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## Systems of conservation laws with third-order Hamiltonian structures

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# Systems of conservation laws with third-order Hamiltonian structures

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## Abstract

We investigate  $n$ -component systems of conservation laws that possess third-order Hamiltonian structures of differential-geometric type. The classification of such systems is reduced to the projective classification of linear congruences of lines in  $\mathbb{P}^{n+2}$  satisfying additional geometric constraints. Algebraically, the problem can be reformulated as follows: for a vector space  $W$  of dimension  $n + 2$ , classify  $n$ -tuples of skew-symmetric 2-forms  $A^\alpha \in \Lambda^2(W)$  such that

$$\phi_{\beta\gamma} A^\beta \wedge A^\gamma = 0,$$

for some non-degenerate symmetric  $\phi$ .

MSC: 37K05, 37K10, 37K20, 37K25.

Keywords: System of Conservation Laws, Linear Congruence, Hamiltonian Operator, WDVV Equation, Projective Group, Reciprocal Transformation, Quadratic Complex, Monge Metric.

*To Professor Boris Konopelchenko on the occasion of his 70th birthday*

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# 1 Introduction

## 1.1 Systems of conservation laws and line congruences in projective space

Systems of conservation laws are  $n$ -component first-order PDEs of the form

$$u_t^i = (V^i(\mathbf{u}))_x, \quad (1)$$

$i = 1, \dots, n$ , where  $V^i(\mathbf{u})$  is a (nonlinear) vector of fluxes. We will assume that the characteristic speeds of system (1), that is, the eigenvalues of the Jacobian matrix of the fluxes  $V^i$ , are real and distinct (condition of strict hyperbolicity). Systems of conservation laws appear in a wide range of applications in continuum mechanics and mathematical physics, see e.g. [21, 25, 35, 32, 34, 33]. Following the geometric approach of [1, 2], with system (1) we associate a congruence (that is,  $n$ -parameter family of lines),

$$y^i = u^i y^{n+1} + V^i y^{n+2}, \quad (2)$$

in auxiliary projective space  $\mathbb{P}^{n+1}$  with homogeneous coordinates  $(y^1 : \dots : y^{n+2})$ . It was demonstrated in [1, 2] that various standard concepts of the theory of conservation laws such as rarefaction curves, shock curves, linear degeneracy, reciprocal transformations, etc, acquire a simple interpretation in the language of the projective theory of congruences. In particular, reciprocally related systems (1) correspond to projectively equivalent congruences (2). Algebro-geometric aspects of this correspondence were investigated in [6, 7, 8, 26].

In this paper we utilise the above geometric correspondence for the classification of systems (1) possessing third-order Hamiltonian structures. We will show that congruences associated with Hamiltonian systems are necessarily *linear*, that is, they are specified by  $n$  linear relations among the Plücker coordinates (in geometric language, codimension  $n$  linear sections of the Grassmannian  $G(1, n+1)$ ). We recall that the lines of a linear congruence in  $\mathbb{P}^{n+1}$  can be characterised geometrically as  $n$ -secants of the *focal variety* (jump locus), which is a codimension two subvariety in  $\mathbb{P}^{n+1}$  (possibly, reducible):

- For  $n = 2$  the focal variety of a linear congruence consists of 2 skew lines in  $\mathbb{P}^3$ .
- For  $n = 3$  the focal variety of a generic linear congruence is a projection of the Veronese surface  $V^2 \subset \mathbb{P}^5$  into  $\mathbb{P}^4$  [5].

- For  $n = 4$  the focal variety is a Palatini threefold in  $\mathbb{P}^5$  [28], etc.

In parametrisation (2), the Plücker coordinates are just  $u^i, V^i, u^i V^j - u^j V^i$ . Imposing  $n$  linear relations among the Plücker coordinates we obtain a linear system for the fluxes  $V^i$  which implies that  $V^i$  are rational in  $u$ . Systems associated with linear congruences are linearly degenerate, and satisfy the Temple property [2].

## 1.2 Third-order Hamiltonian operators

Third-order Hamiltonian operators of differential-geometric type were introduced by Dubrovin and Novikov in [10], and subsequently investigated in [31, 9, 30, 27, 4, 15, 16]. They are defined by the general formula

$$P^{ij} = g^{ij} \partial_x^3 + b_k^{ij} u_x^k \partial_x^2 + (c_k^{ij} u_x^k + c_{km}^{ij} u_x^k u_x^m) \partial_x + d_k^{ij} u_{xxx}^k + d_{km}^{ij} u_{xx}^k u_x^m + d_{kmn}^{ij} u_x^k u_x^m u_x^n,$$

where  $u^i, i = 1, \dots, n$ , are the dependent variables, and the coefficients  $g^{ij}, \dots, d_{kmn}^{ij}$  are functions of  $u^i$  only;  $\partial_x$  stands for the total derivative with respect to  $x$ . The requirement that the corresponding Poisson bracket,

$$\{F, G\} = \int \frac{\delta F}{\delta u^i} P^{ij} \frac{\delta G}{\delta u^j} dx,$$

is skew-symmetric and satisfies the Jacobi identities, imposes strong constraints on the coefficients of  $P$ . We restrict our considerations to the non-degenerate case,  $\det g^{ij} \neq 0$ ; in what follows we use  $g^{ij}$  for raising and lowering indices. It was demonstrated in [31, 9] that there exists a coordinate system (flat coordinates) in which Hamiltonian operator  $P$  takes a simple factorised form [27],

$$P^{ij} = \partial_x \left( g^{ij} \partial_x + c_k^{ij} u_x^k \right) \partial_x. \quad (3)$$

In what follows we will always work in the flat coordinates, and keep for them the notation  $u^i$ ; note that  $u^i$  are nothing but the densities of Casimirs of the corresponding Hamiltonian operator. Introducing  $c_{ijk} = g_{iq} g_{jp} c_k^{pq}$  one can show [30] that the skew-symmetry conditions and the Jacobi identities for operator (3) are equivalent to

$$g_{mn,k} = -c_{mnk} - c_{nmk}, \quad (4a)$$

$$c_{mnk} = -c_{mkn}, \quad (4b)$$

$$c_{mnk} + c_{nkm} + c_{kmn} = 0, \quad (4c)$$

$$c_{mnk,l} = -g^{pq} c_{pml} c_{qnk}. \quad (4d)$$

Equations (4a)–(4c) imply [15]

$$c_{skm} = \frac{1}{3}(g_{sm,k} - g_{sk,m}). \quad (5)$$

The elimination of  $c$  from equations (4) gives a system for the metric  $g$ ,

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0, \quad (6a)$$

$$g_{m[k,n]l} = -\frac{1}{3}g^{pq}g_{p[l,m]}g_{q[k,n]}. \quad (6b)$$

Equations (6a) mean that  $g$  is a Monge metric, and as such is an object of projective differential geometry. Building on the correspondence of Monge metrics to quadratic complexes of lines in  $\mathbb{P}^n$ , in [15, 16] we proposed a classification of Hamiltonian operators (3) for  $n \leq 4$ .

In what follows we will also need the result of Balandin and Potemin [4] according to which the general solution of system (6) is given by the formula

$$g_{ij} = \phi_{\beta\gamma}\psi_i^\beta\psi_j^\gamma, \quad (7)$$

where  $\phi_{\beta\gamma}$  is a non-degenerate constant symmetric matrix, and

$$\psi_k^\gamma = \psi_{km}^\gamma u^m + \omega_k^\gamma; \quad (8)$$

here  $\psi_{km}^\gamma$  and  $\omega_k^\gamma$  are constants such that  $\psi_{km}^\gamma = -\psi_{mk}^\gamma$ . These constants have to satisfy an additional set of quadratic relations,

$$\phi_{\beta\gamma}(\psi_{is}^\beta\psi_{jk}^\gamma + \psi_{js}^\beta\psi_{ki}^\gamma + \psi_{ks}^\beta\psi_{ij}^\gamma) = 0, \quad (9)$$

$$\phi_{\beta\gamma}(\omega_i^\beta\psi_{jk}^\gamma + \omega_j^\beta\psi_{ki}^\gamma + \omega_k^\beta\psi_{ij}^\gamma) = 0, \quad (10)$$

whose algebraic meaning was clarified in [16]. An important invariant of Hamiltonian operator (3) is its singular variety,  $\det g_{ij} = 0$ , which, due to (7), is a double hypersurface of degree  $n - 1$ :

$$\det g = \det \phi (\det \psi)^2;$$

here the degree of  $\det \psi$  equals  $n - 1$  [16].

### 1.3 Hamiltonian systems of conservation laws

In this paper we are interested in *Hamiltonian* systems of conservation laws, namely systems (1) possessing third-order Hamiltonian structures (3):

$$\mathbf{u}_t = (V(\mathbf{u}))_x = P \frac{\delta H}{\delta \mathbf{u}},$$

for some (nonlocal) Hamiltonian functionals  $H$ . Examples of such systems include Monge-Ampère equations, as well as various versions of WDVV equations, see [11] for their geometric treatment based on the theory of Frobenius manifolds. Our main results in this direction can be summarised as follows.

**Theorem 1.** *The necessary and sufficient conditions for a conservative system (1) to possess third-order Hamiltonian operator (3) are the following:*

$$g_{im}V_j^m = g_{jm}V_i^m, \quad (11a)$$

$$c_{mkl}V_i^m + c_{mik}V_l^m + c_{mli}V_k^m = 0, \quad (11b)$$

$$V_{ij}^k = g^{ks}c_{smj}V_i^m + g^{ks}c_{smi}V_j^m, \quad (11c)$$

here lower indices of  $V^m$  denote partial derivatives:  $V_i^m = \partial V^m / \partial u^i$ , etc.

In Theorem 7 of Section 2.5 we present explicit formulae for the corresponding Casimirs, Momentum and Hamiltonian. The proof of Theorem 1 can be found in Section 2.1.

Conditions (11) are analogous to Tsarev's conditions in the theory of first-order homogeneous Hamiltonian operators [36]. System (11) possesses a number of important properties, in particular, in Section 2.2 we establish the following result:

**Theorem 2.** *System (11) is in involution. Its general solution depends on  $\leq \frac{n(n+3)}{2}$  arbitrary constants.*

It is quite remarkable that system (11), which is a linear involutive system with non-constant coefficients, can be integrated in closed form (Section 2.3). This leads to the following result (Section 2.3):

**Theorem 3.** *For Hamiltonian system (1) the following conditions hold:*

- *The associated congruence (2) is linear.*
- *System (1) is linearly degenerate and belongs to the Temple class.*
- *The fluxes  $V^i$  are rational functions of the form*

$$V^i = \frac{Q^i}{\det \psi},$$

where  $\det \psi$  is a polynomial of degree  $n-1$  defining the singular variety, and  $Q^i$  are polynomials of degree  $n$ .

Note that for  $n \geq 4$  systems of conservation laws possessing third-order Hamiltonian structures are neither diagonalisable nor integrable in general.

Based on the classification of linear congruences in  $\mathbb{P}^3$  and  $\mathbb{P}^4$  dating back to the classical work of Castelnuovo [5], this leads to a complete description of Hamiltonian systems (1) for  $n = 2, 3$ , see Section 1.6.

## 1.4 Examples

Here we list examples of conservative systems (1) with third-order Hamiltonian structures (3) that will feature in the classification results below. In order to simplify the expressions for the Hamiltonian densities we introduce potential coordinates  $b^i$  via  $u^i = b^i_x$ . In these coordinates system (1) will no longer be quasilinear, and third-order Hamiltonian operator (3) takes a first-order form, see (21), (22).

**Example 1.** A linear  $n$ -component system of conservation laws,

$$u_t^i = (a^i_j u^j)_x,$$

$a^i_j = \text{const}$ , possesses third-order Hamiltonian formulation

$$u_t^i = \eta^{ij} \partial_x^3 \frac{\delta H}{\delta u^j},$$

$\eta^{ij} = \text{const}$ , with the nonlocal Hamiltonian

$$H = -\frac{1}{2} \int \eta_{jp} a_k^p b^j b^k dx.$$

In this case, conditions (11) reduce to  $\eta_{jp} a_k^p = \eta_{kp} a_j^p$ , which means that the operator  $a$  is symmetric with respect to the metric  $\eta$ .

The associated congruence (2) is the set of lines that intersect  $n$  linear subspaces of codimension two in  $\mathbb{P}^{n+1}$  (the union of these subspaces constitutes the focal variety). These subspaces can be described explicitly: let  $\lambda^k$  be the eigenvalues of  $a$  with the corresponding left eigenvectors  $\xi^k$ , that is,  $a_i^j \xi_j^k = \lambda^k \xi_i^k$ . Then the  $k$ -th focal subspace is defined by two linear equations,  $y^0 = \lambda^k$ ,  $\xi_j^k y^j = 0$ .

**Example 2.** The simplest WDVV equation [11],  $f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$ , can be reduced to a 3-component conservative form,

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^3, \quad u_t^3 = ((u^2)^2 - u^1 u^3)_x, \quad (12)$$



by setting  $u^1 = f_{xxx}$ ,  $u^2 = f_{xxt}$ ,  $u^3 = f_{xtt}$ . System (12) possesses a Hamiltonian formulation  $u_t = P\delta H/\delta u$  [13], with the homogeneous third-order Hamiltonian operator

$$P = \partial_x \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & \partial_x & -\partial_x u^1 \\ \partial_x & -u^1 \partial_x & \partial_x u^2 + u^2 \partial_x + u^1 \partial_x u^1 \end{pmatrix} \partial_x,$$

and the nonlocal Hamiltonian

$$H = - \int \left( \frac{1}{2} u^1 (b^2)^2 + b^2 b^3 \right) dx.$$

Note that system (12) possesses a compatible first-order Hamiltonian formulation, as well as a Lax pair, which elucidate its integrability [13].

The associated congruence (2) was thoroughly investigated in [2]. It consists of trisecant lines of the focal variety which, in this case, is a generic projection of the Veronese surface  $V^2 \subset \mathbb{P}^5$  into  $\mathbb{P}^4$ . Various generalisations of this example can be found in [19, 20, 29, 23].

**Example 3.** The following 4-component conservative system was obtained in [3] in the classification of non-diagonalisable linearly degenerate systems of Temple's class whose characteristic speeds are harmonic (have cross-ratio equal to  $-1$ ):

$$\begin{aligned} u_t^1 &= u_x^3, \\ u_t^2 &= u_x^4, \\ u_t^3 &= \left( \frac{u^1 u^2 u^4 + u^3 ((u^3)^2 + (u^4)^2 - (u^2)^2 - 1)}{u^1 u^3 + u^2 u^4} \right)_x, \\ u_t^4 &= \left( \frac{u^1 u^2 u^3 + u^4 ((u^3)^2 + (u^4)^2 - (u^1)^2 - 1)}{u^1 u^3 + u^2 u^4} \right)_x. \end{aligned} \tag{13}$$

System (13) possesses a Hamiltonian representation  $u_t = P\delta H/\delta u$  where the third-order Hamiltonian operator  $P$  is generated by the Monge metric

$$g_{ij} = \begin{pmatrix} (u^2)^2 + (u^3)^2 + 1 & -u^1 u^2 + u^3 u^4 & -u^1 u^3 + u^2 u^4 & -2u^2 u^3 \\ -u^1 u^2 + u^3 u^4 & (u^1)^2 + (u^4)^2 + 1 & -2u^1 u^4 & u^1 u^3 - u^2 u^4 \\ -u^1 u^3 + u^2 u^4 & -2u^1 u^4 & (u^1)^2 + (u^4)^2 & u^1 u^2 - u^3 u^4 \\ -2u^2 u^3 & u^1 u^3 - u^2 u^4 & u^1 u^2 - u^3 u^4 & (u^2)^2 + (u^3)^2 \end{pmatrix}.$$

Due to  $\det g = (u^1 u^3 + u^2 u^4)^2$ , its singular variety consists of a double quadric and a double plane an infinity. The corresponding nonlocal Hamiltonian  $H$  is given by

$$H = -\frac{1}{2} \int (b^1 b^3 + b^2 b^4 + x(b^1 u^3 - u^1 b^3 + b^2 u^4 - u^2 b^4)) dx,$$

note the explicit dependence on  $x$ . Integrability of system (13) can be demonstrated as follows. Introducing the  $2 \times 3$  matrix

$$Z = \begin{pmatrix} u^1 & u^2 & 1 \\ u^3 & u^4 & 0 \end{pmatrix},$$

one can represent (13) in matrix form (compare with Sect. 4 in [17]),

$$Z_t = (aZZ^T Z + bZ)_x, \quad (14)$$

where  $a = \frac{1}{u^1 u^3 + u^2 u^4}$ ,  $b = -\frac{(u^1)^2 + (u^2)^2 + 1}{u^1 u^3 + u^2 u^4}$ . Introducing the  $5 \times 5$  skew-symmetric matrix

$$S = \begin{pmatrix} 0 & Z \\ -Z^t & 0 \end{pmatrix},$$

one can rewrite (14) as a matrix Hopf-type equation,

$$S_t = (bS - aS^3)_x,$$

with the Lax pair

$$\psi_x = \lambda S \psi, \quad \psi_t = \lambda (bS - aS^3) \psi.$$

Congruence (2) associated with system (13) is related to the Cartan isoparametric hypersurface in  $S^5$ , see [3] for further details. Note that its focal variety is reducible.

**Example 4.** Let us consider a class of conservative 4-component systems of the form

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^3, \quad u_t^3 = u_x^4, \quad u_t^4 = [f(u^1, \dots, u^4)]_x.$$

Under the substitution  $u^1 = f_{xxxx}$ ,  $u^2 = f_{xxx t}$ ,  $u^3 = f_{xxt t}$ ,  $u^4 = f_{xtt t}$  they reduce to a scalar fourth-order PDE for  $f(x, t)$ . One can show that modulo equivalence transformations there exist only two types of such systems possessing third-order Hamiltonian structures:

**Case 1.**  $f = (u^2)^2 - u^1u^3$ . The corresponding system possesses a Hamiltonian formulation  $u_t = P\delta H/\delta u$  with the third-order Hamiltonian operator

$$P = \partial_x \begin{pmatrix} 0 & 0 & 0 & \partial_x \\ 0 & 0 & \partial_x & 0 \\ 0 & \partial_x & 0 & -\partial_x u^1 \\ \partial_x & 0 & -u^1 \partial_x & \partial_x u^2 + u^2 \partial_x \end{pmatrix} \partial_x,$$

and the nonlocal Hamiltonian

$$H = -\frac{1}{2} \int (u^1(b^2)^2 + 2b^2b^4 + (b^3)^2) dx.$$

**Case 2.**  $f = (u^3)^2 - u^2u^4 + u^1$ . The corresponding system possesses a Hamiltonian formulation  $u_t = P\delta H/\delta u$  with the third-order Hamiltonian operator

$$P = \partial_x \begin{pmatrix} \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x \\ 0 & 0 & \partial_x & -\partial_x u^2 \\ 0 & \partial_x & -u^2 \partial_x & \partial_x g + g \partial_x \end{pmatrix} \partial_x,$$

where  $g = u^3 + \frac{1}{2}(u^2)^2$ , and the nonlocal Hamiltonian

$$H = \int (b^2b^3u^3 - b^1b^2 - b^3b^4) dx.$$

We point out that systems from cases 1, 2 are likely to be non-integrable.

## 1.5 Projective invariance

The class of conservative systems (1) is invariant under reciprocal transformations of the form

$$\begin{aligned} d\tilde{x} &= (a_i u^i + a) dx + (a_i V^i + b) dt, \\ d\tilde{t} &= (b_i u^i + c) dx + (b_i V^i + d) dt, \end{aligned} \tag{15}$$

which can be viewed as nonlocal changes of the independent variables  $x, t$ ; here  $a_i, b_i, a, b, c, d$  are arbitrary constants. It was shown in [1, 2] that, along with affine transformations of the dependent variables  $u^i$ , reciprocal transformations generate the group  $SL(n+2)$  which acts by projective transformations on the associated congruence (2). It is remarkable that these transformations preserve the Hamiltonian property.

**Theorem 4.** *The class of conservative systems (1) possessing third-order Hamiltonian formulation (3) is invariant under reciprocal transformations (15).*

We prove this result in Section 2.4. Note that, in contrast to third-order operators (3), first-order Hamiltonian structures of Dubrovin-Novikov type are not reciprocally invariant, and generally become nonlocal [14].

**Remark.** In [15, 16] we have classified third-order Hamiltonian operators/systems modulo the restricted group of reciprocal transformations that change the independent variable  $x$  only,

$$d\tilde{x} = (a_i u^i + a)dx + (a_i V^i + b)dt, \quad d\tilde{t} = dt.$$

In the 3-component case this resulted in the 5 canonical forms. Modulo extended transformations (15), all of them are equivalent to that of Example 2 from Section 1.4.

Ultimately, the classification of Hamiltonian systems of conservation laws (1) up to reciprocal transformations (15) reduces to projective classification of the associated congruences (2).

## 1.6 Classification results

Here we summarise the classification results of Hamiltonian systems of conservation laws with  $n = 2$  and 3 components. The classification is performed modulo reciprocal/projective transformations as discussed in Section 1.5. We always assume that system (1) is strictly hyperbolic, and that the metric  $g$  defining Hamiltonian operator (3) is non-degenerate.

The existing classification of linear congruences in  $\mathbb{P}^3$  and  $\mathbb{P}^4$  readily leads to the classification of 2- and 3-component Hamiltonian systems of conservation laws. Thus, every linear congruence in  $\mathbb{P}^3$  consists of bisecants of two skew lines in  $\mathbb{P}^3$ . This leads to

**Theorem 5.** *For  $n = 2$ , every Hamiltonian system of conservation laws is linearisable (that is, equivalent to 2-component case of Example 1 from Section 1.4).*

Linear congruences in  $\mathbb{P}^4$  were classified by Castelnuovo in [5]: they can be obtained as trisecant lines of suitable projections of the Veronese surface from  $\mathbb{P}^5$  into  $\mathbb{P}^4$ . Thus, all generic linear congruences are projectively equivalent (non-generic projections correspond to systems with degenerate Hamiltonian operators).

**Theorem 6.** *For  $n = 3$ , every Hamiltonian system of conservation laws is either linearisable (that is, equivalent to 3-component case of Example 1 from Section 1.4), or equivalent to the system of WDVV equations (Example 2 from Section 1.4).*

Theorems 5, 6 are proved in Section 2.7. It follows that all 3-component systems of conservation laws with third-order Hamiltonian structures are automatically integrable.

The case  $n = 4$  is far more complicated, primarily, due to the fact that there exists no classification of linear congruences in  $\mathbb{P}^5$ . Only partial results are currently available. In particular, 4-component Hamiltonian systems (1) associated with third-order Hamiltonian operators are not integrable in general.

### 1.7 Algebraic reformulation of the problem

Linear congruences in  $\mathbb{P}^{n+1}$  are defined by  $n$  linear relations in the Plücker coordinates. Setting  $\mathbb{P}^{n+1} = \mathbb{P}(W)$  where  $W$  is a vector space of dimension  $n + 2$ , these linear relations correspond to the choice of an  $n$ -dimensional subspace  $A \subset \Lambda^2(W)$ . Let  $A^1, \dots, A^n$  denote a basis of  $A$ . The condition that the corresponding system (1) is Hamiltonian, is equivalent to the existence of a non-degenerate symmetric matrix  $\phi_{\beta\gamma}$ , the same as in (7), such that

$$\phi_{\beta\gamma} A^\beta \wedge A^\gamma = 0,$$

see Section 2.6. The existence of such relation does not depend on the choice of basis, and imposes strong constraints on  $A$ . Despite its apparent simplicity, the classification of normal forms of such subspaces is an open problem (starting with  $n = 4$ ).

### 1.8 Symbolic computations

Symbolic computations were performed by CDE [38], a Reduce [37] package for integrability of PDEs. CDE (by one of us, RFV) can compute: Fréchet derivatives, formal adjoints, symmetries and conservation laws, Hamiltonian operators, and their brackets. Examples are available from [38], and a User's manual is included in the official Reduce manual; a book with numerous detailed computations is to appear soon [24].

## 2 Proofs

### 2.1 Conditions for a system to be Hamiltonian: proof of Theorem 1

In this section we derive the necessary and sufficient conditions for system (1) to possess Hamiltonian structure (3).

**Theorem 1.** *The necessary and sufficient conditions for a conservative system (1) to possess third-order Hamiltonian operator (3) are the following:*

$$\begin{aligned} g_{im}V_j^m &= g_{jm}V_i^m, \\ c_{mkl}V_i^m + c_{mik}V_l^m + c_{mli}V_k^m &= 0, \\ V_{ij}^k &= g^{ks}c_{smj}V_i^m + g^{ks}c_{smi}V_j^m, \end{aligned}$$

here low indices of  $V^m$  denote partial derivatives,  $V_i^m = \partial V^m / \partial u^i$ , etc.

*Proof.* The proof is based on the Kersten–Krasil’shchik–Verbovetsky approach to Hamiltonian operators [22] which can be summarised as follows. Consider an evolutionary system of the form

$$F^i = u_t^i - f^i(t, x, u, u_x, u_{xx}, \dots) = 0, \quad (17)$$

with the formal linearization (Fréchet derivative)  $\ell_F$ . Let  $P$  be a Hamiltonian operator, that is, a skew-adjoint operator with zero Schouten bracket,  $[P, P] = 0$ . If system (17) possesses  $P$  as a Hamiltonian structure, then  $P$  maps variational derivatives of conserved densities of (17) into generalized (higher) symmetries, that is,

$$\ell_F \circ P = P^* \circ \ell_F^*. \quad (18)$$

Let us introduce the adjoint system (*cotangent covering*) of system (17),

$$\begin{cases} F = 0, \\ \ell_F^*(p) = 0, \end{cases} \quad (19)$$

where  $p$  is an auxiliary (vector) variable. Then (18) is equivalent to

$$\ell_F(P(p)) = 0, \quad (20)$$

which must hold identically modulo (19). Note that the idea of representing Hamiltonian operators  $P$  by linear differential expressions of type  $P(p)$

was used in [18] to compute Hamiltonian cohomology. The advantage of the above formulation is that finding Hamiltonian operators amounts to solving a problem which is computationally the same as finding generalized symmetries.

To apply this technique to system (1) we introduce a potential substitution  $u^i = b_x^i$ , for reasons that will become clear soon, obtaining a *non-quasilinear* system

$$b_t^i = V^i(\mathbf{b}_x). \quad (21)$$

This substitution turns Hamiltonian operator (3) into a first-order operator,

$$P^{ij} = -(g^{ij}(\mathbf{b}_x)\partial_x + c_k^{ij}(\mathbf{b}_x)b_{xx}^k), \quad (22)$$

and the corresponding Hamiltonian can be calculated explicitly (see Section 2.5). Note that the above Hamiltonian operator is **not** of Dubrovin–Novikov type as its coefficients  $g^{is}(\mathbf{b}_x)$  and  $c_k^{is}(\mathbf{b}_x)$  lose their geometric interpretation: their transformation rule is no longer tensorial. The linearisation operator of system (21) is

$$\ell_F(\varphi) = D_t\varphi^i - \frac{\partial V^i}{\partial b_x^j} D_x\varphi^j,$$

with the adjoint

$$\ell_F^*(\psi) = -D_t\psi_k + D_x\left(\frac{\partial V^i}{\partial b_x^k}\psi_i\right).$$

The adjoint system is

$$\begin{aligned} b_t^i &= V^i(\mathbf{b}_x), \\ p_{k,t} &= \frac{\partial^2 V^i}{\partial b_x^k \partial b_x^h} b_{xx}^h p_i + \frac{\partial V^i}{\partial b_x^k} p_{i,x}. \end{aligned}$$

Setting  $P(\mathbf{p}) = -g^{ij}p_{j,x} - c_k^{ij}b_{xx}^k p_j$ , condition (20) takes the form

$$\begin{aligned} \ell_F(P(\mathbf{p})) &= -\frac{\partial g^{ij}}{\partial b_x^k} b_{xt}^k p_{j,x} - g^{ij} p_{j,xt} - \frac{\partial c_k^{ij}}{\partial b_x^h} b_{xt}^h b_{xx}^k p_j - c_k^{ij} b_{xxt}^k p_j - c_k^{ij} b_{xx}^k p_{j,t} \\ &+ \frac{\partial V^i}{\partial b_x^j} \left( \frac{\partial g^{jk}}{\partial b_x^h} b_{xx}^h p_{k,x} + g^{jk} p_{k,xx} + \frac{\partial c_k^{jh}}{\partial b_x^l} b_{xx}^l b_{xx}^k p_h + c_k^{jh} b_{xxx}^k p_h + c_k^{jh} b_{xx}^k p_{h,x} \right). \end{aligned}$$

Using differential consequences of the adjoint system,

$$b_{tx}^i = V_x^i, \quad b_{txx}^i = V_{xx}^i, \quad p_{k,tx} = D_{xx} \frac{\partial V^i}{\partial b_x^k} p_i + 2D_x \frac{\partial V^i}{\partial b_x^k} p_{i,x} + \frac{\partial V^i}{\partial b_x^k} p_{i,xx},$$

we obtain

$$\begin{aligned}
\ell_F(P(\mathbf{p})) &= \left( -g^{ij} \frac{\partial V^h}{\partial b_x^j} + \frac{\partial V^i}{\partial b_x^j} g^{jh} \right) p_{h,xx} \\
&+ \left( -\frac{\partial g^{ih}}{\partial b_x^k} V_x^k - g^{ij} 2D_x \frac{\partial V^h}{\partial b_x^j} - c_k^{ij} b_{xx}^k \frac{\partial V^h}{\partial b_x^j} \right. \\
&\quad \left. + \frac{\partial V^i}{\partial b_x^j} \frac{\partial g^{jh}}{\partial b_x^k} b_{xx}^k + \frac{\partial V^i}{\partial b_x^j} c_k^{jh} b_{xx}^k \right) p_{h,x} \\
&+ \left( -g^{ij} D_{xx} \frac{\partial V^h}{\partial b_x^j} - \frac{\partial c_k^{ih}}{\partial b_x^j} V_x^j b_{xx}^k - c_k^{ih} V_{xx}^k - c_k^{ij} b_{xx}^k \frac{\partial^2 V^h}{\partial b_x^j \partial b_x^l} b_{xx}^l \right. \\
&\quad \left. + \frac{\partial V^i}{\partial b_x^j} \left( \frac{\partial c_k^{jh}}{\partial b_x^l} b_{xx}^l b_{xx}^k + c_k^{jh} b_{xxx}^k \right) \right) p_h.
\end{aligned}$$

The above expression is linear in  $p_h$ ,  $p_{h,x}$ ,  $p_{h,xx}$ , and the coefficients are polynomials in  $b_{xx}^i$ ,  $b_{xxx}^i$ . So, the expression vanishes if and only if

$$-g^{ij} \frac{\partial V^h}{\partial b_x^j} + \frac{\partial V^i}{\partial b_x^j} g^{jh} = 0, \quad (23a)$$

$$-g^{ik} \frac{\partial^2 V^h}{\partial b_x^k \partial b_x^l} - c_k^{ih} \frac{\partial V^k}{\partial b_x^l} + \frac{\partial V^i}{\partial b_x^k} c_l^{kh} = 0, \quad (23b)$$

$$-\frac{\partial g^{ih}}{\partial b_x^k} \frac{\partial V^k}{\partial b_x^l} - g^{ij} 2 \frac{\partial^2 V^h}{\partial b_x^j \partial b_x^l} - c_l^{ij} \frac{\partial V^h}{\partial b_x^j} + \frac{\partial V^i}{\partial b_x^j} \frac{\partial g^{jh}}{\partial b_x^l} + \frac{\partial V^i}{\partial b_x^j} c_l^{jh} = 0, \quad (23c)$$

$$\begin{aligned}
& -g^{ij} \frac{\partial^3 V^h}{\partial b_x^j \partial b_x^l \partial b_x^m} - \frac{1}{2} \left( \frac{\partial c_m^{ih}}{\partial b_x^j} \frac{\partial V^j}{\partial b_x^l} + \frac{\partial c_l^{ih}}{\partial b_x^j} \frac{\partial V^j}{\partial b_x^m} \right) - c_k^{ih} \frac{\partial^2 V^k}{\partial b_x^l \partial b_x^m} \\
& - \frac{1}{2} \left( c_m^{ij} \frac{\partial^2 V^h}{\partial b_x^j \partial b_x^l} + c_l^{ij} \frac{\partial^2 V^h}{\partial b_x^j \partial b_x^m} \right) + \frac{1}{2} \left( \frac{\partial V^i}{\partial b_x^j} \frac{\partial c_m^{jh}}{\partial b_x^l} + \frac{\partial V^i}{\partial b_x^j} \frac{\partial c_l^{jh}}{\partial b_x^m} \right) = 0.
\end{aligned} \quad (23d)$$

The conditions (23a), (23b), (23c) can be simplified by using objects  $g_{ij}$ ,  $c_{ijk}$  with lower indices, leading to

$$g_{ij} \frac{\partial V^j}{\partial b_x^h} - \frac{\partial V^j}{\partial b_x^i} g_{jh} = 0, \quad (24a)$$

$$c_{mkl} \frac{\partial V^m}{\partial b_x^i} + c_{mik} \frac{\partial V^m}{\partial b_x^l} + c_{mli} \frac{\partial V^m}{\partial b_x^k} = 0, \quad (24b)$$

$$\frac{\partial^2 V^k}{\partial b_x^i \partial b_x^j} = g^{ks} c_{smj} \frac{\partial V^m}{\partial b_x^i} + g^{ks} c_{smi} \frac{\partial V^m}{\partial b_x^j}. \quad (24c)$$



Indeed, lowering indices in (23a) leads to (24a). Similarly, lowering indices in (23b) and using (24a) leads to (24c). Using (23b) to eliminate second-order derivatives in (23c) we get

$$-\frac{\partial g^{ih}}{\partial b_x^k} \frac{\partial V^k}{\partial b_x^l} + 2 \left( c_k^{ih} \frac{\partial V^k}{\partial b_x^l} - \frac{\partial V^i}{\partial b_x^k} c_l^{kh} \right) - c_l^{ij} \frac{\partial V^h}{\partial b_x^j} + \frac{\partial V^i}{\partial b_x^j} \frac{\partial g^{jh}}{\partial b_x^l} + \frac{\partial V^i}{\partial b_x^j} c_l^{jh} = 0.$$

Lowering indices and using (24a) again we obtain

$$(g_{ps,k} + 2c_{spk}) \frac{\partial V^k}{\partial b_x^l} + (-g_{sk,l} - c_{skl}) \frac{\partial V^k}{\partial b_x^p} - c_{kpl} \frac{\partial V^k}{\partial b_x^s} = 0.$$

Using (4) we obtain (24b). It remains to show that equation (23d) is a differential consequence of equations (24). In order to prove this statement we will need equation (4d). Let us differentiate (24c) with respect to  $b_x^l$  and lower the index  $k$  by the metric  $g$ . Using (4), (5), (6) we obtain

$$\begin{aligned} g_{km} \frac{\partial^3 V^m}{\partial b_x^j \partial b_x^i \partial b_x^l} &= g^{pq} (c_{kpi} c_{qml} + c_{kpl} c_{qmi}) \frac{\partial V^m}{\partial b_x^j} \\ &+ g^{pq} (c_{kpj} c_{qml} + c_{kpl} c_{qmj}) \frac{\partial V^m}{\partial b_x^i} + g^{pq} (c_{kpi} c_{qmj} + c_{kpj} c_{qmi}) \frac{\partial V^m}{\partial b_x^l}. \end{aligned}$$

One can show that equation (23d) can be brought to this form. Let us first bring (23d) to the form

$$\frac{\partial^3 V^k}{\partial b_x^i \partial b_x^j \partial b_x^l} = -\frac{1}{2} g_{jm} \left( 2c_s^{mk} g^{sq} c_{qpi} + \frac{\partial c_i^{mk}}{\partial b_x^p} + c_i^{ms} g^{kq} c_{qps} \right) \frac{\partial V^p}{\partial b_x^l} \quad (25a)$$

$$-\frac{1}{2} g_{jm} \left( 2c_s^{mk} g^{sq} c_{qpl} + \frac{\partial c_l^{mk}}{\partial b_x^p} + c_l^{ms} g^{kq} c_{qps} \right) \frac{\partial V^p}{\partial b_x^i} \quad (25b)$$

$$-\frac{1}{2} g^{kq} g_{jm} (c_i^{ms} c_{qpl} + c_l^{ms} c_{qpi}) \frac{\partial V^p}{\partial b_x^s} \quad (25c)$$

$$+\frac{1}{2} g_{sm} \left( \frac{\partial c_i^{sk}}{\partial b_x^l} + \frac{\partial c_l^{sk}}{\partial b_x^i} \right) \frac{\partial V^m}{\partial b_x^j}. \quad (25d)$$

Observe that the term (25c) can be rearranged as

$$\begin{aligned} &g_{jm} (c_i^{ms} c_{qpl} + c_l^{ms} c_{qpi}) \frac{\partial V^p}{\partial a^s} \\ &= g^{sb} (c_{bji} c_{qpl} + c_{bjl} c_{qpi}) \frac{\partial V^p}{\partial a^s} \\ &= g^{sp} (c_{bji} c_{qpl} + c_{bjl} c_{qpi}) \frac{\partial V^b}{\partial a^s} \\ &= -c_{qpl} \left( c_{bsj} \frac{\partial V^b}{\partial a^i} + c_{bis} \frac{\partial V^b}{\partial a^j} \right) - c_{qpi} \left( c_{bsj} \frac{\partial V^b}{\partial a^l} + c_{bjs} \frac{\partial V^b}{\partial a^i} \right), \end{aligned}$$

where we used (24a) and (24b). Lowering indices in (25), using

$$g_{hk}g_{jm}\frac{\partial c_l^{mk}}{\partial b_x^p} = \frac{\partial c_{hjl}}{\partial b_x^p} - \frac{\partial g_{hk}}{\partial b_x^p}g_{jm}c_l^{mk} - g_{hk}\frac{\partial g_{jm}}{\partial b_x^p}c_l^{mk}, \quad (26)$$

as well as (4), we arrive at (23d). Replacing in (24)  $b_x^i$  by  $u^i$  we obtain (11). To finish the proof, it remains to note that conditions (11) imply the existence of a (nonlocal) Hamiltonian, see Theorem 7 of Section 2.5 for explicit formulae.  $\square$

## 2.2 Involutivity of system (11): proof of Theorem 2

In this section we establish the involutivity of system (11), and estimate the number of parameters in the general solution.

**Theorem 2.** *System (11) is in involution. Its general solution depends on  $\leq \frac{n(n+3)}{2}$  arbitrary constants.*

*Proof.* We need to show that differentiation of first-order conditions (11a) and (11b) does not lead to new first-order relations. Then, we need to demonstrate consistency of second-order equations (11c). All this is a straightforward tensor algebra. Differentiating

$$g_{ip}V_j^p = g_{jp}V_i^p,$$

we obtain

$$g_{ip}V_{jk}^p + g_{ip,k}V_j^p = g_{jp}V_{ik}^p + g_{jp,k}V_i^p.$$

Using (11c) we get

$$g_{ip}g^{ps}[c_{smj}V_k^m + c_{smk}V_j^m] + g_{ip,k}V_j^p = g_{jp}g^{ps}[c_{smk}V_i^m + c_{smi}V_k^m] + g_{jp,k}V_i^p.$$

Thus,

$$c_{imj}V_k^m + c_{imk}V_j^m + g_{ip,k}V_j^p = c_{jmk}V_i^m + c_{jmi}V_k^m + g_{jp,k}V_i^p,$$

or, relabelling indices,

$$c_{ipj}V_k^p + c_{ipk}V_j^p + g_{ip,k}V_j^p = c_{jpk}V_i^p + c_{jpi}V_k^p + g_{jp,k}V_i^p.$$

This can be rewritten in the form

$$(c_{ipj} - c_{jpi})V_k^p + (c_{ipk} + g_{ip,k})V_j^p - (c_{jpk} + g_{jp,k})V_i^p = 0.$$

Taking into account (4a) we obtain

$$(c_{ipj} - c_{jpi})V_k^P - c_{pik}V_j^P + c_{pjk}V_i^P = 0.$$

Due to (4b) we can rewrite this as

$$(c_{ipj} - c_{jpi})V_k^P + c_{pki}V_j^P + c_{pjk}V_i^P = 0.$$

Using (11b) we obtain

$$(c_{ipj} - c_{jpi})V_k^P = c_{pij}V_k^P.$$

It remains to note that the equality  $c_{ipj} - c_{jpi} = c_{pij}$  holds identically due to the cyclic condition (4c). Thus, differentiation of (11a) does not lead to new first-order relations.

Similarly, differentiating (11b) we obtain

$$c_{mkl,j}V_i^m + c_{mkl}V_{ij}^m + c_{mik,j}V_l^m + c_{mik}V_{lj}^m + c_{mli,j}V_k^m + c_{mli}V_{kj}^m = 0.$$

The substitution of (4d) and (11c) gives

$$\begin{aligned} & -g^{pq}c_{pmj}c_{qkl}V_i^m + c_{mkl}g^{ms}(c_{spi}V_j^P + c_{spj}V_i^P) \\ & -g^{pq}c_{pmj}c_{qik}V_l^m + c_{mik}g^{ms}(c_{spl}V_j^P + c_{spj}V_l^P) \\ & -g^{pq}c_{pmj}c_{qli}V_k^m + c_{mli}g^{ms}(c_{spk}V_j^P + c_{spj}V_k^P) = 0. \end{aligned}$$

Note that all terms apart from those containing  $V_j^P$  cancel, leading to

$$g^{ms}(c_{mkl}c_{spi} + c_{mik}c_{spl} + c_{mli}c_{spk})V_j^P = 0.$$

Due to (4d), (4b) this expression can be rewritten as

$$-(c_{kpi,l} + c_{pik,l} + c_{ikp,l})V_j^P = 0,$$

which is an identity due to (4c).

The compatibility of second-order relations (11c) can be shown as follows. Computation of the consistency condition  $V_{i,j,k}^P = V_{k,j,i}^P$  gives

$$\begin{aligned} & g_{,k}^{ps}[c_{smj}V_i^m + c_{smi}V_j^m] \\ & + g^{ps}[c_{smj}V_{ik}^m + c_{smj,k}V_i^m + c_{smi}V_{jk}^m + c_{smi,k}V_j^m] \\ & = g_{,i}^{ps}[c_{smj}V_k^m + c_{smk}V_j^m] \\ & + g^{ps}[c_{smj}V_{ik}^m + c_{smj,i}V_k^m + c_{smk}V_{ij}^m + c_{smk,i}V_j^m]. \end{aligned}$$

Cancelling terms with  $V_{ik}^m$  results in a simplified expression,

$$\begin{aligned} & g_{,k}^{ps} [c_{smj} V_i^m + c_{smi} V_j^m] + g^{ps} [c_{smj,k} V_i^m + c_{smi} V_{jk}^m + c_{smi,k} V_j^m] \\ &= g_{,i}^{ps} [c_{smj} V_k^m + c_{smk} V_j^m] + g^{ps} [c_{smj,i} V_k^m + c_{smk} V_{ij}^m + c_{smk,i} V_j^m]. \end{aligned}$$

Contraction with  $g_{pq}$  gives

$$\begin{aligned} & g_{pq} g_{,k}^{ps} [c_{smj} V_i^m + c_{smi} V_j^m] + c_{qmj,k} V_i^m + c_{qmi} V_{jk}^m + c_{qmi,k} V_j^m \\ &= g_{pq} g_{,i}^{ps} [c_{smj} V_k^m + c_{smk} V_j^m] + c_{qmj,i} V_k^m + c_{qmk} V_{ij}^m + c_{qmk,i} V_j^m. \end{aligned}$$

Taking into account (11c) along with the identity  $g_{pq} g_{,k}^{ps} = -g^{sp} g_{pq,k}$  we get

$$\begin{aligned} & -g^{sp} g_{pq,k} [c_{smj} V_i^m + c_{smi} V_j^m] + c_{qmj,k} V_i^m \\ & \quad + c_{qpi} g^{ps} [c_{smj} V_k^m + c_{smk} V_j^m] + c_{qmi,k} V_j^m \\ &= -g^{sp} g_{pq,i} [c_{smj} V_k^m + c_{smk} V_j^m] + c_{qmj,i} V_k^m \\ & \quad + c_{qpk} g^{ps} [c_{smj} V_i^m + c_{smi} V_j^m] + c_{qmk,i} V_j^m. \end{aligned}$$

Rearrangement gives

$$\begin{aligned} & [c_{qmj,k} - c_{qpk} g^{ps} c_{smj} - g^{sp} g_{pq,k} c_{smj}] V_i^m \\ & \quad + [c_{qpi} g^{ps} c_{smj} + g^{sp} g_{pq,i} c_{smj} - c_{qmj,i}] V_k^m \\ & \quad + [c_{qmi,k} - g^{sp} g_{pq,k} c_{smi} + c_{qpi} g^{ps} c_{smk} + g^{sp} g_{pq,i} c_{smk} \\ & \quad \quad - c_{qpk} g^{ps} c_{smi} - c_{qmk,i}] V_j^m = 0. \end{aligned}$$

Taking into account (4a) we obtain

$$\begin{aligned} & (c_{qmj,k} + g^{sp} c_{smj} c_{pqk}) V_i^m - (c_{qmj,i} + g^{sp} c_{smj} c_{pqi}) V_k^m \\ & + [(c_{qmi,k} + g^{sp} c_{smi} c_{pqk}) - (c_{qmk,i} + g^{sp} c_{smk} c_{pqi})] V_j^m = 0, \end{aligned}$$

which is an identity due to (4d).

Thus, system (11) is in involution. Since equations (11c) express all second-order partial derivatives of  $V^i$ , the general solution depends on no more than  $n+n^2$  parameters (values of  $V^i$  and first-order derivatives thereof). However, relations (11a) impose  $\frac{n(n-1)}{2}$  independent constraints on first-order derivatives of  $V^i$ . Thus, the general solution depends on no more than  $n + n^2 - \frac{n(n-1)}{2} = \frac{n(n+3)}{2}$  arbitrary constants: the inequality is due to the extra first-order relations (11b) that are not so easy to control. Examples show that solution spaces to equations (11) for different third-order Hamiltonian operators (with the same number of components) may have different dimensions. In particular, the maximal possible dimension,  $\frac{n(n+3)}{2}$ , corresponds to constant-coefficient operators ( $g_{ij} = \text{const}$ ,  $c_{ijk} = 0$ ).  $\square$

### 2.3 Integration of system (11): proof of Theorem 3

It is quite remarkable that system (11) for the fluxes  $V^i$ , which is a linear involutive system with non-constant coefficients, can be integrated in closed form. Let us recall that the metric  $g$  defining Hamiltonian operator (3) can be represented in factorised form (7),  $g_{ij} = \phi_{\beta\gamma}\psi_i^\beta\psi_j^\gamma$ , where  $\phi_{\beta\gamma}$  is a non-degenerate constant symmetric matrix, and  $\psi_k^\gamma = \psi_{km}^\gamma u^m + \omega_k^\gamma$ ; here  $\psi_{km}^\gamma$  and  $\omega_k^\gamma$  are constants such that  $\psi_{km}^\gamma = -\psi_{mk}^\gamma$ . These constants satisfy a set of quadratic relations (9), (10). Using relations (7) – (10), one can show that in the new variables  $W^\gamma$  defined as

$$W^\gamma = \psi_k^\gamma V^k,$$

system (11) takes the form

$$\begin{aligned}\phi_{\beta\gamma}[\psi_{ik}^\beta W^\gamma + \psi_k^\beta W_i^\gamma - \psi_i^\beta W_k^\gamma] &= 0, \\ \phi_{\beta\gamma}[\psi_{ij}^\beta W_k^\gamma + \psi_{jk}^\beta W_i^\gamma + \psi_{ki}^\beta W_j^\gamma] &= 0, \\ W_{ij}^\gamma &= 0,\end{aligned}$$

where lower indices of  $W^\gamma$  denote partial derivatives. The last condition implies that  $W^\gamma$  are linear functions,

$$W^\gamma = \eta_m^\gamma u^m + \xi^\gamma, \tag{29}$$

while the first two conditions imply that the constants  $\eta_m^\gamma$  and  $\xi^\gamma$  satisfy a linear system

$$\begin{aligned}\phi_{\beta\gamma}[\psi_{ij}^\beta \eta_k^\gamma + \psi_{jk}^\beta \eta_i^\gamma + \psi_{ki}^\beta \eta_j^\gamma] &= 0, \\ \phi_{\beta\gamma}[\psi_{ik}^\beta \xi^\gamma + \omega_k^\beta \eta_i^\gamma - \omega_i^\beta \eta_k^\gamma] &= 0.\end{aligned} \tag{30}$$

Thus, finding conservative Hamiltonian systems for a *given* third-order Hamiltonian operator (3) is reduced to linear algebra. Conversely, given conservative system (1), the reconstruction of the associated Hamiltonian representation from system (11) reduces to a linear system for the coefficients of a Monge metric. The above representation implies the following result.

**Theorem 3.** *For Hamiltonian system (1) the following conditions hold:*

- *The corresponding congruence (2) is linear.*
- *System (1) is linearly degenerate and belongs to the Temple class.*

- The fluxes  $V^i$  are rational functions of the form

$$V^i = \frac{Q^i}{\det \psi},$$

where  $\det \psi$  is a polynomial of degree  $n-1$  defining the singular surface, and  $Q^i$  are polynomials of degree  $n$ .

*Proof.* Linearity of the congruence can be demonstrated as follows. Substituting  $W^\gamma = \eta_m^\gamma u^m + \xi^\gamma$  and  $\psi_k^\gamma = \psi_{km}^\gamma u^m + \omega_k^\gamma$  into the formula  $W^\gamma = \psi_k^\gamma V^k$  we obtain a linear relation in the Plücker coordinates (note the skew-symmetry condition  $\psi_{km}^\gamma = -\psi_{mk}^\gamma$ ),

$$\frac{1}{2} \psi_{km}^\gamma (u^m V^k - u^k V^m) + \omega_k^\gamma V^k - \eta_m^\gamma u^m - \xi^\gamma = 0. \quad (31)$$

This proves the linearity. Linear degeneracy and the Temple property of system (1) follows from the linearity of the corresponding congruence [2]. Finally, solving the equations  $W^\gamma = \psi_k^\gamma V^k$  for  $V^k$  implies  $V^k = \psi_\gamma^k W^\gamma$  where  $\psi_\gamma^k$  is the inverse matrix to  $\psi_k^\gamma$ . Thus,  $V^k$  will have  $\det \psi$  in the denominator, while numerators will be polynomials of degree  $n$ .  $\square$

## 2.4 Projective invariance: proof of Theorem 4

In this section we show that third-order Hamiltonian formalism (3) is invariant under reciprocal transformations (15). This is in contrast with the case of first-order Hamiltonian structures of Dubrovin-Novikov type, which generally become nonlocal after a reciprocal transformation [12, 14].

**Theorem 4.** *The class of conservative systems (1) possessing third-order Hamiltonian formulation (3) is invariant under reciprocal transformations (15).*

*Proof.* A general reciprocal transformation (15) can be represented as a composition,

$$(x\text{-transformation}) \circ (x \leftrightarrow t) \circ (x\text{-transformation}),$$

where  $x$ -transformation is a reciprocal transformation changing the variable  $x$  only, and  $x \leftrightarrow t$  denotes the ‘inversion’, that is, the interchange of  $x$  and  $t$ . The invariance of Hamiltonian formalism (3) under  $x$ -transformations was established in [15]. Thus, it remains to show that third-order Hamiltonian formalism (3) is invariant under the inversion. Under this transformation,

the new dependent variables and the new fluxes are defined as  $\tilde{u}^i = V^i$ ,  $\tilde{V}^i = u^i$ , respectively. Recall that system (1) possesses Hamiltonian operator (3) if the following conditions are satisfied:

1. Metric  $g$  of Hamiltonian operator (3) possesses factorised form (7),  $g_{ij} = \phi_{\beta\gamma} \psi_i^\beta \psi_j^\gamma$ , where  $\phi_{\beta\gamma}$  is a constant symmetric matrix and  $\psi_k^\gamma = \psi_{km}^\gamma u^m + \omega_k^\gamma$ ; here the constants  $\omega_k^\gamma$  and skew-symmetric  $\psi_{km}^\gamma$  satisfy relations (9), (10).
2. The functions  $\psi_k^\gamma V^k$  are linear in  $u$ :  $\psi_k^\gamma V^k = \eta_m^\gamma u^m + \xi^\gamma$ , where the constants  $\eta_m^\gamma$ ,  $\xi^\gamma$  satisfy relations (30).

We claim that the ‘inverted’ system is also Hamiltonian, and the metric of the transformed Hamiltonian operator is given by

$$\tilde{g}_{ij} = V_i^m g_{mn} V_j^n, \quad (32)$$

note that this transformation rule is identical to that for first-order Hamiltonian operators of Dubrovin-Novikov type. Thus, we have to demonstrate the following:

1. Metric  $\tilde{g}$  of the transformed Hamiltonian operator possesses factorised form  $\tilde{g}_{ij} = \tilde{\phi}_{\beta\gamma} \tilde{\psi}_i^\beta \tilde{\psi}_j^\gamma$ , where  $\tilde{\phi}_{\beta\gamma}$  is a constant symmetric matrix, and  $\tilde{\psi}_k^\gamma = \tilde{\psi}_{km}^\gamma V^m + \tilde{\omega}_k^\gamma$ ; here  $\tilde{\omega}_k^\gamma$  and skew-symmetric  $\tilde{\psi}_{km}^\gamma$  must satisfy relations (9), (10).
2. The expressions  $\tilde{\psi}_k^\gamma u^k$  are linear in  $V$ :  $\tilde{\psi}_k^\gamma u^k = \tilde{\eta}_m^\gamma V^m + \tilde{\xi}^\gamma$ , where the constants  $\tilde{\eta}_m^\gamma$ ,  $\tilde{\xi}^\gamma$  satisfy relations (30).

We claim that this is indeed the case, furthermore,

$$\tilde{\phi}_{\beta\gamma} = \phi_{\beta\gamma}, \quad \tilde{\psi}_{km}^\gamma = \psi_{km}^\gamma, \quad \tilde{\omega}_k^\gamma = \omega_k^\gamma, \quad \tilde{\eta}_k^\gamma = \omega_k^\gamma, \quad \tilde{\xi}^\gamma = -\xi^\gamma,$$

note that the constants with tilde’s satisfy the same relations (9), (10), (30).

To establish part 1 we proceed as follows. Differentiating the relation  $W^\gamma = \psi_k^\gamma V^k$  with respect to  $u^m$  we obtain  $\eta_m^\gamma = \psi_k^\gamma V_m^k + \psi_{km}^\gamma V^k$ . Solving for  $V_m^k$  gives  $V_m^k = \psi_\gamma^k \eta_m^\gamma - \psi_\gamma^k \psi_{sm}^\gamma V^s$ , where  $\psi_\gamma^k$  is the inverse matrix to  $\psi_k^\gamma$ . Thus, using (32),

$$\begin{aligned} \tilde{g}_{ij} &= V_i^m g_{mk} V_j^k = (\psi_\gamma^m \eta_i^\gamma - \psi_\gamma^m \psi_{ri}^\gamma V^r) g_{mk} (\psi_\tau^k \eta_j^\tau - \psi_\tau^k \psi_{sj}^\tau V^s) \\ &= (\psi_\gamma^m \eta_i^\gamma - \psi_\gamma^m \psi_{ri}^\gamma V^r) \psi_m^\beta \phi_{\beta\gamma} \psi_k^\gamma (\psi_\tau^k \eta_j^\tau - \psi_\tau^k \psi_{sj}^\tau V^s) \end{aligned}$$

$$= (\eta_i^\beta - \psi_{ri}^\beta V^r) \phi_{\beta\gamma} (\eta_j^\gamma - \psi_{sj}^\gamma V^s) = (\eta_i^\beta + \psi_{ir}^\beta V^r) \phi_{\beta\gamma} (\eta_j^\gamma + \psi_{js}^\gamma V^s) = \tilde{\psi}_i^\beta \phi_{\beta\gamma} \tilde{\psi}_j^\gamma,$$

which is the required formula. Finally, for part 2, it is a simple exercise to verify that the relation  $\tilde{\psi}_k^\gamma u^k = \tilde{\eta}_m^\gamma V^m + \tilde{\xi}^\gamma$  follows from  $\psi_k^\gamma V^k = \eta_m^\gamma u^m + \xi^\gamma$ .  $\square$

## 2.5 Casimirs, Momentum, Hamiltonian

Given system (1) satisfying conditions (11), in this section we derive explicit formulae for the corresponding Casimirs, Momentum and the Hamiltonian. To do so we introduce the substitution  $u^i = b_x^i$  transforming system (1) into (non-quasilinear) first-order form (21),

$$b_t^i = V^i(\mathbf{b}_x).$$

In variables  $b^i$ , operator (3) takes first-order form (22). Using  $g_{ij} = \phi_{\beta\gamma} \psi_i^\beta \psi_j^\gamma$  we can rewrite it in factorised form,

$$P^{ij} = -\phi^{\beta\gamma} \psi_\beta^i \partial_x \psi_\gamma^j,$$

recall that  $\psi_\beta^i$  is the inverse matrix to  $\psi_i^\beta$ .

**Theorem 7.** *System (21) can be represented in Hamiltonian form,*

$$b_t^i = V^i(\mathbf{b}_x) = P^{ij} \frac{\delta H}{\delta b^j},$$

with the local Hamiltonian

$$\begin{aligned} H = \int h \, dx = & - \int \phi_{\beta\gamma} \left[ \left( \frac{1}{3} \eta_p^\gamma \psi_{qm}^\beta b_x^m + \frac{1}{2} \omega_p^\beta \eta_q^\gamma \right) b^p b^q \right. \\ & \left. + x \xi^\gamma \left( \frac{1}{2} \psi_{pq}^\beta b^p b_x^q + \omega_q^\beta b^q \right) \right] dx, \end{aligned} \quad (33)$$

note the explicit  $x$ -dependence. The  $n$  Casimirs are given by

$$C^\alpha = \int c^\alpha \, dx = \int \left( \frac{1}{2} \psi_{mk}^\alpha b_x^k + \omega_m^\alpha \right) b^m \, dx. \quad (34)$$

The Momentum has the form

$$M = \int m \, dx = - \int \left( \frac{1}{3} \phi_{\beta\gamma} \omega_q^\beta \psi_{pm}^\gamma b_x^m + \frac{1}{2} \phi_{\beta\gamma} \omega_p^\beta \omega_q^\gamma \right) b^p b^q \, dx. \quad (35)$$



**Remark.** In the particular case  $\xi = 0$ , equations (29), (30), (33) were obtained in [29]. If  $\xi \neq 0$ , the corresponding Hamiltonian density  $h$  has explicit  $x$ -dependence. It may be more than just a curiosity that all known integrable systems (1) with Hamiltonian structure (3) admit a *local* compatible first-order Hamiltonian operator iff  $h$  has no explicit  $x$ -dependence.

*Proof.* Using relations (30), one obtains the following expression for the variational derivative of  $H$ ,

$$\frac{\delta H}{\delta b^j} = -\phi_{\beta\gamma}(\psi_{jp}^\beta b_x^p + \omega_j^\beta)(\eta_q^\gamma b^q + \xi^\gamma x) = -\phi_{\beta\gamma}\psi_j^\beta(\eta_q^\gamma b^q + \xi^\gamma x).$$

Thus,

$$\begin{aligned} b_t^i &= P^{ij} \frac{\delta H}{\delta b^j} = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^j \frac{\delta H}{\delta b^j} = \phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^j \phi_{\mu\nu} \psi_j^\mu (\eta_q^\nu b^q + \xi^\nu x) \\ &= \phi^{\beta\gamma}\psi_\beta^i \partial_x \phi_{\gamma\nu} (\eta_q^\nu b^q + \xi^\nu x) = \phi^{\beta\gamma}\psi_\beta^i \phi_{\gamma\nu} (\eta_q^\nu b_x^q + \xi^\nu) \\ &= \psi_\nu^i (\eta_q^\nu b_x^q + \xi^\nu) = \psi_\nu^i W^\nu = V^i(\mathbf{b}_x), \end{aligned}$$

as required. Similarly, variational derivatives of the Casimirs are

$$\frac{\delta C^\alpha}{\delta b^j} = \psi_{jk}^\alpha b_x^k + \omega_j^\alpha = \psi_j^\alpha,$$

so that

$$P^{ij} \frac{\delta C^\alpha}{\delta b^j} = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^j \frac{\delta C^\alpha}{\delta b^j} = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^j \psi_j^\alpha = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \delta_\gamma^\alpha = 0.$$

Finally, using (10), one computes variational derivatives of the Momentum,

$$\frac{\delta M}{\delta b^j} = -\phi_{\beta\gamma}\psi_j^\beta \omega_m^\gamma b^m = -\phi_{\beta\gamma}\psi_j^\beta \partial_x^{-1} \psi_m^\gamma b_x^m,$$

thus,

$$P^{ij} \frac{\delta M}{\delta b^j} = -\phi^{\beta\gamma}\psi_\beta^i \partial_x \psi_\gamma^j \frac{\delta M}{\delta b^j} = b_x^i,$$

as required. Note that in the original variables  $u^i$ , all of the above densities become nonlocal.  $\square$

## 2.6 Algebraic reformulation of conditions (9), (10), (30)

In this section we demonstrate that algebraic constraints (9), (10), (30) can be represented in a compact invariant form which substantially simplifies their analysis. Let us note that lines (2) pass through the points  $y^i = u^i$ ,  $y^{n+1} = 1$ ,  $y^{n+2} = 0$  and  $y^i = V^i$ ,  $y^{n+1} = 0$ ,  $y^{n+2} = 1$ , respectively. The corresponding Plücker coordinates, which are  $2 \times 2$  minors of the  $2 \times (n+2)$  matrix

$$\begin{pmatrix} u^i & \dots & u^n & 1 & 0 \\ V^i & \dots & V^n & 0 & 1 \end{pmatrix},$$

can be arranged into  $(n+2) \times (n+2)$  skew-symmetric matrix,

$$Y = \left( \begin{array}{ccc|cc} & & & -V^1 & u^1 \\ & U & & \vdots & \vdots \\ & & & -V^n & u^n \\ \hline V^1 & \dots & V^n & 0 & 1 \\ -u^1 & \dots & -u^n & -1 & 0 \end{array} \right),$$

here  $U$  is the skew-symmetric matrix with entries  $u^i V^j - u^j V^i$ . In this notation, relations (31) can be represented as

$$\text{tr} Y A^\gamma = 0,$$

where  $(n+2) \times (n+2)$  skew-symmetric matrices  $A^\gamma$  are defined as

$$A^\gamma = \left( \begin{array}{ccc|cc} & & & \omega_1^\gamma & \eta_1^\gamma \\ & \frac{1}{2}\psi^\gamma & & \vdots & \vdots \\ & & & \omega_n^\gamma & \eta_n^\gamma \\ \hline -\omega_1^\gamma & \dots & -\omega_n^\gamma & 0 & \xi^\gamma \\ -\eta_1^\gamma & \dots & -\eta_n^\gamma & -\xi^\gamma & 0 \end{array} \right),$$

here  $\psi^\gamma$  is the skew-symmetric matrix with entries  $\psi_{ij}^\gamma$ . What is remarkable, relations (9), (10), (30) compactify into a single relation

$$\phi_{\beta\gamma} A^\beta \wedge A^\gamma = 0,$$

where each  $A^\gamma$  is interpreted as a 2-form.

## 2.7 Classification results: proof of Theorems 5, 6

In this Section we summarise the classification of 2- and 3-component Hamiltonian systems of conservation laws based on the classification of linear congruences in  $\mathbb{P}^3$  and  $\mathbb{P}^4$ .

**Theorem 5.** *For  $n = 2$ , every Hamiltonian system of conservation laws is linearisable (that is, equivalent to 2-component case of Example 1 from Section 1.4).*

*Proof.* Every linear congruence in  $\mathbb{P}^3$  consists of bisecants of two skew lines. Modulo projective transformations, any such congruence can be brought to the form

$$y^1 = u^1 y^3 + u^2 y^4, \quad y^2 = u^2 y^3 + u^1 y^4,$$

where  $y^i$  are homogeneous coordinates in  $\mathbb{P}^3$ . In the affine chart  $y^4 = 1$ , the skew lines in question can be defined as  $y^3 = 1$ ,  $y^1 = y^2$  and  $y^3 = -1$ ,  $y^1 = -y^2$ , respectively. The corresponding system of conservation laws is clearly linear,

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^1,$$

which is a particular case of Example 1. □

**Theorem 6.** *For  $n = 3$ , every Hamiltonian system of conservation laws is either linearisable (that is, equivalent to 3-component case of Example 1 from Section 1.4), or equivalent to the system of WDVV equations (Example 2 from Section 1.4).*

*Proof.* Linear congruences in  $\mathbb{P}^4$  were classified by Castelnuovo in [5]. In our presentation we follow [2], and use  $(y^1 : \dots : y^5)$  for homogeneous coordinates in  $\mathbb{P}^4$ . Over  $\mathbb{C}$ , every linear congruence in  $\mathbb{P}^4$  can be brought to one of the four normal forms:

- Generic case: the focal variety is a generic projection of the Veronese surface  $V^2 \subset \mathbb{P}^5$  into  $\mathbb{P}^4$ :

$$y^1 = u^1 y^4 + u^2 y^5, \quad y^2 = u^2 y^4 + u^3 y^5, \quad y^3 = u^3 y^4 + ((u^2)^2 - u^1 u^3) y^5.$$

The corresponding system,

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^3, \quad u_t^3 = ((u^2)^2 - u^1 u^3)_x,$$

does not possess Riemann invariants (Example 2).

- The focal variety is reducible, and consists of a cubic scroll and a plane which intersects the cubic scroll along its directrix:

$$y^1 = u^1 y^4 + u^2 y^5, \quad y^2 = u^2 y^4 + u^3 y^5, \quad y^3 = u^3 y^4 + \frac{u^2 u^3}{u^1} y^5.$$

The corresponding system,

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^3, \quad u_t^3 = \left( \frac{u^2 u^3}{u^1} \right)_x,$$

possesses one Riemann invariant. One can show that this system does not possess non-degenerate third-order Hamiltonian structures.

- The focal variety is reducible, and consists of a two-dimensional quadric and two planes which intersect the quadric along rectilinear generators of different families:

$$y^1 = u^1 y^4 + u^2 y^5, \quad y^2 = u^2 y^4 + u^3 y^5, \quad y^3 = u^3 y^4 + \frac{(u^3)^2 - 1}{u^2} y^5.$$

The corresponding system,

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^3, \quad u_t^3 = \left( \frac{(u^3)^2 - 1}{u^2} \right)_x,$$

possesses two Riemann invariants. One can show that this system does not possess non-degenerate third-order Hamiltonian structures.

- The focal variety consists of 3 planes in general position:

$$y^1 = u^1 y^4 + u^2 y^5, \quad y^2 = u^2 y^4 + u^3 y^5, \quad y^3 = u^3 y^4 + u^2 y^5.$$

The corresponding system is linear:

$$u_t^1 = u_x^2, \quad u_t^2 = u_x^3, \quad u_t^3 = u_x^2,$$

(Example 1).

Note that the number of planar components of the focal variety equals the number of Riemann invariants of the associated system [2].  $\square$

### 3 Concluding remarks

The classification of  $n$ -component Hamiltonian systems of conservation laws has been reduced to the following algebraic problem: for a vector space  $W$  of dimension  $n + 2$ , classify  $n$ -dimensional subspaces  $A \subset \Lambda^2(W)$  satisfying a relation

$$\phi_{\beta\gamma} A^\beta \wedge A^\gamma = 0,$$

where  $A^\alpha$  is a basis of  $A$  and  $\phi$  is symmetric and non-degenerate. This gives rise to the following natural questions:

- Classify normal forms of such subspaces  $A$ , at least for  $n = 4$ . This would provide explicit coordinate representation of Hamiltonian systems of conservation laws.
- Classify subspaces  $A$  corresponding to *integrable* systems of conservation laws (note that for  $n = 2, 3$  all Hamiltonian systems are automatically integrable). We emphasise that for  $n \geq 4$  the integrability is no longer the case in general. We expect that Example 3 from Section 1.4 will play a key role in this classification.

We hope to return to these questions elsewhere.

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