(2p)} < \infty.\end{aligned}$$

This proves the estimate in (ii), and the proof of Theorem 2.1 is complete. □

Theorem 2.2.

Assume the following:

(i) A^{-1} *is a trace class operator, i.e.,* $\sum_{n=1}^{\infty} \mu_n^{-1} < \infty$.

(ii) $T_t \in L(H)$, $t \geq 0$, *is a strongly continuous contraction semigroup.*

Then the mild solution of the linear stochastic evolution equation (2.1) has a version $u : \mathbf{R}^+ \times H \times \Omega \to H$ *which satisfies the assertions (i), (iii) and (iv) of Theorem 2.1. Furthermore, for almost all* $\omega \in \Omega$, *the map* $[0,\infty) \times H \ni (t,x) \to u(t,x,\omega) \in H$ *is jointly continuous, and for any fixed* $a \in \mathbf{R}^+$,

$$E \sup_{0\leq t\leq a} \|u(t,\cdot,\cdot)\|^2_{L(H)} < \infty.$$

Proof.

The proof follows that of Theorem 2.1. We will only highlight the differences.

We assume Hypotheses (i) and (ii). By the proof of Theorem 2.1, Hypothesis (i) implies that the solution of (2.1) admits a version $u : \mathbf{R}^+ \times H \times \Omega \to H$ which satisfies assertions (i) and (iii) of Theorem 2.1.

Use the notation in the proof of Theorem 2.1. In particular, one has

$$u(t,\cdot,\cdot) - T_t = \sum_{n=1}^{\infty} \Phi^n(t), \quad t \geq 0,$$

where the series converges in $L^2(\Omega, L_2(H))$ for each $t \geq 0$. Since A^{-1} is trace-class, then $\int_0^a \|T_s\|^2_{L_2(H)}\, ds < \infty$. Using this, the fact that T_t, $t \geq 0$, is a contraction semigroup, and Theorem 6.10 ([D-Z], p. 160), it follows that $\Phi^1 : [0,\infty) \times \Omega \to L_2(H)$ has a sample-continuous version. Furthermore, there is a positive constant K_3 such that

$$E \sup_{0\leq s\leq t} \|\Phi^1(s)\|^2_{L_2(H)} \leq K_3 \int_0^t \|T_s\|^2_{L_2(H)}\, ds < \infty, \tag{2.25}$$

for all $t \in [0,a]$ ([D-Z], Theorem 6.10, p. 160). We will show that the series $\sum_{n=2}^{\infty} \Phi^n$ converges in $L^{2p}(\Omega, C([0,a], L_2(H))$ for all $p \geq 1$. Therefore, the series $\sum_{n=1}^{\infty} \Phi^n$ converges in $L^2(\Omega, C([0,a], L_2(H)))$.

By Lemma (7.2), ([D-Z], p. 182), we have

$$E\|\Phi^1(t)\|^{2p}_{L_2(H)} \leq K_4 \left[\int_0^t \|T_s\|^2_{L_2(H)}\, ds\right]^p \tag{2.26}$$

for all $t \in [0,a]$ and all $p \geq 1$. The constant K_4 depends on p but is independent of $t \in [0,a]$. Since A^{-1} is trace-class, the above inequality, Proposition 7.3 ([D-Z], p. 184) and an induction argument imply the following inequality:

$$E \sup_{0\leq s\leq t} \|\Phi^n(s)\|^{2p}_{L_2(H)} \leq K_5 \frac{(K_6 t)^{n-1}}{(n-1)!}, \quad t \in [0,a],$$

for all integers $n \geq 2$, and $p \geq 1$, where K_5, K_6 are positive constants depending only on p and a (cf. (2.22′) in the proof of Theorem 2.1). The rest of the proof of the theorem

follows from the above inequality by a similar argument to the one in the proof of Theorem 2.1. $\square$

We will continue to assume the hypotheses of Theorem 2.1 or 2.2. Let $u : \mathbf{R}^+ \times \Omega \to L(H)$ be the regular version of the mild solution of (2.1) given by Theorem 2.1 or 2.2. Our next result in this section identifies u as a *fundamental solution* (or *parametrix*) for (2.1).

Consider the following stochastic integral equation:

$$\left.\begin{aligned} v(t) =& T_t + \int_0^t T_{t-s} B v(s) dW(s), \quad t > 0 \\ v(0) =& I, \end{aligned}\right\} \tag{2.27}$$

where I denotes the identity operator on H and the stochastic integral is interpreted as an Itô integral in the Hilbert space $L_2(H)$.

Remark.

The initial-value problem (2.27) cannot be interpreted strictly in the Hilbert space $L_2(H)$ since $v(0) = I \notin L_2(H)$. On the other hand, one cannot view the equation (2.27) in the Banach space $L(H)$, because the latter Banach space is not sufficiently "smooth" to allow for a satisfactory theory of stochastic integration (cf. [B-E]).

We say that a stochastic process $v : [0, \infty) \times \Omega \to L(H)$ is a *solution* to equation (2.27) if

(i) $v : (0, \infty) \times \Omega \to L_2(H)$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted, and $(\mathcal{B}((0, \infty)) \otimes \mathcal{F}, \mathcal{B}(L_2(H)))$-measurable.

(ii) $v \in L^2((0, a) \times \Omega, L_2(H))$ for all $a \in (0, \infty)$.

(iii) v satisfies (2.27) almost surely.

Theorem 2.3.

Assume the hypotheses of Theorem 2.1 or Theorem 2.2, and let u be the regular version of the mild solution of (2.1) given therein. Then u is the unique solution of (2.27) in $L^2((0,a)\times\Omega, L_2(H))$ for $a>0$.

Proof.

Assume the hypotheses of Theorem 2.1 or 2.2. Let u be the regular version of the mild solution of (2.1) given by these theorems.

Note first that $u:(0,\infty)\times\Omega\to L_2(H)$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted, and $(\mathcal{B}((0,\infty))\otimes\mathcal{F},\mathcal{B}(L_2(H)))$-measurable. This follows from assertion (iv) in Theorem 2.1.

In the proofs of Theorems 2.1, 2.2, we have shown that the series $\sum_{n=1}^{\infty}\Phi^n$ converges absolutely in $L^2(\Omega, C([0,a],L_2(H)))$, and hence also in $L^2(\Omega, L^2((0,a),L_2(H)))$, because of the continuous linear imbedding

$$L^2(\Omega, C([0,a],L_2(H)))\subset L^2(\Omega, L^2((0,a),L_2(H)))\equiv L^2((0,a)\times\Omega, L_2(H)).$$

Thus

$$\int_0^a E\|u(t,\cdot,\cdot)\|^2_{L_2(H)}\,dt\leq 2\int_0^a \|T_t\|^2_{L_2(H)}\,dt+2\Big[\sum_{n=1}^{\infty}\Big\{\int_0^a E\|\Phi^n(t)\|^2_{L_2(H)}\,dt\Big\}^{1/2}\Big]^2<\infty.$$

In particular, the Itô stochastic integral $\int_0^t T_{t-s}Bu(s)\,dW(s)$ is well-defined in $L_2(H)$ for each $t\in[0,a]$ (Lemma 2.2).

We next show that u solves the operator-valued stochastic integral equation (2.27). To see this, use the fact that

$$\Big[\int_0^t T_{t-s}Bu(s,\cdot,\cdot)\,dW(s)\Big](e_n)=\int_0^t T_{t-s}Bu(s,e_n,\cdot)\,dW(s),\quad n\geq 1,$$

and the integral equation (2.2) to conclude that

$$u(t,\omega)(e_n)=T_te_n+\Big[(\omega)\int_0^t T_{t-s}Bu(s,\cdot,\cdot)\,dW(s)\Big](e_n),\quad t\geq 0,\ n\geq 1, \tag{2.28}$$

holds for *all* ω in a sure event $\Omega^* \in \mathcal{F}$ which is independent of n and $t \geq 0$. Since $\{e_n : n \geq 1\}$ is a complete orthonormal system in H, it follows from (2.28) that for all $\omega \in \Omega^*$, one has

$$u(t,\omega)(x) = T_t x + (\omega) \int_0^t \big[T_{t-s} B u(s,\cdot,\cdot)\, dW(s)\big](x), \quad t \geq 0,\, n \geq 1 \tag{2.29}$$

for all $x \in H$. Thus u is a solution of (2.27).

Finally we show that (2.27) has a unique $(\mathcal{F}_t)_{t>0}$-adapted solution in $L^2((0,a) \times \Omega, L_2(H))$. Suppose v_1, v_2 are two such solutions of (2.27). Then

$$E\|v_1(t) - v_2(t)\|^2_{L_2(H)} \leq \|B\|_{L_2(K,H)} \sup_{0 \leq u \leq a} \|T_u\|_{L(H)} \int_0^t E\|v_1(s) - v_2(s)\|^2_{L_2(H)}\, ds \tag{2.30}$$

for all $t \in (0,a]$. The above inequality implies that $E\|v_1(t) - v_2(t)\|^2_{L_2(H)} = 0$ for all $t \geq 0$ and uniqueness holds. $\square$

From now on and throughout this section, we will impose the following

Condition (B):

(i) The operator $B : H \to L_2(K,H)$ can be extended to a bounded linear operator $H \to L(E,H)$, which will also be denoted by B.

(ii) The series $\sum_{k=1}^{\infty} ||B_k^2||_{L(H)}$ converges, where the bounded linear operators $B_k : H \to H$ are defined by $B_k(x) := B(x)(f_k), x \in H$, $k \geq 1$.

Theorem 2.4.

Suppose the hypotheses of Theorems 2.1 or 2.2, and Condition (B) are satisfied. Then the mild solution of (2.1) admits a version $u : \mathbf{R}^+ \times \Omega \to L(H)$ satisfying Theorems 2.1 or 2.2 and is such that

(i) (u,θ) is a perfect $L(H)$-valued cocycle:

$$u(t+s,\omega) = u(t,\theta(s,\omega)) \circ u(s,\omega) \tag{2.31}$$

for all $s, t \geq 0$ *and all* $\omega \in \Omega$*;*

(ii) $\sup_{0 \leq s \leq t \leq a} \|u(t-s, \theta(s,\omega))\|_{L(H)} < \infty$, *for all* $\omega \in \Omega$ *and all* $a > 0$.

Proof.

In view of Theorem 2.3, u satisfies the stochastic integral equation

$$\left.\begin{aligned} u(t) =& T_t + \int_0^t T_{t-s} B u(s)\, dW(s), \quad t > 0 \\ u(0) =& I \end{aligned}\right\} \tag{2.32}$$

with $u(t) \in L_2(H)$ a.s. for all $t > 0$.

Our strategy for proving the cocycle property (2.31) is to approximate the cylindrical Wiener process W in (2.32) by a suitably defined family of smooth processes $W_n : \mathbf{R}^+ \times \Omega \to E$, $n \geq 1$, prove the cocycle property for the corresponding approximating solutions and then pass to the limit in $L_2(H)$ as n tends to ∞.

Define W_n on $\mathbf{R}^+ \times \Omega$, $n \geq 1$, by

$$W_n(t,\omega) := n \int_{t-1/n}^{t} W(u,\omega)\, du - n \int_{-1/n}^{0} W(u,\omega)\, du, \quad t \geq 0,\ \omega \in \Omega. \tag{2.33}$$

It is easy to see that each W_n is a helix:

$$W_n(t, \theta(t_1,\omega)) = W_n(t+t_1,\omega) - W_n(t_1,\omega), \tag{3.34}$$

and

$$W_n'(t, \theta(t_1,\omega)) = W_n'(t+t_1,\omega) \tag{2.35}$$

for all $t, t_1 \geq 0$, $\omega \in \Omega$, $n \geq 1$. In (2.35), the prime $'$ denotes differentiation with respect to t.

For each $k \geq 1$, recall the definition of $B_k : H \to H$ in Condition (B)(ii). For each integer $n \geq 1$, define the process $u_n : \mathbf{R}^+ \times \Omega \to L_2(H)$ to be the unique $(\mathcal{B}((0,\infty)) \otimes$

$\mathcal{F}, \mathcal{B}(L_2(H)))$-measurable, $(\mathcal{F}_t)_{t>0}$-adapted solution of the random integral equation:

$$\left.\begin{aligned} u_n(t,\omega) =& T_t + \int_0^t T_{t-s} \circ \{[B \star u_n(s,\omega)](W_n'(s,\omega))\}\, ds \\ & - \frac{1}{2}\int_0^t \sum_{k=1}^{\infty} T_{t-s} \circ B_k^2 \circ u_n(s,\omega)\, ds, \quad t>0 \\ u_n(0,\omega) =& I, \end{aligned}\right\} \tag{2.27)(n}$$

for $\omega \in \Omega$. Recall that the operation $\star$ is defined by (2.16) in the proof of Lemma 2.2.

Then

$$\lim_{n\to\infty} \sup_{0<t\le a} \|u_n(t) - u(t)\|^2_{L_2(H)} = 0, \tag{2.36}$$

in probability, for each $a > 0$. The convergence (2.36) follows by modifying the proof (in $L_2(H)$) of the Wong-Zakai approximation theorem for stochastic evolution equations in ([Tw], Theorem 3.4.1). (Cf. [I-W], Theorem 7.2, p. 497).

Next, we show that for each $n \ge 1$, (u_n, θ) is a perfect cocycle. Fix $n \ge 1$, $t_1 \ge 0$ and $\omega \in \Omega$. Using (2.27)(n), it follows that

$$\begin{aligned} & u_n(t,\theta(t_1,\omega)) \circ u_n(t_1,\omega) \\ & =T_t \circ u_n(t_1,\omega) \\ & \quad + \int_{t_1}^{t+t_1} T_{t+t_1-s} \circ \{[B \star (u_n(s-t_1,\theta(t_1,\omega)) \circ u_n(t_1,\omega))](W_n'(s-t_1,\theta(t_1,\omega)))\}\, ds \\ & \quad - \frac{1}{2}\int_{t_1}^{t+t_1} \sum_{k=1}^{\infty} T_{t+t_1-s} \circ B_k^2 \circ (u_n(s-t_1,\theta(t_1,\omega)) \circ u_n(t_1,\omega))\, ds \\ & =T_{t+t_1} + \int_0^{t_1} T_{t+t_1-s} \circ \{[B \star u_n(s,\omega)](W_n'(s,\omega))\}\, ds \\ & \quad - \frac{1}{2}\int_0^{t_1} \sum_{k=1}^{\infty} T_{t+t_1-s} \circ B_k^2 \circ u_n(s,\omega)\, ds \\ & \quad + \int_{t_1}^{t+t_1} T_{t+t_1-s} \circ \{[B \star (u_n(s-t_1,\theta(t_1,\omega)) \circ u_n(t_1,\omega))](W_n'(s-t_1,\theta(t_1,\omega)))\}\, ds \\ & \quad - \frac{1}{2}\int_{t_1}^{t+t_1} \sum_{k=1}^{\infty} T_{t+t_1-s} \circ B_k^2 \circ (u_n(s-t_1,\theta(t_1,\omega)) \circ u_n(t_1,\omega))\, ds, \quad t>0. \end{aligned}$$

Hence, using (2.27)(n) and (2.35), we obtain

$$\begin{aligned}
&u_n(t,\theta(t_1,\omega))\circ u_n(t_1,\omega)-u_n(t_1+t,\omega)\\
&\quad=\int_{t_1}^{t+t_1} T_{t+t_1-s}\circ\{[B\star(u_n(s-t_1,\theta(t_1,\omega))\circ u_n(t_1,\omega)-u_n(s,\omega))]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (W_n'(s-t_1,\theta(t_1,\omega)))\}\,ds\\
&\qquad-\frac{1}{2}\int_{t_1}^{t+t_1}\sum_{k=1}^{\infty}T_{t+t_1-s}\circ B_k^2\circ[u_n(s-t_1,\theta(t_1,\omega))\circ u_n(t_1,\omega)-u_n(s,\omega)]\,ds\\
&\quad=\int_0^t T_{t-s}\circ\{[B\star(u_n(s,\theta(t_1,\omega))\circ u_n(t_1,\omega)-u_n(s+t_1,\omega))](W_n'(s,\omega))\}\,ds\\
&\qquad-\frac{1}{2}\int_0^t\sum_{k=1}^{\infty}T_{t-s}\circ B_k^2\circ[u_n(s,\theta(t_1,\omega))\circ u_n(t_1,\omega)-u_n(s+t_1,\omega)]\,ds,
\end{aligned}$$

for all $t>0$. The above identity and a simple application of Gronwall's lemma yields

$$u_n(t,\theta(t_1,\omega))\circ u_n(t_1,\omega)-u_n(t_1+t,\omega)=0 \tag{2.37}$$

for all $t,t_1\geq 0$ and all $\omega\in\Omega$. Hence (u_n,θ) is a perfect cocycle in $L(H)$. Using (2.36) and passing to the limit in $L(H)$ as $n\to\infty$ in the above identity implies that (u,θ) is a crude $L(H)$-valued cocycle. In order to obtain a perfect version of this cocycle, it is sufficient to prove that there is a sure event $\Omega^*\in\mathcal{F}$ (independent of $t_1\in\mathbf{R}^+$) such that $\theta(t,\cdot)(\Omega^*)\subseteq\Omega^*$ for all $t\geq 0$, and there is a subsequence $u_{n'}$ of u_n such that

$$\lim_{n',m'\to\infty}\sup_{0<t\leq a}\|u_{n'}(t,\omega)-u_{m'}(t,\omega)\|^2_{L_2(H)}=0, \tag{2.38}$$

for each $a>0$ and all $\omega\in\Omega^*$. Set $v_n(t_1,t,\omega):=u_n(t-t_1,\theta(t_1,\omega))$, $t\geq t_1\geq 0$. Then v_n solves the integral equation

$$\begin{aligned}
v_n(t_1,t,\omega)=&T_{t-t_1}+\int_{t_1}^t T_{t-s}\circ\{[B\star v_n(t_1,s,\omega)](W_n'(s,\omega))\}\,ds\\
&-\frac{1}{2}\int_{t_1}^t\sum_{k=1}^{\infty}T_{t-s}\circ B_k^2\circ v_n(t_1,s,\omega)\,ds,\\
v_n(t_1,t_1,\omega)=&I,
\end{aligned}$$

for $t \geq t_1 \geq 0$. The above equation implies that $v_n(t_1, t, \omega)$ is continuous in (t_1, t) for each $\omega \in \Omega$. Furthermore, if we apply the approximation scheme (in $L_2(H)$) to the above integral equation, we get a subsequence $\{v_{n'}\}_{n'=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ such that for a.a. $\omega \in \Omega$

$$\lim_{n',m'\to\infty} \sup_{0<t_1\leq t\leq a} \|v_{n'}(t_1, t, \omega) - v_{m'}(t_1, t, \omega)\|^2_{L_2(H)} = 0, \tag{2.39}$$

for each $a > 0$. Now define Ω^* to be the set of all $\omega \in \Omega$ such that the subsequence $\{v_{n'}(t_1, t, \omega) : n' \geq 1\}$ converges in $L(H)$ uniformly in (t_1, t) for $0 < t_1 \leq t \leq a$ and all $a > 0$. Therefore Ω^* is a $\theta(t, \cdot)$-invariant sure event. Define

$$u(t, \omega) := \lim_{n'\to\infty} v_{n'}(0, t, \omega)$$

for all $t \geq 0$ and all $\omega \in \Omega^*$. Hence (u, θ) is a perfect cocycle in $L(H)$. This proves assertion (i) of the theorem.

To prove the second assertion of the theorem, fix $s \geq 0$ and define $\hat{v}_n(s, t, \omega) := \hat{u}_n(t - s, \theta(s, \omega)) = u_n(t - s, \theta(s, \omega)) - T_{t-s}$, $t \geq s \geq 0$. It is easy to see that $\hat{v}_n$ solves the integral equation

$$\begin{aligned}
\hat{v}_n(s, t, \omega) = &\int_s^t T_{t-\lambda} \circ \{[B \star \hat{v}_n(s, \lambda, \omega)](W'_n(\lambda, \omega))\}\, d\lambda + \int_s^t T_{t-\lambda} B T_{\lambda-s}(W'_n(\lambda, \omega))\, d\lambda \\
&- \frac{1}{2} \int_s^t \sum_{k=1}^{\infty} T_{t-\lambda} \circ B_k^2 \circ \hat{v}_n(s, \lambda, \omega)\, d\lambda, \\
\hat{v}_n(s, s, \omega) =& 0 \in L_2(H),
\end{aligned}$$

for $t \geq s \geq 0$. The above equation implies that the map $\Delta \ni (s, t) \mapsto \hat{v}_n(s, t, \omega) \in L_2(H)$ is continuous for each $\omega \in \Omega$. Applying the approximation scheme again, there is a subsequence $\{\hat{v}_{n'}\}_{n'=1}^{\infty}$ of $\{\hat{v}_n\}_{n=1}^{\infty}$ such that for a.a. $\omega \in \Omega$, one has

$$\lim_{n',m'\to\infty} \sup_{0\leq s\leq t\leq a} \|\hat{v}_{n'}(s, t, \omega) - \hat{v}_{m'}(s, t, \omega)\|^2_{L_2(H)} = 0,$$

for each $a > 0$. Define $\hat{\Omega}^*$ to be the set of all $\omega \in \Omega$ such that the subsequence $\{\hat{v}_{n'}(s,t,\omega) : n' \geq 1\}$ converges in $L_2(H)$ uniformly in (s,t) for $0 \leq s \leq t \leq a$ and all $a > 0$. Therefore $\hat{\Omega}^*$ is a $\theta(t,\cdot)$-invariant sure event. Define

$$\hat{u}(t,\omega) := \lim_{n' \to \infty} \hat{v}_{n'}(0,t,\omega)$$

for all $t \geq 0$ and all $\omega \in \hat{\Omega}^*$. Therefore, the map $\Delta \ni (s,t) \mapsto \hat{u}(t-s,\theta(s,\omega)) \in L_2(H)$ is jointly continuous. In particular, $\sup\limits_{0 \leq s \leq t \leq a} \|\hat{u}(t-s,\cdot,\theta(s,\omega))\|_{L(H)} < \infty$, for all $\omega \in \hat{\Omega}^*$ and all $a > 0$. Using the fact that $\sup\limits_{0 \leq s \leq t \leq a} \|T_{t-s}\|_{L(H)} < \infty$, it follows that $u(t,\omega) := \hat{u}(t,\omega) + T_t$, $\omega \in \Omega^* \cap \hat{\Omega}^*$, gives a version of the cocycle that also satisfies assertion (ii) of the theorem. This completes the proof of the theorem. $\square$

Remarks.

(i) Results analogous to Theorem 2.4 hold if B is replaced by the an affine linear map $B(x) := B_0 + B_1(x),\ x \in H$, where $B_0 \in L(E,H)$ and $B_1 : H \to L(E,H)$ satisfies Condition (B). In this case, one gets a cocycle (u,θ) where each map $u(t,\cdot,\omega) : H \to H$ is of the form $u(t,\cdot,\omega) = u_0(t,\cdot,\omega) + u_1(t,\omega)$ with $u_0(t,\cdot,\omega) \in L_2(H)$ and $u_1(t,\omega) \in H$ for $t > 0, \omega \in \Omega$. This follows using minor modifications of the above arguments.

(ii) It is possible to replace B in the see (2.1) by an adapted random field $B : \mathbf{R}^+ \times H \times \Omega \to L(E,H)$ satisfying appropriate integrability and regularity conditions, which is such that $B(t,\cdot,\omega) : H \to L(E,H)$ satisfies Condition (B) for each $t \geq 0$, $\omega \in \Omega$. The conclusions of Theorems 2.1-2.3 will still hold in this case. However, the stochastic semiflow will only satisfy Definition 1.1 (rather than the cocycle property in Definition 1.2). On the other hand if B is stationary, then the cocycle property should hold (on a suitably enlarged probability space) (Theorem 2.4).

(iii) Theorems 2.1-2.4 also hold if the operator A is allowed to have a non-zero discrete spectrum $\{\mu_n : n \geq 1\}$ which is bounded below. This yields a splitting $A = A_0 + A_1$

where $\sigma(A_0)$ consists of positive eigenvalues and $\sigma(A_1)$ of finitely many negative eigenvalues.

(b) Semilinear stochastic evolution equations:

In this section, we continue to assume that the operators A, B, the cylindrical Brownian motion W, the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and the Brownian shift $\theta : \mathbf{R} \times \Omega \to \Omega$ are as defined in part (a) of this section and satisfy the conditions therein. The semigroup generated by $-A$ is denoted as before by $T_t, t \geq 0$. Furthermore, we let $F : H \to H$ be a (Fréchet) C^1 non-linear map satisfying the following locally Lipschitz and linear growth hypotheses:

$$\left.\begin{aligned} |F(v)| &\leq C(1+|v|), \quad v \in H \\ |F(v_1) - F(v_2)| &\leq L_n|v_1 - v_2|, \quad v_i \in H, |v_i| \leq n, i = 1, 2, \end{aligned}\right\} \tag{2.40}$$

for some positive constants $C, L_n, n \geq 1$.

Consider the semilinear stochastic evolution equation:

$$\left.\begin{aligned} du(t) &= -Au(t)dt + F(u(t))dt + Bu(t)\, dW(t), \quad t > 0, \\ u(0) &= x \in H, \end{aligned}\right\} \tag{2.41}$$

where the operators A, B satisfy the hypotheses of Theorem 2.4.

Our main objective in this section is to establish the existence of a C^k perfect cocycle (U, θ) for the above stochastic evolution equation. First we define a *mild solution* of (2.41) as a $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable random field of $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H$, $x \in H$, satisfying the following family of stochastic integral equations:

$$u(t, x, \cdot) = T_t(x) + \int_0^t T_{t-s}(F(u(s, x, \cdot)))\, ds + \int_0^t T_{t-s}Bu(s, x, \cdot)\, dW(s), \quad t \geq 0, \tag{2.42}$$

a.s. ([D-Z], Chapter 7, p. 182).

To fix notation, denote by $\phi : \mathbf{R}^+ \times \Omega \to L(H)$ the perfect cocycle generated by the linear stochastic evolution equation

$$\left.\begin{aligned} d\phi(t) =& - A\phi(t)dt + B\phi(t)\, dW(t), \quad t > 0, \\ \phi(0) =& I \in L(H), \end{aligned}\right\} \tag{2.43}$$

and obtained via Theorem 2.4. That is, $\phi(t,\omega) := u(t,\cdot,\omega) \in L_2(H), t > 0, \omega \in \Omega$, in the notation of part (a) of this section.

Our first step in the construction of a non-linear cocycle of (2.41) is to observe that mild solutions of (2.41) correspond to solutions of a *random* integral equation on H. This is shown in the following theorem:

Theorem 2.5.

Suppose the hypotheses of Theorem 2.4 are satisfied. Then every $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$*-measurable,* $(\mathcal{F}_t)_{t\geq 0}$*-adapted solution field* $U(t,x,\omega)$ *of the* H*-valued random integral equation*

$$U(t,x,\omega) = \phi(t,\omega)(x) + \int_0^t \phi(t-s,\theta(s,\omega))(F(U(s,x,\omega)))\, ds, \quad t \geq 0,\ x \in H, \tag{2.44}$$

is a mild solution of the semilinear stochastic evolution equation (2.41).

Proof.

Let U be a solution of (2.44) with the given measurability properties. It is sufficient to prove that $U(\cdot,x,\cdot)$ satisfies the stochastic integral equation (2.42). Substituting from the identity:

$$\phi(t,\omega)(x) = T_t(x) + (\omega) \int_0^t T_{t-s} B\phi(s,\cdot)(x)\, dW(s), \quad t \geq 0, x \in H,$$

into (2.44), gives the following a.s. relations

$$U(t,x,\cdot) = T_t(x) + \int_0^t T_{t-s} B\phi(s,\cdot)(x)\, dW(s) + \int_0^t T_{t-s}(F(U(s,x,\cdot)))\, ds$$

$$
\begin{aligned}
&+\int_0^t\int_0^{t-s} T_{t-s-s'}B\phi(s',\theta(s,\cdot))(F(U(s,x,\cdot)))\,dW(s',\theta(s,\cdot))\,ds\\
=&T_t(x)+\int_0^t T_{t-s}B\phi(s,\cdot)(x)\,dW(s)+\int_0^t T_{t-s}(F(U(s,x,\cdot)))\,ds\\
&+\int_0^t\int_0^{t-s} T_{t-s-s'}B\phi(s',\theta(s,\cdot))(F(U(s,x,\cdot)))\,dW(s'+s)\,ds\\
=&T_t(x)+\int_0^t T_{t-s}B\phi(s,\cdot)(x)\,dW(s)+\int_0^t T_{t-s}(F(U(s,x,\cdot)))\,ds\\
&+\int_0^t\int_s^t T_{t-\lambda}B\phi(\lambda-s,\theta(s,\cdot))(F(U(s,x,\cdot)))\,dW(\lambda)\,ds\\
=&T_t(x)+\int_0^t T_{t-s}B\phi(s,\cdot)(x)\,dW(s)+\int_0^t T_{t-s}(F(U(s,x,\cdot)))\,ds\\
&+\int_0^t\int_0^{\lambda} T_{t-\lambda}B\phi(\lambda-s,\theta(s,\cdot))(F(U(s,x,\cdot)))\,ds\,dW(\lambda)\\
=&T_t(x)+\int_0^t T_{t-s}(F(U(s,x,\cdot)))\,ds\\
&+\int_0^t T_{t-\lambda}B\{\phi(\lambda)(x)+\int_0^{\lambda}\phi(\lambda-s,\theta(s,\cdot))(F(U(s,x,\cdot)))\,ds\}\,dW(\lambda)\\
=&T_t(x)+\int_0^t T_{t-s}(F(U(s,x,\cdot)))\,ds+\int_0^t T_{t-\lambda}BU(\lambda,x,\cdot)\,dW(\lambda)
\end{aligned}
$$

for $t\geq 0$. Hence U satisfies (2.42) and is therefore a mild solution of (2.41). $\square$

Our next theorem shows that the random integral equation (2.44) admits a unique $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{B}(H)\otimes\mathcal{F},\mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted solution $U:\mathbf{R}^+\times H\times\Omega\to H$. The fact that (U,θ) is a smooth perfect cocycle can be read off from (2.44), as in the proof of Theorem 2.6 below.

For any positive integer j, denote by $L_2^{(j)}(H,H)\subset L^{(j)}(H,H)$ the space of all Hilbert-Schmit j-multilinear maps $A\in L^{(j)}(H,H)$ given the Hilbert-Schmidt norm

$$
\|A\|_{L_2^{(j)}(H,H)}:=\sum_{\substack{n_i\geq 1\\ 1\leq i\leq j}}|A(e_{n_1},e_{n_2},\cdots,e_{n_j})|_H^2<\infty
$$

where $\{e_{n_i}:n_i\geq 1\}$ is a complete orthonormal system in H for each $1\leq i\leq j$.

Theorem 2.6.

Assume that the operators A, B satisfy the hypotheses of Theorem 2.4. Suppose that F satisfies the linear growth and Lipschitz conditions (2.40). Then the mild solution of (2.41) has a $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable version $U : \mathbf{R}^+ \times H \times \Omega \to H$ with the following properties:

(i) For each $x \in H$, $U(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and satisfies (2.42) a.s..

(ii) (U, θ) is a perfect $C^{0,1}$ cocycle (in the sense of Definition 1.2).

(iii) For each $(t, \omega) \in (0, \infty) \times \Omega$, the map $H \ni x \mapsto U(t, x, \omega) \in H$ takes bounded sets into relatively compact sets.

Moreover, if we assume that F is $C^{k,\epsilon}$ on H for a positive integer k and $\epsilon \in (0, 1]$, then the mild solution (U, θ) also enjoys the following properties:

(iv) (U, θ) is a $C^{k,\epsilon}$ perfect cocycle.

(v) For each $(t, x, \omega) \in \mathbf{R}^+ \times H \times \Omega$, the Fréchet derivatives $D^{(j)}U(t, x, \omega) \in L_2^{(j)}(H, H)$, $1 \leq j \leq k$, and each map

$$[0, \infty) \times H \times \Omega \ni (t, x, \omega) \mapsto D^{(j)}U(t, x, \omega) \in L^{(j)}(H, H), \quad 1 \leq j \leq k,$$

is strongly measurable.

(vi) For any positive a, ρ,

$$E \log^+ \left\{ \sup_{\substack{0\leq t_1, t_2 \leq a \\ x\in H}} \frac{|U(t_2, x, \theta(t_1, \cdot))|}{(1 + |x|)} \right\} < \infty$$

and

$$E \log^+ \sup_{\substack{0\leq t_1, t_2 \leq a \\ |x|\leq \rho,\ 1\leq j\leq k}} \left\{ \|D^{(j)}U(t_2, x, \theta(t_1, \cdot))\|_{L^{(j)}(H,H)} \right\} < \infty.$$

Proof.

In view of Theorem 2.5, we construct a version of the mild solution of (2.41) by applying the classical technique of successive approximations to the integral equation (2.44). Define the sequence $U_n : \mathbf{R}^+ \times H \times \Omega \to H, n \geq 1$, by

$$\left.\begin{aligned} U_{n+1}(t,x,\omega) =& \phi(t,\omega)(x) + \int_0^t \phi(t-s,\theta(s,\omega))(F(U_n(s,x,\omega)))\,ds, \\ U_1(t,x,\omega) :=& \phi(t,\omega)(x) \end{aligned}\right\} \tag{2.45}$$

for all $(t,x,\omega) \in \mathbf{R}^+ \times H \times \Omega$. Fix an arbitrary bounded open set S in H. Let $C_b^0(S,H)$ denote the space of all continuous maps $f : S \to H$ such that $f(S)$ is relatively compact in H. Give $C_b^0(S,H)$ the supremum norm

$$\|f\|_{C_b^0} := \sup_{x\in S} |f(x)|_H, \quad f \in C_b^0(S,H).$$

It is not hard to see that $C_b^0(S,H)$ is a Banach space. For fixed $\omega \in \Omega$ and any $a > 0$, we will view the sequence (2.45) as a uniformly convergent sequence of bounded measurable paths $[0,a] \ni t \mapsto U_n(t,\cdot,\omega) \in C_b^0(S,H)$ in the Banach space $C_b^0(S,H)$. To see this, we use induction on n. In view of Theorem 2.4 (ii), define the finite random constant $\|\phi\|_\infty := \sup_{0\leq s\leq t\leq a} \|\phi(t-s,\theta(s,\omega))\|_{L(H)}, \omega \in \Omega$. Let C be the positive constant appearing in (2.40). Define

$$M_1 = \sup_{x\in S}[|x| + Ca]\|\phi\|_\infty e^{C\|\phi\|_\infty a}, \quad \omega \in \Omega.$$

For some integer $n \geq 1$, consider the following induction hypothesis:

Hypotheses H(n):

(i) For each $(t,\omega) \in (0,a] \times \Omega$, $U_n(t,\cdot,\omega) \in C_b^0(S,H)$;

(ii) $|U_n(t,x,\omega) \leq [|x| + Ca]\|\phi\|_\infty e^{C\|\phi\|_\infty t}$ for all $(t,x,\omega) \in [0,a] \times H \times \Omega$;

(iii) $|U_{n+1}(t,x,\omega) - U_n(t,x,\omega)| \leq C[1+\|\phi\|_\infty |x|]L^{n-1}\|\phi\|_\infty^n \dfrac{t^n}{n!}, \quad (t,x,\omega) \in [0,a]\times H\times\Omega,$ where L is the Lipschitz constant of F on the ball $B(0,M_1) \subset H$.

(iv) $U_n : \mathbf{R}^+ \times H \times \Omega \to H$ is $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, and for each $x \in H$, $U_n(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted.

We will first check that $H(1)$ is satisfied. Since $\phi(t, \cdot, \omega) : H \to H$ is continuous linear for each $(t, \omega) \in [0, a] \times \Omega$, it is clear that $H(1)(i)$ and $H(1)(ii)$ are satisfied. Using (2.45) and the linear growth property of F, it follows that

$$|U_2(t, x, \omega) - U_1(t, x, \omega)| \leq C\|\phi\|_\infty \int_0^t [1 + |\phi(s, \omega)(x)|_H]\, ds \leq C[1 + \|\phi\|_\infty |x|]\|\phi\|_\infty t,$$

for all $(t, x, \omega) \in [0, a] \times H \times \Omega$. Therefore, $H(1)(iii)$ holds. To see the measurability inductive hypothesis $H(1)(iv)$, use the definition of U_1 in (2.45) and Theorem 2.1.

Now assume that $H(n)$ holds for some integer $n \geq 1$. In particular, for each $(t, \omega) \in (0, a] \times \Omega$, $U_n(t, \cdot, \omega)$ maps S into a relatively compact set in H. Therefore, the map

$$H \ni x \mapsto \int_0^t \phi(t - s, \theta(s, \omega))(F(U_n(s, x, \omega))\, ds \in H$$

takes S into a relatively compact set in H, because, for fixed $(t, \omega) \in (0, a] \times \Omega$, the integrand

$$H \ni x \mapsto \phi(t - s, \theta(s, \omega))(F(U_n(s, x, \omega))) \in H$$

has the same property. Hence, $U_{n+1}(t, \cdot, \omega)(S)$ is relatively compact in H for each $(t, \omega) \in (0, a] \times \Omega$. Since $U_n(t, \cdot, \omega) : H \to H$ is continuous, it is easy to see from (2.45) that $U_{n+1}(t, \cdot, \omega) : H \to H$ is also continuous for each $(t, \omega) \in (0, a] \times \Omega$. Hence, $H(n + 1)(i)$ is satisfied. Using $H(n)(ii)$, the Lipschitz property of F and (2.45), a straightforward computation shows that $H(n + 1)(iii)$ is satisfied. A similar argument, using $H(n)(ii)$, the linear growth property of F and (2.45), shows that $H(n + 1)(ii)$ also holds. To check $H(n + 1)(iv)$, note first that for fixed $s \in [0, t]$, the map $\Omega \ni \omega \mapsto \phi(t - s, \theta(s, \omega)) \in L(H)$ is $\mathcal{F}_t$-measurable. This follows from the approximation argument at the end of the proof of Theorem 2.4. Hence by $H(n)(iv)$, it follows that for fixed $s \in [0, t]$, the map $\Omega \ni \omega \mapsto \phi(t - s, \theta(s, \omega))(F(U_n(s, x, \omega))) \in L(H)$ is $\mathcal{F}_t$-measurable. Hence by (2.45), it is easy to see

that $U_{n+1}(t,x,\cdot)$ is $\mathcal{F}_t$-measurable for fixed $(t,x) \in \mathbf{R}^+ \times H$. Furthermore, the integrand on the right-hand-side of (2.45) is jointly-measurable in (s,x,ω), and therefore $U_{n+1}(t,\cdot,\cdot)$ is jointly measurable for any fixed $t > 0$. By continuity of the path $\mathbf{R}^+ \ni t \mapsto U_{n+1}(t,x,\omega)$ for fixed $(x,\omega) \in H \times \Omega$, the joint measurability of $U_{n+1} : \mathbf{R}^+ \times H \times \Omega \to H$ follows. Hence $H(n+1)(iv)$ is satisfied. Therefore, $H(n)$ holds by induction for all integers $n \geq 1$.

The inequality $H(n)(iii)$ implies that the series $\sum_{n=1}^{\infty}[U_{n+1}(t,\cdot,\omega) - U_n(t,\cdot,\omega)]$ converges in $C_b^0(S,H)$ uniformly in $t \in [0,a]$ for each $\omega \in \Omega$. Therefore, the sequence $\{U_n(t,\cdot,\omega)\}_{n=1}^{\infty}$ converges in $C_b^0(S,H)$ uniformly in $t \in [0,a]$ for each $\omega \in \Omega$. Its limit

$$\lim_{n\to\infty} U_n(t,\cdot,\omega) = U_1(t,\cdot,\omega) + \sum_{n=1}^{\infty}[U_{n+1}(t,\cdot,\omega) - U_n(t,\cdot,\omega)], \quad (t,\omega) \in [0,a] \times \Omega,$$

is a solution of the random integral equation (2.44). Call this limit $U(t,\cdot,\omega) \in C_b^0(S,H)$ for $(t,\omega) \in \mathbf{R}^+ \times \Omega$. It is immediately clear from $H(n)(iv)$ and Theorem (2.5) that U satisfies the measurability requirements and assertion (i) of the theorem.

We next show that $U(t,\cdot,\omega) : H \to H$ is C^1 for fixed $(t,\omega) \in \mathbf{R}^+ \times \Omega$. For each $(x,y,\omega) \in H \times H \times \Omega$, denote by $z(\cdot,x,y,\omega)$ the unique solution of the random linear integral equation:

$$\begin{aligned} z(t,x,y,\omega) = &\int_0^t \phi(t-s,\theta(s,\omega))DF(U(s,x,\omega))z(s,x,y,\omega)\,ds \\ &+ \int_0^t \phi(t-s,\theta(s,\omega))DF(U(s,x,\omega))\phi(s,\omega)(y)\,ds, \quad t > 0. \end{aligned} \tag{2.46}$$

If we suppress $y \in H$, we can view (2.46) as a linear integral equation in $L_2(H)$ with a unique solution $[0,\infty) \ni t \mapsto z(t,x,\cdot,\omega) \in L_2(H)$ for fixed $(x,\omega) \in H \times \Omega$. This holds easily (by successive approximations) because DF is bounded on bounded subsets of H and $\{U(t,x,\omega); 0 \leq t \leq a, |x| \leq M\}$ is bounded for any $M > 0$, and $\|\phi\|_\infty$ is finite. We claim that $U(t,\cdot,\omega)$ is Fréchet differentiable with Fréchet derivative $DU(t,x,\omega) \in L_2(H)$ given by

$$DU(t,x,\omega)(y) = z(t,x,y,\omega) + \phi(t,\omega)(y), \quad y \in H \tag{2.47}$$

for each $(t,x,\omega)\in\mathbf{R}^+\times H\times\Omega$. To prove our claim, define

$$\mu(t,x,y,h,\omega):=U(t,x+hy,\omega)-U(t,x,\omega)-h[z(t,x,y,\omega)+\phi(t,\omega)(y)] \tag{2.48}$$

for each $(t,x,y,h,\omega)\in\mathbf{R}^+\times H\times H\times\mathbf{R}\times\Omega$. Using (2.48), (2.44) and (2.46), we obtain:

$$\begin{aligned}\mu(t,x,y,h,\omega)=&\int_0^t\phi(t-s,\theta(s,\omega))DF(U(s,x,\omega))\mu(s,x,y,h,\omega)\,ds\\&+\int_0^t\phi(t-s,\theta(s,\omega))\Big\{\int_0^1 DF[\lambda U(s,x+hy,\omega)+(1-\lambda)U(s,x,\omega)]\\&\qquad-DF(U(s,x,\omega))\,d\lambda\Big\}(U(s,x+hy,\omega)-U(s,x,\omega))\,ds\end{aligned} \tag{2.49}$$

for all $(t,x,y,h,\omega)\in\mathbf{R}^+\times H\times H\times\Omega$. Set

$$M_2=\sup_{s\le a,|h|\le1,|y|\le1}\{|U(s,x+hy,\omega)|\},\quad \omega\in\Omega.$$

Then M_2 is finite for each $\omega\in\Omega$, because of H(n)(ii). Let $L_1>0$ be the Lipschitz constant of DF on the ball $B(0,M_2)$, and $\|DF\|$ be the bound of DF on $B(0,M_2)$. Then (2.49) implies the following inequality:

$$\begin{aligned}|\mu(t,x,y,h,\omega)|\le&\|\phi\|_\infty\|DF\|\int_0^t|\mu(s,x,y,\omega)|\,ds\\&+L_1\|\phi\|_\infty\int_0^t|U(s,x+hy,\omega)-U(s,x,\omega)|^2\,ds\end{aligned} \tag{2.50}$$

for all $t\in[0,a],x,y\in H,h\in\mathbf{R},|y|,|h|\le1,\omega\in\Omega$. Using (2.44) and Gronwall's lemma, it is easy to see that

$$|U(t,x+hy,\omega)-U(t,x,\omega)|\le|h|\|\phi\|_\infty|y|e^{\|\phi\|_\infty\|DF\|t} \tag{2.51}$$

for all $t\in[0,a],x,y\in H,h\in\mathbf{R},|y|,|h|\le1,\omega\in\Omega$. By (2.50), (2.51) and another simple application of Gronwall's lemma, we obtain

$$|\mu(t,x,y,h,\omega)|\le\frac{|h|^2|y|^2\|\phi\|_\infty^2L_1}{2\|DF\|}\Big[e^{2\|\phi\|_\infty\|DF\|a}-1\Big]e^{\|\phi\|_\infty\|DF\|t} \tag{2.52}$$

for all $t \in [0,a], x, y \in H, h \in \mathbf{R}, |y|, |h| \leq 1, \omega \in \Omega$. Thus,

$$\lim_{h \to 0} \frac{1}{h} \sup_{\substack{|y| \leq 1 \\ 0 \leq t \leq a}} |\mu(t,x,y,h,\omega)| = 0 \tag{2.53}$$

for all $x \in H, \omega \in \Omega$. The above relation shows that $U(t,\cdot,\omega) : H \to H$ is Fréchet differentiable at any $x \in H$ and our claim (2.47) holds. Now combining (2.46) and (2.47), it follows that $DU(t,x,\omega)$ satisfies the $L(H)$-valued integral equation:

$$DU(t,x,\omega) = \phi(t,\omega) + \int_0^t \phi(t-s,\theta(s,\omega)) DF(U(s,x,\omega)) DU(s,x,\omega)\, ds \tag{2.54}$$

for each $(t,x,\omega) \in \mathbf{R}^+ \times H \times \Omega$. In the above integral equation, the "coefficients"

$$[0,\infty) \times \Omega \ni (t,\omega) \mapsto \phi(t,\omega) \in L(H)$$

$$\Delta \times H \times \Omega \ni (s,t,x,\omega) \mapsto \phi(t-s,\theta(s,\omega)) DF(U(s,x,\omega)) \in L(H)$$

are jointly measurable, where $\Delta = \{(s,t) \in \mathbf{R}^2 : 0 \leq s \leq t\}$. Therefore, the solution map

$$[0,\infty) \times H \times \Omega \ni (t,x,\omega) \mapsto DU(t,x,\omega) \in L(H)$$

is jointly measurable. Furthermore, by continuity of the map $H \ni x \mapsto DF(U(s,x,\omega)) \in L(H,\mathbf{R})$ it follows from (2.54) that the map $H \ni x \mapsto DU(t,x,\omega) \in L(H)$ is continuous for fixed $t > 0$ and $\omega \in \Omega$. Thus $U(t,\cdot,\omega) : H \to H$ is C^1. (In fact, the map $H \ni x \mapsto DU(t,x,\omega) \in L_2(H), t > 0$, is continuous because of the continuity of the map $H \ni x \mapsto z(t,x,\cdot,\omega) \in L_2(H)$ in the $L_2(H)$-valued integral equation underlying (2.46).)

Suppose further that F is $C^{k,\epsilon}, k \geq 1, \epsilon \in (0,1]$. For $k = 1$, assertion (vi) of the theorem follows from (2.44), the linear growth property of F, (2.54), Gronwall's lemma and the fact that $E\|\phi\|_\infty < \infty$. By suppressing y in (2.46) and taking higher-order Fréchet derivatives with respect to x of the underlying $L_2(H)$-valued integral equation, assertions (v) and (vi) can be established by induction on $k > 1$.

It remains to prove that (U,θ) is a perfect cocycle on H. We use uniqueness of solutions of (2.44). Fix $t_1 \geq 0$, $\omega \in \Omega$ and $x \in H$. It is sufficient to prove that

$$U(t+t_1,x,\omega) = U(t,U(t_1,x,\omega),\theta(t_1,\omega)) \tag{2.55}$$

for all $t \geq 0$. Define the two mappings $y, z : [0,\infty) \to H$ by

$$y(t) := U(t,U(t_1,x,\omega),\theta(t_1,\omega)), \quad z(t) := U(t+t_1,x,\omega) \tag{2.56}$$

for all $t \geq 0$. Since U satisfies (2.44), it follows that

$$\begin{aligned} y(t) =& \phi(t,\theta(t_1,\omega))(U(t_1,x,\omega)) + \int_0^t \phi(t-s,\theta(s,\theta(t_1,\omega)))(F(U(s,U(t_1,x,\omega),\theta(t_1,\omega)))\,ds \\ =& \phi(t,\theta(t_1,\omega))(\phi(t_1,\omega)(x)) + \int_0^{t_1} \phi(t,\theta(t_1,\omega))\{\phi(t_1-s,\theta(s,\omega))(F(U(s,x,\omega)))\}\,ds \\ &+ \int_{t_1}^{t+t_1} \phi(t+t_1-s,\theta(s,\omega))(F(y(s-t_1)))\,ds \\ =& \phi(t+t_1,\omega)(x) + \int_0^{t_1} \phi(t+t_1-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds \\ &+ \int_{t_1}^{t+t_1} \phi(t+t_1-s,\theta(s,\omega))(F(y(s-t_1)))\,ds \end{aligned}$$

for all $t \geq 0$. Making the substitution $t' := t+t_1$, the above relation yields

$$\begin{aligned} y(t'-t_1) =& \phi(t',\omega)(x) + \int_0^{t_1} \phi(t'-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds \\ &+ \int_{t_1}^{t'} \phi(t'-s,\theta(s,\omega))(F(y(s-t_1)))\,ds, \quad t' > t_1. \end{aligned} \tag{2.57}$$

Using (2.44) and the definition of z, it follows that

$$\begin{aligned} z(t) =& \phi(t+t_1,\omega)(x) + \int_0^{t_1} \phi(t+t_1-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds \\ &+ \int_{t_1}^{t+t_1} \phi(t+t_1-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds \quad t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} z(t'-t_1) =& \phi(t',\omega)(x) + \int_0^{t_1} \phi(t'-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds \\ &+ \int_{t_1}^{t'} \phi(t'-s,\theta(s,\omega))(F(z(s-t_1)))\,ds, \quad t' \geq t_1. \end{aligned} \tag{2.58}$$

It is easy to see that (2.57) and (2.58) imply

$$\begin{aligned}|y(t'-t_1)-z(t'-t_1)| &\le \int_{t_1}^{t'} \|\phi(t'-s,\theta(s,\omega))\| \cdot |F(y(s-t_1))-F(z(s-t_1))|\,ds \\ &\le L\|\phi\|_\infty \int_{t_1}^{t'} |y(s-t_1))-z(s-t_1)|\,ds, \quad t_1 \le t' \le t_1+a,\end{aligned} \tag{2.59}$$

where L is the Lipschitz constant of F on the bounded set $\{y(s), z(s), 0 \le s \le a\}$. From the above inequality, we get $y(t'-t_1)-z(t'-t_1)=0$ for all $t' \ge t_1$. Hence, $y(t)=z(t)$ for all $t \ge 0$. This implies the perfect cocycle property (2.55) and completes the proof of the theorem. □

Remark.

From the proof of Theorem 2.6, it is easy to see that the assertions of the theorem still hold if one replaces the linear growth condition F by the condition that F carries bounded sets in H into bounded sets, and $U(\cdot,\cdot,\omega)$ is bounded on bounded subsets of $[0,\infty)\times H$.

3. Semilinear stochastic partial differential equations: Lipschitz nonlinearity.

Let $\mathcal{D}$ be a smooth bounded domain in $\mathbf{R}^d$. Consider the Laplacian operator:

$$\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial \xi_i^2} \tag{3.1}$$

defined on $\mathcal{D}$. Let $H := H_0^k(\mathcal{D})$ be the Sobolev space of order $k > d/2$, i.e., the completion of $C_0^\infty(\mathcal{D})$ under the Sobolev norm

$$||u||^2_{H_0^k} := \sum_{|\alpha|\le k} \int_{\mathcal{D}} |D^\alpha u(\xi)|^2 d\xi,$$

where $d\xi$ denotes d-dimensional Lebesgue measure on $\mathbf{R}^d$.

Consider the spde

$$\left.\begin{aligned} du(t) =& \frac{1}{2}\Delta u(t)dt + f(u(t))dt + \sum_{i=1}^{\infty} \sigma_i u(t)\, dW_i(t), \quad t > 0 \\ u(0) =& \psi \in H_0^k(\mathcal{D}) \\ u(t)|_{\partial\mathcal{D}} =& 0, \quad t \geq 0, \end{aligned}\right\} \tag{3.2}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a C_b^∞ function, the $\sigma_i : \mathcal{D} \to \mathbf{R}, i \geq 1$, are functions in the Sobolev space $H_0^s(\mathcal{D})$ with $s > k + \frac{d}{2}$, and the $W_i, i \geq 1$, are standard independent one-dimensional Brownian motions. Assume that the coefficients σ_i in (3.2) satisfy the following condition

$$\sum_{i=1}^{\infty} \|\sigma_i\|_{H_0^s}^2 < \infty. \tag{3.3}$$

Denote by $C_0^\infty(\mathcal{D})$ the set of all smooth test functions $\phi : \mathcal{D} \to \mathbf{R}$ which vanish on $\partial\mathcal{D}$. Let $L^\infty(\mathcal{D})$ stand for all essentially bounded measurable functions $\psi : \mathcal{D} \to \mathbf{R}$ with the usual norm

$$\|\psi\|_\infty := \text{essup}_{\xi\in\mathcal{D}}|\psi(\xi)|.$$

An $(\mathcal{F}_t)_{t\geq 0}$-adapted random field $u : \mathbf{R}^+ \times \mathcal{D} \times \Omega \to \mathbf{R}$ is a *weak solution* of (3.2) if $u(t, \cdot, \omega) \in H_0^k(\mathcal{D})$ for a.a. $\omega \in \Omega$ and the following identity holds:

$$\left.\begin{aligned} d < u(t), \phi >_{L^2} =& \nu < u(t), \Delta\phi >_{L^2} \ dt + < f(u(t)), \phi >_{L^2} \ dt + \sum_{i=1}^{\infty} < \sigma_i u(t), \phi >_{L^2} \ dW_i(t), \\ u(0) =& \psi \in H_0^k(\mathcal{D}), \\ u(t)|_{\partial\mathcal{D}} =& 0, \quad t > 0, \end{aligned}\right\}$$

for all $\phi \in C_0^\infty(\mathcal{D})$ a.s.. In the above equality, $< \cdot, \cdot >_{L^2}$ denotes the inner product on the Hilbert space $L^2(\mathcal{D})$ of all square-integrable functions $\psi : \mathcal{D} \to \mathbf{R}$, viz.

$$< \psi_1, \psi_2 >_{L^2} := \int_{\mathcal{D}} \psi_1(\xi)\psi_2(\xi)\, d\xi, \quad \psi_1, \psi_2 \in L^2(\mathcal{D}),$$

where $d\xi$ stands for d-dimensional Lebesgue measure.

We will show that (3.2) admits a unique weak solution $u(t) \in H$ a.s., for each $\psi \in H$. Furthermore, the ensemble of all weak solutions of (3.2) generates a a C^∞ perfect

cocycle (*also denoted by the same symbol*) $u : \mathbf{R}^+ \times H \times \Omega \to H$ satisfying the assertions of Theorem 3.5 below. In particular, the stochastic semiflow $u(t, \cdot, \omega) : H \to H$ takes bounded sets into relatively compact sets in H.

In this section and for the rest of the article, we should emphasize that although the weak *solution* $u : \mathbf{R}^+ \times \mathcal{D} \times \Omega \to \mathbf{R}$ of (3.2) and the associated stochastic *semiflow* $u : \mathbf{R}^+ \times H \times \Omega \to H$ are denoted by the same symbol u, the distinction between the two notions should be clear from the context.

Set $A := -\frac{1}{2}\Delta$ with Dirichlet boundary conditions on $\partial\mathcal{D}$. We will view the spde (3.2) as a semilinear stochastic evolution equation in H of the form (2.41) (Section 2). First, define the Nymetskii operator

$$F(u)(\xi) := f(u(\xi)), \quad \xi \in \mathcal{D}, u \in H. \tag{3.3$'$}$$

In Lemma 3.3 below, we will show that F is a C^∞ map $H \to H$. Secondly, consider a Hilbert space K and assume $\{f_k\}_{k=1}^\infty$ is a complete orthonormal system in K. Then

$$W(t) = \sum_{k=1}^\infty W^k(t) f_k, \quad t \in \mathbf{R},$$

is a cylindrical Brownian motion with covariance space K. As in section 2, $W(t)$ is a E-valued Brownian motion on the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}}, P)$ for a separable Hilbert space E such that $K \subset E$ is a Hilbert-Schmidt embedding. Denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ be the standard P-preserving (ergodic) Brownian shift. It is easy to see that (W, θ) is a perfect helix on E:

$$W(t_1 + t_2, \omega) = W(t_2, \theta(t_1, \omega)) - W(t_1, \omega), \quad t_1, t_2 \in \mathbf{R}, \omega \in \Omega.$$

Define the linear operator $B : H \to L_2(K, H)$ by setting

$$B(u)(f_i) := \sigma_i u, \quad u \in H = H_0^k(\mathcal{D}),\ i \geq 1.$$

In view of the continuous linear (Sobolev) imbedding

$$H_0^s(\mathcal{D}) \hookrightarrow C^k(\mathcal{D}),$$

it is easy to see that $B \in L(H, L_2(K, H))$ and satisfies Condition (B) of section 2(a). Thirdly, observe that weak solutions of the spde (3.2) correspond to mild solutions of the semilinear see:

$$\left.\begin{aligned} du(t) =& - Au(t)dt + F(u(t))dt + Bu(t)\, dW(t), \quad t > 0 \\ u(0) =& \psi \in H := H_0^k(\mathcal{D}) \end{aligned}\right\} \tag{3.2$'$}$$

([D-Z], p. 156).

Finally, we will establish a perfect C^∞-cocycle on the Sobolev space $H = H_0^k(\mathcal{D})$ for mild solutions of the semilinear see (3.2$'$), and hence for weak solutions of the spde (3.2).

We begin with some preparations. Following standard notation, let α be a d-tuple of non-negative integers, viz. $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_d)$ and denote $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$. For any $\phi \in C^{|\alpha|}(\mathcal{D})$, denote

$$(D^{(\alpha)}\phi)(\xi) \equiv \phi^{(\alpha)}(\xi) := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}\phi(\xi), \quad \xi \in \mathcal{D},$$

and for any integer $l > 0$, define

$$\|D^l\phi\|_{L^2} := \sum_{|\alpha|=l} \|D^{(\alpha)}\phi\|_{L^2}.$$

Lemma 3.1.

Let $\beta_1, \cdots, \beta_\mu$ be d-tuples and $|\alpha| = |\beta_1| + |\beta_2| + \cdots + |\beta_\mu|$, then there exists a constant $c > 0$ such that

$$\begin{aligned} &||f_1^{(\beta_1)} f_2^{(\beta_2)} \cdots f_\mu^{(\beta_\mu)}||_{L^2} \\ &\quad \le c^\mu ||f_1||_{L^\infty}^{1-\frac{|\beta_1|}{|\alpha|}} ||f_2||_{L^\infty}^{1-\frac{|\beta_2|}{|\alpha|}} \cdots ||f_\mu||_{L^\infty}^{1-\frac{|\beta_\mu|}{|\alpha|}} ||D^{|\alpha|}f_1||_{L^2}^{\frac{|\beta_1|}{|\alpha|}} ||D^{|\alpha|}f_2||_{L^2}^{\frac{|\beta_2|}{|\alpha|}} ||D^{|\alpha|}f_\mu||_{L^2}^{\frac{|\beta_\mu|}{|\alpha|}}. \end{aligned}$$

A proof of this lemma is given in [Ta], using Gagliardo-Nirenberg-Moser estimates.

Lemma 3.2.

Let F be smooth and assume $F(0) = 0$. Then for $u \in H_0^k(\mathcal{D}) \cap L^\infty$,

$$||F(u)||_{H_0^k(\mathcal{D})} \leq cC_k(||u||_{L^\infty})(1 + ||u||_{L^\infty})^{k-1}||u||_{H_0^k(\mathcal{D})},$$

where

$$C_k(\lambda) = \sup_{|u|\leq\lambda,\quad 1\leq\mu\leq k} |F^{(\mu)}(u)|,$$

and c is a constant.

Proof.

We need only prove the assertion of the lemma for $u \in C_0^\infty(\mathcal{D})$. The chain rule gives for any d-tuple α with $1 \leq |\alpha| \leq k$,

$$D^\alpha F(u) = \sum_{\beta_1+\beta_2+\cdots+\beta_\mu=\alpha,\mu\geq 1} c_\beta u^{(\beta_1)}u^{(\beta_2)}\cdots u^{(\beta_\mu)}F^{(\mu)}(u).$$

Hence

$$||D^\alpha F(u)||_{L^2} \leq C_k(||u||_{L^\infty}) \sum_{\beta_1+\beta_2+\cdots+\beta_\mu=\alpha,\mu\geq 1} c_\beta||u^{(\beta_1)}u^{(\beta_2)}\cdots u^{(\beta_\mu}||_{L^2}.$$

Applying Lemma 3.1 to $f_i = u, i = 1, 2, \cdots, \mu$, we have

$$||u^{(\beta_1)}u^{(\beta_2)}\cdots u^{(\beta_\mu)}||_{L^2} \leq c^\mu||u||_{L^\infty}^{\mu-1}||D^{|\alpha|}u||_{L^2}.$$

Therefore,

$$\begin{aligned}\sum_{\beta_1+\beta_2+\cdots+\beta_\mu=\alpha,\mu\geq 1} c_\beta||u^{(\beta_1)}u^{(\beta_2)}\cdots u^{(\beta_\mu)}||_{L^2} \leq & \sum_{\beta_1+\beta_2+\cdots+\beta_\mu=\alpha,\mu\geq 1} c_\beta c^\mu||u||_{L^\infty}^{\mu-1}||D^{|\alpha|}u||_{L^2} \\ \leq & c||D^{|\alpha|}u||_{L^2} \sum_{1\leq\mu\leq|\alpha|} C_\mu^{|\alpha|}||u||_{L^\infty}^{\mu-1} \\ = & c||D^{|\alpha|}u||_{L^2}(1 + ||u||_{L^\infty})^{|\alpha|-1} \\ \leq & c||u||_{H^k}(1 + ||u||_{L^\infty})^{k-1},\end{aligned}$$

for a constant $c > 0$. Note also that

$$||F(u)||_{L^2} \leq C_k||u||_{L^2} \leq C_k||u||_{H^k}, \quad u \in C_0^\infty(\mathcal{D}).$$

The assertion of the lemma follows easily from the above inequality. □

Lemma 3.3.

Suppose $k > \frac{d}{2}$, and $f : \mathbf{R} \to \mathbf{R}$ is a C^∞ function. Then the function $F : H_0^k(\mathcal{D}) \to H_0^k(\mathcal{D})$ defined by (3.3′) is a C^∞ map from $H_0^k(\mathcal{D})$ into $H_0^k(\mathcal{D})$.

Proof.

Recall the following Sobolev imbeddings

$$H_0^r(\mathcal{D}) \hookrightarrow L^{\frac{2d}{d-2r}}(\mathcal{D}), \qquad r < \frac{d}{2},$$

$$H_0^r(\mathcal{D}) \hookrightarrow L^\infty(\mathcal{D}), \qquad r > \frac{d}{2}.$$

Let us first prove that $F \in C^1(H, H)$, where $H := H_0^k(\mathcal{D})$. Fix $u \in H$. We will show that F is Fréchet differentiable and $DF(u)(h)(\xi) \equiv S_u(h)(\xi) = f'(u(\xi))h(\xi),\ h \in H, \xi \in \mathcal{D}$. To prove this, note that only functions in some ball $B(0, \delta) \subset H$ centered at 0 are involved. By the Sobolev imbedding theorem, the range of functions in $B(0, \delta)$ is contained in a compact interval in $\mathbf{R}$. Thus, we can assume $f \in C_b^\infty$ in the sequel. We start by proving that $S_u(h) \in H$ for $h \in H$. Let $r \leq k$. By the chain and product rules, it follows that $(S_u(h))^{(r)}$ can be written as a finite sum whose general term is of the form: $C(\xi)u^{(l_1)}(\xi) \cdot \cdots u^{(l_m)}(\xi)h^{(j_1)}(\xi) \cdots h^{(j_n)}(\xi)$, where $C(\cdot) \in L^\infty(\mathcal{D})$, and $l_1 + \cdots + l_m + j_1 + \cdots j_n = r$. Since $u^{(l)} \in H_0^{k-l}(\mathcal{D})$ and $h^{(j)} \in H_0^{k-j}(\mathcal{D})$, the Sobolev imbedding theorem implies that $u^{(l)} \in L^{\frac{2d}{d-2k+2l}}(\mathcal{D})$ and $h^{(j)} \in L^{\frac{2d}{d-2k+2j}}(\mathcal{D})$. As

$$\sum_{i=1}^m (d - 2k + 2l_i) + \sum_{i=1}^n (d - 2k + 2j_i) - d \leq (m + n - 1)(d - 2k) < 0,$$

we have

$$\frac{\sum_{i=1}^m (d - 2k + 2l_i)}{2d} + \frac{\sum_{i=1}^n (d - 2k + 2j_i)}{2d} \leq \frac{1}{2}.$$

By Hölder's inequality and the Sobolev imbedding theorem, this implies that

$$|C(\cdot)u^{(l_1)}(\cdot) \cdots u^{(l_m)}(\cdot)h^{(j_1)}(\cdot) \cdots h^{(j_n)}(\cdot)|_{L^2(\mathcal{D})} \leq c|u|_H^m |h|_H^n.$$

where c is a positive constant. Thus $S_u(h)$ is not only in H, but the map $H \ni h \mapsto S_u(h) \in H$ is a continuous linear operator. Now

$$F(u+th)(\xi) - F(u)(\xi) - tS_u(h)(\xi) = \int_0^t [f'(u(\xi)+sh(\xi)) - f'(u(\xi))]h(\xi)ds$$

for each $\xi \in \mathcal{D}, u, h \in H, t \geq 0$. To show that $DF(u) = S_u$, we need to prove that

$$\lim_{t\to 0} \sup_{|h|_H \leq 1} \left| \frac{1}{t} \int_0^t [f' \circ (u+sh) - f' \circ (u)] \cdot h \, ds \right|_H = 0.$$

It is sufficient to establish

$$\lim_{s\to 0} \sup_{|h|_H \leq 1} |[f' \circ (u+sh) - f' \circ (u)] \cdot h|_H = 0.$$

The above relation will hold if we show that

$$\lim_{s\to 0} \sup_{|h|_H \leq 1} |[(f' \circ (u+sh) - f' \circ (u)) \cdot h]^{(r)}|_{L^2(\mathcal{D})} = 0$$

for $r \leq k$.

Elementary computations show that $[(f' \circ (u+sh) - f' \circ (u)) \cdot h]^{(r)}(\xi)$ is a finite sum consisting of terms which are either of the form

$$G_1(\xi) := (f^{(l)} \circ (u+sh) - f^{(l)} \circ (u))(\xi) u^{(l_1)}(\xi) ... u^{(l_m)}(\xi) h^{(j_1)}(\xi) ... h^{(j_n)}(\xi), l \leq r+1,$$

or of the form

$$G_2(\xi) := s^a C(\xi) u^{(l_1)}(\xi) ... u^{(l_m)}(\xi) h^{(j_1)}(\xi) ... h^{(j_n)}(\xi), a \geq 1, C(\cdot) \in L^\infty(\mathcal{D}),$$

where $l_1 + ... + l_m + j_1 + ... + j_n = r$. For terms like G_1, using the Lipschitz continuity of $f^{(l)}$ it follows that

$$|G_1(\xi)| \leq cs|h|_{L^\infty(\mathcal{D})} |u^{(l_1)}(\xi) ... u^{(l_m)}(\xi) h^{(j_1)}(\xi) ... h^{(j_n)}(\xi)|.$$

Using Hölder's inequality and the Sobolev imbedding theorem, and arguing as in the proof of $S_u(h) \in H$, we obtain the following estimate

$$|G_1|_{L^2(\mathcal{D})} \leq cs|u|_H^m |h|_H^{n+1}$$

where c is a positive constant. Hence,

$$\lim_{s\to 0} \sup_{|h|_H \leq 1} |G_1|_{L^2(\mathcal{D})} = 0.$$

Similar arguments lead also to

$$\lim_{s\to 0} \sup_{|h|_H \leq 1} |G_2|_{L^2(\mathcal{D})} = 0.$$

Therefore,

$$\lim_{s\to 0} \sup_{|h|_H \leq 1} |[(f' \circ (u+sh) - f' \circ (u)) \cdot h]^{(k)}|_{L^2(\mathcal{D})} = 0, \quad r \leq k,$$

which completes the proof that $F : H \to H$ is Fréchet differentiable. The fact that F is r-times differentiable for $r \geq 2$ can be proved inductively using similar but lengthier computations. Details are left to the reader. $\square$

Using Itô's formula, it is easy to see that the solution of the following H-valued linear stochastic differential equation

$$du^* = Bu^* dW(t), \quad u^*(0) = \psi \in H := H_0^k(\mathcal{D})$$

is given by

$$u^*(t, \psi, \omega)(\xi) := Q(t, \xi, \omega)\psi(\xi), \quad \xi \in \mathcal{D}, \psi \in H, t \geq 0,$$

where the process $Q : \mathbf{R}^+ \times \mathcal{D} \times \Omega \to \mathbf{R}$ is defined by

$$Q(t, \xi, \omega) := \exp\left\{\sum_{i=1}^{\infty} \sigma_i(\xi) W_i(t, \omega) - \frac{1}{2}\sum_{i=1}^{\infty} \sigma_i^2(\xi) t\right\}, \; t \geq 0, \xi \in \mathcal{D}, \omega \in \Omega.$$

Using the perfect helix property of (W, θ), the reader may easily check the following cocycle identity for Q:

$$Q(t_1 + t_2, \xi, \omega) = Q(t_2, \xi, \theta(t_1, \omega))Q(t_1, \xi, \omega) \quad t_1, t_2 \geq 0, \omega \in \Omega.$$

The above identity immediately implies that $u^* : \mathbf{R}^+ \times H \times \Omega \to H$ is a perfect linear cocycle with respect to the Brownian shift θ.

We now prove the following proposition:

Proposition 3.4.

Assume $f \in C_b^k(R)$, $k > \frac{d}{2}$, and the forgoing conditions on the coefficients of the spde (3.2). Let S be a bounded subset of $H_0^k(\mathcal{D})$. Then for any $T > 0$ and almost all $\omega \in \Omega$, the weak solution $u(t, \psi)$ of the spde (3.2) satisfies

$$\sup_{\psi \in S} \sup_{0 \leq t \leq a} \|u(t, \psi)\|_{H_0^k(\mathcal{D})} \leq C(\omega, a),$$

for any $a \in \mathbf{R}^+$, where $C(\omega, a)$ is a random positive constant.

Proof.

Let $u(t, \psi)$ be the weak solution of the spde (3.2) with initial function $\psi \in H_0^k(\mathcal{D})$. Pick a sequence $\{\psi_n : n \geq 1\}$ of smooth functions in $C_b^\infty(\mathcal{D})$ such that $\psi_n \to \psi$ as $n \to \infty$ in $H_0^k(\mathcal{D})$. Let $u_n(t, \xi) := u(t, \psi_n)(\xi)$, $t \geq 0, \xi \in \mathcal{D}, n \geq 1$. Then each u_n, $n \geq 1$, is a strong solution of the spde (3.2). Define $v_n(t, \xi) := Q(t, \xi)^{-1} u_n(t, \xi)$, $t \geq 0, \xi \in \mathcal{D}$. Using the relations

$$dQ(t, \xi) = \sum_{i=1}^{\infty} \sigma_i(\xi) Q(t, \xi)\, dW_i(t), \quad t > 0, \xi \in \mathcal{D},$$

$$dQ(t, \xi)^{-1} = \sum_{i=1}^{\infty} \sigma_i^2(\xi) Q(t, \xi)^{-1}\, dt - \sum_{i=1}^{\infty} \sigma_i(\xi) Q(t, \xi)^{-1}\, dW_i(t), \quad t > 0, \xi \in \mathcal{D},$$

and Itô's formula, it follows that

$$\begin{aligned} dv_n(t,\xi) =& Q(t,\xi)^{-1}\frac{1}{2}\Delta u_n(t,\xi)dt + Q(t,\xi)^{-1}f(u_n(t,\xi))dt + Q(t,\xi)^{-1}u_n(t,\xi)\sum_{i=1}^{\infty}\sigma_i(\xi)dW_i(t) \\ &+ u_n(t,\xi)Q(t,\xi)^{-1}\sum_{i=1}^{\infty}\sigma_i^2(\xi)dt - u_n(t,\xi)Q(t,\xi)^{-1}\sum_{i=1}^{\infty}\sigma_i(\xi)dW_i(t) \\ &- u_n(t,\xi)Q(t,\xi)^{-1}\sum_{i=1}^{\infty}\sigma_i^2(\xi)dt \end{aligned}$$

a.s. for all $t > 0, \xi \in \mathcal{D}$,.

Therefore, for each $n \geq 1$, $v_n(t,\xi,\omega)$ satisfies the following parabolic equation with random coefficients:

$$\begin{aligned} \frac{\partial v_n}{\partial t} =& \frac{1}{2}\Delta v_n + < \nabla lnQ(t,\xi), \nabla v_n >_{\mathbf{R}^d} - [\frac{1}{2}Q(t,\xi)\Delta Q(t,\xi)^{-1} \\ &+ < \nabla Q(t,\xi), \nabla Q(t,\xi)^{-1} >_{\mathbf{R}^d}]v_n + Q(t,\xi)^{-1}f(Q(t,\xi)v_n), \quad t > 0, \end{aligned} \tag{3.4^n}$$

$$v_n(0,\xi) = \psi_n(\xi).$$

Let v denote the unique weak solution of the parabolic random pde

$$\begin{aligned} \frac{\partial v}{\partial t} =& \frac{1}{2}\Delta v + < \nabla lnQ(t,\xi), \nabla v >_{\mathbf{R}^d} - [\frac{1}{2}Q(t,\xi)\Delta Q(t,\xi)^{-1} \\ &+ < \nabla Q(t,\xi), \nabla Q(t,\xi)^{-1} >_{\mathbf{R}^d}]v + Q(t,\xi)^{-1}f(Q(t,\xi)v), \quad t > 0, \end{aligned} \tag{3.4}$$

$$v(0,\xi) = \psi(\xi)$$

for $t > 0, \xi \in \mathcal{D}$, with $\psi \in H_0^k(\mathcal{D})$. Since the coefficients of (3.4) are smooth, it is well known that

$$\lim_{n\to\infty}\sup_{0\leq t\leq a}\|v(t,\cdot,\omega) - v_n(t,\cdot,\omega)\|_{H_0^k(\mathcal{D})} = 0$$

for each $\omega \in \Omega$ and any $a \in \mathbf{R}^+$. By rewriting (3.4), it is easy to see that v satisfies the random pde

$$\frac{\partial v}{\partial t} = \frac{1}{2}Q(t,\xi)^{-1}\Delta(Q(t,\xi)v) + Q(t,\xi)^{-1}f(Q(t,\xi)v), \quad t > 0. \tag{3.5}$$

Since $\psi \in H_0^k(\mathcal{D}), k > \frac{d}{2}$, then by virtue of the Sobolev imbedding of $H_0^k(\mathcal{D})$ into $L^\infty(\mathcal{D})$, we can view (3.4) as a random reaction diffusion equation in $L^\infty(\mathcal{D})$ whose non-linear term has linear growth and is globally Lipschitz. Hence, using standard heat-kernel estimates, it follows that

$$\sup_{t\in[0,a]} \|v(t,\cdot,\omega)\|_\infty < \infty \tag{3.6}$$

for all $\omega \in \Omega$.

Now put $f \equiv 0$ in the random pde (3.5) and use uniqueness of solutions together with the identity

$$Q(t_1+t_2,\xi,\omega) = Q(t_2,\xi,\theta(t_1,\omega))Q(t_1,\xi,\omega) \quad t_1,t_2 \geq 0, \omega \in \Omega$$

in order to conclude that weak solutions of the linear spde

$$du(t,\xi) = \frac{1}{2}\Delta u(t,\xi)dt + Bu(t,\xi)dW(t), \quad u(0,\xi) = \psi(\xi) \tag{3.7}$$

yield a stochastic linear semiflow $\phi : \mathbf{R}^+ \times H_0^k(\mathcal{D}) \times \Omega \to H_0^k(\mathcal{D})$ such that (ϕ,θ) is a perfect $L(H_0^k(\mathcal{D}))$-valued cocycle. Full details of the argument are given in the proof of Theorem 4.1 in the next section.

It is easy to see that the weak solution u of the spde (3.2) satisfies the following random integral equation:

$$u(t,\xi,\omega) = \phi(t,\psi,\omega)(\xi) + \int_0^t \phi(t-s,\theta(s,\omega))F(u(s,\xi,\omega))\,ds, \tag{3.8}$$

for $t \geq 0, \psi \in H_0^k(\mathcal{D}), \xi \in \mathcal{D}$.

Now, using Lemma 3.2 together with (3.6), one gets a positive random constant C_k^1 such that

$$||F(u(t,\cdot,\omega))||_{H_0^k} \leq C_k^1(\omega,a)||u(t,\cdot,\omega)||_{H_0^k}$$

for all $t \in [0,a], \omega \in \Omega$ and any $a \in \mathbf{R}^+$.

Finally, the assertion of the proposition follows from (3.8) and a simple application of Gronwall's lemma. $\square$

Theorem 3.5.

Suppose $k > \frac{d}{2}$. Assume $f : R \to R$ is a C_b^∞ function. Assume all the forgoing conditions on the coefficients and the noise term in the spde (3.2). Then for each $\psi \in H_0^k(\mathcal{D}))$ the spde (3.2) has a unique weak $(\mathcal{F}_t)_{t\geq 0}$-adapted solution $u(\cdot,\psi,\cdot) : \mathbf{R}^+\times\Omega \to H_0^k(\mathcal{D}))$. Furthermore, the family of weak solutions $u(\cdot,\psi,\cdot)$, $\psi \in H_0^k(\mathcal{D})$, admits a $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{B}(H_0^k(\mathcal{D}))\otimes\mathcal{F},\mathcal{B}(H_0^k(\mathcal{D})))$-measurable version $u : \mathbf{R}^+ \times H_0^k(\mathcal{D}) \times \Omega \to H_0^k(\mathcal{D})$ having the following properties:

(i) For each $\psi \in H_0^k(\mathcal{D})$, $u(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to H_0^k(\mathcal{D})$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted.

(ii) (u,θ) is a C^∞ perfect cocycle on $H_0^k(\mathcal{D})$ (in the sense of Definition 1.2).

(iii) For each $(t,\omega) \in (0,\infty)\times\Omega$, the map $H_0^k(\mathcal{D}) \ni \psi \mapsto u(t,\psi,\omega) \in H_0^k(\mathcal{D})$ takes bounded sets into relatively compact sets.

(iv) For each $(t,\psi,\omega) \in (0,\infty)\times H_0^k(\mathcal{D})\times\Omega$, and any integer $r \geq 1$, the Fréchet derivative $D^{(r)}u(t,\psi,\omega) \in L_2^{(r)}(H_0^k(\mathcal{D}),H_0^k(\mathcal{D}))$, and the map

$$[0,\infty)\times H_0^k(\mathcal{D})\times\Omega \ni (t,\psi,\omega) \mapsto D^{(r)}u(t,\psi,\omega) \in L^{(r)}(H_0^k(\mathcal{D}),H_0^k(\mathcal{D}))$$

is strongly measurable.

(v) For any positive a,ρ and any positive integer r,

$$E\log^+\left\{\sup_{\substack{0\leq t_1,t_2\leq a\\ \psi\in H_0^k(\mathcal{D})}}\frac{\|u(t_2,\psi,\theta(t_1,\cdot))\|_{H_0^k(\mathcal{D})}}{(1+\|\psi\|_{H_0^k(\mathcal{D})})}\right\} < \infty$$

and

$$E\log^+\sup_{\substack{0\leq t_1,t_2\leq a\\ \|\psi\|_{H_0^k(\mathcal{D})}\leq\rho}}\left\{\|D^{(r)}u(t_2,\psi,\theta(t_1,\cdot))\|_{L^{(r)}(H_0^k(\mathcal{D}),H_0^k(\mathcal{D}))}\right\} < \infty.$$

Proof.

It is easy to see that the linear cocycle (ϕ,θ) of the spde (3.7) in the proof of Proposition 3.4 satisfies all the assertions in Theorem 2.1 and Theorem 2.2. The theorem now holds because of Proposition 3.4 and the remark following Theorem 2.6. □

4. Semilinear stochastic partial differential equations: Non-Lipschitz nonlinearity.

In this section, we will study two types of semilinear stochastic partial differential equations with non-Lipschitz nonlinearities and infinite dimensional noise.

The two classes of spde's considered are *stochastic reaction diffusion equations* and *stochastic Burgers equation with additive noise.* We prove the existence of a compacting C^1-cocycle in each case.

(a) Stochastic reaction diffusion equations

This class of spde's has dissipative nonlinear terms and infinite dimensional spatially smooth white noise. We prove the existence of a compacting $C^{0,1}$-cocycle satisfying appropriate regularity properties (Theorem 4.1). It appears that the cocycle is in general not Fréchet differentiable over the space of all L^2 functions on the domain (cf. [Te], p. 298). However, for a subclass of dissipative non-linearities with a certain dimension requirement, we further prove that the cocycle is C^1 and possesses Oseledec-type integrability properties (Theorem 4.2).

In [F.2], Flandoli studied the existence of *continuous* semi-flows for a class of spde's with finite dimensional noise and polynomial nonlinearities of odd degree and with negative leading coefficients.

Consider the following stochastic reaction diffusion equation in a smooth bounded domain $\mathcal{D} \subset \mathbf{R}^d$,

$$\left.\begin{aligned} du =& \nu \Delta u \, dt + f(u(t)) \, dt + \sum_{i=1}^{\infty} \sigma_i u \, dW_i(t), \quad t > 0 \\ u(0) =& \psi \\ u(t)|_{\partial \mathcal{D}} =& 0, \quad t > 0. \end{aligned}\right\} \tag{4.1}$$

where Δ is the Laplacian on $\mathcal{D}$, $\nu > 0$ is a real constant. The initial function $\psi : \mathcal{D} \to \mathbf{R}$ is square-integrable with respect to Lebesgue measure on $\mathcal{D}$, and a Dirichlet boundary

condition is assumed on the boundary $\partial\mathcal{D}$. The noise term $\sum_{i=1}^{\infty} \sigma_i u \, dW_i(t)$ is very similar to the one in (3.2) of section 3, but we assume here that $\sigma_i : \mathcal{D} \to R$, $i \geq 1$, are functions in the Sobolev space $H_0^s(\mathcal{D})$ with $s > 2 + \frac{d}{2}$, and

$$\sum_{i=1}^{\infty} ||\sigma_i||^2_{H_0^s} < \infty.$$

The nonlinearity $f : \mathbf{R} \to \mathbf{R}$ satisfies the following classical dissipativity conditions:

Conditions (D).

The function f is C^2, and there are positive constants $c_i, i = 1, 2, 3, 4$, and a positive integer p such that

$$-c_2 - c_3 s^{2p} \leq f(s)s \leq c_2 - c_1 s^{2p}$$

$$f'(s) \leq c_4$$

for all $s \in \mathbf{R}$.

A typical example of a function $f : \mathbf{R} \to \mathbf{R}$ satisfying Conditions (D) is the polynomial $f(s) := \sum_{k=1}^{2p-1} a_k s^k$, $s \in \mathbf{R}$, where $a_{2p-1} < 0$. (See e.g. [Te], pp. 83-85.)

Solutions of (4.1) are to be understood in a *weak* sense as defined below.

Consider the Hilbert space $H := L^2(\mathcal{D})$ of all square-integrable functions $\psi : \mathcal{D} \to \mathbf{R}$ furnished with the L^2 inner product

$$< \psi_1, \psi_2 >_H := \int_{\mathcal{D}} \psi_1(\xi)\psi_2(\xi) \, d\xi, \quad \psi_1, \psi_2 \in H,$$

where $d\xi$ stands for Lebesgue measure on $\mathcal{D}$. Denote the induced norm on H by

$$|\psi|_H := \left[\int_{\mathcal{D}} |\psi(\xi)|^2 \, d\xi \right]^{1/2}, \quad \psi \in H.$$

Recall $C_0^\infty(\mathcal{D})$, the set of all smooth test functions $\phi : \mathcal{D} \to \mathbf{R}$ which vanish on $\partial\mathcal{D}$. Let $L^\infty(\mathcal{D})$ stand for all essentially bounded measurable functions $\psi : \mathcal{D} \to \mathbf{R}$ with the usual norm

$$\|\psi\|_\infty := \text{essup}_{\xi \in \mathcal{D}} |\psi(\xi)|.$$

An $(\mathcal{F}_t)_{t\geq 0}$-adapted random field $u : \mathbf{R}^+ \times \mathcal{D} \times \Omega \to \mathbf{R}$ is a *weak solution* of (4.1) if $u(t,\cdot,\omega) \in H$ for a.a. $\omega \in \Omega, t > 0$, and the following identity holds:

$$\left.\begin{aligned} d < u(t), \phi >_H &= \nu < u(t), \Delta\phi >_H \ dt + < f(u(t)), \phi >_H \ dt + \sum_{i=1}^{\infty} < \sigma_i u(t), \phi >_H \ dW_i(t), \\ u(0) &= \psi \in L^2(\mathcal{D}), \\ u(t)|_{\partial\mathcal{D}} &= 0, \quad t > 0, \end{aligned}\right\}$$

for all $\phi \in C_0^\infty(\mathcal{D})$ a.s..

Note that, unless f has linear growth ($p = 1$ in Conditions (D)), the Nemytskii operator $F(u)(\xi) := f(u(\xi)),\ \xi \in \mathcal{D}$, does not even map $H = L^2(\mathcal{D})$ into itself. Thus one cannot view (4.1) as a semilinear see on H. even Nevertheless, we will show that for each $\psi \in H$, (4.1) admits a unique weak solution $u(t) \in H$ a.s., for all $t > 0$. Furthermore, the ensemble of all weak solutions of (4.1) generates a a globally Lipschitz cocycle (*also denoted by the same symbol*) $u : \mathbf{R}^+ \times H \times \Omega \to H$ satisfying the assertions of Theorem 4.1 below. In particular, the stochastic semiflow $u(t,\cdot,\omega) : H \to H$ takes bounded sets into relatively compact sets in H, and its global Lipschitz constant has moments of all orders.

As for Fréchet differentiability of the cocycle $u : \mathbf{R}^+ \times H \times \Omega \to H$ on the whole of H, it appears to be *not true* when f is smooth and satisfies Conditions (D) (cf. [Te], p. 298). However, under a stronger dimension requirement on the polynomial growth rate p of f, we are able to establish that the cocycle u is C^1 on H (Theorem 4.2). Furthermore, it satisfies similar assertions to those of Theorem 2.6. In particular, its Fréchet derivatives $Du(t,\psi,\omega) : H \to H$ are compact for all $(t,\psi,\omega) \in (0,\infty) \times H \times \Omega$.

In (4.1), the special case $f(s) := s(1-s)$, $s \in \mathbf{R}$, corresponds to the well-known stochastic KPP equation. It is not covered by the analysis in this section since it only admits positive solutions for all time. Its random travelling wave and ergodic properties were considered in [E-Z], [D-T-Z.1] and [O-V-Z]. For the KPP equation with additive noise, the reader may refer to [E-H] for the existence of the invariant measure.

The following lemma reduces (4.1) to a random family of reaction-diffusion equations.

Lemma 4.1.

Recall the process $Q : \mathbf{R}^+ \times \mathcal{D} \times \Omega \to \mathbf{R}$ *defined by* $Q(t,\xi) := \exp\left\{ \sum_{i=1}^{\infty} \sigma_i(\xi) W_i(t) - \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i^2(\xi) t \right\}$, $t \geq 0$. *Let* u *be a weak solution of (4.1) and set* $v(t) := Q(t)^{-1} u(t)$, $t \geq 0$. *Define* $\tilde{f} : \mathbf{R}^+ \times \mathcal{D} \times \mathbf{R} \times \Omega \to \mathbf{R}$ *by* $\tilde{f}(t,\xi,s,\omega) := Q(t,\xi,\omega)^{-1} f(Q(t,\xi,\omega)s)$, $t \in \mathbf{R}^+, \xi \in \mathcal{D}, s \in \mathbf{R}, \omega \in \Omega$. *Then* v *is a weak solution of the random reaction-diffusion equation*

$$\left.\begin{aligned} \frac{\partial v}{\partial t} =& \nu Q(t)^{-1} \Delta(Q(t)v) + \tilde{f}(t, v(t)), \quad t > 0 \\ v(0) =& \psi \in L^2(\mathcal{D}) \\ v(t)|_{\partial \mathcal{D}} =& 0, \quad t > 0. \end{aligned}\right\} \tag{4.2}$$

Conversely, every weak solution v *of (4.2) corresponds to a weak solution* u *of (4.1) given by* $u(t) := Q(t)v(t)$, $t \geq 0$.

Proof.

Suppose u is a weak solution of (4.1) with initial function $\psi \in L^2(\mathcal{D})$. Define

$$v(t,\xi,\omega) := Q(t,\xi,\omega)^{-1} u(t,\xi,\omega),\ t \geq 0, \xi \in \mathcal{D}, \omega \in \Omega. \tag{4.3}$$

Assume first that the initial function $\psi : \mathcal{D} \to \mathbf{R}$ is smooth. Then u is a strong solution of (4.1). Hence by Itô's formula (as in the proof of Proposition 3.4), it follows that v is a (strong) solution of the random reaction-diffusion equation (4.2). The case of a general $\psi \in L^2(\mathcal{D})$ can be handled by approximating ψ in the L^2-norm by a sequence of smooth functions $\psi_n : \mathcal{D} \to \mathbf{R}$, $n \geq 1$, as in the proof of Proposition 3.4.

A similar argument, using Itô's formula and the relation

$$dQ(t,\xi) = \sum_{i=1}^{\infty} \sigma_i(\xi) Q(t,\xi)\, dW_i(t), \quad t > 0, \tag{4.4}$$

proves the second assertion of the lemma. □

The next lemma shows that the non-linear term $\tilde{f}$ in (4.2) inherits the dissipativity properties of the original non-linear term f in (4.1).

Lemma 4.2.

Suppose f satisfies Conditions (D). Let $0 < a < \infty$. Then there exist $\mathcal{F}$-measurable positive random variables $\tilde{c}_i \in \bigcap_{k=1}^{\infty} L^k(\Omega, \mathbf{R}), i = 1, 2, 3,$ such that the following is true:

$$\left.\begin{aligned} -\tilde{c}_2(\omega) - \tilde{c}_3(\omega)s^{2p} \leq \tilde{f}(t,\xi,s,\omega)s \leq -\tilde{c}_1(\omega)s^{2p} + \tilde{c}_2(\omega), \\ \frac{\partial \tilde{f}(t,\xi,s,\omega)}{\partial s} \leq c_4 \end{aligned}\right\} \tag{4.5}$$

for all $t \in [0, a], s \in \mathbf{R}, \omega \in \Omega$.

Proof.

Fix $a \in (0, \infty), 0 \leq t \leq a, \xi \in \mathcal{D}, s \in \mathbf{R}, \omega \in \Omega$. Then Conditions (D) imply that

$$\begin{aligned} -c_2 Q(t,\xi,\omega)^{-2} - c_3 Q(t,\xi,\omega)^{(2p-2)} s^{2p} \leq & \tilde{f}(t,\xi,s,\omega)s \\ \leq & -c_1 Q(t,\xi,\omega)^{(2p-2)} s^{2p} + c_2 Q(t,\xi,\omega)^{-2}, \\ \frac{\partial \tilde{f}(t,s,\omega)}{\partial s} \leq & c_4. \end{aligned}$$

Define

$$\tilde{c}_1(\omega) := c_1 \inf_{0 \leq t \leq a, \xi \in \mathcal{D}} Q(t,\xi,\omega)^{(2p-2)}, \ \tilde{c}_2(\omega) := c_2 \sup_{0 \leq t \leq a, \xi \in \mathcal{D}} Q(t,\xi,\omega)^{-2}$$
$$\tilde{c}_3(\omega) := c_3 \sup_{0 \leq t \leq a, \xi \in \mathcal{D}} Q(t,\xi,\omega)^{(2p-2)}$$

for all $\omega \in \Omega$. By sample continuity of $Q(t,\xi)$ and $Q(t,\xi)^{-1}$, it is clear that each $\tilde{c}_i(\omega), i = 1, 2, 3,$ is finite for a.a. $\omega \in \Omega$. The estimates of the lemma follow immediately from the above inequalities and the definition of $Q(t,\xi)$. The existence of all moments of $\tilde{c}_i, i = 1, 2, 3,$ follows from Burkholder-Davis-Gundy inequality and the fact that $Q(t,\xi)$ and $Q(t,\xi)^{-1}$ satisfy the linear sde's (4.3) and (4.4). □

In view of Lemmas 4.1 and 4.2, we can now adapt standard methods from deterministic pde's in order to prove Theorem 4.1 below. In particular, the existence of the stochastic semiflow for weak solutions of the spde (4.1) follows from the regularity properties of solutions to the random reaction diffusion equation (4.2). For the existence of the semiflow of (4.2), its global Lipschitz continuity and compactness, we refer the reader to [Te], pp. 80-102, 371-374. Note that Lemma 4.2 ensures that the non-linear time-dependent random term $\tilde{f}$ in (4.2) satisfies appropriate dissipativity estimates which carry sufficient uniformity in t to allow for the apriori estimates in [Te] to work. This renders the proof of Theorem 4.1 below an adaptation of the corresponding arguments in [Te]. Thus, we will only sketch the proof and leave many of the details to the reader.

Theorem 4.1.

Assume that f in (4.1) satisfies Conditions (D). Then for each $\psi \in H := L^2(\mathcal{D})$, the spde (4.1) admits a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted weak solution $u(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to H$ such that $u(\cdot,\psi,\omega) \in L^{2p}((0,T), L^{2p}(\mathcal{D})) \cap C(\mathbf{R}^+, H)$ for a.a. $\omega \in \Omega$. The family of all weak solutions of (4.1) has a $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{B}(H)\otimes\mathcal{F}, \mathcal{B}(H))$-measurable version $u : \mathbf{R}^+ \times H \times \Omega \to H$ with the following properties:

(i) For each $\psi \in H$, $u(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to H$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted weak solution of (4.1). H

(ii) (u,θ) is a $C^{0,1}$ perfect cocycle on H (in the sense of Definition 1.2).

(iii) For each $(t,\omega) \in (0,\infty) \times \Omega$, the map $H \ni \psi \mapsto u(t,\psi,\omega) \in H$ is globally Lipschitz and takes bounded sets in H into relatively compact sets.

(iv) For any positive a, ρ,

$$E \log^+ \sup_{\substack{0\leq t\leq a \\ |\psi|_H \leq \rho}} |u(t,\psi,\cdot)|_H < \infty.$$

(v) For each $\omega \in \Omega$,

$$\limsup_{t\to\infty} \frac{1}{t} \log^+ \sup_{\substack{\psi_1\neq\psi_2, \\ \psi_1,\psi_2\in H}} \frac{|u(t,\psi_1,\omega) - u(t,\psi_2,\omega)|_H}{|\psi_1 - \psi_2|_H} \leq \frac{1}{2}\Big(c_4 - \nu\lambda_1 - \sigma^2\Big)$$

where $\sigma^2 := \inf_{\xi\in\mathcal{D}} \sum_{i=1}^{\infty} \sigma_i^2(\xi)$. *In particular, if*

$$\sup_{s\in\mathbf{R}} f'(s) - \nu\lambda_1 - \sigma^2 < 0,$$

then the stochastic flow $u(t,\cdot,\omega) : H \to H$ *is a uniform contraction for sufficiently large* $t > 0$.

Proof.

The existence and uniqueness of a weak solution of (4.1) follows from the corresponding result for the random reaction-diffusion equation (4.2) ([Te], pp. 89-91). Using the dissipativity estimates (4.5) on $\tilde{f}$, a straightforward modification of the Galerkin approximation technique in [Te] (pp. 89-91) gives the existence of a weak solution $u(\cdot,\psi,\omega) : \mathbf{R}^+ \to H$ of the random reaction-diffusion equation (4.2) for each fixed $\omega \in \Omega$ (cf. also [Ro], pp. 221-227). The joint measurability and $(\mathcal{F}_t)_{t\geq 0}$-adaptedness of the solution are also immediate consequences of the Galerkin approximations. This completes the proof of assertion (i) of the theorem.

To prove assertion (iii), denote by $v(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to H$, $\psi \in H$, the family of all weak solutions of the random pde (4.2). We will show that for each $\omega \in \Omega$, the map $H \ni \psi \mapsto u(t,\psi,\omega) \in H$ is globally Lipschitz uniformly in t over bounded sets in $\mathbf{R}^+$. To see this, let $\psi_i \in H, i = 1,2$. Denote by $v_i(t) := v(t,\psi_i), t \geq 0, i = 1,2$, the weak solutions of the random pde (4.2) starting at $\psi_i \in H, i = 1,2$. Then multiplying both sides of the equation

$$\frac{\partial(v_1(t) - v_2(t))}{\partial t} = \nu Q(t)^{-1}\Delta(Q(t)(v_1(t) - v_2(t))) + \tilde{f}(t,v_1(t)) - \tilde{f}(t,v_2(t)), \quad t > 0,$$

by $v_1(t) - v_2(t)$ and integrating over $\mathcal{D}$, we obtain

$$\int_{\mathcal{D}} (v_1(t) - v_2(t)) \frac{\partial(v_1(t) - v_2(t))}{\partial t}\, d\xi = \nu < Q(t)^{-1}\Delta(Q(t)(v_1(t) - v_2(t))), v_1(t) - v_2(t) >_H + \int_{\mathcal{D}} \tilde{f}(t,v_1(t)) - \tilde{f}(t,v_2(t))(v_1(t) - v_2(t))\, d\xi,$$

for all $t > 0$. Using the Mean-Value Theorem and the second estimate in (4.5), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_1(t)-v_2(t)|_H^2 =& \nu < Q(t)^{-1}\Delta(Q(t)(v_1(t)-v_2(t))), v_1(t)-v_2(t) >_H \\ &+ \int_{\mathcal{D}}\int_0^1 \frac{\partial \tilde{f}}{\partial s}(t, \lambda v_1(t)+(1-\lambda)v_2(t))\, d\lambda (v_1(t)-v_2(t))^2\, d\xi \\ \leq& -\nu\lambda_1|v_1(t)-v_2(t)|_H^2 + c_4|v_1(t)-v_2(t)|_H^2 \end{aligned} \tag{4.6}$$

for all $t > 0$. In the above inequality, λ_1 is the smallest eigenvalue of $-Q(t)^{-1}\Delta(Q(t)\cdot)$. This turns out to be the same as the smallest eigenvalue of $-\Delta$. Applying Gronwall's lemma to (4.6), we get

$$|v_1(t,\omega)-v_2(t,\omega)|_H^2 \leq |\psi_1-\psi_2|_H^2 \exp\{(c_4-\nu\lambda_1)t\} \tag{4.7}$$

for all $t \geq 0$ and all $\omega \in \Omega$. Using the relations $u(t,\psi_i,\omega) = Q(t,\xi,\omega)v_i(t,\omega), i = 1,2$, in (4.7), we deduce that

$$|u(t,\psi_1,\omega)-u(t,\psi_2,\omega)|_H \leq |\psi_1-\psi_2|_H \exp\left\{\frac{1}{2}\Big(c_4-\nu\lambda_1\Big)t\right\} \sup_{\xi\in\mathcal{D}} Q(t,\xi,\omega)$$

for all $t \geq 0$, $\omega \in \Omega$, $\psi_1, \psi_2 \in H$. For any $a > 0$, define the random variable

$$c_5(\omega) := \sup_{0\leq t\leq a} \exp\left\{\frac{1}{2}\Big(c_4-\nu\lambda_1\Big)t\right\} \sup_{\xi\in\mathcal{D}} Q(t,\xi,\omega), \quad \omega\in\Omega.$$

Then it is easy to see that $E\log^+ c_5 < \infty$, and

$$|u(t,\psi_1,\omega)-u(t,\psi_2,\omega)|_H \leq c_5(\omega)|\psi_1-\psi_2|_H \tag{4.8}$$

for all $t \in [0,a]$, $\omega \in \Omega$, $\psi_1, \psi_2 \in H$. This proves the first assertion in (iii). (Note that (4.8) implies pathwise uniqueness of the weak solution to the spde (4.1): Just put $\psi_1 = \psi_2 = \psi$, a given initial function in H.) The local compactness of the semiflow $H \ni \psi \mapsto u(t,\psi,\omega) \in H$, $t > 0, \omega$, follows from the fact that $H \ni \psi \mapsto v(t,\psi,\omega) \in H$, $t > 0, \omega$, takes bounded sets in H to relatively compact sets.

We next prove the perfect cocycle property in (ii). To this end, fix $\psi \in H, \omega \in \Omega, t_1, t_2 \geq 0$. Define

$$Y(t) := v(t+t_1, \psi, \omega), \quad Z(t) := Q(t_1,\omega)^{-1} v(t, Q(t_1,\omega)v(t_1,\psi,\omega), \theta(t_1,\omega))$$

for all $t \geq 0$. Recall the perfect cocycle identity:

$$Q(t_1+t_2, \xi, \omega) = Q(t_2, \xi, \theta(t_1,\omega))Q(t_1,\xi,\omega) \quad t_1, t_2 \geq 0, \xi \in \mathcal{D}. \tag{4.9}$$

By the definition of $\tilde{f}$ in Lemma 4.1 and the above cocycle property, one gets

$$\begin{aligned}\tilde{f}(t,\xi,s,\theta(t_1,\omega)) =& Q(t,\xi,\theta(t_1,\omega))^{-1} f(Q(t,\xi,\theta(t_1,\omega))s) \\ =& Q(t_1,\xi,\omega)Q(t+t_1,\xi,\omega)^{-1} f(Q(t+t_1,\xi,\omega)Q(t_1,\xi,\omega)^{-1}s) \\ =& Q(t_1,\xi,\omega)\tilde{f}(t+t_1,\xi,Q(t_1,\xi,\omega)^{-1}s,\omega)\end{aligned} \tag{4.10}$$

for all $s \in \mathbf{R}, t \geq 0, \xi \in \mathcal{D}$.

We now claim that the weak solution of the random reaction-diffusion equation (4.2) satisfies the following identity

$$v(t+t_1,\psi,\omega) = Q(t_1,\omega)^{-1} v(t, Q(t_1,\omega)v(t_1,\psi,\omega), \theta(t_1,\omega)) \tag{4.11}$$

for all $t \geq 1$. This says that $Y(t) = Z(t)$ for all $t \geq 0$. Using (4.11) and the relation between u and v, it is easy to check that (u,θ) is a perfect cocycle. So we need only prove (4.11). By the definition of Z and (4.10), it follows that

$$\begin{aligned}\frac{\partial Z}{\partial t} =& \nu Q(t_1,\omega)^{-1} Q(t,\theta(t_1,\omega))^{-1} \Delta(Q(t,\theta(t_1,\omega))v(t,Q(t_1,\omega)v(t_1,\psi,\omega),\theta(t_1,\omega))) \\ &+ Q(t_1,\omega)^{-1}\tilde{f}(t, v(t,Q(t_1,\omega)v(t_1,\psi,\omega),\theta(t_1,\omega)),\theta(t_1,\omega)) \\ =& \nu Q(t+t_1,\omega)^{-1}\Delta(Q(t+t_1,\omega)Z(t)) \\ &+ \tilde{f}(t+t_1, Q(t_1,\omega)^{-1} v(t,Q(t_1,\omega)v(t_1,\psi,\omega),\theta(t_1,\omega)),\omega) \\ =& \nu Q(t+t_1,\omega)^{-1}\Delta(Q(t+t_1,\omega)Z(t)) + \tilde{f}(t+t_1, Z(t),\omega), \quad t>0;\end{aligned}$$

and $Z(0) = v(t_1, \psi, \omega)$. Now from its definition, Y also satisfies the same random pde:

$$\frac{\partial Y}{\partial t} = \nu Q(t+t_1,\omega)^{-1}\Delta(Q(t+t_1,\omega)Y(t)) + \tilde{f}(t+t_1, Y(t), \omega), \quad t > 0$$

with the same initial condition $Y(0) = v(t_1, \psi, \omega)$. Therefore, by uniqueness of weak solutions to the above pde, we must have $Y(t) = Z(t)$ for all $t \geq 0$. This proves our claim, and hence (u, θ) is a perfect cocycle on H.

Assertion (v) of the theorem follows easily from (4.8). This completes the proof of the theorem. $\square$

Our next result establishes Fréchet differentiability of the cocycle generated by the reaction diffusion equation:

$$\left.\begin{aligned} du =& \nu\Delta u\, dt + (1-|u|^\alpha)u\, dt + \sum_{i=1}^{\infty}\sigma_i(\xi)u\, dW_i(t), \quad t>0 \\ u(0) =& \psi \in H := L^2(\mathcal{D}) \\ u(t)|_{\partial\mathcal{D}} =& 0, \quad t > 0. \end{aligned}\right\} \tag{4.12}$$

where $\nu > 0$ is a positive constant and Δ is the Laplacian on a smooth bounded domain $\mathcal{D}$ with Dirichlet boundary conditions. The result is established under the dimension requirement $\alpha < \frac{4}{d}$. It is not clear whether this condition is necessary for Fréchet differentiability of the cocycle.

Theorem 4.2.

In (4.12), assume that $\alpha < \frac{4}{d}$. Then for each $\psi \in H := L^2(\mathcal{D})$, the spde (4.12) admits a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted weak solution $u(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to H$ such that $u(\cdot,\psi,\omega) \in L^{2p}((0,T), L^{2p}(\mathcal{D})) \cap C(\mathbf{R}^+, H)$ for a.a. $\omega \in \Omega$. The family of all weak solutions of (4.12) has a $(\mathcal{B}(R^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable version $u : \mathbf{R}^+ \times H \times \Omega \to H$ with the following properties:

(i) For each $\psi \in H$, $u(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to H$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted weak solution of (4.12).

(ii) (u, θ) *is a* C^1 *perfect cocycle on* H *(in the sense of Definition 1.2).*

(iii) For each $(t, \omega) \in (0, \infty) \times \Omega$*, the map* $H \ni \psi \mapsto u(t, \psi, \omega) \in H$ *is globally Lipschitz and takes bounded sets in* H *into relatively compact sets.*

(iv) For each $(t, \psi, \omega) \in (0, \infty) \times H \times \Omega$*, the Fréchet derivative* $Du(t, \psi, \omega) \in L(H)$ *is compact, and the map*

$$[0, \infty) \times H \times \Omega \ni (t, \psi, \omega) \mapsto Du(t, \psi, \omega) \in L(H)$$

is strongly measurable.

(v) For any positive a, ρ*,*

$$E \log^+ \sup_{\substack{0 \le t \le a \\ |\psi|_H \le \rho}} \left\{ |u(t, \psi, \cdot)|_H + \|Du(t, \psi, \cdot)\|_{L(H)} \right\} < \infty.$$

Proof.

Fix any $\psi \in H = L^2(\mathcal{D})$. The existence and uniqueness of the solution to (4.12) in $L^2(\mathcal{D})$ is well-known as the nonlinear term satisfies the dissipativity condition ([D-Z]). This also follows by a similar argument to the proof of Theorem 4.1. So assertion (i) follows easily. The main purpose is to prove assertions (ii), (iii) and (iv). Recall that $Q(t, \xi) := \exp\left\{ \sum_{i=1}^{\infty} \sigma_i(\xi) W_i(t) - \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i(\xi)^2 t \right\}$, $t \ge 0, \xi \in \mathcal{D}$, and let $v(t) = u(t) Q^{-1}(t), t \ge 0$.

For simplicity of notation and till further notice, we will suppress the dependence of the random fields u, v, etc.. on ω.

Observe that $v(t, \psi)$ is a weak solution of the random reaction diffusion equation

$$\frac{\partial v}{\partial t} = \nu \Delta v + 2\nu Q(t)^{-1} \nabla Q(t) \nabla v + v(\nu Q(t)^{-1} \Delta Q(t) + 1 - Q^{\alpha}(t) |v|^{\alpha}), \quad t > 0. \tag{4.13}$$

By the Feynman-Kac formula, we have

$$v(t, \psi)(\xi) = \hat{E}[\chi_{\tau_t = t} \psi(x_t) e^{\int_0^t (\nu Q(t-s, x_s)^{-1} \Delta Q(t-s, x_s) + 1 - Q^{\alpha}(t-s, x_s) |v|^{\alpha}(t-s, \psi)(x_s)) ds}] \tag{4.14}$$

where x is the solution of the following stochastic differential equation

$$dx(s) = \sqrt{2\nu}dB(s) + 2\nu\nabla \log Q(t-s, x_s)ds, \quad x_0 = \xi \in \mathcal{D},$$

and B is a Brownian motion in $\mathbf{R}^d$ independent of the $W_i, i \geq 1$. In (4.14), $\tau_t := \min(\tau, t)$, where τ is the first time the diffusion x hits $\partial\mathcal{D}$. Define $\beta := \nu \sup\limits_{0\leq s\leq t\leq a, \xi\in\mathcal{D}} \frac{\Delta Q(t-s,\xi)}{Q(t-s,\xi)}$ for any $a > 0$. It follows from Jensen's inequality and (4.14) that

$$\begin{aligned}
|v(t,\psi)|_H^2 \leq & \int_{\mathcal{D}} (\hat{E}\chi_{\tau_t=t}|\psi(x_t)|e^{(\beta+1)t})^2 d\xi \\
\leq & e^{2(\beta+1)t} \int_{\mathcal{D}} \left(\int_{\mathcal{D}} p_t(\xi, y)|\psi(y)|dy \right)^2 d\xi \\
\leq & e^{2(\beta+1)t} \int_{\mathcal{D}} \int_{\mathcal{D}} p_t(\xi, y)(\psi(y))^2 dy d\xi \\
\leq & e^{2(\beta+1)t}|\psi|_H^2, \qquad 0 \leq t \leq a.
\end{aligned} \tag{4.15}$$

In the above inequalities, $p_t(\xi, y)$ denotes the heat kernel associated with $\nu\Delta + 2\nu(\nabla \log Q(t))\nabla$ on $\mathcal{D}$ with Dirichlet boundary condition. Define the induced heat semigroup $T_t; H \to H, t \geq 0$, by

$$(T_t\psi)(\xi) := \int_{\mathcal{D}} p_t(\xi, y)\psi(y)dy, \quad \psi \in H, \xi \in \mathcal{D}, t \geq 0.$$

Note that there exists a constant $c > 0$ such that

$$p_t(\xi, y) \leq \frac{c}{t^{\frac{d}{2}}}, \quad \xi, y \in \mathcal{D}, t > 0. \tag{4.16}$$

It is easy to see, using Jensen's inequality and (4.16), that

$$\begin{aligned}
|v(t,\psi)(\xi)|^2 \leq & (\hat{E}\chi_{\tau_t=t}|\psi(x_t)|e^{(\beta+1)t})^2 \\
\leq & e^{2(\beta+1)t}\hat{E}\chi_{\tau_t=t}(\psi(x_t))^2 \\
\leq & e^{2(\beta+1)t} \int_{\mathcal{D}} p_t(\xi, y)(\psi(y))^2 dy \\
\leq & e^{2(\beta+1)t} \frac{c}{t^{\frac{d}{2}}} \int_{\mathcal{D}} \psi^2(y)\, dy
\end{aligned}$$

for all $\xi \in \mathcal{D}$ and $t > 0$. Hence

$$\|v(t,\psi)\|_\infty \leq \frac{\sqrt{c}\, e^{(\beta+1)t}}{t^{\frac{d}{4}}} |\psi|_H \tag{4.17}$$

for $t > 0$.

Now let $\psi, g \in L^2(\mathcal{D})$ with $|g|_H \leq 1$, and h be a small real number. Since v is a mild solution of (4.13), it follows that $v(t,\psi+hg) - v(t,\psi)$ satisfies the following convolution equation in H:

$$\begin{aligned} v(t,\psi+hg) - v(t,\psi) =& hT_t g + \int_0^t T_{t-s}(v(s,\psi+hg) - v(s,\psi))(1 + \frac{\Delta Q(s)}{Q(s)})ds \\ &+ \int_0^t T_{t-s} Q^\alpha(s)[v(s,\psi)|v(s,\psi)|^\alpha - v(s,\psi+hg)|v(s,\psi+hg)|^\alpha]ds, t > 0. \end{aligned} \tag{4.18}$$

Define $m(x) := x|x|^\alpha$ for each $x \in \mathbf{R}$. Then $m'(x) = (\alpha+1)|x|^\alpha$, $x \in \mathbf{R}$. By the Mean-Value Theorem, we have

$$\begin{aligned} &v(s,\psi+hg)|v(s,\psi+hg)|^\alpha - v(s,\psi)|v(s,\psi)|^\alpha \\ &\quad =(\alpha+1)\int_0^1 |rv(s,\psi+hg) + (1-r)v(s,\psi)|^\alpha dr(v(s,\psi+hg) - v(s,\psi)) \end{aligned} \tag{4.19}$$

for all $s \in \mathbf{R}^+$. Combining (4.18) and (4.19) we obtain

$$\begin{aligned} &|v(t,\psi+hg) - v(t,\psi)|_H \\ \leq& h|g|_H + c_1(\omega)\int_0^t |v(s,\psi+hg) - v(s,\psi)|_H\, ds \\ &+ c_1(\omega)(\alpha+1)\int_0^t (\|v(s,\psi+hg)\|_{L^\infty} + \|v(s,\psi)\|_{L^\infty})^\alpha |v(s,\psi+hg) - v(s,\psi)|_H\, ds, \end{aligned}$$

for all $0 \leq t \leq a$, where $c_1(\omega)$ is a positive random constant. By virtue of (4.17), we get

$$\begin{aligned} |v(t,\psi+hg) - v(t,\psi)|_H \leq& h|g|_H + c_1(\omega)\int_0^t |v(s,\psi+hg) - v(s,\psi)|_H\, ds \\ &+ c_2(\omega)(\alpha+1)\int_0^t \frac{1}{s^{\frac{d}{4}\alpha}} |v(s,\psi+hg) - v(s,\psi)|_H\, ds, \end{aligned} \tag{4.20}$$

for all $0 \le t \le a$, where $c_2(\omega) > 0$ is a random constant depending on the ball $\{g \in H : |g|_H \le 1\}$. Using Gronwall's lemma and the requirement $\alpha d < 4$, we obtain

$$\sup_{0\le t\le a} \sup_{\substack{g\in H\\ |g|_H\le 1}} |v(t,\psi+hg) - v(t,\psi)|_H \le C(\omega)h.$$

Consider the $L(H)$-valued integral equation:

$$G_t(\psi) = T_t + \int_0^t T_{t-s}\Big(Q(s)^{-1}\Delta Q(s) + 1 - (\alpha+1)Q^\alpha(s)|v(s,\psi)|^\alpha\Big)G_s(\psi)ds, t \ge 0, \quad (4.21)$$

where $1-(\alpha+1)Q^\alpha(s)|v(s,\psi)|^\alpha$ is regarded as a multiplication operator on $L^2(\mathcal{D})$ whose operator norm satisfies the inequality

$$||Q(s)^{-1}\Delta Q(s) + 1 - (\alpha+1)Q^\alpha(s)|v(s,\psi)|^\alpha||_\infty \le C_a(\omega)\frac{1}{s^{\frac{d}{4}\alpha}}|\psi|_H^\alpha, \quad 0 < s \le a, \quad (4.22)$$

for a positive random constant $C_a(\omega)$. *Claim:* There exists a unique, continuous solution $[0,\infty) \ni t \mapsto G_t(\psi) \in L(H)$ to equation (4.21). Moreover for $t > 0$, $G_t(\psi) : H \to H$ is compact.

Proof of claim: Let $G_t^1(\psi) = T_t, t \ge 0$. Define for $n \ge 1$

$$G_t^{n+1}(\psi) = T_t + \int_0^t T_{t-s}\Big(Q(s)^{-1}\Delta Q(s) + 1 - (\alpha+1)Q^\alpha(s)|v(s,\psi)|^\alpha\Big)G_s^n(\psi)ds. \quad (4.23)$$

Then (4.22) and (4.23) imply that

$$||G_t^{n+1}(\psi)||_{L(H)} \le 1 + C_a(\omega)||\psi||^\alpha \int_0^t \frac{1}{s^{\frac{d}{4}\alpha}}||G_s^n(\psi)||_{L(H)}ds., \quad 0 < t \le a \quad (4.24)$$

Since $\frac{\alpha d}{4} < 1$, then by the standard successive approximation technique it follows that the sequence $\{G_\cdot^n(\psi)\}_{n=1}^\infty$ converges to the unique solution of (4.12). Next we prove that $G_t(\psi)$ is compact for each $t > 0$. It suffices to show that a Cauchy sequence can be extracted from the set $\{G_t(\psi)(g) : |g|_H \le 1\}$ for each $t > 0$. Let $\delta_m, m \ge 1$ be a sequence of positive numbers decreasing to zero. Since T_t is compact for every $t > 0$, by a diagonal process there exists a sequence $g_n \in H$ with $|g_n|_H \le 1$ such that $T_{\delta_m}g_n, n \ge 1$ is a Cauchy

sequence for every m. Since $T_t, t \geq 0$, is a contraction semigroup on H, it is easy to see that $T_t g_n, n \geq 1$ is a Cauchy sequence for every $t > 0$. Now consider

$$G_t(\psi)(g_n) - G_t(\psi)(g_m) = T_t(g_n - g_m) + \int_0^t T_{t-s}\Big(Q(s)^{-1}\Delta Q(s) + 1 - (\alpha+1)Q^\alpha(s)|v(s,\psi)|^\alpha\Big)(G_s(\psi)(g_n) - G_s(\psi)(g_m))ds, t \geq 0. \tag{4.25}$$

Hence,

$$\begin{aligned} &|G_t(\psi)(g_n) - G_t(\psi)(g_m)|_H \\ &\quad \leq |T_t g_n - T_t g_m|_H + C_a(\omega)|\psi|_H^\alpha \int_0^t \frac{1}{s^{\frac{d}{4}\alpha}} |G_s(\psi)(g_n) - G_s(\psi)(g_m)|_H \, ds. \end{aligned} \tag{4.26}$$

for all $t \in [0,a]$. Set $l(t) := \limsup_{n,m\to\infty} |G_t(\psi)(g_n) - G_t(\psi)(g_m)|_H$, $0 \leq t \leq a$. Taking $\limsup_{n,m\to\infty}$ on both sides of (4.26) we obtain

$$l(t) \leq C_a(\omega)|\psi|_H^\alpha \int_0^t \frac{1}{s^{\frac{d}{4}\alpha}} l(s)ds, \quad 0 < t \leq a.$$

This implies that $l(t) = 0$ for all $t > 0$, and completes the proof of the claim.

Next we show that v is Fréchet differentiable and $Dv(t,\psi) = G_t(\psi)$ for all $t \geq 0$. First we note that by using the Feynman-Kac formula and a similar argument as in the proof of (4.15) and (4.17), one has

$$\int_{\mathcal{D}} G_t(\psi)(g)(\xi)d\xi \leq e^{2(\beta+1)t}|g|_H^2, \tag{4.27}$$

and

$$G_t(\psi)(g)(\xi) \leq \frac{\sqrt{c}\, e^{(\beta+1)t}}{t^{\frac{d}{4}}}|g|_H^2, \quad 0 < t \leq a. \tag{4.28}$$

Denote

$$\mu_t(\psi, g) = \frac{1}{h}(v(t,\psi+hg) - v(t,\psi)) - G_t(\psi)(g), \quad t > 0. \tag{4.29}$$

It is easy to see that μ satisfies the following integral equation:

$$\begin{aligned} \mu_t(\psi,g) = &\int_0^t T_{t-s}(1 + \frac{\Delta Q(s)}{Q(s)})\mu_s(\psi,g)ds \\ &- \int_0^t T_{t-s}Q^\alpha(s)[\frac{1}{h}(v(s,\psi+hg)|v(s,\psi+hg)|^\alpha - v(s,\psi)|v(s,\psi)|^\alpha)]ds \\ &+ (\alpha+1)\int_0^t T_{t-s}Q^\alpha(s)[|v(s,\psi)|^\alpha G_s(\psi)(g)]ds, \quad t \geq 0. \end{aligned} \tag{4.30}$$

Using $m(y)-m(x)=\int_0^1 m'(ry+(1-r)x)dr(y-x)$ it follows from (4.30) that

$$\begin{aligned}\mu_t(\psi,g)=&\int_0^t T_{t-s}(1+\frac{\Delta Q(s)}{Q(s)})\mu_s(\psi,g)ds\\&-(\alpha+1)\int_0^t T_{t-s}Q^\alpha(s)[\int_0^1|rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha dr\mu_s(\psi,g)]ds\\&+(\alpha+1)\int_0^t T_{t-s}Q^\alpha(s)[\int_0^1(|v(s,\psi)|^\alpha-|rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr\\&\qquad G_s(\psi)(g)]ds,\quad t\geq 0.\end{aligned}\tag{4.31}$$

Set $D(t):=\sup_{|g|_H\leq 1}|\mu_t(\psi,g)|_H,\ t\geq 0$. Using the L^∞ bound on $v(s,\psi+hg)$ this implies that for $0<t\leq a$, one has

$$\begin{aligned}D(t)\leq&C(\omega)\int_0^t D(s)ds+C(\omega)\int_0^t\frac{1}{s^{\frac{d}{4}\alpha}}D(s)ds\\&+C(\omega)\sup_{|g|_H\leq 1}\int_0^t|\int_0^1(|v(s,\psi)|^\alpha\\&\qquad-|rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)drG_s(\psi)(g)|_H\,ds.\end{aligned}\tag{4.32}$$

Again by Gronwall's lemma, it follows that there is a random constant $C(\omega)$ such that

$$\begin{aligned}&D(t)\\&\leq C(\omega)\sup_{|g|_H\leq 1}\int_0^t|(\int_0^1(|v(s,\psi)|^\alpha-|rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr)G_s(\psi)(g)|_H\,ds.\end{aligned}\tag{4.33}$$

for all $t\in[0,a]$. To complete the proof of assertions (ii) and (iv), it suffices to show that

$$\lim_{h\to 0}\sup_{|g|_H\leq 1}\int_0^t|(\int_0^1(|v(s,\psi)|^\alpha-|rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr)G_s(\psi)(g)|_H\,ds=0\tag{4.34}$$

for all $t\in[0,a]$. Let us prove (4.24) for $\alpha\leq 1$ and $\alpha>1$ separately. Assume first $\alpha\leq 1$. By Hölder inequality,

$$\begin{aligned}&\sup_{|g|_H\leq 1}\int_0^t|(\int_0^1(|v(s,\psi)|^\alpha-|rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr)G_s(\psi)(g)|_H\,ds\\&\leq\sup_{|g|_H\leq 1}\int_0^t|(|v(s,\psi)-v(s,\psi+hg)|^\alpha)G_s(\psi)(g)|_H\,ds\\&\leq\sup_{|g|_H\leq 1}\int_0^t\|G_s(\psi)(g)\|_{L^\infty}^\alpha(|v(s,\psi)-v(s,\psi+hg)|_H^\alpha)|G_s(\psi)(g)|^{1-\alpha}||ds\\&\leq\sup_{|g|_H\leq 1}\int_0^t\|G_s(\psi)(g)\|_{L^\infty}^\alpha|v(s,\psi)-v(s,\psi+hg)|_H^\alpha|G_s(\psi)(g)|_H^{1-\alpha}\,ds.\end{aligned}$$

By virtue of (4.20) and (4.28), we get

$$\sup_{|g|_H\le 1}\int_0^t |(\int_0^1 (|v(s,\psi)|^\alpha - |rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr)G_s(\psi)(g)||ds$$
$$\le C_\alpha(\omega)h^\alpha\int_0^t \frac{1}{s^{\frac{d}{4}\alpha}}ds = C_\alpha(\omega)\frac{1}{1-\frac{\alpha d}{4}}t^{1-\frac{\alpha d}{4}}h^\alpha, \quad 0<t\le a,$$

where $C_\alpha(\omega)$ is a random constant depending on the set $\{g\in L^2(\mathcal{D}) : ||g||_{L^2}\le 1\}$. This implies (4.24).

Assume now that $\alpha>1$. Then

$$\begin{aligned}
&\int_0^t |(\int_0^1 (|v(s,\psi)|^\alpha - |rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr)G_s(\psi)(g)|_H\,ds\\
&\le \int_0^t ds\int_0^1 dr|(|v(s,\psi)|^\alpha - |rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)G_s(\psi)(g)|_H\\
&\le \int_0^t ds\int_0^1 dr\int_0^1 dk\alpha|(k|v(s,\psi)|+(1-k)|rv(s,\psi+hg)+(1-r)v(s,\psi)|)^{\alpha-1}\\
&\qquad\qquad |v(s,\psi+hg)-v(s,\psi)|G_s(\psi)(g)|_H\\
&\le \int_0^t ds\int_0^1 dr\int_0^1 dk\alpha\|(k|v(s,\psi)|+(1-k)|rv(s,\psi+hg)+(1-r)v(s,\psi)|)^{\alpha-1}\|_{L^\infty}\\
&\qquad\qquad ||G_s(\psi)(g)||_{L^\infty}|v(s,\psi+hg)-v(s,\psi)|_H, \quad 0<t\le a.
\end{aligned} \tag{4.35}$$

By (4.17), (4.20) and (4.28) it follows from (4.35) that

$$\sup_{|g|_H\le 1}\int_0^t |(\int_0^1 (|v(s,\psi)|^\alpha - |rv(s,\psi+hg)+(1-r)v(s,\psi)|^\alpha)dr)G_s(\psi)(g)|_H\,ds$$
$$\le C_\alpha(\omega)h\int_0^t \frac{1}{s^{\frac{d}{4}(\alpha-1)}}\frac{1}{s^{\frac{d}{4}}}ds, 0<t\le a. \tag{4.36}$$

This implies (4.34). So assertion (iv) holds.

To establish (iii), use (4.18), (4.19) and a similar argument to the proof of (4.20), to obtain the following inequality

$$\begin{aligned}
|v(t,\psi_m)-v(t,\psi_n)|_H \le &|T_t\psi_m - T_t\psi_n|_H + C_1(\omega)\int_0^t |v(s,\psi_m)-v(s,\psi_n)|_H\,ds\\
&+ C_2(\omega)(\alpha+1)\int_0^t \frac{1}{s^{\frac{d}{4}\alpha}}|v(s,\psi_m)-v(s,\psi_n)|_H\,ds, \quad 0<t\le a,
\end{aligned} \tag{4.37}$$

for $\psi_n, \psi_m \in H$ such that $|\psi_m|_H, |\psi_n|_H \leq 1$. As in the proof of the compactness of $Dv(t,\psi)$, we can select a subsequence denoted also by $\{\psi_n\} \subset \{\psi : |\psi|_H \leq 1\}$ such that for each $t > 0$, $|T_t\psi_n - T_t\psi_m|_H \to 0$ as $n, m \to \infty$. One then can prove from (4.32) that $\lim_{n,m\to\infty} |v(t,\psi_n) - v(t,\psi_m)|_H = 0$. Therefore $v(t,\cdot) : H \to H$ is compact or each $t > 0$.. This implies the compactness of each Fréchet derivative $Du(t,\psi,\omega) : H \to H$, $t > 0, \omega \in \Omega$. Hence the first assertion in (iv) holds.

To prove the strong measurability assertion in (iv), we now highlight the dependence of u on ω. Note first that the map

$$[0,\infty) \times H \times \Omega \ni (t,\psi,\omega) \mapsto u(t,\psi,\omega) \in H$$

is jointly measurable. This is a consequence of the (uniform) continuity of

$$[0,a] \times H \ni (t,\psi) \mapsto u(t,\psi,\omega) \in H, \quad \omega \in \Omega,$$

and the measurability of

$$\Omega \ni \omega \mapsto u(t,\psi,\omega) \in H, \quad (t,\psi) \in \mathbf{R}^+ \times H.$$

Secondly, the joint strong measurability of

$$[0,\infty) \times H \times \Omega \ni (t,\psi,\omega) \mapsto Du(t,\psi,\omega) \in L(H)$$

follows from the relation

$$Du(t,\psi,\omega)(\eta) = \lim_{h\to 0} \frac{1}{h}[u(t,\psi+h\eta,\omega) - u(t,\psi,\omega)], \quad (t,\omega) \in \mathbf{R}^+ \times \Omega, \psi, \eta \in H. \quad (4.38)$$

Finally, note that the integrability estimate in (v) follows from the Lipschitz property of $u(t,\cdot,\omega) : H \to H$, $(t,\omega) \in \mathbf{R}^+ \times \Omega$. In particular, (4.38) and the above Lipschitz property give

$$\|Du(t,\psi,\omega)\|_{L(H)} \leq c_5(\omega)$$

for all $(t, \psi, \omega) \in [0, a] \times H \times \Omega$, with $E \log^+ c_5 < \infty$. $\square$

Remarks.

(i) It is easy to see that above proof is also valid for initial boundary value problem with Neumann boundary condition. Note the exact formula of the heat kernel was not needed in the proof. Only estimates such as (4.6) and (4.7) were actually needed. These kind of estimate holds for Laplacian operator on a bounded domain with smooth boundary and Neumann boundary condition. The generalized solution of (4.1) can be defined following Freidlin [Fr]:

$$\begin{aligned} &u(t,\psi)(\xi) \\ &=\hat{E}\left[\psi(x_t^*)e^{\int_0^t(\frac{\Delta(t-s,x_s^*)}{Q(t-s,x_s^*)}+1-|u|^\alpha(t-s,\psi)(x_s^*))ds-\frac{1}{2}\sum\limits_{i=1}^{\infty}\int_0^t\sigma_i^2(x_s^*)ds+\sum\limits_{i=1}^{\infty}\sigma_i(x_s^*)dW_i(t-s)}\right] \end{aligned}$$

a.s. Here x_t^* is a diffusion process starting at $\xi \in \mathcal{D}$ with reflection on the boundary $\partial\mathcal{D}$ generated with the operator $\nu\Delta + 2\nu\nabla \log Q(t)\nabla$. One can see that the analysis in the proof of Theorem 4.1 carries through for this case as well.

(ii) The dimension restriction is used only to guarantee the Fréchet differentiability of the semiflow in Theorem 4.2. This condition is not needed for the existence of the globally Lipschitz flow in Theorem 4.1. The conditions in Theorem 4.2 are stronger than those in Theorem 4.1, and accordingly the result.

(b) Burgers equation with additive noise

The stochastic Burgers equation has been considered intensively by many researchers in recent years ([B-C-J], [B-C-F], [D-T-Z], [D-D-T], [E-K-M-S], [E-V], [H-L-O-U-Z], [T-Za], [T-Z]). Here we consider the following stochastic Burgers equation on interval $[0, 1]$,

$$\begin{aligned} du + u\frac{\partial u}{\partial \xi}\,dt =& \frac{1}{2}\Delta u dt + \sum_{i=1}^{\infty}\sigma_i(\xi)dW_i(t), \quad t > 0, \\ u(0,\psi)(\xi) =& \psi(\xi), \\ u(t,\psi)(0) =& u(t,\psi)(1) = 0. \end{aligned} \tag{4.39}$$

Cf. [E-K-M-S]. Here W_i, $i \geq 1$, are independent one dimensional standard Brownian motions. We assume the same conditions as in [E-K-M-S] i.e.

$$\sigma_i'(\xi) \in C^3([0,1]), \quad ||\sigma_i'||_{C^3} \leq \frac{C}{i^2}, \quad i \geq 1,$$

where C is a positive constant. Define

$$b(t,\xi) = \sum_{i=1}^{\infty} \sigma_i(\xi) W_i(t), \quad t \geq 0, \xi \in [0,1].$$

Under this assumption, b, $\frac{\partial b}{\partial \xi}$, $\frac{\partial^2 b}{\partial \xi^2}$ are bounded in $(t,\xi) \in [0,a] \times [0,1]$ for each ω and any $a \in \mathbf{R}^+$. Define

$$v(t,\xi) := u(t,\xi) - b(t,\xi), \quad t > 0, \xi \in [0,1].$$

It is easy to see that $v(t,\xi)$ satisfies the following equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial \xi} = \frac{1}{2}\Delta v - b\frac{\partial v}{\partial \xi} - v\frac{\partial b}{\partial \xi} - b\frac{\partial b}{\partial \xi} + \frac{1}{2}\frac{\partial^2 b}{\partial \xi^2}, \tag{4.40}$$

Viewing equation (4.40) as a random Burgers equation, it is not hard to see that, for each initial $\psi \in L^2([0,1])$, it has a unique global solution $v(\cdot,\psi,\omega) \in C((\mathbf{R}^+ \times [0,1], \mathbf{R})$ for each $\omega \in \Omega$; and for any $a > 0$ and any bounded set $S \subset L^2([0,1])$, the following holds

$$\sup_{\substack{t \in [0,a] \\ \psi \in S}} ||v(t,\psi,\omega)||_{L^2([0,1])} < \infty \tag{4.41}$$

for all $\omega \in \Omega$ (cf. [Ta], Chapter 15, Proposition 1.3).

A continuous semi-flow for (4.39) was obtained in [B-C-F]. However, this is not sufficient for our purposes, since we seek to construct *differentiable* families of stable/unstable manifolds near hyperbolic stationary solutions of (4.39). In the following theorem, we establish the existence of a perfect C^1 cocycle. In Part II of this work ([M-Z-Z]), this fact will enable us to use multiplicative ergodic theory techniques in order to prove a local stable/unstable manifold theorem near stationary solutions of the stochastic Burgers equation (4.39).

The existence of a stationary solution for (4.39) is a highly non-trivial result, obtained by E, Khanin, Mazel and Sinai in [E-K-M-S] using variational methods.

Theorem 4.3.

Consider the stochastic Burgers equation (4.39) and assume that b, $\frac{\partial b}{\partial \xi}$, $\frac{\partial^2 b}{\partial \xi^2}$ *are bounded in* $(t,\xi) \in [0,a] \times [0,1]$ *for each* ω *and any* $a > 0$. *Then equation (4.39) has a mild solution with a* $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(L^2([0,1])) \otimes \mathcal{F}, \mathcal{B}(L^2([0,1])))$*-measurable version* $u : \mathbf{R}^+ \times L^2([0,1]) \times \Omega \to L^2([0,1])$ *having the following properties:*

(i) For each $\psi \in L^2([0,1])$, $u(\cdot,\psi,\cdot) : \mathbf{R}^+ \times \Omega \to L^2([0,1])$ *is* $(\mathcal{F}_t)_{t\geq 0}$*-adapted.*

(ii) (u,θ) *is a* C^1 *perfect cocycle on* $L^2([0,1])$ *(in the sense of Definition 1.2).*

(iii) For each $(t,\omega) \in (0,\infty) \times \Omega$, *the map* $L^2([0,1]) \ni \psi \mapsto u(t,\psi,\omega) \in L^2([0,1])$ *takes bounded sets into relatively compact sets.*

(iv) For each $(t,\psi,\omega) \in (0,\infty) \times L^2([0,1]) \times \Omega$, *the Fréchet derivative* $Du(t,\psi,\omega) \in L(L^2([0,1]))$ *is compact. Furthermore, the map*

$$[0,\infty) \times L^2([0,1]) \times \Omega \ni (t,\psi,\omega) \mapsto Du(t,\psi,\omega) \in L(L^2([0,1]))$$

is strongly measurable and for each $(t,\omega) \in (0,\infty) \times \Omega$, *the map* $L^2([0,1]) \ni \psi \mapsto Du(t,\psi,\omega) \in L^2([0,1])$ *takes bounded sets into relatively compact sets.*

(v) For any positive a, ρ,

$$E \log^+ \sup_{\substack{0\leq t\leq a \\ \|\psi\|_{L^2([0,1])}\leq \rho}} \left\{ |u(t,\psi,\cdot)|_{L^2([0,1])} + \|Du(t,\psi,\cdot)\|_{L(L^2([0,1]))} \right\} < \infty.$$

Proof.

Assertion (i) follows easily from the global existence of solutions to (4.40).

To prove (ii), consider the Burgers equation (4.40). Let $\psi, g \in L^2([0,1])$ and h be a small real number. Denote again by $p_t(\xi,y)$ the heat kernel for the Laplacian $\frac{1}{2}\Delta$ on $[0,1]$ with Dirichlet boundary conditions. Let $T_t : L^2([0,1]) \to L^2([0,1])$, $t \geq 0$, be the corresponding heat semi-group. There are positive constants c_1, c_2 such that

$$\left| \frac{\partial p_t(\xi,y)}{\partial y} \right| \leq \frac{c_1}{t} \mathrm{e}^{-\frac{(\xi-y)^2}{2c_2 t}} \tag{4.42}$$

for all $t > 0$, $\xi, y \in [0,1]$ (c.f. [L-S-U], p. 413). Then pick a positive constant c_3 such that

$$\int_0^1 \frac{c_3}{\sqrt{t}} \mathrm{e}^{-\frac{y^2}{2c_2 t}}\, dy \le 1 \tag{4.43}$$

for all $t > 0$.

Using (4.40), variation of parameters, and integration by parts, we get

$$\begin{aligned}
v(t,\psi)(\xi) =& T_t\psi(\xi) - \frac{1}{2}\int_0^t T_{t-s}\nabla v^2(s,\psi)(\xi)ds \\
&+ \int_0^t T_{t-s}(-\nabla(b(s,y)v(s,\psi)) - b(s,y)\nabla b(s,y) + \frac{1}{2}\Delta b(s))(\xi)ds \\
=& \int_0^1 p_t(\xi,y)\psi(y)dy - \frac{1}{2}\int_0^t\int_0^1 p_{t-s}(\xi,y)\nabla v^2(s,\psi)(y)dyds \\
&+ \int_0^t\int_0^1 p_{t-s}(\xi,y)(-\nabla(b(s,y)v(s,\psi)) - b(s,y)\nabla b(s,y) + \frac{1}{2}\Delta b(s,y)dyds \\
=& \int_0^1 p_t(\xi,y)\psi(y)dy + \frac{1}{2}\int_0^t\int_0^1 \nabla p_{t-s}(\xi,y)v^2(s,\psi)(y)dyds \\
&+ \int_0^t\int_0^1 \nabla p_{t-s}(\xi,y)b(s,y)v(s,\psi)(y)dyds \\
&+ \int_0^t\int_0^1 p_{t-s}(\xi,y)(-b(s,y)\nabla b(s,y) + \frac{1}{2}\Delta b(s,y)\, dy\, ds
\end{aligned}$$

for all $t \ge 0$. Thus

$$\begin{aligned}
&v(t,\psi+hg)(\xi) - v(t,\psi)(\xi) \\
&\quad= h\int_0^1 p_t(\xi,y)g(y)dy + \frac{1}{2}\int_0^t\int_0^1 \nabla p_{t-s}(\xi,y)(v^2(s,\psi+hg)(y) - v^2(s,\psi)(y))dyds \\
&\quad\quad+ \int_0^t\int_0^1 \nabla p_{t-s}(\xi,y)(b(s,y)(v(s,\psi+hg)(y) - v(s,\psi)(y))dyds, \quad t > 0.
\end{aligned} \tag{4.44}$$

Squaring both sides of the above equality and integrating with respect $\xi \in [0,1]$, we obtain

$$\begin{aligned}
&||v(t,\psi+hg) - v(t,\psi)||^2_{L^2([0,1])} \\
&\quad\le 3h^2||g||^2_{L^2([0,1])} + \frac{3}{4}\int_0^1\left(\int_0^t\int_0^1 \nabla p_{t-s}(\xi,y)(v^2(s,\psi+hg)(y) - v^2(s,\psi)(y))dyds\right)^2 d\xi \\
&\quad\quad+ 3\int_0^1\left(\int_0^t\int_0^1 \nabla p_{t-s}(\xi,y)b(s,y)(v(s,\psi+hg)(y) - v(s,\psi)(y))dyds\right)^2 d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq 3h^2||g||^2_{L^2([0,1])} \\
&\quad + \frac{3}{4}\int_0^1 \left(\int_0^t \frac{1}{\sqrt{t-s}}\int_0^1 \frac{c_1}{\sqrt{t-s}} \mathrm{e}^{-\frac{(\xi-y)^2}{2c_2(t-s)}} (v^2(s,\psi+hg)(y) - v^2(s,\psi)(y))dyds\right)^2 d\xi \\
&\quad + 3\int_0^1 \left(\int_0^t \frac{1}{\sqrt{t-s}}\int_0^1 \frac{c_1}{\sqrt{t-s}} \mathrm{e}^{-\frac{(\xi-y)^2}{2c_2(t-s)}} b(s,y)(v(s,\psi+hg)(y) - v(s,\psi)(y))\, dyds\right)^2 d\xi,
\end{aligned}
$$

for all $t > 0$. Now use Cauchy-Schwartz inequality, the heat kernel estimate (4.42) and Fubini's theorem to obtain

$$
\begin{aligned}
||v(t,\psi+hg) - v(t,\psi)||^2_{L^2([0,1])} &\leq 3h^2||g||^2_{L^2([0,1])} + \frac{3}{4}\int_0^1\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}ds \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} \\
&\quad \left(\int_0^1 \frac{c_1}{\sqrt{t-s}} \mathrm{e}^{-\frac{(\xi-y)^2}{2c_2(t-s)}} (v^2(s,\psi+hg)(y) - v^2(s,\psi)(y))dy\right)^2 dsd\xi \\
&\quad + 3\int_0^1\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}ds \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} \\
&\quad \left(\int_0^1 \frac{c_1}{\sqrt{t-s}} \mathrm{e}^{-\frac{(\xi-y)^2}{2c_2(t-s)}} b(s,y)(v(s,\psi+hg) - v(s,\psi))(y)dy\right)^2 dsd\xi \\
&\leq 3h^2||g||^2_{L^2([0,1])} + Ct^{\frac{1}{4}}\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}||v(s,\psi+hg) - v(s,\psi)||^2_{L^2([0,1])}ds \\
&\quad + Ct^{\frac{1}{4}}\int_0^t \frac{1}{(t-s)^{\frac{1}{4}}}||v(s,\psi+hg) - v(s,\psi)||_{L^2([0,1])}ds, \quad t > 0
\end{aligned}
$$

for all $t > 0$ and some positive (random) constant C. Iterating the above computation, we get

$$
\begin{aligned}
&||v(t,\psi+hg) - v(t,\psi)||^2_{L^2([0,1])} \\
&\quad \leq 3h^2||g||^2_{L^2([0,1])} + Ct^{\frac{1}{4}}\int_0^t \left(\frac{1}{(t-s)^{\frac{3}{4}}} + \frac{1}{(t-s)^{\frac{1}{4}}}\right)3h^2||g||^2_{L^2([0,1])}ds \\
&\quad + C^2t^{\frac{1}{4}}\int_0^t\int_0^s s^{\frac{1}{4}}\left(\frac{1}{(t-s)^{\frac{3}{4}}} + \frac{1}{(t-s)^{\frac{1}{4}}}\right)\left(\frac{1}{(s-r)^{\frac{3}{4}}} + \frac{1}{(s-r)^{\frac{1}{4}}}\right) \\
&\qquad ||v(r,\psi+hg) - v(r,\psi)||^2_{L^2([0,1])}drds, \quad t > 0.
\end{aligned}
$$

Consider now the elementary estimate

$$
\int_r^t \frac{s^\alpha}{(t-s)^\beta(s-r)^\gamma}ds = \int_0^{t-r} \frac{(s+r)^\alpha}{(t-r-s)^\beta s^\gamma}ds \leq \frac{C_1}{(t-r)^{\beta+\gamma-1}}, \quad t \geq r > 0,
$$

which holds for any $\alpha \geq 0$, $0 \leq \beta < 1$, $0 \leq \gamma < 1$, and where $C_1 > 0$ is a positive (deterministic) constant. Using the above estimate together with Fubini's theorem, gives

$$\begin{aligned}
&||v(t,\psi+hg)-v(t,\psi)||^2_{L^2([0,1])} \\
&\qquad \leq 3h^2||g||^2_{L^2([0,1])} + 3C(4t^{\frac{1}{2}} + \frac{4}{3}t)h^2||g||^2_{L^2([0,1])} \\
&\qquad + C^2 t^{\frac{1}{4}} \int_0^t ||v(r,\psi+hg)-v(r,\psi)||^2_{L^2([0,1])} \times \\
&\qquad\qquad \int_r^t s^{\frac{1}{4}} \left(\frac{1}{(t-s)^{\frac{3}{4}}} + \frac{1}{(t-s)^{\frac{1}{4}}} \right) \left(\frac{1}{(s-r)^{\frac{3}{4}}} + \frac{1}{(s-r)^{\frac{1}{4}}} \right) dsdr \\
&\leq 3h^2||g||^2_{L^2([0,1])} + 3C(4t^{\frac{1}{2}} + \frac{4}{3}t)h^2||g||^2_{L^2([0,1])} \\
&\qquad + C_2(t+t^{\frac{1}{4}}) \int_0^t \frac{1}{(t-r)^{1/2}} ||v(r,\psi+hg)-v(r,\psi)||^2_{L^2([0,1])} dr
\end{aligned} \tag{4.45}$$

for all $t \in (0,a]$, $a \in \mathbf{R}^+$. Iterating the above process once more and applying Gronwall's lemma, we obtain

$$\sup_{\substack{0 \leq t \leq a, g \in L^2([0,1]) \\ \|g\|_{L^2} \leq 1}} ||v(t,\psi+hg)-v(t,\psi)||^2_{L^2([0,1])} \leq Mh^2, \tag{4.46}$$

for any $a \in \mathbf{R}^+$, where M is a positive random constant depending on a.

For fixed $\psi, g \in L^2([0,1])$, define $G := G(t,\psi)(g)(\xi)$, $t > 0, \xi \in [0,1]$, to be the weak solution of the "linearized" Burgers equation

$$\frac{\partial G}{\partial t} + \frac{\partial (v(t,\psi)G)}{\partial \xi} = \frac{1}{2}\Delta G - \frac{\partial (bG)}{\partial \xi}, \quad G(0,\psi)(g) = g \in L^2([0,1]).$$

Set

$$\mu_t(\psi,g) := v(t,\psi+hg) - v(t,\psi) - hG(t,\psi)(g), \quad |h| < 1, t \geq 0.$$

Then it is easy to see that

$$\begin{aligned}
\mu_t(\psi,g)(\xi) = -\int_0^t \int_0^1 p_{t-s}(\xi,y)(&\frac{1}{2}\nabla(v(s,\psi+hg)(y) - v(s,\psi)(y))^2 \\
&+ \nabla(v(s,\psi)(y)\mu_s(\psi,g)(y)) + \nabla(b(s,y)\mu_s(\psi,g)(y)))dyds
\end{aligned}$$

$$= \int_0^t \int_0^1 \nabla p_{t-s}(\xi, y)(\frac{1}{2}(v(s, \psi + hg)(y) - v(s, \psi)(y))^2 + v(s, \psi)(y)\mu_s(\psi, g)(y) + b(s, y)\mu_s(\psi, g)(y))dyds.$$

for all $t > 0$. So using the Cauchy-Schwartz inequality and (4.46), we obtain

$$\begin{aligned}
&||\mu_t(\psi, g)||^2_{L^2([0,1])} \\
&\leq \frac{3}{4}\int_0^1 (\int_0^t \int_0^1 \nabla p_{t-s}(\xi, y)(v(s, \psi + hg)(y) - v(s, \psi)(y))^2 dyds)^2 d\xi \\
&\quad + 3\int_0^1 (\int_0^t \int_0^1 \nabla p_{t-s}(\xi, y)(v(s, \psi)(y)\mu_s(\psi, g)(y)dyds)^2 d\xi \\
&\quad + 3\int_0^1 (\int_0^t \int_0^1 \nabla p_{t-s}(\xi, y)(b(s, y)\mu_s(\psi, g)(y))dyds)^2 d\xi \\
&\leq \frac{3}{4}\int_0^1 \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} ds \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} \Bigg(\int_0^1 \frac{c_1}{\sqrt{t-s}} e^{-\frac{(y-\xi)^2}{c_2(t-s)}} \\
&\qquad\qquad (v(s, \psi + hg)(y) - v(s, \psi)(y))^2 dy \Bigg)^2 dsd\xi \\
&\quad + 3\int_0^1 \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} ds \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} \Bigg(\int_0^1 \frac{c_1}{\sqrt{t-s}} e^{-\frac{(y-\xi)^2}{c_2(t-s)}} (v(s, \psi)(y)\mu_s(\psi, g)(y)dy \Bigg)^2 dsd\xi \\
&\quad + 3\int_0^1 \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} ds \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} \Bigg(\int_0^1 \frac{c_1}{\sqrt{t-s}} e^{-\frac{(y-\xi)^2}{c_2(t-s)}} (b(s, y)\mu_s(\psi, g)(y))dy \Bigg)^2 dsd\xi.
\end{aligned}$$

for all $t > 0$. Thus

$$\begin{aligned}
||\mu_t(\psi, g)||^2_{L^2([0,1])} \leq & 3c_1 t^{\frac{1}{4}} \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \Bigg(\int_0^1 (v(s, \psi + hg)(y) - v(s, \psi)(y))^2 dy \Bigg)^2 ds \\
& + 12c_1 t^{\frac{1}{4}} \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \int_0^1 v^2(s, \psi)(y)dy \int_0^1 \mu_s^2(\psi, g)(y)dyds \\
& + 12t^{\frac{1}{4}} ||b||^2_\infty \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} \int_0^1 \mu_s^2(\psi, g)(y)dyds \\
\leq & Ch^4 + C\int_0^t (\frac{1}{(t-s)^{\frac{3}{4}}} + \frac{1}{(t-s)^{\frac{1}{4}}}) ||\mu_s(\psi, g||^2_{L^2([0,1])} ds, \quad 0 \leq t \leq a,
\end{aligned}$$

where $C = C(\omega, a)$ is a positive random constant. Using the previous iteration argument followed by Gronwall's lemma, we obtain the following estimate

$$\sup_{\substack{g \in L^2([0,1]), ||g||_{L^2} \leq 1 \\ 0 \leq t \leq a}} ||\mu_t(\psi, g)||^2_{L^2([0,1])} \leq M_1 h^4, \quad |h| < 1$$

for some positive random constant $M_1 = M_1(\omega, a)$. This implies that $\dfrac{\mu_t(\psi, g)}{h}$ converges to 0 as $h \to 0$ in $L^2([0,1])$, uniformly in $(t, g) \in [0, a] \times \{g \in L^2([0,1]) : ||g||_{L^2} \leq 1\}$. Therefore, $\dfrac{v(t, \psi + hg) - v(t, \psi)}{h} \to G_t(\psi, g)$ as $h \to 0$, uniformly for $g \in \{g; ||g||_{L^2([0,1])} \leq 1\}$. Hence, v is Fréchet differentiable at $\psi \in L^2([0,1])$, with Fréchet derivative $Dv(t, \psi) : L^2([0,1]) \to L^2([0,1])$ satisfying the $L(L^2([0,1]))$-valued linear equation

$$\begin{aligned} Dv(t,\psi) =& T_t - \int_0^t T_{t-s}\left(\frac{\partial v(s,\psi)}{\partial \xi} Dv(s,\psi) + v(s,\psi)\frac{\partial Dv(s,\psi)}{\partial \xi}\right) ds \\ & - \int_0^t T_{t-s}\left(Dv(s,\psi)\frac{\partial b(s)}{\partial \xi} + b(s)\frac{\partial Dv(s,\psi)}{\partial \xi}\right) ds \end{aligned} \tag{4.47}$$

for $(t, \psi) \in \mathbf{R}^+ \times L^2([0,1])$.

In order to complete the proof of assertion (ii) of the theorem, it remains to prove that (u, θ) is a perfect cocycle in $L^2([0,1])$. It is easy to see from (4.39) that

$$u(t,\psi)(\omega) = T_t\psi - \int_0^t T_{t-s}u(s,\psi,\omega)\nabla u(s,\psi,\omega)\, ds + (\omega)\int_0^t T_{t-s}db(s)$$

for $t > 0, \omega \in \Omega, \psi \in L^2([0,1])$. We need to prove that

$$u(t, u(t_1,\psi,\omega), \theta(t_1,\omega)) = u(t + t_1, \psi, \omega). \tag{4.48}$$

for $t, t_1 \geq 0, \omega \in \Omega, \psi \in L^2([0,1])$. To see this, fix $t_1 \geq 0, \omega \in \Omega, \psi \in L^2([0,1])$, and denote

$$Y(t) := u(t, u(t_1,\psi,\omega), \theta(t_1,\omega)), \quad Z(t) := u(t + t_1, \psi, \omega), \quad t > 0.$$

Then

$$\begin{aligned} Y(t) =& T_t u(t_1,\psi,\omega) - \int_0^t T_{t-s}u(s, u(t_1,\psi,\omega), \theta(t_1,\omega))\frac{\partial u(s, u(t_1,\psi,\omega), \theta(t_1,\omega))}{\partial y} ds \\ & + \int_{t_1}^{t+t_1} T_{t+t_1-s}db(s) \\ =& T_{t+t_1}\psi - \int_0^{t_1} T_{t+t_1-s}u(s,\psi,\omega)\frac{\partial u(s,\psi,\omega)}{\partial y} ds \\ & - \int_{t_1}^{t+t_1} T_{t+t_1-s}Y(s-t_1)\frac{\partial Y(s-t_1)}{\partial y} ds + \int_0^{t+t_1} T_{t+t_1-s}db(s), \quad t > 0. \end{aligned}$$

Also,

$$\begin{aligned}Z(t) =&T_{t+t_1}\psi - \int_0^{t_1} T_{t+t_1-s}u(s,\psi,\omega)\frac{\partial u(s,\psi,\omega)}{\partial y}ds\\ &- \int_{t_1}^{t+t_1} T_{t+t_1-s}Z(s-t_1)\frac{\partial Z(s-t_1)}{\partial y}ds + \int_0^{t+t_1} T_{t+t_1-s}db(s), \quad t>0.\end{aligned}$$

Therefore,

$$\begin{aligned}Y(t)(\xi) - Z(t)(\xi) =& -\frac{1}{2}\int_{t_1}^{t+t_1} T_{t+t_1-s}\left(\frac{\partial Y^2(s-t_1)}{\partial y} - \frac{\partial Z^2(s-t_1)}{\partial y}\right)(\xi)ds\\ =&\frac{1}{2}\int_{t_1}^{t+t_1} ds\int_0^1 \frac{\partial p_{t+t_1-s}(\xi,y)}{\partial y}(Y^2(s-t_1)(y) - Z^2(s-t_1)(y))dy\\ =&\frac{1}{2}\int_0^t ds\int_0^1 \frac{\partial p_{t-s}(\xi,y)}{\partial y}(Y^2(s)(y) - Z^2(s)(y))dy,\end{aligned}$$

for $t>0, \xi\in\mathcal{D}$. Using Cauchy-Schwartz inequality and (4.41), we have

$$\begin{aligned}&||Y(t)-Z(t)||^2_{L^2([0,1])} = \int_0^1 (Y(t)(\xi)-Z(t)(\xi))^2 d\xi\\ \le& C\int_0^1 d\xi(\frac{1}{2}\int_0^t ds\int_0^1 \frac{1}{\sqrt{t-s}}\frac{c_1}{\sqrt{t-s}}e^{-\frac{(\xi-y)^2}{2c_2(t-s)}}|Y^2(s)(y)-Z^2(s)(y)|dy)^2\\ \le& C\int_0^1\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}ds\int_0^t \frac{1}{(t-s)^{\frac{1}{4}}}\left(\int_0^1 \frac{c_1}{\sqrt{t-s}}e^{-\frac{(\xi-y)^2}{2c_2(t-s)}}|Y^2(s)(y)-Z^2(s)(y)|dy\right)^2 dsd\xi\\ \le& C\int_0^1\int_0^t \frac{1}{(t-s)^{\frac{1}{4}}}\int_0^1 \frac{c_1}{\sqrt{t-s}}e^{-\frac{(\xi-y)^2}{2c_2(t-s)}}(Y(s)(y)-Z(s)(y))^2dy\\ &\times\int_0^1 \frac{c_1}{\sqrt{t-s}}e^{-\frac{(\xi-y)^2}{2c_2(t-s)}}(Y(s)(y)+Z(s)(y))^2dydsd\xi\\ \le& C\sup_{0\le s\le T}[||Y(s)+Z(s)||^2_{L^2([0,1])}]\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}||Y(s)-Z(s)||^2_{L^2([0,1])}ds\\ \le& M(\omega)\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}||Y(s)-Z(s)||^2_{L^2([0,1])}ds, \qquad (4.49)\end{aligned}$$

for all $t\in(0,a]$, where C is a generic constant that may change from line to line. Note that in the above computation, we have also used the fact that

$$\sup_{0\le s\le a}[||Y(s)+Z(s)||^2_{L^2([0,1])}] < \infty, \quad a\in\mathbf{R}^+.$$

As in the proof of (4.45) it follows from (4.49) and Gronwall's lemma that $Y(t) = Z(t)$ for all $t \geq 0$. This completes the proof of assertion (ii) of the theorem.

To prove assertion (iii), it is easy to see from the proof of (4.45) that

$$\begin{aligned}
&||v(t,\psi_m,\omega) - v(t,\psi_n,\omega)||^2_{L^2([0,1])} \\
&\quad \leq 3||T_t\psi_m - T_t\psi_n||^2_{L^2([0,1])} + Ct\int_0^t \left(\frac{1}{\sqrt{t-s}} + 1\right)||v(s,\psi_m,\omega) - v(s,\psi_n,\omega)||^2_{L^2([0,1])}ds \\
&\quad + Ct^{\frac{1}{4}}\int_0^t \left(\frac{1}{(t-s)^{\frac{3}{4}}} + \frac{1}{(t-s)^{\frac{1}{4}}}\right)||v(s,\psi_m,\omega) - v(s,\psi_n,\omega)||^2_{L^2([0,1])}ds,
\end{aligned} \tag{4.50}$$

for $t \in [0,a]$ where C is a positive random constant. Now using (4.50) and the same argument as in the proof of compactness of $Dv(t,\psi,\omega)$ in Theorem 4.1, one can show $v(t,\cdot,\omega) : L^2([0,1]) \to L^2([0,1])$, $t > 0$, takes bounded sets into relatively compact sets. The only difference is that we have to iterate (4.50) once before we can use Gronwall's lemma. Details of the proof are omitted. □

Remark.

All the results in this section hold for Burgers equation

$$\begin{aligned}
du + u\frac{\partial u}{\partial \xi}\,dt =& \nu\Delta u dt + \sum_{i=1}^{\infty}\sigma_i(\xi)dW_i(t), \quad t > 0, \\
u(0,\psi)(\xi) =& \psi(\xi), \\
u(t,\psi)(0) =& u(t,\psi)(1) = 0.
\end{aligned}$$

where ν is a positive constant.

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