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# Multidimensional Integrable Schrödinger Operators With Matrix Potential

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**Abstract.** The Schrödinger operators with matrix rational potential, which are D-integrable, i.e. can be intertwined with the pure Laplacian, are investigated. Corresponding potentials are uniquely determined by their singular data which is a configuration of the hyperplanes in  $\mathbf{C}^n$  with prescribed matrices. We describe some algebraic conditions (matrix locus equations) on these data, which are sufficient for D-integrability. As the examples some matrix generalisations of the Calogero-Moser operators are considered.

## Introduction.

Let us consider a Schrödinger operator

$$L = -\Delta + U(z),$$

where  $z \in \mathbf{R}^n$  or  $\mathbf{C}^n$  and  $U(z)$  is a matrix-valued meromorphic function. We will call such an operator as *D-integrable* if there exists a differential operator

$\mathcal{D}$  with meromorphic matrix coefficients and constant scalar highest term, such that

$$L\mathcal{D} = \mathcal{D}L_0 \tag{1}$$

where  $L_0 = -\Delta$  is the pure Laplacian acting on the vector-valued functions (cf. [1-3], where the scalar case  $d = 1$  has been considered).

In dimension  $n = 1$  all such operators with rational potentials can be described as the results of matrix Darboux transformations (see [4]), which explains the terminology.

In dimension  $n > 1$  the situation is much more complicated even in the scalar case. For the review of the known results in this direction we refer to the recent paper [5]. In particular, as it has been shown in [1, 2] the singularities of the potential  $U(z)$  of any D-integrable Schrödinger operator have to be located on a union of the hyperplanes. The proof given in [1, 2] works also in the matrix case under an additional assumption of the regularity (see below theorem 3), so in the rational case the potential  $U(z)$  should have a form

$$U(z) = \sum_{i=1}^N \frac{(\alpha_i, \alpha_i) A_i}{((\alpha_i, z) + c_i)^2}.$$

Such a potential is determined by a configuration of the hyperplanes  $\Pi_i$  in  $\mathbf{C}^n$  given by the equations  $(\alpha_i, z) + c_i = 0$  with prescribed constant matrices  $A_i$ .

In the present paper we describe the conditions (so-called *matrix locus equations*) on these data which guarantee D-integrability. This generalises to the matrix case the main result of the paper [3]. Locus equations can be interpreted as the conditions of the local trivial monodromy for the corresponding Schrödinger equations (cf. [4-6]). This allows us to construct the examples of such configurations and related matrix D-integrable Schrödinger operators. Our proof of the existence of the intertwining operator  $\mathcal{D}$  is effective; the corresponding formula is a matrix version of the Berest's formula [7].

Some important examples of such operators were known in the theory of the generalised matrix Calogero-Moser systems (see [8-12]), although the fact of their D-integrability seems to have not been emphasized. The corresponding operators have the form

$$L = -\Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{m_\alpha(m_\alpha I - s_\alpha)(\alpha, \alpha)}{(\alpha, z)^2},$$

where  $\mathcal{R}$  is a root system in  $\mathbf{R}^n$  related to some Coxeter group  $G$ ,  $m_\alpha$  is an integer-valued  $G$ -invariant function on  $\mathcal{R}$ ,  $s_\alpha$  is the matrix of reflection with respect to the hyperplane  $(\alpha, z) = 0$ .

We show that the operator  $L$  is D-integrable also for some non-Coxeter configurations  $\mathcal{R}$  discovered in the scalar case in [13, 14]. Remarkably enough the matrix locus equations in this case turned out to coincide with the condition of the existence of the rational Baker-Akhiezer function in the so-called "old axiomatics" proposed in [15, 16] (see [5] for the detailed discussion of this notion).

This explains the appearance in the matrix case of the same configurations as in the scalar situation. Another interesting relation between the scalar and matrix generalisations of the Calogero-Moser system has been proposed recently by P.Bracken and N.Kamran [12] (see also [11]).

We should mention that in the classical case matrix generalisation of the Calogero-Moser system were introduced first by J.Gibbons and Th.Hermesen in [17] (see [18] for further results in this direction). Our results show that the quantum situation is actually much richer than the classical one.

## 1 Monodromy of the matrix Schrödinger equations in the complex domain.

Let's start with the one-dimensional case following essentially [4]. Let

$$L = -D^2 + U(z), \quad z \in \mathbf{C}, \quad D = \frac{d}{dz} \quad (2)$$

be a Schrödinger operator with meromorphic  $d \times d$ -matrix potential  $U(z)$ . Let  $z = 0$  be a regular singular point, i.e. a pole of the second order of  $U(z)$ . Consider a formal solution of the Schrödinger equation

$$L\psi = \lambda\psi \quad (3)$$

in the form

$$\psi = z^{-m} \sum_{i \geq 0} \psi_{-m+i} z^i. \quad (4)$$

Substituting (4) into the Schrödinger equation (3) with

$$U(z) = \frac{C_{-2}}{z^2} + \frac{C_{-1}}{z} + \sum_{r \geq 0} C_r z^r \quad (5)$$

we obtain that  $\psi_{-m}$  is an eigenvector of  $C_{-2}$ :

$$C_{-2}\psi_{-m} = m(m+1)\psi_{-m}.$$

If for any  $\lambda$  we can construct a basis of solutions of (3) with integer  $m$  (i.e.  $\psi$  is single-valued) then we say that the operator  $L$  has *local trivial monodromy* around  $z = 0$ . This is equivalent to the fact that all the solutions of the corresponding matrix Schrödinger equation (3) are single-valued near  $z = 0$  for all  $\lambda$ . In this case one can prove that  $C_{-2}$  is diagonalizable with eigenvalues  $\lambda_i = m_i(m_i + 1)$ ,  $i = 1, 2, \dots$  where  $m_i \in \mathbf{Z}$  (see [4]). Thus  $C_{-2}$  has a form

$$C_{-2} = \sum_{i=1}^k m_i(m_i + 1)P_i, \quad (6)$$

where  $P_i$  are commuting projectors to the corresponding eigenspaces:

$$P_i P_j = \delta_{ij} P_i, \quad \sum_{i=1}^k P_i = I,$$

where  $I$  is identity operator. We assume that  $0 \leq m_1 < m_2 < \dots < m_k = M$ . The following result [4] gives the conditions on the coefficients  $C_j$  of the expansion of the potential (5) which are equivalent to the local trivial monodromy of  $L$ .

**Theorem 1.** *A matrix Schrödinger operator (2) with a meromorphic potential (5) has local trivial monodromy around  $z = 0$  if and only if  $C_{-2}$  has a form (6) and the coefficients  $C_l$  with  $l = -1, 0, \dots, 2M - 1$  satisfy the relation*

$$P_i C_l P_j = 0 \tag{7}$$

when  $|m_i - m_j| \geq l + 1$  or  $m_i + m_j = l + 1, l + 3, \dots, l + 2k + 1, \dots$  (i.e. when  $m_i + m_j - l$  is a positive odd number). In particular, the matrix residue  $C_{-1} = 0$ .

The coefficients  $\psi_{-M}, \psi_{-M+1}, \dots, \psi_{M-1}$  of the corresponding expansions of the vector-eigenfunctions

$$\psi = z^{-M} (\psi_{-M} + z \psi_{-M+1} + \dots + z^k \psi_{-M+k} + \dots)$$

satisfy the conditions

$$P_i \psi_l = 0$$

if  $m_i + l < 0$  or  $m_i + l = 1, 3, \dots, 2k + 1, \dots, 2m_i - 1$  for  $m_i \geq 1$ .

Notice that for  $l = 0$  the conditions (7) are equivalent to the commutativity relation

$$[C_{-2}, C_0] = 0. \tag{8}$$

Now let us consider the multidimensional case. We will assume that the potential  $U(z)$  of the Schrödinger operator

$$L = -\Delta + U(z), \tag{9}$$

where  $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ ,  $\Delta = \frac{\partial^2}{\partial z_1^2} + \dots + \frac{\partial^2}{\partial z_n^2}$ , is a meromorphic  $d \times d$  matrix-valued function having a pole of the second order along the hyperplane  $\Pi_\alpha : (\alpha, z) = 0$ , which is assumed to be non-isotropic:  $(\alpha, \alpha) \neq 0$  (cf. [1, 2, 5]). We will suppose for simplicity that  $(\alpha, \alpha) = 1$ . The Laurent expansion of the potential in the normal direction  $\alpha$  at the vicinity of  $\Pi_\alpha$  can be written in the form

$$U(z) = \sum_{r \geq -2} C_r(\alpha, z)^r, \tag{10}$$

where  $C_r = C_r(z^\perp)$  are some analytic  $d \times d$  matrix-valued functions on the hyperplane  $\Pi_\alpha$  and  $z^\perp$  is orthogonal projection of  $z$  onto  $\Pi_\alpha$ . Let us suppose that there exists a formal solution of the Schrödinger equation

$$L\psi = \lambda\psi \tag{11}$$

of the form

$$\psi(z) = (\alpha, z)^{-m} \sum_{s \geq 0} \psi_{-m+s}(\alpha, z)^s \quad (12)$$

for some  $m$ , where the coefficients  $\psi_r = \psi_r(\lambda, z^\perp)$  are analytic vector-functions on  $\Pi_\alpha$ . Substituting series (10) and (12) into the equation (11) one can see that

$$C_{-2}\psi_{-m} = m(m+1)\psi_{-m},$$

i.e.  $\psi_{-m}$  is an eigenvector of  $C_{-2}$  with the eigenvalue  $m(m+1)$ .

**Definition.** We say that a Schrödinger operator (9) with the potential (10) has *local trivial monodromy around the hyperplane  $\Pi_\alpha$*  if

1) at any point of  $\Pi_\alpha$  matrix  $C_{-2}$  is diagonalizable with eigenvalues having the form  $m(m+1)$ ,  $m \in \mathbf{Z}$ ,

2) for any  $m \in \mathbf{Z}$  such that  $m(m+1)$  is an eigenvalue of  $C_{-2}$  and for any choice of the corresponding eigenvector  $\psi_{-m}(z^\perp)$  there exists a formal solution (12) of the equation (11) for any  $\lambda$  (notice that any such eigenvalue can be represented in the form  $\mu(\mu+1)$  in two different ways:  $\mu = m$  or  $\mu = -m-1$ ).

In principle,  $C_{-2}$  might depend on the point at the hyperplane  $\Pi_\alpha$ , but this is not the case.

**Lemma 1.** *If the operator (9) has a local trivial monodromy around the hyperplane  $\Pi_\alpha$  then  $C_{-2}$  is a constant matrix.*

To prove the lemma we can assume without loss of generality that  $\alpha = (1, 0, \dots, 0)$ . Then the Schrödinger operator can be written in the form

$$L = -\frac{\partial^2}{\partial z_1^2} - \tilde{\Delta} + U(z), \quad (13)$$

where  $\tilde{\Delta} = \frac{\partial^2}{\partial z_2^2} + \dots + \frac{\partial^n}{\partial z_n^2}$ . Thus we can consider (13) as a one-dimensional Schrödinger operator with the matrix operator-valued "potential"

$$\tilde{U}(z) = -\tilde{\Delta} + U(z).$$

Applying formally the theorem 1 and, in particular (8), we have

$$\left[ C_0 - \tilde{\Delta}, C_{-2} \right] \equiv 0$$

or

$$[C_0, C_{-2}] - 2 \cdot \sum_{k=2}^n (\partial_k C_{-2}) \partial_k - \tilde{\Delta}(C_{-2}) \equiv 0.$$

Therefore,  $\partial_k C_{-2} = 0$  for all  $k = 2, \dots, n$ , i.e.  $C_{-2}$  is a constant. Alternative, more rigorous way to prove the lemma is to repeat the arguments of the proof of the theorem 1 (see [4-6]).

Now similarly to the one-dimensional case [4] (see theorem 1 above) one can prove the following

**Theorem 2.** *A matrix Schrödinger operator (9) with a meromorphic potential (10) has local trivial monodromy around the hyperplane  $\Pi_\alpha$  if and only if*

1.  $C_{-2}$  is a constant diagonalisable matrix

$$C_{-2} = \sum_{i=1}^k m_i(m_i + 1)P_i,$$

where  $0 \leq m_1 < m_2 < \dots < m_k = M$  are some integers,  $P_i$  are commuting projectors

$$P_i P_j = \delta_{ij} P_i, \quad \sum_{i=1}^k P_i = I.$$

2. The coefficients  $C_l$  with  $l = -1, 0, \dots, 2M - 1$  satisfy the following relations

$$P_i C_l P_j \equiv 0 \tag{14}$$

if  $|m_i - m_j| \geq l + 1$  or  $m_i + m_j = l + 1, l + 3, \dots, l + 2k + 1, \dots$ . In particular,  $C_{-1} \equiv 0$  and  $[C_0, C_{-2}] \equiv 0$ .

The coefficients  $\psi_{-M}, \psi_{-M+1}, \dots, \psi_{M-1}$  of the corresponding expansions of the vector-eigenfunctions

$$\psi = (\alpha, z)^{-M} (\psi_{-M} + (\alpha, z)\psi_{-M+1} + \dots + (\alpha, z)^k \psi_{-M+k} + \dots)$$

satisfy the conditions

$$P_i \psi_l \equiv 0 \tag{15}$$

if  $m_i + l < 0$  or  $m_i + l = 1, 3, \dots, 2k + 1, \dots, 2m_i - 1$  for  $m_i \geq 1$ .

## 2 Matrix locus equations and D-integrability.

Let's consider a matrix Schrödinger operator (9) with a rational potential  $U(z)$  decaying at infinity. We will assume that all the singularities are regular, i.e.  $U(z)$  has the poles of the second order at most.

We would like to show that in this case the trivial monodromy property implies D-integrability, i.e. the existence of the intertwining operator (1). First of all, the potential must have the form

$$U(z) = \sum_{i=1}^N \frac{(\alpha_i, \alpha_i) A_i}{((\alpha_i, z) + c_i)^2} \tag{16}$$

due to the following result.

**Theorem 3.** *The regular singularities of the matrix potential of any D-integrable Schrödinger operator  $L$  are located on a union of non-isotropic hyperplanes. If such a potential is rational and decaying at infinity it should have a form (16).*

The proof essentially repeats the arguments of the scalar case investigated in [1, 2]. The coefficient  $(\alpha_i, \alpha_i)$  is written at the numerator of the expression (16) for the convenience, as this makes the matrices  $A_i$  independent on the choice of the equation of the corresponding hyperplane.

Let us assume now that the operator  $L$  with the potential (16) has local trivial monodromy around all the hyperplanes  $\Pi_i: (\alpha_i, z) + c_i = 0$ . We will say in this case that  $L$  has *trivial monodromy*. The local trivial monodromy conditions (14) around all the hyperplanes form a highly-overdetermined algebraic system on the configuration of the hyperplanes with prescribed matrices  $A_i$ . We will call this system a *matrix locus equations*.

**Theorem 4.** *Let  $L$  be a matrix Schrödinger operator (9) with a rational potential (16) satisfying the matrix locus equations. Then  $L$  is  $D$ -integrable.*

*Proof.* From the theorem 2 it follows that

$$A_s = \sum_{i=1}^{k_s} m_i^{(s)}(m_i^{(s)} + 1)P_i^{(s)}, \quad 0 \leq m_1^{(s)} < m_2^{(s)} < \dots < m_{k_s}^{(s)} = M_s$$

with some projectors  $P_i^{(s)}$ :  $P_i^{(s)}P_j^{(s)} = \delta_{ij}P_i^{(s)}$ ,  $\sum_{i=1}^{k_s} P_i^{(s)} = I$ .

Following the main idea of [3] let us introduce a linear space  $V$  consisting of the  $d \times d$  matrix-valued functions  $\Psi(z)$ ,  $z \in \mathbf{C}^n$  which satisfy the conditions:

- 1)  $\Psi(z) \prod_{s=1}^N ((\alpha_s, z) + c_s)^{M_s}$  is holomorphic in  $\mathbf{C}^n$ ;
- 2) the coefficients of the series expansion of  $\Psi(z)$  at the vicinity of hyperplanes  $(\alpha_s, z) + c_s = 0$ ,  $s = 1, \dots, N$  satisfy the conditions (15) with  $M = M_s$ .

The crucial observation is that the matrix locus equations (14) imply that the space  $V$  is invariant under  $L$  (cf. [3],[4]).

Let's consider the matrix function  $\Psi_0 = \prod_{s=1}^N ((\alpha_s, z) + c_s)^{M_s} e^{(k,z)} I$ , where  $I$  is the identity matrix. Evidently,  $\Psi_0 \in V$  and, therefore, all the functions

$$\Psi_i = (L + k^2)^i \Psi_0, \quad i = 1, 2, \dots$$

belong to  $V$  as well. These functions have the form

$$\Psi_i = \frac{P_i(k, z) e^{(k,z)}}{\prod_{s=1}^N ((\alpha_s, z) + c_s)^{M_s}},$$

where  $P_i(k, z)$  are some matrix polynomials in  $k, z$ . Since

$$P_{i+1} = \phi(-\Delta - 2(k, \frac{\partial}{\partial z}) + U(z))\phi^{-1}P_i, \quad \phi = \prod_{s=1}^N ((\alpha_s, z) + c_s)^{M_s},$$

the degrees of  $P_i$  in  $z$  are decreasing with  $i$ . So, there exists such  $j$  that  $(L + k^2)\Psi_j = 0$ . It is easy to see that for  $M = \sum_{s=1}^N M_s$

$$\Psi_M = [(-2)^M M! \prod_{s=1}^N (\alpha_s, k)^{M_s} I + \dots] e^{(k,z)} \neq 0, \quad (17)$$



where the dots mean the terms decaying while  $z \rightarrow \infty$ . We claim that  $\Psi_{M+1} = (L + k^2)\Psi_M = 0$ . Indeed, assume that this is not true. Then for some  $j > M$  we have

$$\Psi_{j+1} = (L + k^2)\Psi_j = 0$$

with  $\Psi_j \neq 0$ . Since

$$P_{j+1} = \phi(-\Delta - 2(k, \frac{\partial}{\partial z}) + U(z))\phi^{-1}P_j = 0$$

and  $P_j$  is polynomial in  $k$  its highest coefficient  $P_j^{(0)}$  has to satisfy the condition

$$(k, \frac{\partial}{\partial z})P_j^{(0)} = 0.$$

One can show that this implies that  $P_j^{(0)}$  must be polynomial in  $z$  (see [19], lemma 2.5). On the other hand one can see from (17) that  $\Psi_j$  for  $j > M$  decays as  $z \rightarrow \infty$ . This contradiction means that  $L\Psi_M = -k^2\Psi_M$ . Presenting  $\Psi_M$  in the form  $\Psi_M = \mathcal{D}e^{(k,z)}I$  for a proper matrix differential operator  $\mathcal{D}(z, \frac{\partial}{\partial z})$  we have

$$L\Psi = L\mathcal{D}e^{(k,z)}I = -k^2\mathcal{D}e^{(k,z)}I = -\mathcal{D}k^2e^{(k,z)}I = \mathcal{D}L_0e^{(k,z)}I$$

and, therefore,

$$L\mathcal{D} = \mathcal{D}L_0.$$

The theorem is proved.

*Remark.* Notice that our proof gives an explicit formula for the intertwining operator

$$\mathcal{D}\left(e^{(k,z)}I\right) = (L + k^2)^M \left( \prod_{s=1}^N ((\alpha_s, z) + c_s)^{M_s} e^{(k,z)}I \right).$$

Such a formula has been discovered in the scalar case by Yu.Berest [7].

### 3 Generalised matrix Calogero-Moser system.

Let us consider the following matrix Schrödinger operator

$$L = -\Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{m_\alpha(m_\alpha I - \hat{s}_\alpha)(\alpha, \alpha)}{(\alpha, z)^2}. \quad (18)$$

Here  $\mathcal{R}$  is any Coxeter root system in  $\mathbf{R}^n$ ,  $\mathcal{R}_+$  is its positive part consisting of the normals to the reflection hyperplanes of the corresponding Coxeter group  $G$ ,  $m(\alpha) = m_\alpha$  is a  $G$ -invariant function on  $\mathcal{A}$ ,  $\hat{s}_\alpha$  stands for the reflection with respect to  $\alpha$  in an arbitrary matrix representation  $\pi$  of the group  $G$ :  $\hat{s}_\alpha = \pi(s_\alpha)$

For the trivial one-dimensional representation we have a scalar Schrödinger operator which is the well-known generalised Calogero-Moser operator related

to the Coxeter group  $G$  (see [20]). Thus (18) can be considered as a natural matrix generalisation of these operators.

I. Cherednik [8] seems to be the first to consider such generalisations in the case when  $G$  is a Weyl group of any semisimple Lie algebra. He showed that the corresponding quantum system has  $n$  commuting quantum integrals and, therefore, it is integrable in a usual quantum mechanical sense.

Let us show that if all  $m_\alpha$  are integers then the operator (18) is D-integrable. This implies the usual integrability and even more stronger property known as algebraic integrability (see theorem 7 below).

Let

$$s_\alpha(z) = z - 2 \frac{(\alpha, z)}{(\alpha, \alpha)} \alpha \quad (19)$$

be the orthogonal reflection with respect to the hyperplane  $(\alpha, z) = 0$ . The matrix potential of the operator (18) has the following equivariance property for any  $\alpha \in \mathcal{R}$

$$\hat{s}_\alpha U(z) = U(s_\alpha(z)) \hat{s}_\alpha. \quad (20)$$

This can be easily checked using  $G$ -invariance of  $m_\alpha$  and the property

$$\hat{s}_\alpha \hat{s}_\beta = \hat{s}_{s_\alpha(\beta)} \hat{s}_\alpha.$$

From (20) it follows that the coefficients  $C_l$  of the Laurent expansion of the potential  $U$  near the hyperplane  $(\alpha, z) = 0$  satisfy the following relation

$$\begin{aligned} \hat{s}_\alpha C_{2k} &= C_{2k} \hat{s}_\alpha, \\ \hat{s}_\alpha C_{2k-1} + C_{2k-1} \hat{s}_\alpha &= 0 \end{aligned} \quad (21)$$

for any  $k$ . Comparing the formula (18) with (16) we see that the corresponding matrices  $A_i$  have two eigenvalues:  $m_\alpha(m_\alpha + 1)$  and  $m_\alpha(m_\alpha - 1)$ . Using this it is easy to check that the relations (21) imply the local trivial monodromy conditions (14) and, therefore, D-integrability of the operator (18) due to the theorem 4. Thus, we have proved

**Theorem 5.** *The generalised matrix Calogero-Moser operator (18) with integer  $G$ -invariant  $m_\alpha$  is D-integrable.*

In the scalar case the Calogero-Moser operator admits integrable deformations related to the non-Coxeter configurations of the hyperplanes [13, 14, 5]. It is interesting that these deformations admit a matrix generalisation as well.

Let  $\mathfrak{A}$  be a finite set of the hyperplanes  $\Pi_\alpha$  in a complex Euclidean space  $\mathbf{C}^n$  given by the equations  $(\alpha, z) = 0$ , taken with some multiplicities  $m_\alpha \in \mathbf{Z}_+$ . Here  $\alpha \in \mathcal{A}$ ,  $\mathcal{A}$  is a finite set of noncollinear vectors. Consider the matrix Schrödinger operator

$$L = -\Delta + U(z)$$

with

$$U(z) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha - s_\alpha)(\alpha, \alpha)}{(\alpha, z)^2} \quad (22)$$

where  $s_\alpha$  is the  $n \times n$  matrix of the reflection (19).

**Theorem 6.** *Operator  $L$  has trivial monodromy if and only if the following conditions for the configuration  $\mathcal{A}$  hold for each  $\alpha \in \mathcal{A}$*

$$A_j = \sum_{\beta \neq \alpha} \frac{m_\beta(m_\beta + 1)(\beta, \beta)(\alpha, \beta)^{2j-1}}{(\beta, z)^{2j+1}} \Big|_{(\alpha, z)=0} \equiv 0, \quad j = 1, 2, \dots, m_\alpha, \quad (23)$$

$$B_j = \sum_{\beta \neq \alpha} \frac{m_\beta(\alpha, \beta)^{2j-1}}{(\beta, z)^{2j-1}} \Big|_{(\alpha, z)=0} \equiv 0, \quad j = 1, 2, \dots, m_\alpha. \quad (24)$$

*Proof.* Let us consider first the case  $m_\alpha = 1$ . Then we have two locus conditions (see (14)) for  $L$ :

$$C_0 C_{-2} = C_{-2} C_0 \quad \text{or} \quad C_0 s_\alpha = s_\alpha C_0 \quad (25)$$

and

$$(C_1 \alpha, \alpha) = 0. \quad (26)$$

From (22) we can calculate

$$C_j = \sum_{\beta \neq \alpha} \frac{m_\beta(m_\beta - s_\beta)(\beta, \beta)(\alpha, \beta)^j}{(\beta, z)^{2+j}} \Big|_{(\alpha, z)=0},$$

and condition (25) reduces to

$$\sum_{\beta \neq \alpha} \frac{m_\beta(\beta, \beta)(s_\beta s_\alpha - s_\alpha s_\beta)}{(\beta, z)^2} \Big|_{(\alpha, z)=0} \equiv 0. \quad (27)$$

Let us choose some  $\gamma \neq \alpha$  and consider the subsum in (27) corresponding to the 2-dimensional plane  $\langle \alpha, \gamma \rangle$ . Since  $s_\beta$  acts trivially on the orthogonal complement to the plane  $\pi = \langle \alpha, \gamma \rangle$  for any  $\beta \in \pi$  we may assume that  $\alpha = (1, 0)$ ,  $\beta = (\cos \phi_\beta, \sin \phi_\beta)$ . Then

$$s_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_\beta = \begin{pmatrix} -\cos 2\phi_\beta & -\sin 2\phi_\beta \\ -\sin 2\phi_\beta & \cos 2\phi_\beta \end{pmatrix}$$

and

$$s_\beta s_\alpha = \begin{pmatrix} \cos 2\phi_\beta & \sin 2\phi_\beta \\ -\sin 2\phi_\beta & \cos 2\phi_\beta \end{pmatrix},$$

$$s_\alpha s_\beta = (s_\beta s_\alpha)^{-1}.$$

Since  $(\beta, z)|_{(\alpha, z)=0} = \beta_2 z_2 = z_2 \sin \phi_\beta$  we have to check that

$$\sum_{\beta \in \langle \alpha, \gamma \rangle, \beta \neq \alpha} \frac{\sin 2\phi_\beta}{\sin^2 \phi_\beta} = 0.$$

But it easily follows from the identity (24)  $B_1 \equiv 0$ .

The second locus condition (26) reduces to

$$\sum_{\beta \neq \alpha} \frac{m_\beta (m_\beta + \cos 2\phi_\beta) \cos \phi_\beta}{\sin^3 \phi_\beta} = 0.$$

This is equivalent to the combination  $A_1 - 2B_1 = 0$  of the identities (27–28).

Now let us consider the case  $m_\alpha > 1$ . Locus equations (14) take the form

$$C_j s_\alpha = (-1)^j s_\alpha C_j, \quad j = 0, 1, \dots, 2m_\alpha - 2, \quad (28)$$

$$(C_{2m_\alpha - 1} \alpha, \alpha) = 0. \quad (29)$$

As above, everything reduces to the two-dimensional case and we will use the same notations. The relations (28) reduce to

$$\sum_{\beta \neq \alpha} \frac{m_\beta \cos^j \phi_\beta \sin 2\phi_\beta}{\sin^{2+j} \phi_\beta} = 0, \quad (30)$$

for  $j = 2l, l = 0, 1, \dots, m_\alpha - 1$ , and

$$\sum_{\beta \neq \alpha} \frac{2s_\alpha m_\beta^2 \cos^j \phi_\beta - (s_\alpha s_\beta + s_\beta s_\alpha) m_\beta \cos^j \phi_\beta}{\sin^{2+j} \phi_\beta} = 0, \quad (31)$$

for  $j = 2l - 1, l = 1, \dots, m_\alpha - 1$ . The relations (30) are equivalent to

$$\sum_{\beta \neq \alpha} \frac{m_\beta \cos^{2l+1} \phi_\beta}{\sin^{2l+1} \phi_\beta} = 0,$$

which coincide with  $B_{l+1} \equiv 0$ . Conditions (31) are equivalent to

$$\left\{ \begin{array}{l} \sum_{\beta \neq \alpha} \frac{m_\beta^2 \cos^{2l-1} \phi_\beta}{\sin^{2l+1} \phi_\beta} = 0, \quad l = 1, \dots, m_\alpha - 1 \\ \sum_{\beta \neq \alpha} \frac{m_\beta \cos^{2l-1} \phi_\beta \cos 2\phi_\beta}{\sin^{2l+1} \phi_\beta} = 0, \quad l = 1, \dots, m_\alpha - 1. \end{array} \right.$$

The first part of the last equations due to the identity  $A_l \equiv 0$  reduces to

$$\sum_{\beta \neq \alpha} \frac{m_\beta \cos^{2l-1} \phi_\beta}{\sin^{2l+1} \phi_\beta} = 0,$$

the left hand side of which equals  $B_l + B_{l+1}$ . The second one is equivalent to  $B_{l+1} - B_l \equiv 0$  and is satisfied for  $l = 1, \dots, m_\alpha - 1$ .

Finally, the condition  $(C_{2m_\alpha - 1} \alpha, \alpha) = 0$  is equivalent to

$$\sum_{\beta \neq \alpha} \frac{m_\beta^2 \cos^{2m_\alpha - 1} \phi_\beta + m_\beta \cos 2\phi_\beta \cos^{2m_\alpha - 1} \phi_\beta}{\sin^{2m_\alpha + 1} \phi_\beta} = 0$$

or, using  $A_{m_\alpha} \equiv 0$ , to

$$\sum_{\beta \neq \alpha} \frac{m_\beta \cos^{2m_\alpha - 1} \phi_\beta (\cos 2\phi_\beta - 1)}{\sin^{2m_\alpha + 1} \phi_\beta} = 0,$$

which coincides with  $B_{m_\alpha} \equiv 0$ . Theorem 6 is proved.

*Remark.* It is interesting to note that the conditions (23, 24) are equivalent to the existence of the so-called Baker-Akhiezer function in "old axiomatics" (see [16, 5]). Indeed, the A-conditions (23) coincide with the locus equations for the scalar case and, therefore, guarantee the existence of the Baker-Akhiezer function in "new axiomatics" ([5]). The B-conditions (24) mean that the function  $\phi = \prod_{\beta \neq \alpha} (\beta, z)^{m_\beta}$  has zero odd normal derivatives at the hyperplane  $\Pi_\alpha$ :

$$\left( \frac{\partial}{\partial \alpha} \right)^{2j-1} \prod_{\beta \neq \alpha} (\beta, z)^{m_\beta} \Big|_{(\alpha, z)=0} = 0,$$

which together with the new axiomatics provide the old one (see section 1 in [5]).

In particular, the conditions (23, 24) are satisfied for the following non-Coxeter configurations  $A_n(m)$  and  $C_{n+1}(m, l)$  discovered in [13, 14, 5].

Configuration  $A_n(m)$  consists of the following vectors in  $\mathbf{R}^{n+1}$ :  $e_i - e_j$  with multiplicity  $m$  ( $1 \leq i < j \leq n$ ) and  $e_i - \sqrt{m}e_{n+1}$  with multiplicity 1 ( $i = 1, \dots, n$ ) (for  $m = 1$  we have the root system  $A_n$ ). Parameter  $m$  is also allowed to be negative. Then one should consider vectors  $e_i - e_j$  with the multiplicity  $-1 - m$ . In the last case we have a complex configuration in  $\mathbf{C}^{n+1}$ .

Configuration  $C_{n+1}(m, l)$  consists of the following set of vectors in  $\mathbf{R}^{n+1}$ :

$$C_{n+1}(m, l) = \begin{cases} e_i \pm e_j & \text{with multiplicity } k \\ 2e_i & \text{with multiplicity } m \\ 2\sqrt{k}e_{n+1} & \text{with multiplicity } l \\ e_i \pm \sqrt{k}e_{n+1} & \text{with multiplicity } 1 \end{cases}$$

where  $l$  and  $m$  are integer parameters such that  $k = \frac{2m+1}{2l+1} \in \mathbf{Z}$ ,  $1 \leq i < j \leq n$ . If  $l = m = k = 1$  the system  $C_{n+1}(m, l)$  coincides with the classical root system  $C_{n+1}$ . As before, the parameters  $k, m, l$  may be negative, in that case the corresponding multiplicities should be  $-1 - k$ ,  $-1 - m$  or  $-1 - l$  respectively.

**Corollary.** *The matrix Schrödinger operators with potentials (22) corresponding to the configurations  $A_n(m)$  and  $C_{n+1}(m, l)$  are D-integrable.*

Let us prove now that in the considered cases D-integrability implies usual quantum integrability and even more – so-called algebraic integrability.

We say that a matrix Schrödinger operator  $L$  in  $\mathbf{R}^n$  is *integrable* if there exist  $n$  pairwise commuting matrix differential operators  $L_1 = L, L_2, \dots, L_n$  having the algebraically independent constant scalar highest symbols  $P_j(k)$ ,  $j = 1, \dots, n$ . If there exists one more commuting matrix differential operator  $L_{n+1}$

with the highest constant scalar symbol  $P_{n+1}(k)$  such that  $P_{n+1}(k)$  takes different values on the solutions of the system  $P_i(k) = c_i$  ( $i = 1, \dots, n$ ) for generic  $c_1, \dots, c_n$  the operator  $L$  is called *algebraically integrable* (see [21, 15, 9]).

Let's assume that the matrix potential  $U$  is symmetric:  $U = U^*$ .

**Theorem 7.** *Any  $D$ -integrable matrix Schrödinger operator  $L$  with a rational symmetric potential (16) is algebraically integrable.*

*Proof.* We follow here the idea of the paper [22]. Let  $A^*$  denote a formal conjugate to a matrix differential operator  $A$  then taking a conjugation of the relation  $LD = \mathcal{D}L_0$  we have  $\mathcal{D}^*L^* = L_0^*\mathcal{D}^* = L_0\mathcal{D}^*$ . If  $U = U^*$  then  $L = L^*$  and we obtain  $\mathcal{D}^*L = L_0\mathcal{D}^*$ . Now define the operators  $L_1 = L$ ,  $L_{1+i} = \mathcal{D}\partial_i\mathcal{D}^*$  ( $i = 1, \dots, n$ ). We claim that they are pairwise commuting. Indeed,  $LL_{1+i} = L\mathcal{D}\partial_i\mathcal{D}^* = \mathcal{D}L_0\partial_i\mathcal{D}^* = \mathcal{D}\partial_iL_0\mathcal{D}^* = \mathcal{D}\partial_i\mathcal{D}^*L = L_{1+i}L$ , so  $[L_1, L_k] = 0$  for all  $k = 2, \dots, n+1$ . Consider now the commutator  $[L_l, L_k]$ ,  $l > 1$ . From the previous relations and Jacobi identity it follows that  $[[L_k, L_l], L] = 0$ . Berezin's lemma (see lemma 2.5 in [19]) says that the highest symbol of  $[L_k, L_l]$  has to be polynomial in  $z$ , but from the definition of  $L_k$  and the construction of  $\mathcal{D}$  it follows that it decays as  $z \rightarrow \infty$ . This means that  $[L_k, L_l] = 0$  for any  $k, l = 1, \dots, n+1$ . One can check that the highest symbols of  $L_k$  satisfies the property demanded at the definition of algebraic integrability. The theorem is proved.

*Remark.* The statement of the theorem seems to be true without the assumption of the symmetry of the potential.

## 4 Two-dimensional case.

Let us consider the matrix locus configurations on the plane in the case when all the lines pass through the origin. In the scalar case essentially all such configurations have been described by Yu.Berest and I.Lutsenko [23] (see [5] for details).

In the matrix case in the polar coordinates  $(r, \phi)$  the corresponding potential  $U$  has a form

$$U(r, \phi) = \frac{1}{r^2}V(\phi), \quad (32)$$

where

$$V(\phi) = \sum_{i=1}^k \frac{A_i}{\sin^2(\phi - \phi_i)}. \quad (33)$$

Here  $A_i$  are some matrices,  $\phi_i$  are the angles corresponding to the lines of configurations. Strictly speaking this is true only on the real plane  $\mathbf{R}^2$  but this can be easily generalised to  $\mathbf{C}^2$  (see [5]).

**Proposition 1.** *Two-dimensional matrix Schrödinger operator*

$$L = -\Delta + U \quad (34)$$

with the potential (32) has trivial monodromy if and only if the same is true for the one-dimensional Schrödinger operator

$$\mathcal{L} = -\frac{d^2}{d\phi^2} + V(\phi) \quad (35)$$

with trigonometric potential (33).

The proof is a simple check that the local trivial monodromy conditions for these two operators are equivalent.

**Proposition 2.** *If the operator (35) is D-integrable then the same is true for the two-dimensional Schrödinger operator (34).*

Indeed, in one-dimensional case D-integrability implies the trivial monodromy for the operator (35) and, therefore, for the two-dimensional operator (34). According to the theorem 4 this guarantees the D-integrability of (34).

In dimension 1 D-integrability is equivalent to the fact that the operator  $L$  (35) is the result of so-called matrix Darboux transformation applied to  $\mathcal{L}_0 = -\frac{d^2}{d\phi^2}$  (see e.g. [4]). All such operators can be described using the notion of quasideterminants introduced by I.Gelfand and V.Retakh (see [24]).

Let  $k$  be the order of the intertwining operator  $\mathcal{D}$ . Consider any solution  $\Phi$  of the simple matrix differential equation  $-\frac{d^2}{d\phi^2}\Phi = \Phi C$  where  $\Phi$  is  $d \times kd$  matrix,  $C$  is any diagonalisable  $kd \times kd$  matrix with the eigenvalues of the form  $\lambda = p^2$  with  $p \in \mathbf{Z}$ . Let  $\Phi = (\Psi_1, \dots, \Psi_k)$  where  $\Psi_i$  are the corresponding  $d \times d$  matrices. Then the intertwining operator  $\mathcal{D}$  can be written as quasideterminant

$$\mathcal{D}(\Psi) = |W(\Psi_1, \dots, \Psi_k, \Psi)|_{k+1, k+1},$$

where

$$W(\Psi_1, \dots, \Psi_k, \Psi) = \begin{pmatrix} \Psi_1 & \dots & \Psi_k & \Psi \\ \vdots & \ddots & \vdots & \vdots \\ \Psi_1^{(k-1)} & \dots & \Psi_k^{(k-1)} & \Psi^{(k-1)} \\ \Psi_1^{(k)} & \dots & \Psi_k^{(k)} & \Psi^{(k)} \end{pmatrix},$$

(see [4] for the details). The potential  $U$  has a form

$$U = 2a_1'(\phi), \quad (36)$$

where  $a_1(\phi)$  is the first matrix coefficient of  $\mathcal{D}$ :

$$\mathcal{D} = D^k + a_1(\phi)D^{k-1} + \dots + a_k(\phi), \quad D = \frac{d}{d\phi}.$$

Under some assumptions on  $\Phi$  one can give more explicit formula for the potential (see [4]).

**Theorem 8.** *Two-dimensional matrix Schrödinger operator (34) with the potential of the form (32) related to any result (36) of the one-dimensional matrix Darboux transformation described above has trivial monodromy and therefore*

*D-integrable. Conversely, for any D-integrable operator (34) the corresponding one-dimensional operator (35) is related to the operator  $\mathcal{L}_0 = -d^2/d\phi^2$  by a matrix Darboux transformation.*

The proof of the inverse statement follows from

**Lemma 2.** *Any one-dimensional Schrödinger operator (34) with trigonometric potential (33) which satisfies local trivial monodromy conditions at all the singularities is D-integrable.*

Proof of the lemma essentially combines the arguments of the matrix rational case (see [4] or theorem 3 above) and the scalar trigonometric case investigated in [3].

It is worthy to derive the explicit formula for such operators in the simplest case of three lines with prescribed  $2 \times 2$  matrices with the eigenvalues 0 and 2. In this case it seems to be more suitable to use matrix locus equations rather than Darboux transformation. Thus, let  $V(\phi)$  be of the form

$$V(\phi) = \frac{2P_\alpha}{\sin^2(\phi - \alpha)} + \frac{2P_\beta}{\sin^2(\phi - \beta)} + \frac{2P_\gamma}{\sin^2(\phi - \gamma)}, \quad (37)$$

where  $\phi, \alpha, \beta, \gamma \in \mathbf{C}$ ;  $P_\alpha, P_\beta, P_\gamma$  are some projector matrices of rank 1. According to the theorem 1 the operator (34) has trivial monodromy if its Laurent expansion at pole  $\phi = \phi_0$

$$V(\phi) = \frac{C_{-2}}{(\phi - \phi_0)^2} + \frac{C_{-1}}{(\phi - \phi_0)} + C_0 + C_1(\phi - \phi_0) + \dots$$

satisfies the conditions

$$C_{-1} = 0,$$

$$[C_{-2}, C_0] = 0, \quad (38)$$

$$C_{-2}C_1C_{-2} = 0. \quad (39)$$

Conditions  $C_{-1} = 0$  are, obviously, fulfilled. Expanding  $V(\phi)$  near  $\phi = \alpha$

$$\begin{aligned} V(\phi) &= \frac{2P_\alpha}{(\phi - \alpha)^2} + \left( \frac{2P_\beta}{\sin^2(\alpha - \beta)} + \frac{2P_\gamma}{\sin^2(\alpha - \gamma)} + \frac{2P_\alpha}{3} \right) + \\ &+ \left( \frac{-4P_\beta \cos(\alpha - \beta)}{\sin^3(\alpha - \beta)} + \frac{-4P_\gamma \cos(\alpha - \gamma)}{\sin^3(\alpha - \gamma)} \right) (\phi - \alpha) + \dots, \end{aligned}$$

and then near  $\phi = \beta$  and  $\phi = \gamma$  we get the following system of the equations using (38)

$$\left[ P_\alpha, \frac{P_\beta}{\sin^2(\alpha - \beta)} + \frac{P_\gamma}{\sin^2(\alpha - \gamma)} \right] = 0, \quad (40)$$

$$\left[ P_\beta, \frac{P_\alpha}{\sin^2(\beta - \alpha)} + \frac{P_\gamma}{\sin^2(\beta - \gamma)} \right] = 0, \quad (41)$$



$$\left[ P_\gamma, \frac{P_\alpha}{\sin^2(\gamma - \alpha)} + \frac{P_\gamma}{\sin^2(\gamma - \beta)} \right] = 0. \quad (42)$$

It is easy to see that (42) follows from (40) and (41). Conditions (39) give

$$P_\alpha \left( \frac{P_\beta \cos(\alpha - \beta)}{\sin^3(\alpha - \beta)} + \frac{P_\gamma \cos(\alpha - \gamma)}{\sin^3(\alpha - \gamma)} \right) P_\alpha = 0, \quad (43)$$

$$P_\beta \left( \frac{P_\alpha \cos(\beta - \alpha)}{\sin^3(\beta - \alpha)} + \frac{P_\gamma \cos(\beta - \gamma)}{\sin^3(\beta - \gamma)} \right) P_\beta = 0, \quad (44)$$

$$P_\gamma \left( \frac{P_\alpha \cos(\gamma - \alpha)}{\sin^3(\gamma - \alpha)} + \frac{P_\beta \cos(\gamma - \beta)}{\sin^3(\gamma - \beta)} \right) P_\gamma = 0. \quad (45)$$

Solving of the system of the equations (40 - 45) and making a suitable transformation

$$V(\phi) \longrightarrow C V(\phi) C^{-1} \quad (46)$$

we arrive at the formula

$$P_\alpha = \frac{1}{\sin(\alpha - \beta) \sin(\alpha - \gamma)} \cdot \xi_\alpha^t \eta_\alpha, \quad (47)$$

where

$$\begin{aligned} \xi_\alpha &= (-\cos \alpha, \sin \alpha), \eta_\alpha = (s(\alpha; \beta, \gamma), c(\alpha; \beta, \gamma)), \\ s(\alpha; \beta, \gamma) &= \sin^2 \alpha \cos(\beta + \gamma - \alpha) - \cos \alpha \sin \beta \sin \gamma, \\ c(\alpha; \beta, \gamma) &= \cos^2 \alpha \sin(\beta + \gamma - \alpha) - \sin \alpha \cos \beta \cos \gamma. \end{aligned}$$

Projectors  $P_\beta$  and  $P_\gamma$  can be obtained by corresponding permutations of  $\alpha$ ,  $\beta$  and  $\gamma$ .

**Theorem 9.** *Any three lines on the plane with prescribed matrices  $P_\alpha$ ,  $P_\beta$  and  $P_\gamma$  (47) where  $\alpha$ ,  $\beta$ ,  $\gamma$  are the corresponding angles form a matrix locus configuration. Modulo (46) this describes all three lines  $2 \times 2$  matrix locus configurations with prescribed matrices having the eigenvalues 0 and 2.*

It is interesting to note that the potential (37, 47) is symmetric if and only if  $\alpha = \vartheta$ ,  $\beta = \vartheta + \frac{\pi}{3}$  and  $\gamma = \vartheta + \frac{2\pi}{3}$  for some  $\vartheta$  which corresponds to the matrix Calogero-Moser system (18) related to  $A_2$  root system.

**Concluding remarks.** Similarly to the scalar case [5] one can introduce the notion of the multidimensional matrix Baker-Akhiezer function. This would lead to the proof of the algebraic integrability for the corresponding Schrödinger operators. The bispectral properties of these functions and the relations to the Huygens' Principle we are planning to discuss in a separate paper.

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## References

- [1] Yu.Yu. Berest, A.P. Veselov *On the singularities of the potentials of exactly solvable Schrödinger operators and Hadamard's problem*. Russian Math. Surveys **53** (1) (1998), 211–212.
- [2] Yu.Yu. Berest, A.P. Veselov *On the structure of singularities of integrable Schrödinger operators*. Submitted to Letters in Math. Physics.
- [3] O.A. Chalykh *Darboux transformations for multidimensional Schrödinger operators*. Russian Math. Surveys **53** (2) (1998), 167–168.
- [4] V.M. Goncharenko, A.P. Veselov *Monodromy of the matrix Schrödinger equations and Darboux transformations*. J.Phys. A: Math.Gen **31** (1998), 5315–5326.
- [5] O.A. Chalykh, M.V. Feigin, A.P. Veselov. *Multidimensional Baker-Akhiezer Functions and Huygens' Principle*. Submitted to Commun. Math. Physics.
- [6] J.J. Duistermaat, F.A. Grünbaum *Differential equations in the spectral parameter*. Commun. Math. Phys. **103** (1986), 177–240.
- [7] Yu.Yu. Berest *Huygens' Principle and the bispectral problem*. CRM, Proceedings and Lecture Notes **14** (1998), 11–30.
- [8] I. Cherednik *Integration of quantum many-body problems by affine Knizhnik-Zamolodchikov equations*. Adv. Math. **106** (1) (1994), 65–95.
- [9] P. Etingof *Quantum integrable systems and representations of Lie algebras*. J. Math. Phys. **36** (6) (1995), 2636–2651.
- [10] P. Etingof, K. Styrkas *Algebraic integrability of Schrödinger operators and representations of Lie algebras*. Compositio Math. **98** (1) (1995), 91–112.
- [11] A. González-López, N. Kamran. *The multidimensional Darboux transformation*. J. Geom. Phys. **26** (3-4) (1998), 202–226.
- [12] P. Bracken, N. Kamran *Matrix Calogero-Moser hamiltonians and the multidimensional Darboux transformation*. Preprint, 1998.
- [13] A.P. Veselov, M.V. Feigin, O.A. Chalykh *New integrable deformations of quantum Calogero - Moser problem*. Usp. Mat. Nauk **51** (3) (1996), 185–186.
- [14] O.A. Chalykh, M.V. Feigin, A.P. Veselov *New integrable generalizations of Calogero-Moser quantum problem*. J. Math. Phys **39** (2) (1998), 695–703.
- [15] O.A. Chalykh, A.P. Veselov *Commutative rings of partial differential operators and Lie algebras*. Commun. Math. Phys. **126** (1990), 597–611.

- [16] A.P. Veselov, K.L. Styrkas, O.A. Chalykh *Algebraic integrability for Schrödinger equation and finite reflection groups*. Theor. Math. Phys. **94** (2) (1993), 253–275.
- [17] J. Gibbons, T. Hermsen *A generalization of the Calogero-Moser system*. Physica D **11** (1984), 337–348.
- [18] I. Krichever, O. Babelon, E. Billey, M. Talon *Spin generalization of the Calogero-Moser system and the matrix KP equation*. Amer. Math. Soc. Transl. **170** (2) (1995), 83–119.
- [19] F.A. Berezin *Laplace operators on semisimple Lie groups*. Proc. Moscow Math. Soc. **6** (1957), 371–463.
- [20] M.A. Olshanetsky, A.M. Perelomov *Quantum integrable systems related to Lie algebras*. Phys. Rep. **94** (1983), 313–404.
- [21] I.M. Krichever *Methods of algebraic geometry in the theory of nonlinear equations*. Usp. Mat. Nauk **32** (6) (1977), 183–208.
- [22] O.A. Chalykh *Additional integrals of the generalized quantum Calogero-Moser problem*. Theoret. and Math. Phys. **109** (1) (1996), 1269–1273.
- [23] Yu. Yu. Berest, I. M. Lutsenko *Huygens' principle in Minkowski spaces and soliton solutions of the Korteweg-de Vries equation* Commun. Math. Phys. **190** (1997), 113–132.
- [24] I. Gelfand, V. Retakh *Determinants of matrices over noncommutative rings* Funct. Anal. Appl. **25** (2) (1991), 91–102.