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**Modulational instabilities and Fermi – Pasta – Ulam recurrence
in a coupled long wave – short wave system, with a mismatch in
group velocity**

by

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ABSTRACT

The resonance of two envelopes of short (capillary) waves with a common long (gravity) wave component is considered. A mismatch in group velocity is incorporated and the resonance conditions need not be satisfied exactly. This slight detuning permits a wider choice of modes and consequently, a much richer set of dynamics. The linear instability of plane waves is studied, and the dominant unstable wave numbers are identified. The subsequent fully nonlinear evolution of these perturbed plane waves is investigated by direct numerical simulations. The Fermi – Pasta – Ulam recurrence phenomenon is observed.

1. Introduction

Modulational instabilities of plane waves and their subsequent evolution may have profound influence on the dynamics of many nonlinear systems (e.g. Fujimura et al, 1988). These studies are relevant in many fields, e.g., anharmonic lattices (Burlakov et al, 1996), fluid dynamics (Janssen, 1981; Xu and Wei, 1992), optics (Cambournac et al, 2002; Simaey et al, 2002) and plasma physics (Sharma et al 2005). In many instances, this linear, modulational instability will lead to a time-periodic return of the perturbed system into a state very close to the original, primary state. This phenomenon is commonly termed the Fermi – Pasta – Ulam recurrence (FPU) in the literature. A comprehensive analytical approach is lacking, but a first attempt restricts attention to the lowest order Fourier modes. Reasonable approximations of the period of recurrence can then be obtained (Infeld, 1981).

Fluid dynamics is probably the field where FPU and related questions have been studied most intensively. Surface waves on deep water constitute the platform where a combination of theoretical, computational and experimental techniques provides a penetrating investigation of the interplay among triad and quartet resonances, FPU, and the long time dynamics (Yuen and Lake, 1982).

The focus of the present work is on the dynamics of interacting long wave with short waves, which will be seen to provide another platform for studying FPU. Earlier works have demonstrated that the necessary condition for such ‘long – short’ resonance is that the group velocity of the short wave is equal to the phase velocity of the long wave; such a resonance can be treated as a degenerate case of a triad resonance (Benney, 1977; Grimshaw, 1977; Grimshaw, 1981).

More precisely, we shall study coupled systems where two short waves are in resonance with a common long waves. Some special cases, e.g., modes with equal group velocity, are amenable to analytical treatment (Ma, 1981). A main goal of the present study is

to remove this constraint. The inclusion of a difference in group velocity allows a wider choice in the selection of modes, and permits a richer set of dynamics.

Modulational instability for a single component, or uncoupled, long – short system has been studied earlier. The plane wave will generally be unstable for even fairly small amplitudes of the primary wave, if the dispersion coefficient is sufficiently small, a common occurrence in stratified fluid flows in the environment (Ma and Redekopp, 1979). These properties were further explored by Koop and Redekopp (1981). The instability is independent of the initial magnitude of the long wave, and a long wave can be generated by an unstable short wave. Such instability is ‘unidirectional’ in the sense that a long wave cannot generate the short wave if the short wave is absent initially. Experimental aspects were also considered in that work. Yoshinaga et al (1991) performed numerical studies on the associated recurrence phenomena using a slightly modulated short wave as an initial condition. Both recurrence and chaotic motions are possible, depending on the magnitude of certain control parameters of the governing system.

2. Theory

2.1 Background

By choosing the resonance conditions in an appropriate manner, the nonlinear evolution equations take the form

$$iA_T - iVA_X + \delta_1 A_{XX} = \lambda_1 LA, \quad (1)$$

$$iB_T + iVB_X + \delta_2 B_{XX} = \lambda_2 LB, \quad (2)$$

$$L_T = \mu_1 (|A|^2)_X + \mu_2 (|B|^2)_X. \quad (3)$$

A , B , L are the two short waves and the long wave respectively, and V measures the mismatch between the group velocity of the short waves, and the phase velocity of the long wave. T and X are the properly scaled slow time and space coordinates in the averaged –

group velocity reference frame. The linear part of such an evolution operator is quite generic in many nonlinear evolution systems (Aranson and Kramer, 2002). The values for the other parameters will depend on the exact configurations of the fluid flow under consideration (Funakoshi and Oikawa, 1983).

Plane waves for this system are readily obtained:

$$A = A_o \exp(-i\lambda_1 L_o T) , \quad B = B_o \exp(-i\lambda_2 L_o T) , \quad L = L_o , \quad (4)$$

where A_o, B_o, L_o are the initial amplitudes of the unperturbed plane waves. Linear stability is studied by introducing small perturbations, and searching for modal solutions with a factor of $\exp[i(p x - \Omega t)]$. The dispersion relation is:

$$\begin{aligned} & \Omega^5 + (-2p^2 V^2 - p^4 \delta_1^2 - p^4 \delta_2^2) \Omega^3 + (2p^5 V \delta_1^2 - 2p^5 V \delta_2^2 + 2p^3 A_o^2 \delta_1 \lambda_1 \mu_1 + 2p^3 B_o^2 \delta_2 \lambda_2 \mu_2) \Omega^2 \\ & + (p^4 V^4 - p^6 V^2 \delta_1^2 - p^6 V^2 \delta_2^2 + p^8 \delta_1^2 \delta_2^2 - 4p^4 V A_o^2 \delta_1 \lambda_1 \mu_1 + 4p^4 V B_o^2 \delta_2 \lambda_2 \mu_2) \Omega + \\ & (2p^5 V^2 A_o^2 \delta_1 \lambda_1 \mu_1 - 2p^7 A_o^2 \delta_1 \delta_2^2 \lambda_1 \mu_1 + 2p^5 V^2 B_o^2 \delta_2 \lambda_2 \mu_2 - 2p^7 B_o^2 \delta_1^2 \delta_2 \lambda_2 \mu_2) = 0 \end{aligned} , \quad (5)$$

p is the wave number. Ω is the angular frequency, and complex Ω implies modulational instability, with the imaginary part representing a rate of growth of the disturbance.

2.2 Numerical simulations

Linear instability is valid provided the nonlinear terms are negligible. To study the eventual fate of the instability for sufficiently large disturbances, we solve the system (1 – 3) numerically with these initial conditions:

$$A = A_o [1 - \varepsilon \cos pX] , \quad (6a)$$

$$B = B_o [1 - \varepsilon \cos pX] , \quad (6b)$$

$$L = 0 , \quad (6c)$$

where A_o, B_o are the initial amplitudes of the two unperturbed plane waves. The strength of the perturbation ε is set to be 0.1 for all the simulations reported in this study, and p is the wave number of the perturbation. The Hopscotch finite difference technique will be used

(Taha and Ablowitz 1984; Tsang and Chow, 2004). This uses both ‘implicit’ and ‘explicit’ schemes to march forward in time. By using a special arrangement of the grid points, all the intermediate variables can be computed ‘explicitly’ and these schemes are stable.

3. Numerical results

The dispersion relation for capillary – gravity waves dictates that relatively long waves are dominated by gravity, while relatively short waves are influenced mainly by capillary effects. An inspection of the relevant dispersion relation shows that only capillary waves can satisfy the resonance condition with a long gravity wave. Rescaling in (1) allows one to set $\delta_1 = 1$, $\lambda_1 = 1$, and further rescaling of A and B permits one to put $\mu_1 = 1$ and $\mu_2 = \pm 1$, leaving V, δ_2, λ_2 as the actual free parameters. The wave mode B can be categorized as a capillary wave if δ_2 (related to the second derivative of the dispersion relation) is positive, or a gravity wave if δ_2 is negative. Since we note that a gravity wave cannot satisfy the long – short resonance condition with another long gravity wave, we shall focus on the case δ_2 being positive. We shall, however, still present one case of negative δ_2 , since we can treat (1 – 3) as a generic system for long – short wave interaction, without any particular reference to surface waves in mind. For simplicity of presentation we fix all the other parameters as plus or minus one except for the mismatch in group velocity V . A fully three dimensional search in the parameter space can be performed but is unlikely to change the main physics. The cases we will consider are summarized in Table 1.

Case	Wave mode A	Wave Mode B	δ_1	λ_1	μ_1	δ_2	λ_2	μ_2
I	Capillary	Capillary	1	1	1	1	1	1
II	Capillary	Gravity	1	1	1	-1	-1	-1

3.1 Capillary-capillary short waves interacting with long gravity waves

Dependence on the perturbation wave number p

In the first case, we will study the interactions among two short capillary wave envelopes (A and B) and one long wave component (L), which are effectively two degenerate resonant triads with the long wave in common. We first treat the case where the initial amplitudes for both waveguides A and B are the same ($A_o = B_o = 0.2$), and there is no group velocity mismatch ($V = 0$) between A and B . There is a single interval of linear instability (Figure 1a), and perturbation modes in that range will grow initially. Here we only show the long wave component, since the behavior of the short wave envelopes is similar. Nonlinear effects will soon come into play and a sequence of growth and decay results (Fermi – Pasta – Ulam (FPU) recurrence, Figure 1b). For comparison purpose, the results for an initial condition with wave number ($p = 2$) outside the band of instability are also reported. For small times the long wave component is almost unchanged, as that wave number corresponds to stability. However, discretization errors of the finite difference schemes and noise in the computations will eventually generate wave modes in the unstable band, and hence the FPU property again manifests itself (Figure 1c).

The dynamics is even more intriguing if one introduces a mismatch in group velocity ($V = 1$) between the two short waves. The unstable region for modulational instability splits into three bands (Figure 2a). The recurrence behavior becomes much more complex as there are three local maxima for the perturbation wave number p (Figure 2b). Observe that each ‘hill’ or ‘blob’ in the recurrence has a wavelength of about 4 to 6, which is consistent with a wave number p of about 1 to 1.25 (since wave number = $2 \pi / \text{wavelength}$).

Dependence on the initial amplitudes

When the initial amplitudes A_o , B_o are varied and a group velocity difference ($V = 1$) is present, the dynamics is even more complex and will depend on the relative magnitude of

A_o , B_o as the introduction of the mismatch V breaks the symmetry of the system. First we consider the case of A_o being smaller. The unstable region (Figure 3a) diminishes dramatically both in magnitude and extent. The impact on the dynamics is clearly illustrated in Figure 3b for the typical case of the amplitude of A_o being small (0.02) and B_o being large (0.2). Recurrence behavior is again observed. The period of about 500 time units is fairly well defined, and the fluctuations in the long wave component are smaller, again consistent with the smaller magnitude of the linear instability.

On the other hand, the case of a relatively smaller amplitude B might still have a fairly large band of instability. Figure 4a illustrates the situation for A_o being 0.2 and B_o being 0.02. Numerical simulations again shows some recurrence phenomena, with each ‘hill’ or ‘blob’ having a wavelength of about 5, consistent with the dominant unstable wave number p of about 1 (Figure 4b).

An extreme case is the scenario where the amplitude B is almost zero initially. The remarkable fact is that the magnitude of B can still grow to order one through this ‘long – short’ resonance mechanism. Further details are described in section 3.3 below.

Dependence on (further) increase in mismatch in group velocity

As we increase the mismatch in group velocity between the two short wave envelopes further ($V = 2$), the instability bands shift to higher wave numbers, with two closely clustered bands, and a third, negligible one near the origin (Figure 5a). The dynamical implication is that the predominant feature in any Fermi – Pasta – Ulam recurrence will have smaller wavelength. This feature is clearly revealed in Figure 5b, where the cycle of ‘self localization into a sequence of peaks, demodulation to the near original state, and self localization again’ can be observed. The ‘hills’ or ‘blobs’ have wavelength of about 3, a manifestation of the maximum instability of about $p = 2$. Some ‘hills’ exhibit twin peaks, probably as a result of the closely clustered bands of instability. The whole structure is rather robust as more or less

the same feature is demonstrated again with a different initial condition. Here we choose a smaller wave number ($p = 0.15$) as the starting point. Even though this choice is far away from the bands of dominant instabilities, discretization errors and noise, as argued above, will eventually generate modes with unstable wavelength. Thus the recurrence behavior is again observed after a fairly quiet period for small time (Figure 5c).

When the mismatch in group velocity is increased still further ($V = 3$), the bands of wave numbers for instability continue to shift higher (Figure 6a). Again full numerical simulations of the long – short interaction equations (1 – 3) show the FPU recurrence (Figure 6b). Not surprisingly, the wave lengths of the recurring states (about 2) correlate strongly with the unstable wave number of about 3 (again wave number = 2π /wavelength). The region or time interval of demodulation to nearly original state is now much longer.

3.2 Dispersive terms of different signs

We now consider the case where the dispersive effects in the two waveguides are of opposite signs, which corresponds to the signs of the second derivative terms in equations (1) and (2) being different. Although this situation cannot be realized for gravity-capillary waves, it can arise in stratified fluid flows, where there are several branches for the dispersion relation. Furthermore, we can consider (1 – 3) as a generic system of long – short waves, without reference to any fluid flow setting, and then it is always conceivable that a medium will exist with the desired conditions. In fact, in nonlinear optics, such dual-channel systems exist and are applicable. One channel, which supports bright solitons, will act as the payload channel, while the other one with an opposite sign of dispersion, will serve as the support channel by providing the necessary stability property (Tsang et al, 2005). Here for simplicity we shall choose the coefficients as $\mu_1 = 1$ and $\mu_2 = -1$. There is no linear instability for $V = 0$, and hence we shall focus on nonzero V .

With the introduction of a mismatch in group velocity ($V = 1$), a band of wave numbers corresponding to modulational instabilities appears (Figure 7a). Full scale numerical simulations of the governing equations (1 – 3) again show a form of recurrence behavior (Figure 7b). The wavelength of in the growth cycle correlates very well with the predominant mode in the linear instability regime (Figure 7b). Hence we can conclude that this FPU recurrence is fairly robust for this type of evolution equations involving long wave – short waves. Furthermore, on changing the amplitude ratios (whether A large, B small or vice versa), the unstable regions do not change drastically, unlike the previous section. Recurrence behavior can again be found, but the details will be omitted.

3.3 Generation of a short wave by another short wave

Even if the amplitude B is negligibly small (0.000002) initially, a suitable set of parameters can be chosen where B can grow through this resonance mechanism with a common long wave component. Figure 8 illustrates this situation where only the amplitude A is present initially, but the amplitude B eventually attains a magnitude of order one, and then affects the FPU phenomena of A in a significant manner.

4. Discussions and conclusions

A system of two short waves in resonance with a common long wave is considered. The novel feature here is that a mismatch in group velocity is allowed, and hence a wider choice of modes is permitted. Linear, modulational instability of plane waves is identified, and the fate of these instabilities is studied by full numerical simulations. In general some forms of Fermi – Pasta – Ulam recurrence (FPU) exist for this coupled system. The FPU recurrence is altered accordingly in the sense that the wavelengths of the underlying structures shorten, as the dominant wave numbers of the instabilities shift to higher values. The period of recurrence seems to lengthen with an increasing difference in group velocity.

Furthermore, such modulational instabilities and FPU continue to exist in systems of long – short waves with opposite dispersions, and thus such phenomena are fairly robust and probably universal for a generic long – short interaction system.

In a single component, or uncoupled, system, short wave instability can generate long waves, but not vice versa, i.e., short waves cannot be generated if they are absent initially. While this is still broadly true for the coupled system, a short wave can generate another short wave via their common long wave partner. Such a scenario could be especially relevant in oceanic applications.

One question which has not been properly addressed in this work is the existence of chaotic motion, which has been demonstrated earlier in the literature (Yoshinaga et al, 1991). For the special classes of initial conditions employed here, no chaotic motions are observed but the existence of such chaos cannot be ruled out. Further work along these lines will be fruitful in the future.

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Figures

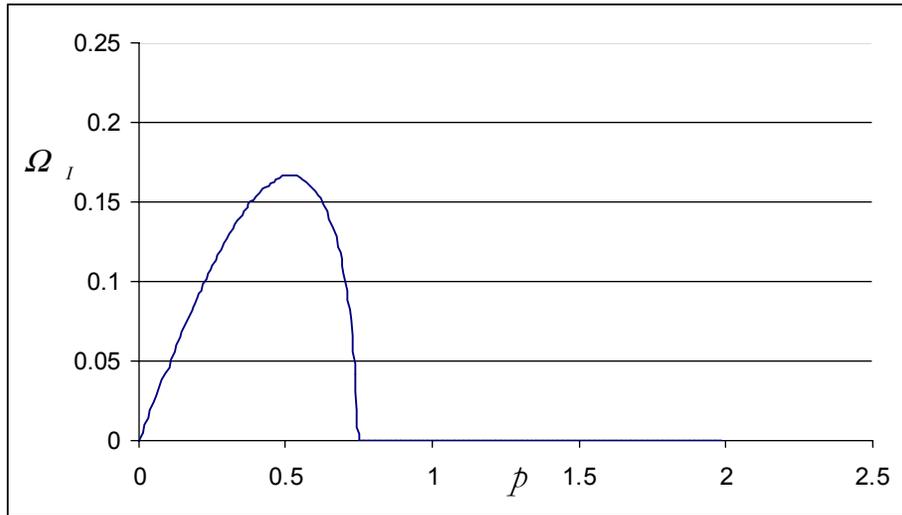


Figure 1a: The growth rate for modulational instabilities with group velocity mismatch $V = 0$ and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

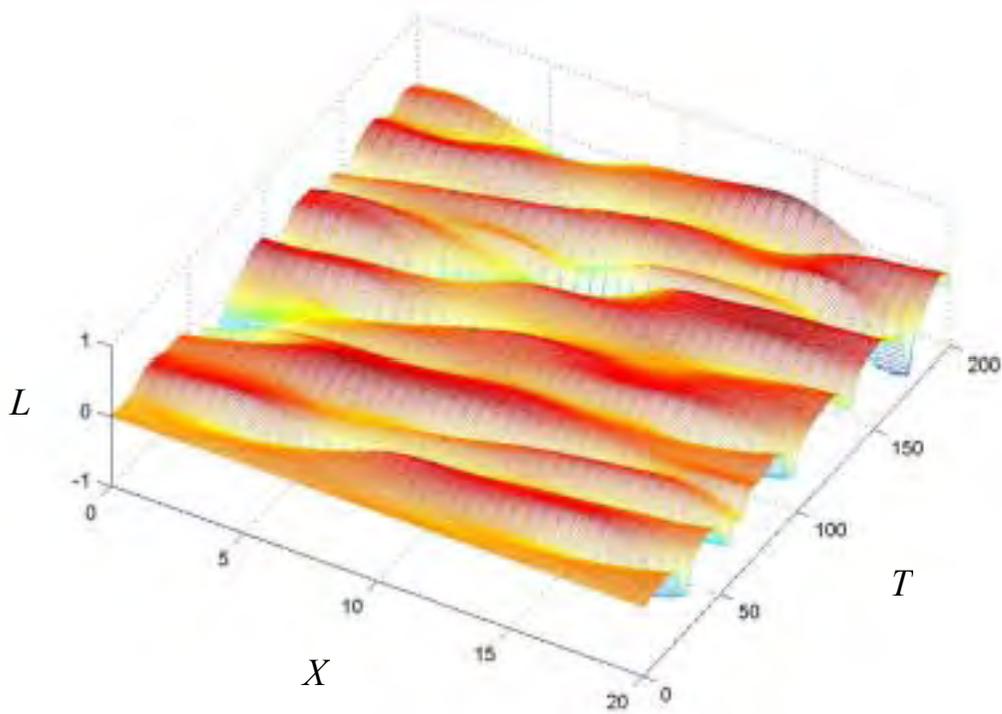


Figure 1b: Long-time evolution of the long wave L with group velocity mismatch $V = 0$, wave number of perturbation $p = 0.5$, and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

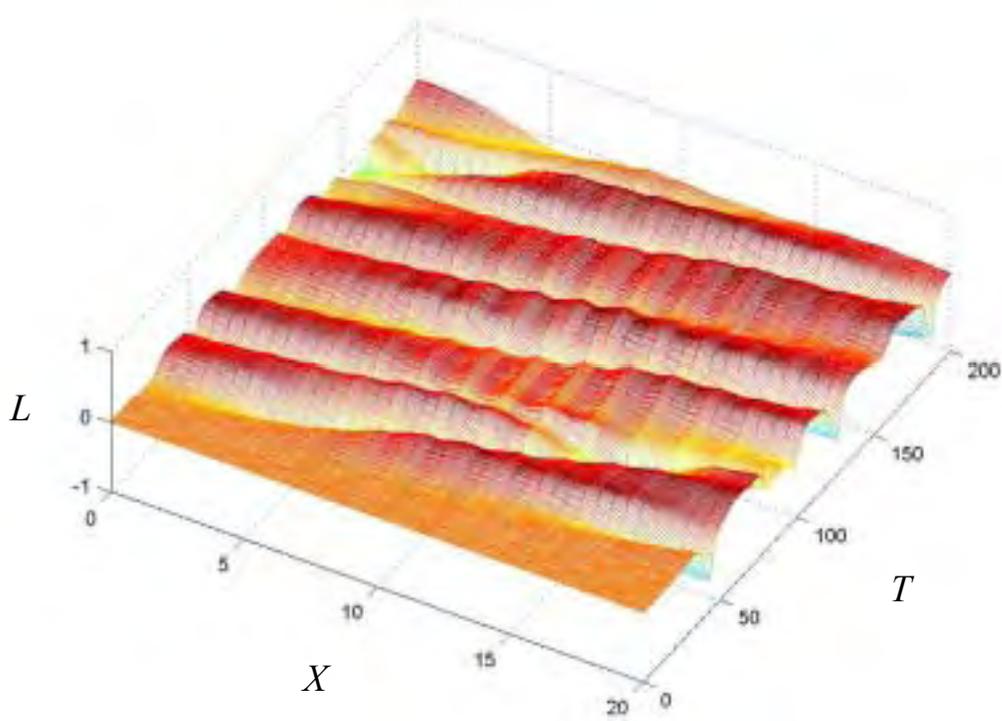


Figure 1c Long-time evolution of the long wave L with group velocity mismatch $V = 0$, wave number of perturbation $p = 2$, and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

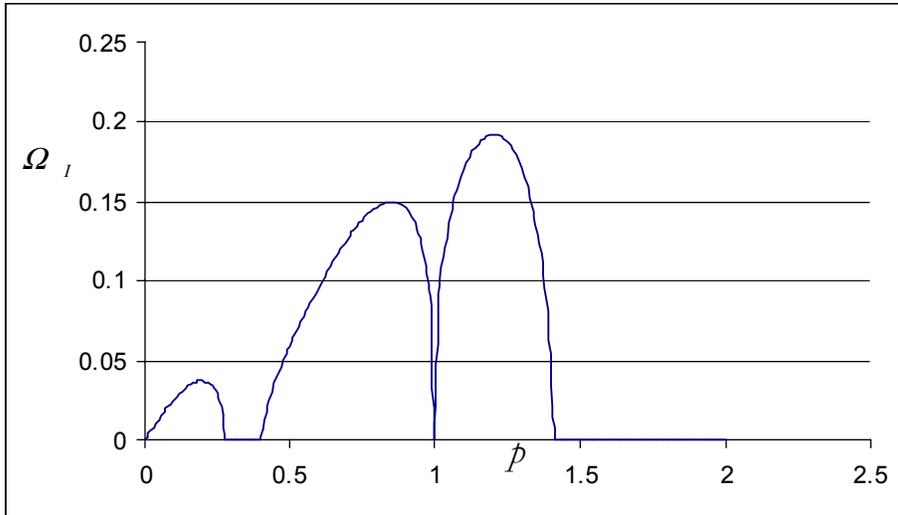


Figure 2a: The growth rate for modulational instabilities with group velocity mismatch $V = 1$ and initial amplitudes $A_0=0.2$, $B_0=0.2$

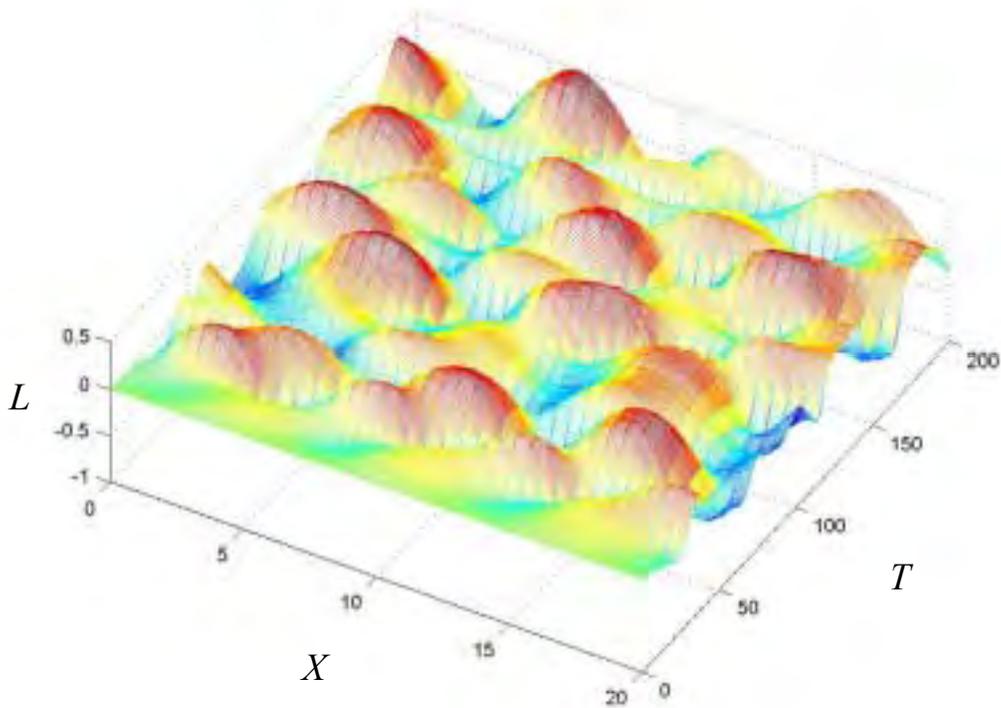


Figure 2b Long-time evolution of the long wave L with group velocity mismatch $V = 1$, wave number of perturbation $p = 1$, and initial amplitudes $A_0=0.2$, $B_0=0.2$

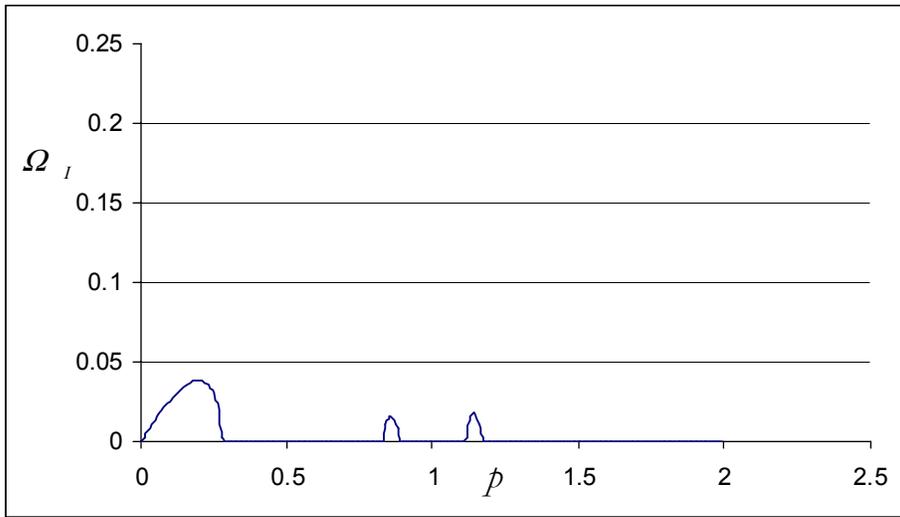


Figure 3a: The growth rate for modulational instabilities with group velocity mismatch $V = 1$ and initial amplitudes $A_0 = 0.02$, $B_0 = 0.2$

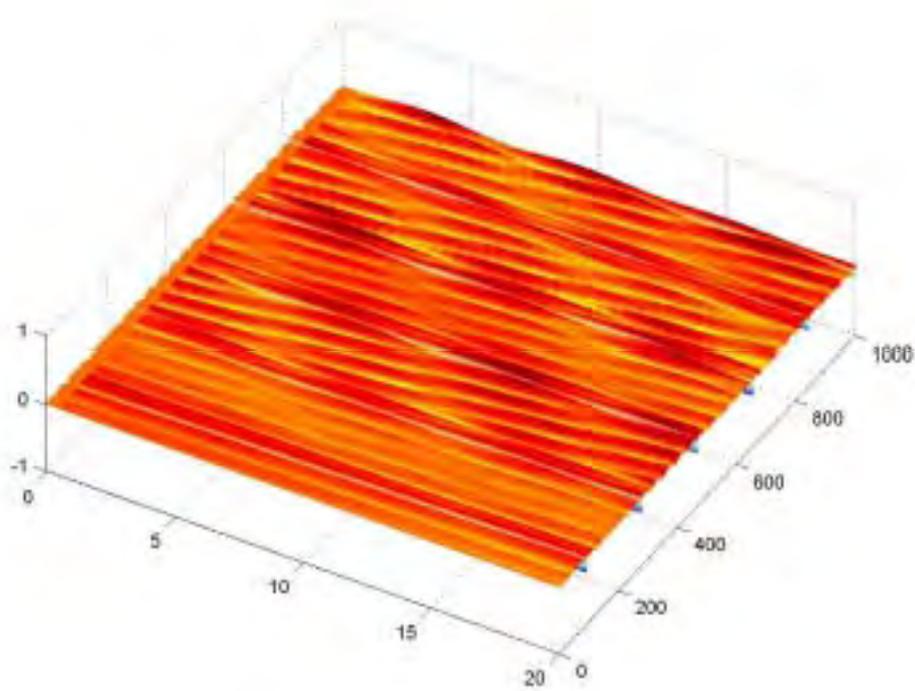


Figure 3b: Long-time evolution of the long wave L with group velocity mismatch $V = 1$, wave number of perturbation $p = 0.2$, and initial amplitudes $A_0 = 0.02$, $B_0 = 0.2$

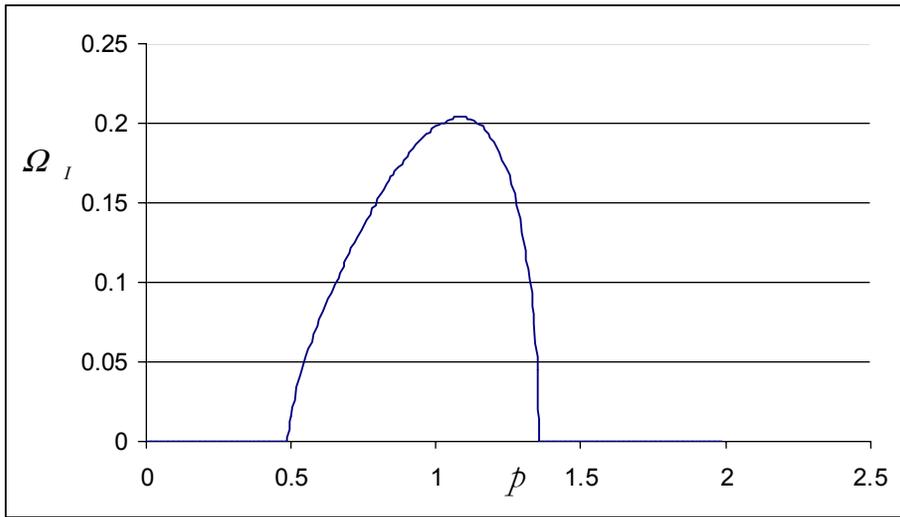


Figure 4a: The growth rate for modulational instabilities with group velocity mismatch $V = 1$ and initial amplitudes $A_0=0.2$, $B_0=0.02$

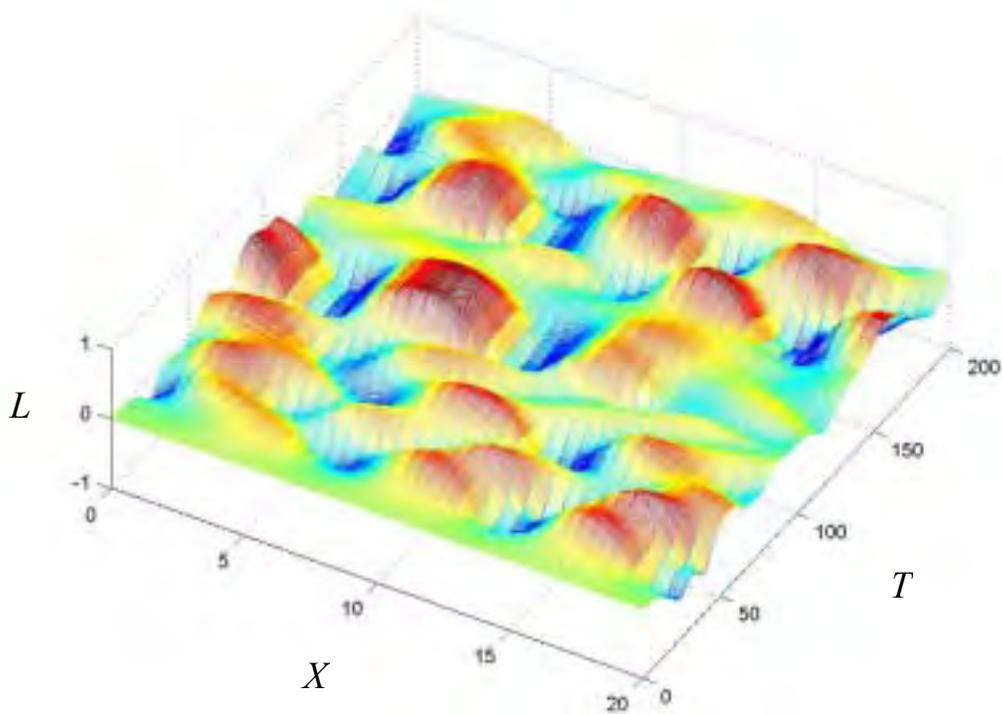


Figure 4b: Long-time evolution of the long wave L with group velocity mismatch $V = 1$, wave number of perturbation $p = 1$, and initial amplitudes $A_0=0.2$, $B_0=0.02$

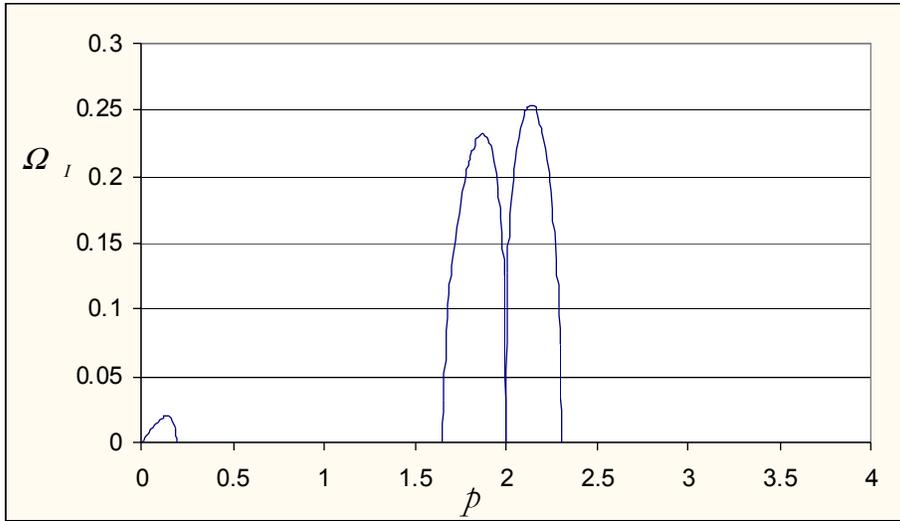


Figure 5a: The growth rate for modulational instabilities with group velocity mismatch $V = 2$ and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

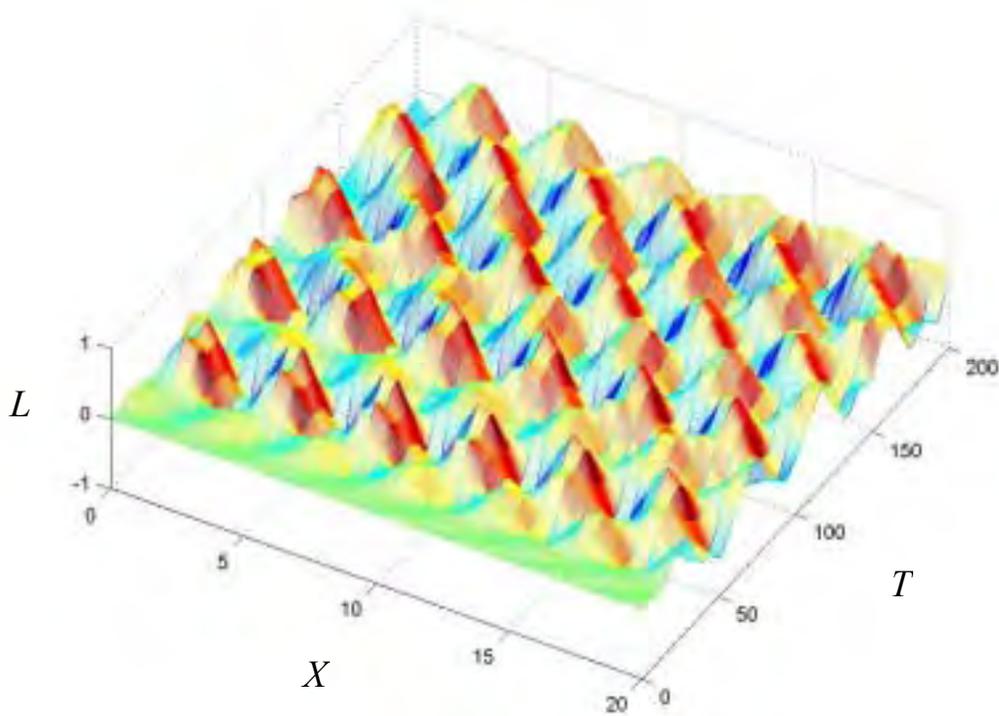


Figure 5b: Long-time evolution of the long wave L with group velocity mismatch $V = 2$, wave number of perturbation $p = 2$, and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

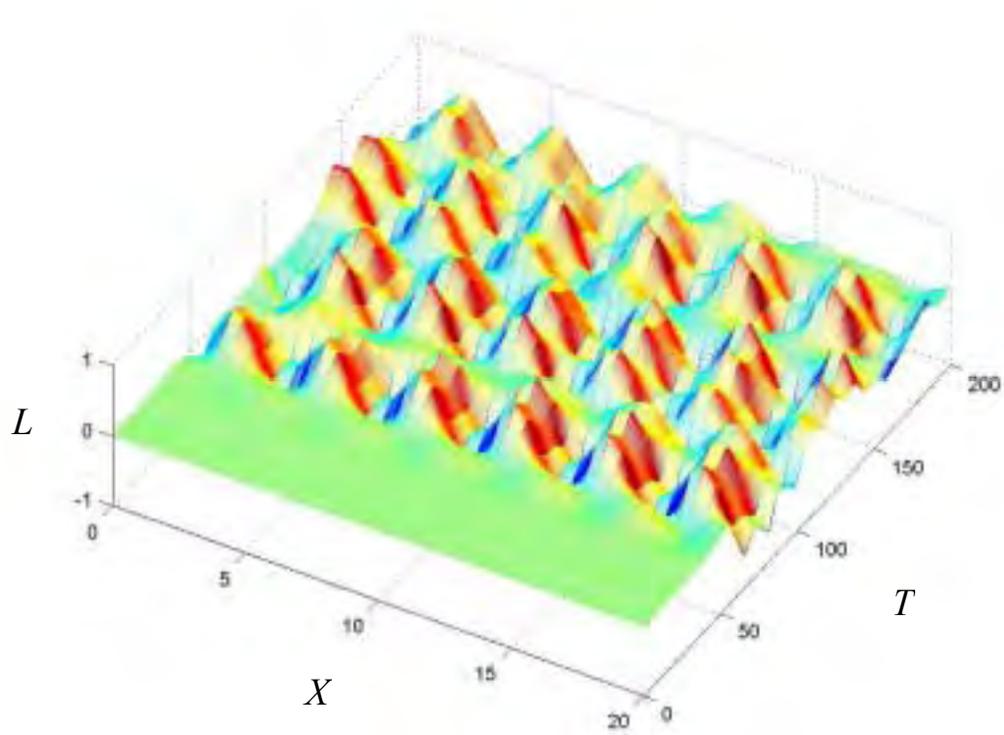


Figure 5c: Long time evolution of the long wave L with group velocity mismatch $V = 2$, wave number of perturbation $p = 0.15$, and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

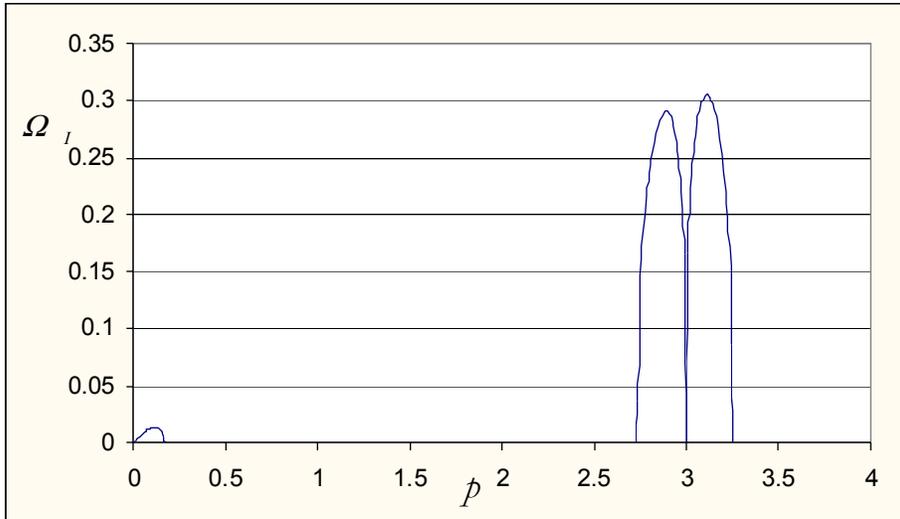


Figure 6a: The growth rate for modulational instabilities with group velocity mismatch $V = 3$ and initial amplitudes $A_0=0.2$, $B_0=0.2$

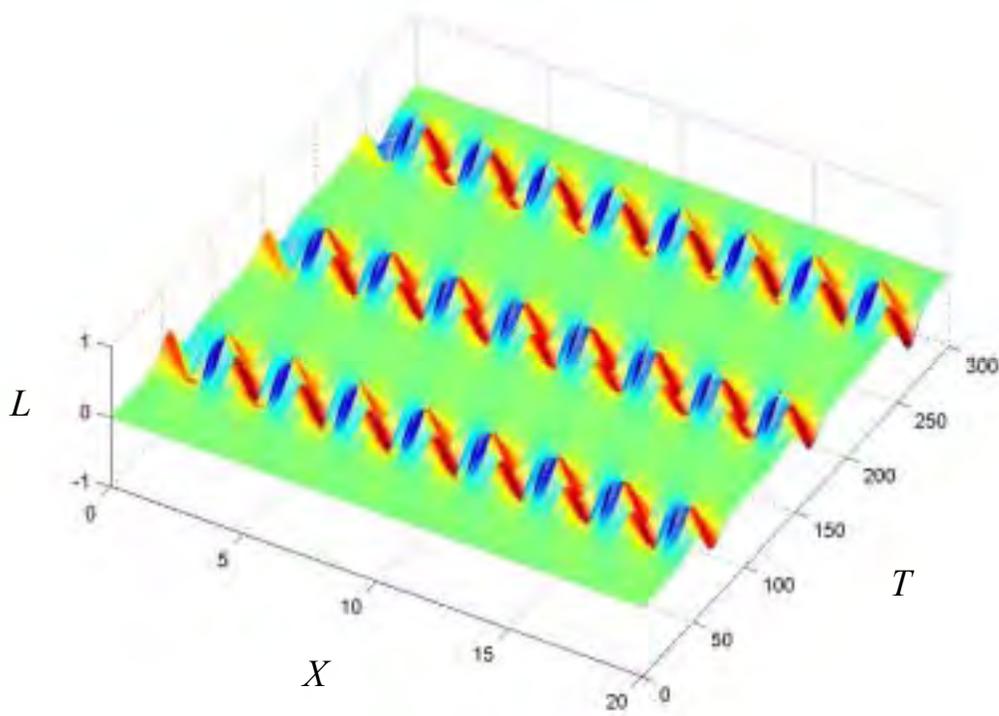


Figure 6b: Long-time evolution of the long wave L with group velocity mismatch $V = 3$, wave number of perturbation $p = 3$, and initial amplitudes $A_0=0.2$, $B_0=0.2$

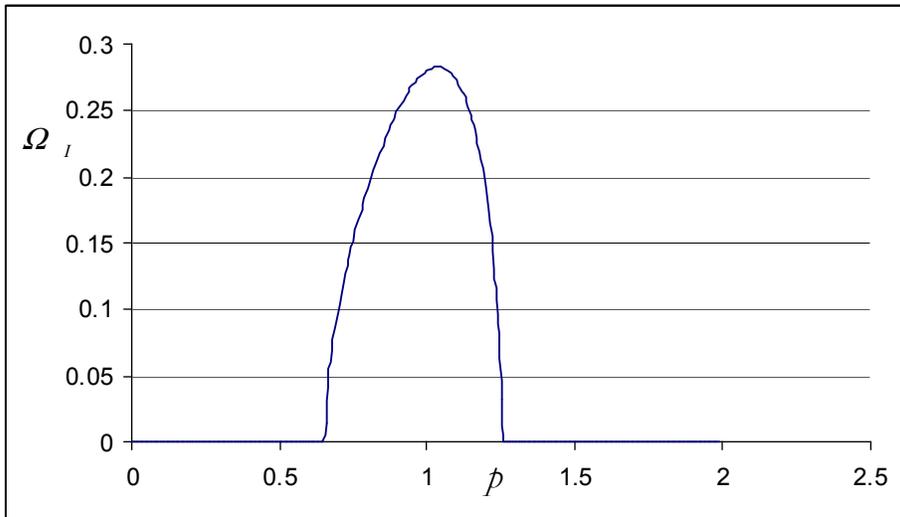


Figure 7a: The growth rate for modulational instabilities for opposite dispersions, with group velocity mismatch $V = 1$ and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

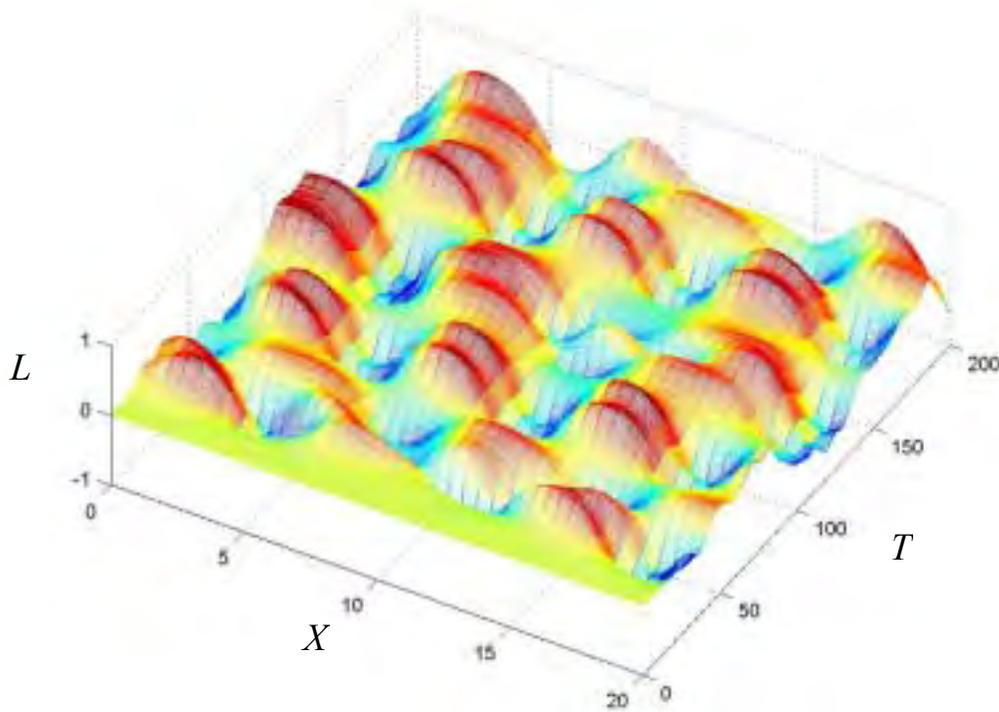


Figure 7b: Long-time evolution of the long wave L for opposite dispersions, with group velocity mismatch $V = 1$, wave number of perturbation $p = 1$, and initial amplitudes $A_0 = 0.2$, $B_0 = 0.2$

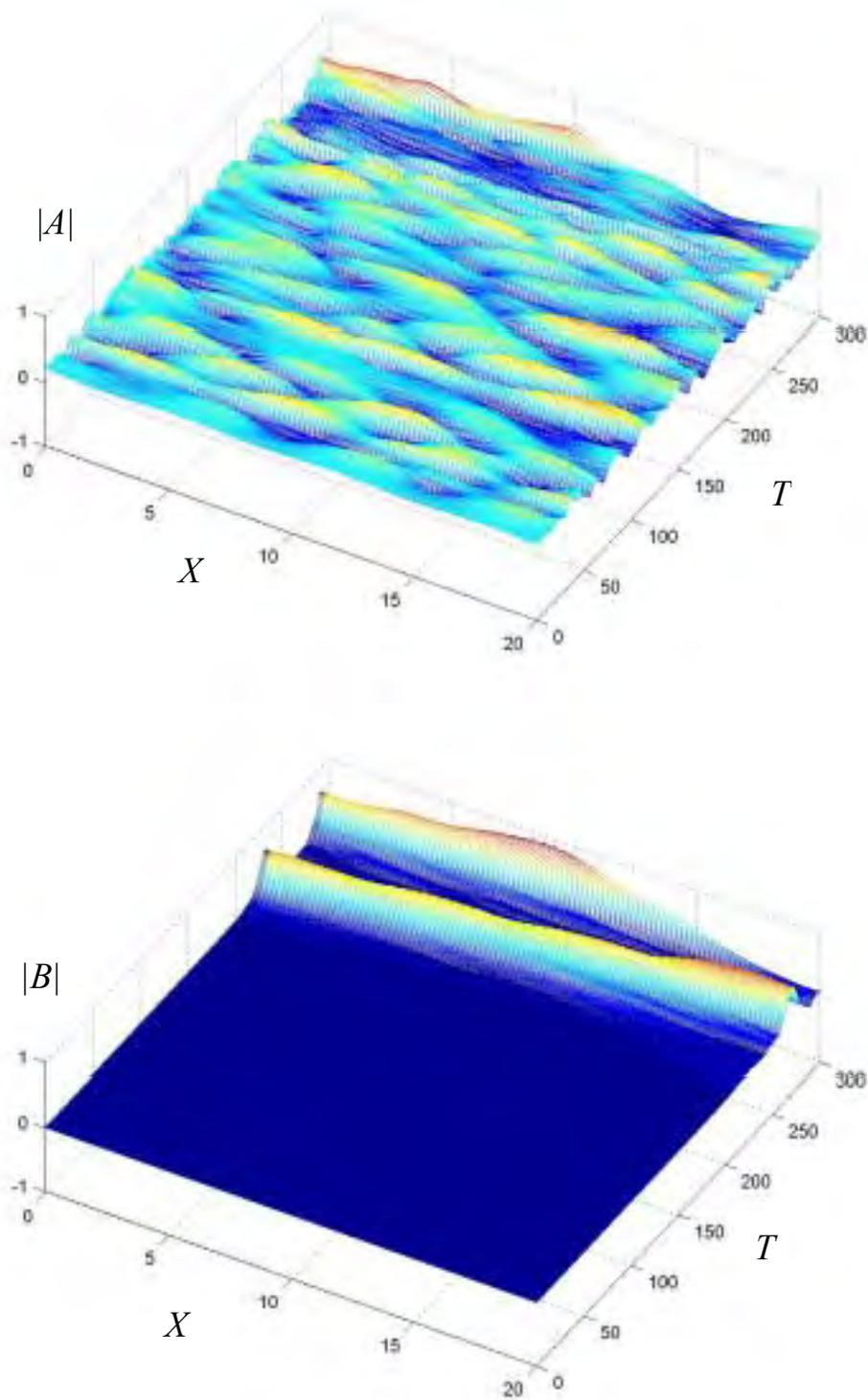


Figure 8: Long time evolution of the short wave envelopes A (top) and B (bottom) with parameters $\delta_1 = \lambda_1 = \mu_1 = 1$, $\delta_2 = 2, \lambda_2 = 0.5, \mu_2 = 1$, and group velocity mismatch $V = 1$, wave number of perturbation $p = 0.5$, and amplitude $A_0 = 0.2$, $B_0 = 0.000002$.