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PLEASE CITE THE PUBLISHED VERSION

<http://dx.doi.org/10.1007/s00208-012-0890-1>

PUBLISHER

© Springer Verlag (Germany)

VERSION

AM (Accepted Manuscript)

PUBLISHER STATEMENT

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Laface, Antonio, and Elisa Postinghel. 2019. "Secant Varieties of Segre-veronese Embeddings of $(p^1)^r$ ". figshare. <https://hdl.handle.net/2134/25236>.

SECANT VARIETIES OF SEGRE-VERONESE EMBEDDINGS OF $(\mathbb{P}^1)^r$.

ANTONIO LAFACE AND ELISA POSTINGHEL

ABSTRACT. We use a double degeneration technique to calculate the dimension of the secant variety of any Segre-Veronese embedding of $(\mathbb{P}^1)^r$.

INTRODUCTION

The problem of determining the dimension of linear systems through double points in general position on an algebraic variety X is a very hard one to be solved in general. Complete results are known for varieties of small dimension [BD10, CC02, Laf02, CGG07, VT05] and in any dimension just for \mathbb{P}^n by the Alexander-Hirschowitz theorem [AH95, Pos12].

Secant varieties of Segre-Veronese varieties are not well-understood, so far, and many efforts have been made, see for example [CGG05], [CGG08], [Bal06], [Abr08], [AB09], [AB11]. In this paper we determine the dimension of any linear system $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ of hypersurfaces of $(\mathbb{P}^1)^r$ of multi-degree (d_1, \dots, d_r) through n double points in general position. Solving this problem is equivalent, via Terracini's Lemma, to calculate the dimension of the secant variety of any Segre-Veronese embedding $(\mathbb{P}^1)^r \rightarrow \mathbb{P}^N$, defined by the complete linear system $|\mathcal{O}(d_1, \dots, d_r)|$ of $(\mathbb{P}^1)^r$. Our proof is based on a double induction, on the dimension r and on the degree $d_1 + \dots + d_r$. A basic step for our induction is represented by the fundamental paper [CGG11], where the authors show that, if all the $d_i = 1$, then $\mathcal{L}_{(1, \dots, 1)}(2^n)$ has always but in one case ($r = 4$) the expected dimension.

Our approach consists in degenerating $(\mathbb{P}^1)^r$ to a union of two varieties both isomorphic to $(\mathbb{P}^1)^r$ and simultaneously degenerating the linear system to a linear system obtained as fibered product of linear systems on the two components over the restricted system on their intersection. The limit linear system is somewhat easier than the original one, in particular this degeneration argument allows to use induction on the multi-degree and on r . This construction is a generalization of the technique introduced by Ciliberto and Miranda in [CM98] and [CM00] to study higher multiplicity interpolation problems in \mathbb{P}^2 and recently generalized by the second author in [Pos12] to the higher dimensional case to study linear systems of d -hypersurfaces of \mathbb{P}^r with a general collection of nodes.

The paper is organized as follows. In Section 1 we provide the basic notation. Section 2 contains the statement of our main theorem together with its counterpart on secant varieties. We enter the double degeneration technique in Section 3, while

2000 *Mathematics Subject Classification.* 14C20.

Key words and phrases. Double points, secant varieties, birational transformations.

The first author was supported by Proyecto FONDECYT Regular 2011, N. 1110096. The second author was supported by Marie-Curie IT Network SAGA, [FP7/2007-2013] grant agreement PITN-GA- 2008-214584. Both authors were partially supported by Institut Mittag-Leffler.

Section 4 is devoted to apply this technique to compute the dimension of linear systems. In Section 5 we prove our theorem and finally we conclude in Section 6 by describing all the linear systems whose dimension is not the expected one.

1. NOTATION

Let $\mathcal{L} := \mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ be the linear system of multi-degree (d_1, \dots, d_r) hypersurfaces of $(\mathbb{P}^1)^r$ which are singular at n points in general position. Its *virtual dimension* is defined to be

$$v(\mathcal{L}) := \prod_{i=1}^r (d_i + 1) - 1 - (r + 1)n,$$

i.e. the dimension of the linear system $|\mathcal{O}(d_1, \dots, d_r)|$ of multi-degree (d_1, \dots, d_r) hypersurfaces of $(\mathbb{P}^1)^r$ minus the number of conditions imposed by the double points. The dimension of \mathcal{L} cannot be less than -1 , hence we define the *expected dimension* to be

$$e(\mathcal{L}) := \max\{v(\mathcal{L}), -1\}.$$

If the conditions imposed by the assigned points are not linearly independent, the dimension of \mathcal{L} is greater than the expected one: in that case we say that \mathcal{L} is *special*. Otherwise, if the dimension and the expected dimension of \mathcal{L} coincide, we say that \mathcal{L} is *non-special*.

We are interested in investigating if a given linear system \mathcal{L} is non-special. The dimension of \mathcal{L} is *upper-semicontinuous* in the position of the points in $(\mathbb{P}^1)^n$ and it achieves its minimum value when they are in *general position*. Let Z be the zero-dimensional scheme of length $(r + 1)n$ given by n double points in general position. With abuse of notation we will sometimes adopt the same symbol $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$, or simply \mathcal{L} , for denoting the linear system and the sheaf $\mathcal{O}(d_1, \dots, d_r) \otimes \mathcal{I}_Z$. With this in mind, consider the following restriction exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{(d_1, \dots, d_r)} \longrightarrow \mathcal{L}_{(d_1, \dots, d_r)|_Z}.$$

Taking cohomology, being $h^1((\mathbb{P}^1)^r, \mathcal{L}_{(d_1, \dots, d_r)}) = 0$ we get the following cohomological characterization for the speciality of the linear system: \mathcal{L} is non-special if and only if

$$h^0((\mathbb{P}^1)^r, \mathcal{L}) \cdot h^1((\mathbb{P}^1)^r, \mathcal{L}) = 0.$$

2. THE CLASSIFICATION THEOREM

In this section we state our main theorem and recall its connection with the dimension of the secant varieties of the Segre-Veronese embeddings of $(\mathbb{P}^1)^r$.

Theorem 2.1. *The linear system $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ of $(\mathbb{P}^1)^r$ is non-special except in the following cases.*

r	degrees	n	$v(\mathcal{L})$	$\dim(\mathcal{L})$
2	(2, 2a)	2a + 1	-1	0
3	(1, 1, 2a)	2a + 1	-1	0
	(2, 2, 2)	7	-2	0
4	(1, 1, 1, 1)	3	0	1

All the exception in Theorem 2.1 where previously known. Moreover a proof of the two dimensional case can be found in [Laf02, VT05]. The three dimensional case was treated in [BD10, CGG07]. The special system in dimension 4 has been found in [CGG05].

This theorem has an equivalent reformulation in terms of higher secant varieties of Segre-Veronese embeddings of products of \mathbb{P}^1 's. Let X be a projective variety of dimension r embedded in \mathbb{P}^N . The n -secant variety $S_n(X)$ of X is defined to be the Zariski closure of the union of all the linear spans in \mathbb{P}^N of n -tuples of independent points of X . We have, counting parameters, that

$$\dim(S_n(X)) \leq \min\{nr + n - 1, N\}$$

The n -secant variety $S_n(X)$ is *defective* if strict inequality holds; it is *non-defective* if equality holds.

Let $N := \prod_{i=1}^r (d_i + 1) - 1$ and let $\nu : (\mathbb{P}^1)^r \rightarrow \mathbb{P}^N$ be the *Segre-Veronese* embedding of multi-degree (d_1, \dots, d_r) . Denote by X the image of ν .

Theorem 2.2. *Let X be defined as above. The n -secant variety of X is non-defective except in the following cases:*

r	degrees	n	Exp. codim.	Codim.	Equations
2	(2, 2a)	2a + 1	0	1	[CGG08]
3	(1, 1, 2a)	2a + 1	0	1	[CGG08]
	(2, 2, 2)	7	0	1	(6.2)
4	(1, 1, 1, 1)	3	1	2	[CGG08]

A hypersurface S of $(\mathbb{P}^1)^r$ of multi-degree (d_1, \dots, d_r) corresponds via the Segre-Veronese embedding to a hyperplane section H of X . Moreover S has a double point at p if and only if H is tangent to X at $\nu(p)$. Now, fix p_1, \dots, p_n general points in $(\mathbb{P}^1)^r$ and consider the linear system $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ of multi-degree (d_1, \dots, d_r) hypersurfaces singular at p_1, \dots, p_n . It corresponds to the linear system of hyperplanes H in \mathbb{P}^N tangent to X at $\nu(p_1), \dots, \nu(p_n)$. This linear system has as base locus the general tangent space to $S_n(X)$. The following classical result, known as *Terracini Lemma*, proves the equivalence between Theorem 2.1 and Theorem 2.2.

Lemma 2.3 (Terracini's Lemma). *Let $X \subseteq \mathbb{P}^N$ be an irreducible, non-degenerate, projective variety of dimension r . Let p_1, \dots, p_n be general points of X , with $n \leq N + 1$. Then*

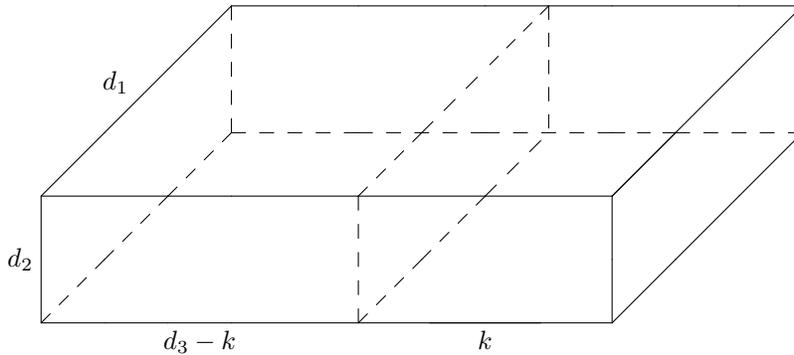
$$T_{S_n(X), p} = \langle T_{X, p_1}, \dots, T_{X, p_n} \rangle,$$

where $p \in \langle p_1, \dots, p_n \rangle$ is a general point in $S_n(X)$.

3. THE DEGENERATION TECHNIQUE

We begin by constructing a toric flat degeneration of the variety X into a union of two toric varieties X_1 and X_2 both isomorphic to X .

3.1. A toric degeneration of $(\mathbb{P}^1)^r$. Let $P = P_{(d_1, \dots, d_r)}$ be the convex lattice polytope $[0, d_1] \times \dots \times [0, d_r] \subseteq \mathbb{R}^r$. Assume, without loss of generality, that $1 \leq d_1 \leq \dots \leq d_r$. Assume moreover that $d_r \geq 2$. Its integer points define the toric map which is the Segre-Veronese embedding $\nu : (\mathbb{P}^1)^r \rightarrow \mathbb{P}^N$ given by the line bundle

FIGURE 1. A regular subdivision of $P_{(d_1, d_2, d_3)}$

$\mathcal{O}(d_1, \dots, d_r)$. As before we will denote by X the image of ν . For any integer k such that $1 \leq k \leq d_r - 1$, consider the function $\phi_k : P \cap \mathbb{Z}^r \rightarrow \mathbb{Z}$ defined by

$$\phi_k(v) = \begin{cases} 0 & \text{if } v_r \leq k \\ v_r - k & \text{if } v_r > k. \end{cases}$$

It defines a *regular subdivision* of P in the following way. Consider the convex hull of the half lines $\{(v, t) \in P \times \mathbb{R}_{\geq 0} : t \geq \phi_k(v)\}$. This is an unbounded polyhedra with two lower faces. By projecting these faces on P we obtain the regular subdivision

$$\mathcal{T} = \{P_{(d_1, \dots, d_{r-1}, d_r - k)}, P_{(d_1, \dots, d_{r-1}, k)}\},$$

with $P_{(d_1, \dots, d_{r-1}, d_r - k)} \cup P_{(d_1, \dots, d_{r-1}, k)} = P$ and $P_{(d_1, \dots, d_{r-1}, d_r - k)} \cap P_{(d_1, \dots, d_{r-1}, k)} = P_{(d_1, \dots, d_{r-1})}$, where $P_{(d_1, \dots, d_{r-1})} = [0, d_1] \times \dots \times [0, d_{r-1}]$. We show the configuration of this toric degeneration in Figure 1 for $r = 3$. The regular subdivision defines a 1-dimensional embedded degeneration in the following way. Let \mathcal{X} be the toric subvariety of $\mathbb{P}^N \times \mathbb{A}^1$ which is image of the toric morphism

$$(\mathbb{C}^*)^r \times \mathbb{C}^* \rightarrow \mathbb{P}^N \times \mathbb{A}^1 \quad (x, t) \mapsto (\{t^{\phi_k(v)} x^v : v \in P \cap \mathbb{Z}^r\}, t).$$

Then \mathcal{X} admits two morphisms, induced by the projections, onto \mathbb{P}^N and \mathbb{A}^1 . Denote by $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$ the second morphism. The fiber X_t of π over t is isomorphic to X if t is not null and to the union $X_1 \cup X_2$ if $t = 0$. Both the X_i 's are isomorphic to $(\mathbb{P}^1)^r$ and their intersection R is isomorphic to $(\mathbb{P}^1)^{r-1}$. More precisely, X_1 is the Segre-Veronese embedding of $(\mathbb{P}^1)^r$ defined by the complete linear system $|\mathcal{O}(d_1, \dots, d_{r-1}, d_r - k)|$, while X_2 is the Segre-Veronese embedding of $(\mathbb{P}^1)^r$ defined by $|\mathcal{O}(d_1, \dots, d_{r-1}, k)|$. The intersection R is the Segre-Veronese embedding of $(\mathbb{P}^1)^{r-1}$ given by $|\mathcal{O}(d_1, \dots, d_{r-1})|$.

3.2. The (k, n_2) -degeneration of \mathcal{L} . A line bundle on X_0 corresponds to two line bundles, respectively on X_1 and on X_2 , which agree on the intersection R . We consider the linear system $\mathcal{L}_t := \mathcal{L}$ of multi-degree (d_1, \dots, d_r) hypersurfaces of X with n assigned general points $p_{1,t}, \dots, p_{n,t}$ of multiplicity 2.

Fix a non-negative integer $n_1 \leq n$ and specialize n_1 points generically on X_1 and the other $n_2 = n - n_1$ points generically on X_2 , i.e. take a flat family $\{p_{1,t}, \dots, p_{n,t}\}_{t \in \mathbb{A}^1}$ such that $p_{1,0}, \dots, p_{n_1,0} \in X_1$ and $p_{n_1+1,0}, \dots, p_{n,0} \in X_2$. The

limiting linear system \mathcal{L}_0 on X_0 is formed by the divisors in the flat limit of the bundle $\mathcal{O}(d_1, \dots, d_r)$ on the general fiber X_t , singular at $p_{1,0}, \dots, p_{n,0}$. Consider the following linear systems:

$$(3.1) \quad \begin{aligned} \mathcal{L}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k)}(2^{n_1}) & \mathcal{L}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k)}(2^{n_2}) \\ \hat{\mathcal{L}}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k - 1)}(2^{n_1}) & \hat{\mathcal{L}}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k - 1)}(2^{n_2}), \end{aligned}$$

where $\mathcal{L}_i, \hat{\mathcal{L}}_i$ are defined on X_i and $\hat{\mathcal{L}}_i$ is the kernel of the restriction of \mathcal{L}_i to R . This is given by the exact sequence:

$$0 \longrightarrow \hat{\mathcal{L}}_i \longrightarrow \mathcal{L}_i \longrightarrow \mathcal{L}_i|_R =: \mathcal{R}_i \longrightarrow 0$$

The kernel $\hat{\mathcal{L}}_i$ consists of those divisors of \mathcal{L}_i which vanish identically on R , i.e. the divisors in \mathcal{L}_i containing R as component. An element of \mathcal{L}_0 consists either of a divisor on X_1 and a divisor on X_2 , both satisfying the conditions imposed by the multiple points, which restrict to the same divisor on R , or it is a divisor corresponding to a section of the bundle which is identically zero on X_1 (or on X_2) and which gives a general divisor in \mathcal{L}_2 (or in \mathcal{L}_1 respectively) containing R as a component. We have, by upper-semicontinuity, that $\dim(\mathcal{L}_0) \geq \dim(\mathcal{L})$.

Lemma 3.1. *In the above notation, if $\dim(\mathcal{L}_0) = e(\mathcal{L})$, then the linear system \mathcal{L} has the expected dimension, i.e. it is non-special.*

We will say that \mathcal{L}_0 is obtained from \mathcal{L} by a (k, n_2) -degeneration.

3.3. The (k, n_2, β) -degeneration of \mathcal{L} . Fix a non-negative integer $\beta < \min(r, n_2)$. Suppose that we have already performed a (k, n_2) -degeneration of \mathcal{L} . We perform a further degeneration of the linear system \mathcal{L}_0 on the central fiber by sending β among the n_2 double points of X_2 to the intersection R of X_1 and X_2 . As a result we obtain the following systems

$$(3.2) \quad \begin{aligned} \mathcal{L}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k)}(2^{n_1}) & \mathcal{L}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k)}(2^{n_2}) \\ \hat{\mathcal{L}}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k - 1)}(2^{n_1}) & \hat{\mathcal{L}}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k - 1)}(2^{n_2 - \beta}, 1^\beta), \end{aligned}$$

On the intersection R we have

$$\mathcal{R}_2 \subseteq \mathcal{L}_{(d_1, \dots, d_{r-1})}(2^\beta).$$

Observe that we are abusing our original notation for \mathcal{L}_2 and $\hat{\mathcal{L}}_2$ since the n_2 double points on X_2 are no longer in general position. We will say that \mathcal{L}_0 is obtained from \mathcal{L} by a (k, n_2, β) -degeneration, implying that if $\beta > 0$ we perform the double degeneration, while if $\beta = 0$ we do not need to perform it.

4. COMPUTING THE DIMENSION OF THE LIMIT SYSTEM

Our aim is to compute $\dim(\mathcal{L}_0)$ by recursion. The simplest cases occurs when all the divisors in \mathcal{L}_0 come from a section which is identically zero on one of the two components: in those cases the matching sections of the other system must lie in the kernel of the restriction map.

Lemma 4.1. *If \mathcal{L}_2 is empty, then $\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}_1)$.*

If, on the contrary, the divisors on \mathcal{L}_0 consist of a divisor on X_1 and a divisor on X_2 , both not identically zero, which match on R , then the dimension of \mathcal{L}_0 depends on the dimension of the intersection $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$ of the restricted systems.

Lemma 4.2. $\dim(\mathcal{L}_0) = \dim(\mathcal{R}) + \dim(\hat{\mathcal{L}}_1) + \dim(\hat{\mathcal{L}}_2) + 2$.

Proof. A section of $H^0(X_0, \mathcal{L}_0)$ is obtained by taking an element in $H^0(R, \mathcal{R})$ and choosing preimages of such an element: $h^0(X_0, \mathcal{L}_0) = h^0(R, \mathcal{R}) + h^0(X_1, \hat{\mathcal{L}}_1) + h^0(X_2, \hat{\mathcal{L}}_2)$. Thus, at the linear system level we get the formula. \square

4.1. Transversality of the restricted systems. The crucial point is to compute the dimension of \mathcal{R} . If the systems $\mathcal{R}_1, \mathcal{R}_2 \subseteq |\mathcal{O}_R(d_1, \dots, d_{r-1})|$ are *transversal*, i.e. if they intersect properly, then the dimension of the intersection \mathcal{R} is easily computed. It is immediate to see that if $\hat{\mathcal{L}}_1$ (or $\hat{\mathcal{L}}_2$) is non-special with virtual dimension ≥ -1 , and moreover \mathcal{L}_1 (resp. \mathcal{L}_2) is non-special, then the restricted system \mathcal{R}_1 (resp. \mathcal{R}_2) is the complete linear system $|\mathcal{O}_R(d_1, \dots, d_{r-1})|$ and transversality trivially holds. If none of the two kernel systems satisfies this property, so that $\mathcal{R}_1, \mathcal{R}_2$ are both proper subsets of $|\mathcal{O}_R(d_1, \dots, d_{r-1})|$, then we need to prove that they intersect properly.

4.2. Applying the double degeneration technique. In what follows we will make use of just two types of degenerations, so we consider them in detail now. Choose

$$(4.1) \quad k = r \quad n_2 = \prod_{i=1}^{r-1} (d_i + 1).$$

Then $v(\mathcal{L}_2) = -1$. Since transversality trivially holds, and in particular $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$, the problem of studying \mathcal{L} is recursively translated to the problem of studying the lower degree linear system $\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - r - 1)}(2^{n_1})$ which has the same virtual dimension. This method is simple and useful if one wants to focus on some particular r and apply induction on the multi-degree (d_1, \dots, d_r) , provided that the base steps of the induction, i.e. any case (d_1, \dots, d_r) with $d_1 \leq \dots \leq d_r \leq r + 1$ are analysed in advance.

However, the aim of this paper is to cover all cases of linear systems of divisors of any multi-degree $(\mathbb{P}^1)^r$, for any r . To this end, we want to exploit induction not only on the multi-degree but also on r in order to have a more compact and powerful method. This can be done by choosing $k = 1$ and applying a double degeneration as described in Section 3.3. This argument consists in an ad hoc adaptation of the degeneration technique implied in [Pos12] to prove the non-speciality of linear systems of divisors of degree d in \mathbb{P}^r with a general collection of double points.

Choose integers k, n_2, β as follows

$$(4.2) \quad k = 1 \quad \prod_{i=1}^{r-1} (d_i + 1) = r(n_2 - \beta) + \beta, \quad \beta \in \{0, \dots, r - 1\}$$

and perform the double degeneration of X, \mathcal{L} described above. Observe that elements of $\hat{\mathcal{L}}_2$ are in bijection with elements of $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2 - \beta}, 1^\beta)$, that is the linear system of hypersurfaces of $|\mathcal{O}(d_1, \dots, d_{r-1})|$ on $(\mathbb{P}^1)^{r-1}$ singular at $n_2 - \beta$ points and passing through β points, all of them in general position:

$$\hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1}, 0)}(2^{n_2 - \beta}, 1^\beta) \cong \mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2 - \beta}, 1^\beta).$$

Observe that the last system has virtual dimension -1 by the definition of n_2 and β . Let $\pi_r : (\mathbb{P}^1)^r \rightarrow (\mathbb{P}^1)^{r-1}$ be the projection on the first $r - 1$ factors. Observe that

$$\mathcal{R}_2 \subseteq \mathcal{L}_{(d_1, \dots, d_{r-1})}(1^{n_2 - \beta}, 2^\beta)$$

since elements of \mathcal{L}_2 contain the lines $\pi_r^{-1}(p_i)$ through each one of the n_2 double points p_i of X_2 .

Lemma 4.3. *Let \mathcal{R}_2 be as above. If $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2})$ is non-special, then $\mathcal{R}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1})}(1^{n_2-\beta}, 2^\beta)$.*

Proof. Since $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2})$ is non-special, then $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2}, 1^\beta)$ is non-special due to the fact that the β simple points are in general position on R . Consider the exact sequence of linear systems:

$$0 \longrightarrow \hat{\mathcal{L}}_2 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{R}_2 \longrightarrow 0.$$

Since $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2}, 1^\beta)$ is non-special of virtual dimension -1 , then it is empty. Thus $\hat{\mathcal{L}}_2$ is empty so that \mathcal{R}_2 is the complete linear system $\mathcal{L}_{(d_1, \dots, d_{r-1})}(1^{n_2-\beta}, 2^\beta)$ obtained by restricting \mathcal{L}_2 to R . \square

In order to prove that transversality holds, it is enough to prove that β double points supported on R impose independent conditions to the linear system \mathcal{R}_1 .

Lemma 4.4 (Transversality Lemma). *In the same notation as above, if the linear system $\mathcal{L}_{(d_1, \dots, d_r-k)}(2^{n_1+\beta})$ is non-special, for points in general position, then \mathcal{R}_1 and \mathcal{R}_2 intersect transversally on R . In particular*

$$\dim(\mathcal{R}) = \max\{\dim(\mathcal{R}_1) - (n_2 - \beta) - r\beta, -1\}.$$

Proof. Since $\beta < r$, the scheme formed by n_1 double points on X_1 and β additional double points on R is general in X_1 . If β nodes in general position impose independent conditions to $\mathcal{L}_1 = \mathcal{L}_{(d_1, \dots, d_r-k)}(2^{n_1})$, namely if $\mathcal{L}_{(d_1, \dots, d_r-k)}(2^{n_1+\beta})$ is non-special, then the β nodes supported on R give independent conditions to \mathcal{R}_1 . Moreover, the intersection \mathcal{R} is formed by those elements in \mathcal{R}_1 that are singular at β points and pass through $n_2 - \beta$ points in general position. This proves the formula for $\dim(\mathcal{R})$. \square

5. THE PROOF OF THE CLASSIFICATION THEOREM

Aim of this section is to prove Theorem 2.1 by induction on the multi-degree and on the dimension r of the variety.

Define the integers

$$n^- := \left\lfloor \frac{1}{r+1} \prod_{i=1}^r (d_i + 1) \right\rfloor, \quad n^+ := \left\lceil \frac{1}{r+1} \prod_{i=1}^r (d_i + 1) \right\rceil.$$

Notice that if non-speciality holds for a collection of n^- double points, then it holds for a smaller number of double points. On the other hand, if there are no hypersurfaces of multi-degree (d_1, \dots, d_r) with n^+ general nodes, the same is true adding other nodes. It is enough to analyse the cases $n^- \leq n \leq n^+$.

In what follows we will make use of the following fact proved in [CGG05]:

$$(5.1) \quad \dim \mathcal{L}_{(d_1, \dots, d_r)}(2^n) = \dim \mathcal{L}_d(d - d_1, \dots, d - d_r, 2^n),$$

where $d = d_1 + \dots + d_r$ and the elements of the right hand side system are hypersurfaces of \mathbb{P}^r with r points of multiplicity $d - d_1, \dots, d - d_r$ and n double points, all in general position.

Proposition 5.1. *If d_1, \dots, d_r be positive integers as in the following table, then $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ is non-special for any value of n .*

r	$degrees$	$bound$
2	$(d_1, d_2) \neq (2, 2a)$	$d_1, d_2 \leq 6$
3	$(d_1, d_2, d_3) \neq (1, 1, 2a), (2, 2, 2)$	$d_1, d_2, d_3 \leq 6$
4	$(d_1, d_2, d_3, d_4) \neq (1, 1, 1, 1)$	$d_1, \dots, d_4 \leq 4$
	$(1, 1, d_3, d_4)$	$d_3, d_4 \leq 6$
	$(2, 2, 2, 5)$	
5	$(1, 1, 1, 1, d_5)$	$d_5 \leq 5$

Proof. By applying (5.1) we reduce to consider the linear system of \mathbb{P}^r given by $\mathcal{L}_d(d - d_1, \dots, d - d_r, 2^n)$. By means of the programs at this url: <http://www2.udec.cl/~alaface/software/pin.html> we analyze these systems by extracting $n + r$ random points of \mathbb{P}^r on the finite field \mathbb{F}_{307} and calculate the degree d part of the ideal with the assigned fat points. \square

5.1. The case $r = 2$. This case is well known in literature, see for example [Laf02, Proposition 5.2] or [VT05]. In any case, for the sake of completeness, we provide a complete proof of this case as well.

Proposition 5.2. *Let d_1, d_2 be positive integers. Then $\mathcal{L}_{(d_1, d_2)}(2^n)$ is special only in the cases described by Theorem 2.1.*

Proof. By Proposition 5.1 it is enough to concentrate on the case $d_2 \geq 6$. Apply a $(2, d_1 + 1)$ -degeneration. Then $v(\mathcal{L}_2) = -1$ and $v(\hat{\mathcal{L}}_1) = v(\mathcal{L})$. By induction hypothesis \mathcal{L}_2 is non-special so that it is empty and $\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}_1)$ by Lemma 4.1. If (d_1, d_2) is not equal to $(1, 2a)$, then $\hat{\mathcal{L}}_1$ is non-special by induction, so we conclude by Lemma 3.1. \square

5.2. The case $r = 3$. We will prove the non-speciality of $\mathcal{L}_{(d_1, d_2, d_3)}(2^n)$, with all the d_i positive and $(d_1, d_2, d_3, n) \neq (2, 2, 2, 7), (1, 1, 2a, 2a + 1)$. This case also was previously known, see [CGG07] or [BD10].

Proposition 5.3. *Let d_1, d_2, d_3 be positive integers. Then $\mathcal{L}_{(d_1, d_2, d_3)}(2^n)$ is special only in the cases described by Theorem 2.1.*

Proof. By Proposition 5.1 it is enough to concentrate on the case $d_3 \geq 6$. Apply a $(3, n_2)$ -degeneration with $n_2 := (d_1 + 1)(d_2 + 1)$. Then $v(\mathcal{L}_2) = -1$ and $v(\hat{\mathcal{L}}_1) = v(\mathcal{L})$. Observe that by induction hypothesis \mathcal{L}_2 is non-special so that it is empty and $\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}_1)$ by Lemma 4.1. If (d_1, d_2, d_3) is not equal to $(1, 1, 2a)$, then $\hat{\mathcal{L}}_1$ is non-special by induction, so we conclude by Lemma 3.1. \square

5.3. The case $r = 4$. We proceed with our investigation by proving the non-speciality of $\mathcal{L}_{(d_1, \dots, d_4)}(2^n)$ for (d_1, d_2, d_3, d_4) distinct from $(1, 1, 1, 1)$.

Proposition 5.4. *Let d_1, d_2, d_3, d_4 be positive integers. Then $\mathcal{L}_{(d_1, \dots, d_4)}(2^n)$ is special only in the cases described by Theorem 2.1.*

Proof. We begin by analyzing four distinct cases.

- (1) $(d_1, d_2, d_3, d_4) = (1, 1, 1, d_4)$ with $d_4 \geq 7$. In this case we perform a $(4, 8)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(1, 1, 1, d_4 - 4)}(2^{n-8}) \quad \mathcal{L}_2 = \mathcal{L}_{(1, 1, 1, 4)}(2^8).$$

Since \mathcal{L}_2 is empty by Proposition 5.1 and $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1, 1, 1, d_4 - 5)}(2^{n-8})$ is non-special by induction on d_4 , we conclude using Lemma 4.1.

- (2) $(d_1, d_2, d_3, d_4) = (1, 1, 4, d_4)$, with $d_4 \geq 6$. In this case we perform a $(4, 20)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(1,1,4,d_4-4)}(2^{n-20}) \quad \mathcal{L}_2 = \mathcal{L}_{(1,1,4,4)}(2^{20}).$$

Since \mathcal{L}_2 is empty by Proposition 5.1 and $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1,1,4,d_4-5)}(2^{n-20})$ is non-special by induction on d_4 , we conclude using Lemma 4.1.

- (3) $(d_1, d_2, d_3, d_4) = (1, 1, d_3, d_4)$, with $6 \leq d_3 \leq d_4$. In this case we perform a $(4, 4d_4 + 4)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(1,1,d_3,d_4-4)}(2^{n-4d_4-4}) \quad \mathcal{L}_2 = \mathcal{L}_{(1,1,d_3,4)}(2^{4d_4+4}).$$

Since \mathcal{L}_2 is empty by the previous case and $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1,1,d_3,d_4-5)}(2^{n-4d_4-4})$ is non-special by induction on d_4 , we conclude using Lemma 4.1.

- (4) $(d_1, d_2, d_3, d_4) = (2, 2, 2, d_4)$, with $d_4 \geq 6$. In this case we perform a $(4, 27)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(2,2,2,d_4-4)}(2^{n-27}) \quad \mathcal{L}_2 = \mathcal{L}_{(2,2,2,4)}(2^{27}).$$

Since \mathcal{L}_2 is empty by Proposition 5.1 and $\hat{\mathcal{L}}_1 = \mathcal{L}_{(2,2,2,d_4-5)}(2^{n-27})$ is non-special by induction on d_4 , we conclude using Lemma 4.1.

Suppose now that (d_1, d_2, d_3, d_4) is distinct from $(1, 1, 1, d_4)$, $(2, 2, 2, d_4)$ and $(1, 1, 2a, d_4)$. We perform a $(1, n_2, \beta)$ -degeneration, with n_2 and β defined as in (4.2) obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(d_1,d_2,d_3,d_4-1)}(2^{n_1}) \quad \mathcal{L}_2 = \mathcal{L}_{(d_1,d_2,d_3,1)}(2^{n_2}).$$

Recall that exactly β of the n_2 double points of X_2 are sent to R . Thus the kernels are:

$$\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1,d_2,d_3,d_4-2)}(2^{n_1}) \quad \hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1,d_2,d_3,0)}(2^{n_2-\beta}, 1^\beta).$$

Observe that $\hat{\mathcal{L}}_2 \cong \mathcal{L}_{(d_1,d_2,d_3)}(2^{n_2-\beta}, 1^\beta)$ and the last system is empty by induction since it has virtual dimension -1 . Also $\hat{\mathcal{L}}_1$ is empty since it is non-special by induction and has negative virtual dimension. Thus \mathcal{L} is non-special by the assumption on the d_i and by Lemma 4.2 and Lemma 4.4. \square

5.4. **The case $r = 5$.** We show that there are no special systems in dimension 5.

Proposition 5.5. *Let d_1, \dots, d_5 be positive integers. Then $\mathcal{L}_{(d_1, \dots, d_5)}(2^n)$ is non-special.*

Proof. If $(d_1, d_2, d_3, d_4, d_5) = (1, 1, 1, 1, d_5)$, with $d_5 \geq 6$, we perform a $(5, 16)$ -degeneration obtaining the linear systems

$$\mathcal{L}_1 = \mathcal{L}_{(1,1,1,1,d_5-5)}(2^{n-16}) \quad \mathcal{L}_2 = \mathcal{L}_{(1,1,1,1,5)}(2^{16}).$$

Since \mathcal{L}_2 is empty by Proposition 5.1 and $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1,1,1,1,d_5-6)}(2^{n-16})$ is non-special by induction, we conclude using Lemma 4.1.

Suppose now $(d_1, d_2, d_3, d_4, d_5) \neq (1, 1, 1, 1, d_5)$. Like in the four dimensional case we perform a $(1, n_2, \beta)$ -degeneration, with n_2 and β defined as in (4.2) obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(d_1, \dots, d_4, d_5-1)}(2^{n_1}) \quad \mathcal{L}_2 = \mathcal{L}_{(d_1, \dots, d_4, 1)}(2^{n_2}).$$

Recall that exactly β of the n_2 double points of X_2 are sent to R . Thus the kernels are:

$$\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1, \dots, d_4, d_5-2)}(2^{n_1}) \quad \hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1, \dots, d_4, 0)}(2^{n_2-\beta}, 1^\beta).$$

Observe that $\hat{\mathcal{L}}_2 \cong \mathcal{L}_{(d_1, d_2, d_3, d_4)}(2^{n_2 - \beta}, 1^\beta)$ and the last system is empty by induction since it has virtual dimension -1 . Also $\hat{\mathcal{L}}_1$ is empty since it is non-special by induction and has negative virtual dimension. Thus \mathcal{L} is non-special by the assumption on the d_i and by Lemma 4.2. \square

5.5. The cases $r \geq 6$. We show that there are no special systems in dimension ≥ 6 .

Proposition 5.6. *Let $r \geq 6$. Let d_1, \dots, d_r be positive integers. Then the linear system $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ is non-special.*

Proof. We proceed by induction on $d := d_1 + \dots + d_r$. If $d = r$, then $d_i = 1$ for any i . This case is non-special as proved in [CGG11]. If $d > r$, then after possibly reordering the d_i we can assume that $d_r > 1$. We perform a $(1, n_2, \beta)$ -degeneration, with n_2 and β defined as in (4.2) obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - 1)}(2^{n_1}) \quad \mathcal{L}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1}, 1)}(2^{n_2}).$$

Recall that exactly β of the n_2 double points of X_2 are sent to R . Thus the kernels are:

$$\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - 2)}(2^{n_1}) \quad \hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1}, 0)}(2^{n_2 - \beta}, 1^\beta).$$

Observe that $\hat{\mathcal{L}}_2 \cong \mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2 - \beta}, 1^\beta)$ and the last system is empty by induction since it has virtual dimension -1 . The kernel system $\hat{\mathcal{L}}_1$ has negative virtual dimension for any (d_1, \dots, d_r) , as one can easily check. Furthermore the dimension of the limit system equals the dimension of the intersection $\mathcal{R}_1 \cap \mathcal{R}_2$ by Lemma 4.2. Indeed, since \mathcal{R}_1 and \mathcal{R}_2 intersect transversally by Lemma 4.4, we have

$$\begin{aligned} \dim(\mathcal{R}_1 \cap \mathcal{R}_2) &= \max\{\dim(\mathcal{R}_1) - (n_2 - \beta) - r\beta, -1\} \\ &= \max\left\{\prod_{i=1}^r (d_i + 1) - 1 - (r + 1)n, -1\right\} = e(\mathcal{L}). \end{aligned}$$

\square

6. SPECIAL LINEAR SYSTEMS

In this section we complete the proof of Theorem 2.1 by calculating the dimension of each special system.

Proposition 6.1. *The following linear systems have the stated virtual and effective dimensions. In each case the system \mathcal{L} is singular along a smooth rational variety.*

r	degrees	n	$v(\mathcal{L})$	$\dim(\mathcal{L})$
2	(2, 2a)	2a + 1	-1	0
3	(1, 1, 2a)	2a + 1	-1	0
	(2, 2, 2)	7	-2	0
4	(1, 1, 1, 1)	3	0	1

Proof. Observe that Proposition 5.1 already provide a computer based proof of the first part of the statement. Anyway we prefer to give an alternative proof also for the calculation of the dimension in order to make things more explicit. The dimension of each such system can be found by repeated use of formula (5.1) together with the following one (see [LU10]). Let $\mathcal{L}^{\mathbb{P}^n} := \mathcal{L}_d(m_1, \dots, m_{n+1}, m_{n+2}, \dots, m_r)$, then

$$(6.1) \quad \dim \mathcal{L}^{\mathbb{P}^n} = \dim \mathcal{L}_{d+k}(m_1 + k, \dots, m_{n+1} + k, m_{n+2}, \dots, m_r),$$

where $k = (n-1)d - m_1 - \dots - m_{n+1}$. Our analysis begins with the two dimensional case. If $r = 2$, by applying (5.1) first and (6.1) after, we get

$$\begin{aligned} \dim \mathcal{L}_{(2,2a)}(2^{2a+1}) &= \dim \mathcal{L}_{2a+2}(2a, 2^{2a+2}) \\ &= \dim \mathcal{L}_{2a}(2(a-1), 2^{2a}) \\ &= \dim \mathcal{L}_2(2^2) = 0. \end{aligned}$$

The linear system is singular along the unique smooth rational curve in $\mathcal{L}_{(1,a)}(1^{2a+1})$. If $r = 3$ we have to analyze two cases. The first one is

$$\begin{aligned} \dim \mathcal{L}_{(1,1,2a)}(2^{2a+1}) &= \dim \mathcal{L}_{2a+2}((2a+1)^2, 2^{2a+2}) \\ &= \dim \mathcal{L}_{2a}(2a-1, 2^{2a}) \\ &= \dim \mathcal{L}_2(2^2, 1^2) = 0. \end{aligned}$$

Observe that the dimension of the last system is 0 since it contains just a pair of planes which intersect along a line through the first two points. These two planes can be seen also in the original system. They are $D_1 + D_2 \in \mathcal{L}_{(1,1,2a)}(2^{2a+1})$, with $D_1 \in \mathcal{L}_{(1,0,a)}(1^{2a+1})$ and $D_2 \in \mathcal{L}_{(0,1,a)}(1^{2a+1})$. Thus the system is singular along the curve $D_1 \cap D_2$ which is smooth by Bertini's theorem and rational by the adjunction formula. The second case in dimension $r = 3$ is

$$\begin{aligned} \dim \mathcal{L}_{(2,2,2)}(2^7) &= \dim \mathcal{L}_6(4^3, 2^7) \\ &= \dim \mathcal{L}_4(2^9) = 0. \end{aligned}$$

This last system is well known from Alexander-Hirschowitz theorem, see [AH95, Pos12]. Observe that $\mathcal{L}_{(2,2,2)}(2^7)$ is singular along the smooth rational surface defined by $\mathcal{L}_{(1,1,1)}(1^7)$. Our last case is when $r = 4$. Here we have

$$\begin{aligned} \dim \mathcal{L}_{(1,1,1,1)}(2^3) &= \dim \mathcal{L}_4(3^4, 2^3) \\ &= \dim \mathcal{L}_2(2^2, 1^4) \\ &= \dim \mathcal{L}_1(1^3) = 1. \end{aligned}$$

Consider now the linear system $\mathcal{L}_{(1,1,1,1)}(2^3)$, where the three double points have coordinates: $p_1 := ([1 : 0], [1 : 0], [1 : 0], [1 : 0])$, $p_2 := ([0 : 1], [0 : 1], [0 : 1], [0 : 1])$, $p_3 := ([1 : 1], [1 : 1], [1 : 1], [1 : 1])$. An easy calculation shows that the elements of the linear system are zero locus of the sections of the pencil:

$$\alpha_0 (x_0 y_1 - x_1 y_0)(z_0 w_1 - z_1 w_0) + \alpha_1 (x_0 z_1 - x_1 z_0)(y_0 w_1 - y_1 w_0).$$

In particular the general element of the pencil is singular along the curve $V(x_0 y_1 - x_1 y_0) \cap V(z_0 w_1 - z_1 w_0) \cap V(x_0 z_1 - x_1 z_0) \cap V(y_0 w_1 - y_1 w_0)$. This is exactly the image of the diagonal morphism $\Delta : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, hence it is a smooth rational curve. \square

6.1. The 7-secant variety of the $(2, 2, 2)$ -embedding of $(\mathbb{P}^1)^3$. Our list of special systems contains the case $\mathcal{L}_{(2,2,2)}(2^7)$ which has virtual dimension -2 and dimension 0. In this section we will show that the corresponding secant variety enjoys a symmetry which allows us to determine its equation.

Let $\varphi : (\mathbb{P}^1)^3 \rightarrow \mathbb{P}^{26}$ be the Segre-Veronese embedding of $(\mathbb{P}^1)^3$ defined by the complete linear system $\mathcal{L}_{(2,2,2)}$, and let S be the 7-secant variety to the image of φ .

Denote, by abuse of notation, with the same letter the corresponding linear map of vector spaces

$$\begin{aligned} \varphi : V \otimes V \otimes V &\longrightarrow \text{Sym}^2(V) \otimes \text{Sym}^2(V) \otimes \text{Sym}^2(V) \\ a \otimes b \otimes c &\longmapsto a^2 \otimes b^2 \otimes c^2, \end{aligned}$$

where each copy of V represents the vector space of degree 1 homogeneous polynomials in two variables. Let V^* be the dual of V . Let $W := \text{Sym}^2(V)^{\otimes 3}$ be the codomain of φ . Given a polynomial $f \in W$, it belongs to the 7-secant variety S if and only if

$$f = a_1^2 b_1^2 c_1^2 + \cdots + a_7^2 b_7^2 c_7^2,$$

where we are adopting the short notation abc for $a \otimes b \otimes c$. Consider the catalecticant map

$$C_f : (V^*)^{\otimes 3} \rightarrow V^{\otimes 3} \quad a^* b^* c^* \mapsto a^* b^* c^*(f).$$

We have that if $f \in S$, then $\text{rk}(C_f) \leq 7$. Hence, as f varies in W , the determinant of C_f gives an equation that vanishes on S . One can prove that this equation is irreducible by showing that it defines a hypersurface of \mathbb{P}^{26} which is singular in codimension two. Since this equation turns out to be irreducible and S is an irreducible hypersurface, this will provide us the equation of S . In coordinates, consider the reverse lexicographical order defined by $a_0 < a_1 < b_0 < b_1 < c_0 < c_1$. Let $\{a_0 b_0 c_0, \dots, a_1 b_1 c_1\}$ be a basis of $V^{\otimes 3}$ and let $\{a_0^* b_0^* c_0^*, \dots, a_1^* b_1^* c_1^*\}$ be the corresponding dual basis of the dual vector space $(V^*)^{\otimes 3}$. An element $f \in W$ can be uniquely written as a linear combination of ordered monomials:

$$f = (1/8)z_0 a_0^2 b_0^2 c_0^2 + (1/4)z_1 a_0 a_1 b_0^2 c_0^2 + \cdots + (1/8)z_{26} a_1^2 b_1^2 c_1^2,$$

where the coefficients are the inverses of the products of the exponents of the a_i, b_i, c_i 's. We have, for example, $a_0^* b_0^* c_0^*(f) = z_0 a_0 b_0 c_0 + z_1 a_1 b_0 c_0 + z_3 a_0 b_1 c_0 + z_4 a_1 b_1 c_0 + z_9 a_0 b_0 c_1 + z_{10} a_1 b_0 c_1 + z_{12} a_0 b_1 c_1 + z_{13} a_1 b_1 c_1$. Thus the equation of S is given by the vanishing of the determinant of the following matrix

$$(6.2) \quad \begin{bmatrix} z_0 & z_1 & z_3 & z_4 & z_9 & z_{10} & z_{12} & z_{13} \\ z_1 & z_2 & z_4 & z_5 & z_{10} & z_{11} & z_{13} & z_{14} \\ z_3 & z_4 & z_6 & z_7 & z_{12} & z_{13} & z_{15} & z_{16} \\ z_4 & z_5 & z_7 & z_8 & z_{13} & z_{14} & z_{16} & z_{17} \\ z_9 & z_{10} & z_{12} & z_{13} & z_{18} & z_{19} & z_{21} & z_{22} \\ z_{10} & z_{11} & z_{13} & z_{14} & z_{19} & z_{20} & z_{22} & z_{23} \\ z_{12} & z_{13} & z_{15} & z_{16} & z_{21} & z_{22} & z_{24} & z_{25} \\ z_{13} & z_{14} & z_{16} & z_{17} & z_{22} & z_{23} & z_{25} & z_{26} \end{bmatrix}.$$

Observe that the previous 8×8 matrix is the tensor product of the three 2×2 matrices, each of which defines the quadratic Veronese embedding of \mathbb{P}^1 .

ACKNOWLEDGEMENTS

It is a pleasure to thank Giorgio Ottaviani for suggesting us the idea of how to calculate the equation of the secant variety of the $(2, 2, 2)$ -embedding of $(\mathbb{P}^1)^3$. We are pleased as well to thank Chiara Brambilla for many useful conversations.

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