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\vee -Systems, Holonomy Lie Algebras, and Logarithmic Vector Fields

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It is shown that the description of certain class of representations of the holonomy Lie algebra \mathfrak{g}_Δ associated with hyperplane arrangement Δ is essentially equivalent to the classification of \vee -systems associated with Δ . The flat sections of the corresponding \vee -connection can be interpreted as vector fields, which are both logarithmic and gradient. We conjecture that the hyperplane arrangement of any \vee -system is free in Saito's sense and show this for all known \vee -systems and for a special class of \vee -systems called harmonic, which includes all Coxeter systems. In the irreducible Coxeter case the potentials of the corresponding gradient vector fields turn out to be Saito flat coordinates, or their one-parameter deformations. We give formulas for these deformations as well as for the potentials of the classical families of harmonic \vee -systems.

1 Introduction

The theory of the hyperplane arrangements has a rich history and is known to be related to many areas of mathematics: combinatorics, topology, singularity theory, and classical analysis, see the Introduction in [37].

It was the link with the singularity theory, which was the motivation to study the logarithmic vector fields for Saito, who introduced an important notion of a free

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arrangement [41]. However, in spite of substantial work done since then in this area (mainly by Terao and his school, see, e.g., recent paper [1]) there is not much known about how large the class of free arrangements is. As Orlik and Terao wrote in [37]: “It is an enduring mystery of the subject just what makes an arrangement free,” which is still valid now. In particular, there is still not much evidence in favour of Terao’s conjecture that freeness is a combinatorial property and depends only on the intersection lattice [37].

One of our main messages is that this problem may be closely related to the classification of the logarithmic Frobenius structures and \vee -systems, which is also a largely open problem, see the most recent discussion in [45]. The \vee -systems are special finite covector configurations introduced in [50, 51] in relation with certain class of solutions of the generalised Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations, playing a fundamental role in 2D topological field theory, $N = 2$ SUSY Yang–Mills theory, and the theory of Frobenius manifolds introduced by Dubrovin [10, 11, 31]. They can be defined as follows.

Let V be a complex vector space and $\mathcal{A} \subset V^*$ be a finite set of pairwise non-collinear vectors in the dual space V^* (covectors) spanning V^* . To such a set one can associate the following *canonical form* $G_{\mathcal{A}}$ on V :

$$G_{\mathcal{A}}(x, y) = \sum_{\alpha \in \mathcal{A}} \alpha(x)\alpha(y), \quad (1)$$

where $x, y \in V$. Let us assume that this form is non-degenerate and thus establishes the isomorphism

$$\varphi_{\mathcal{A}} : V \rightarrow V^*.$$

Let $\alpha^{\vee} = \varphi_{\mathcal{A}}^{-1}(\alpha)$ be the corresponding inverse image of $\alpha \in \mathcal{A}$. Because of the choice of the canonical form α^{\vee} is a complicated function of all $\alpha \in \mathcal{A}$.

The system \mathcal{A} is called *\vee -system* if the following *\vee -conditions*

$$\sum_{\beta \in \Pi \cap \mathcal{A}} \beta(\alpha^{\vee})\beta^{\vee} = \nu \alpha^{\vee} \quad (2)$$

are satisfied for any $\alpha \in \mathcal{A}$ and any two-dimensional plane $\Pi \subset V^*$ containing α and some ν , which may depend on Π and α . If Π contains more than two covectors then (2) implies that ν does not depend on $\alpha \in \Pi$ and

$$\sum_{\beta \in \Pi \cap \mathcal{A}} \beta^{\vee} \otimes \beta|_{\Pi} = \nu(\Pi)Id. \quad (3)$$

If Π contains only two covectors from \mathcal{A} , say α and β , then (2) implies that

$$G_{\mathcal{A}}(\alpha^{\vee}, \beta^{\vee}) = 0. \quad (4)$$

The examples of \vee -systems include all two-dimensional systems, Coxeter systems, and the so-called deformed root systems [32, 46, 50], but the full classification is still an open problem (see the latest results in [16, 17, 28, 45]). The combinatorial (or matroidal) structure of all known \vee -systems is quite special, but there are no general results known so far.

In this paper we aim to link this problem with Saito's theory of logarithmic vector fields and free arrangements [41] and with Kohno's theory of holonomy Lie algebras and logarithmic connections [25].

For any finite set of non-collinear covectors $\mathcal{A} \subset V^*$ one can consider the *associated arrangement* of complex hyperplanes $\Delta = \Delta_{\mathcal{A}} := \cup_{\alpha \in \mathcal{A}} H_{\alpha}$ in V given by $\alpha(x) = 0$, $\alpha \in \mathcal{A}$ and the corresponding *holonomy Lie algebra* \mathfrak{g}_{Δ} with generators $\{t_{\alpha}\}_{\alpha \in \mathcal{A}}$ and the relations

$$[t_{\alpha}, \sum_{\beta \in \mathcal{A} \cap \Pi} t_{\beta}] = 0, \quad \alpha \in \mathcal{A} \cap \Pi, \quad (5)$$

where Π is any two-dimensional subspace of V^* (see [22, 25]). This Lie algebra coincides with the Lie algebra of the unipotent completion of the fundamental group of the corresponding complement $\Sigma = V \setminus \Delta$ [22]. Its enveloping algebra is the quadratic dual of the cohomology algebra $H^*(\Sigma, \mathbb{C})$ in the cases when the latter is quadratic [53]. The relations (5) are equivalent to the flatness of the universal logarithmic connection [25]

$$\nabla_{\xi} = \partial_{\xi} - \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi)}{\alpha(x)} t_{\alpha}, \quad \xi \in V, x \in \Sigma. \quad (6)$$

In particular, for the standard arrangement of hyperplanes H_{ij} in \mathbb{C}^n given by $z_i - z_j = 0$, $1 \leq i < j \leq n$ we have the Kohno–Drinfeld Lie algebra \mathfrak{t}_n with generators $t_{ij} = t_{ji}$, $1 \leq i < j \leq n$ and relations [22]

$$[t_{ij}, t_{kl}] = 0, \quad [t_{ij}, t_{ik} + t_{jk}] = 0 \quad (7)$$

for all distinct i, j, k, l .

The first result of this paper is a one-to-one correspondence between certain linear representations of holonomy Lie algebras and \vee -systems (see Theorem 1). It is

essentially a reformulation of the known equivalence of the \vee -conditions and the flatness of the corresponding \vee -connection [51]

$$\nabla_{\xi}^{\vee} = \partial_{\xi} - \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi)}{\alpha(x)} \alpha^{\vee} \otimes \alpha, \quad (8)$$

where $\xi \in V$, $x \in \Sigma$, and $\kappa \in \mathbb{C}$ is a parameter. Similar result was also pointed out recently by Arsie and Lorenzoni in [5].

By identifying $T_x \Sigma$ with V we can view the flat sections of the \vee -connection

$$\nabla_{\xi}^{\vee} \psi = 0, \quad \psi, \xi \in V, x \in \Sigma \quad (9)$$

as the vector fields on Σ , which are parallel with respect to ∇_{ξ}^{\vee} (\vee -parallel vector fields). The monodromy of the system (9) gives a linear representation of the corresponding fundamental group $\pi_1(\Sigma)$ in V .

Important examples of \vee -systems are the following classical series found in [8]:

$$A_n(c) = \{ \sqrt{c_i c_j} (e_i - e_j), 0 \leq i < j \leq n \}, \quad (10)$$

$$B_n(c) = \left\{ \sqrt{c_i c_j} (e_i \pm e_j), 1 \leq i < j \leq n; \quad \sqrt{2c_i (c_i + c_0)} e_i, 1 \leq i \leq n \right\}, \quad (11)$$

respectively, with non-zero parameters c_0, \dots, c_n with non-zero sum.

In the $A_n(c)$ case the corresponding system (9) is equivalent to the classical Jordan–Pochhammer system with the solutions, which can be given by the Pochhammer type integrals (see [3, 39] and Section 3). The monodromy of this system is closely related to the classical Gassner representation of the pure braid group [7] (see the precise statement and the relation with bending of polygons in [20]). For a review of the higher rank representations of the braid group in relation with Knizhnik–Zamolodchikov (KZ) equation we refer to Kohno [23, 24].

In the main part of the paper we study the polynomial solutions of the systems (9), which are polynomial \vee -parallel vector fields, in relation with the theory of logarithmic vector fields and free arrangements [41]. Such solutions may exist only for special values of parameter κ , which can be shown to be equal to the degree of the corresponding solution.

We call \vee -system \mathcal{A} *harmonic* if there are $n = \text{rank } \mathcal{A}$ linearly independent polynomial \vee -parallel vector fields of degrees $\kappa_1, \dots, \kappa_n$ such that

$$\kappa_1 + \dots + \kappa_n = |\mathcal{A}|$$

is the number of covectors in \mathcal{A} . We show that for any harmonic \vee -system the corresponding vector fields are gradient and freely generate all logarithmic vector fields $Der(\log \Delta)$ as a module over polynomial algebra, which means that the corresponding arrangements are free in Saito's sense [37]. As a corollary by Terao's factorisation theorem [37] the Poincaré polynomial, which is the generating function $P_\Sigma(t) = \sum b_i t^i$ of Betti numbers of Σ , in that case has the form

$$P_\Sigma(t) = \prod_{i=1}^n (1 + \kappa_i t).$$

We conjecture that all the arrangements of \vee -systems are free, so the corresponding Poincaré polynomials are always factorisable in such a form with integer κ_i . We prove this for all known \vee -systems [16, 17], by showing that in dimension $n > 2$ the corresponding arrangements are equivalent to Coxeter arrangements or their restrictions, which are known to be free [38].

In Section 4 we prove that the classical series of \vee -systems (10), (11) are harmonic and present the residue formulae for the potentials of the corresponding gradient vector fields (see Theorems 4 and 5). This fact seems to be remarkable since as we show even the restrictions of Coxeter systems in general may not be harmonic.

In the last section we discuss the Coxeter case and the relation of harmonic \vee -systems with Saito flat coordinates on the orbit space of Coxeter groups [42, 43]. We prove that all Coxeter \vee -systems are harmonic and find the corresponding potentials. In the case when all the roots are normalised to have the same length these potentials are known to be precisely the Saito flat coordinates [14], so in the non-simply laced cases we have one-parameter deformations of these coordinates, which we describe explicitly.

2 \vee -Systems and Representations of Holonomy Lie Algebras

Let Δ be a central hyperplane arrangement in V . For any hyperplane $H \in \Delta$ we choose $\alpha_H \in V^*$ such that $H = \{x \in V : \alpha_H(x) = 0\}$. We will call the corresponding set

$$\mathcal{A} = \{\alpha_H, H \in \Delta\} \subset V^*$$

an *equipment* of Δ . We will assume that the set \mathcal{A} generates V^* . Arrangement Δ is called *irreducible* if one cannot decompose $V^* = V_1 \oplus V_2$ such that $\mathcal{A} = (\mathcal{A} \cap V_1) \cup (\mathcal{A} \cap V_2)$.

Assume now that V is a complex Euclidean space with symmetric non-degenerate bilinear form G . Denote by $\hat{\alpha} = G^{-1}\alpha$ the vector corresponding to $\alpha \in V^*$ and look for

representations $\rho : \mathfrak{g}_\Delta \rightarrow \text{End}(V)$ of holonomy Lie algebra \mathfrak{g}_Δ of the form

$$\rho(t_\alpha) = \hat{\alpha} \otimes \alpha, \quad \alpha \in \mathcal{A} \quad (12)$$

for some equipment \mathcal{A} of Δ . In general, there are no such equipments, so these representations exist only for special hyperplane arrangements.

To state the theorem we will need the following notion of complex Euclidean \vee -system introduced in [17].

Let \mathcal{A} be a finite set of non-collinear vectors in a complex Euclidean vector space $V \cong V^*$. We say that the set \mathcal{A} is *well-distributed* in V if the canonical form (1) is proportional to the Euclidean form G . The set \mathcal{A} is called *complex Euclidean \vee -system* if it is well-distributed in V and any its two-dimensional subsystem is either reducible (consists of two orthogonal vectors) or well-distributed in the corresponding plane.

We allow here the canonical form to be zero. If the canonical form (1) is non-degenerate, then we can use it to define the Euclidean structure on V and we have the definition of the usual \vee -system.

Theorem 1. For any \vee -system \mathcal{A} the formula

$$\rho(t_\alpha) = \alpha^\vee \otimes \alpha, \quad \alpha \in \mathcal{A}$$

defines a representation of the associated holonomy Lie algebra \mathfrak{g}_Δ . The same is true for complex Euclidean \vee -systems and representation (12).

Conversely, if (12) is a representation of the holonomy Lie algebra \mathfrak{g}_Δ for an irreducible arrangement Δ with equipment \mathcal{A} then \mathcal{A} is a complex Euclidean \vee -system. \square

Thus for given hyperplane arrangement Δ the description of all representations of holonomy Lie algebra \mathfrak{g}_Δ of the form (12) is essentially equivalent to the classification of all \vee -systems \mathcal{A} associated with Δ . Note that ρ depends not only on the arrangement but also on the choice of the equations of the hyperplanes.

Recall first that due to Kohno [25] the flatness conditions of the universal logarithmic connection (6)

$$[\nabla_{\xi_i}, \nabla_{\eta_j}] = 0$$

are equivalent to the relations (5).

A similar interpretation of the ∇-conditions as flatness of the corresponding ∇-connection on the tangent bundle $T(\Sigma) \approx \Sigma \times V$

$$\nabla_{\xi}^{\vee} = \partial_{\xi} - \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi)}{\alpha(x)} \alpha^{\vee} \otimes \alpha, \quad \xi \in V, x \in \Sigma \quad (13)$$

was pointed out in [51]. Indeed, it is easy to see that the relation $[\nabla_{\xi}^{\vee}, \nabla_{\eta}^{\vee}] = 0$ is equivalent to the identity

$$\sum_{\alpha, \beta \in \mathcal{A}} \frac{\alpha \wedge \beta}{\alpha(x)\beta(x)} [\alpha^{\vee} \otimes \alpha, \beta^{\vee} \otimes \beta] = 0,$$

which in its turn is equivalent to the commutation relations

$$[\alpha^{\vee} \otimes \alpha, \sum_{\beta \in \mathcal{A} \cap \Pi} \beta^{\vee} \otimes \beta] = 0 \quad (14)$$

for all $\alpha \in \mathcal{A}$ and all two-dimensional subspaces $\Pi \subset V^*$ containing α . Now if Π contains only two covectors α and β then we have

$$[\alpha^{\vee} \otimes \alpha, \beta^{\vee} \otimes \beta] = \alpha(\beta^{\vee})\alpha^{\vee} \otimes \beta - \beta(\alpha^{\vee})\beta^{\vee} \otimes \alpha$$

which is zero for non-proportional α and β only if

$$\alpha(\beta^{\vee}) = \beta(\alpha^{\vee}) = G_{\mathcal{A}}(\alpha^{\vee}, \beta^{\vee}) = 0,$$

which is ∇-condition (4). If Π contains more than two covectors then the commutation relations (14) are equivalent to the property that the restriction of the operator $\sum_{\beta \in \mathcal{A} \cap \Pi} \beta^{\vee} \otimes \beta$ on Π is proportional to the identity, which coincides with ∇-condition (3).

Similarly, for any complex Euclidean ∇-system the corresponding connection

$$\widehat{\nabla}_{\xi} = \partial_{\xi} - \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi)}{\alpha(x)} \widehat{\alpha} \otimes \alpha$$

is flat, which is equivalent to the relations

$$[\widehat{\alpha} \otimes \alpha, \sum_{\beta \in \mathcal{A} \cap \Pi} \widehat{\beta} \otimes \beta] = 0 \quad (15)$$

for all $\alpha \in \mathcal{A}$ and all two-dimensional subspaces $\Pi \subset V^*$ containing α (cf. [52]).

To prove the first part of the theorem we note that substitution of (12) into the holonomy Lie algebra relations (5) gives the relations (15).

Now fixing α and summing these relations over all two-dimensional Π containing α we have

$$[\widehat{\alpha} \otimes \alpha, \sum_{\beta \in \mathcal{A}} \widehat{\beta} \otimes \beta] = 0. \quad (16)$$

Since this is true for all $\alpha \in \mathcal{A}$, the set \mathcal{A} generates V^* , and the arrangement is irreducible this implies that the operator $\sum_{\beta \in \mathcal{A}} \widehat{\beta} \otimes \beta$ is proportional to the identity, or equivalently, that

$$G_{\mathcal{A}} = \sum_{\beta \in \mathcal{A}} \beta \otimes \beta = \mu G.$$

If $\mu \neq 0$ then $G_{\mathcal{A}}$ is non-degenerate and $\alpha^\vee = \mu^{-1} \widehat{\alpha}$ satisfy \vee -conditions (14). If $\mu = 0$ then we have complex Euclidean \vee -system. This completes the proof.

3 \vee -Systems and Gradient Logarithmic Vector Fields

One of the main problems in the theory of \vee -systems is the characterisation of the corresponding hyperplane arrangements (see, e.g., [45]). Since in dimension 2 any covector system is a \vee -system, the problem starts from dimension 3.

We would like to link this problem with the characterisation of free arrangements in the theory of logarithmic vector fields initiated by Kyoji Saito [41]. We start with a brief review of this theory, mainly following Orlik and Terao [37].

Consider a hyperplane arrangement $\Delta \subset \mathbb{C}^n$. A polynomial vector field $X = \xi_i(z) \frac{\partial}{\partial z_i}$ on \mathbb{C}^n is called *logarithmic* if it is tangent to every hyperplane $H \in \Delta$. The hyperplane arrangement Δ is *free* if the space of all logarithmic vector fields $Der(\log \Delta)$ is free as the module over polynomial algebra $P_n = \mathbb{C}[z_1, \dots, z_n]$ (see [37, 41]). The degrees b_1, \dots, b_n of the corresponding homogeneous generators X_1, \dots, X_n are called the *exponents* of the arrangement:

$$\exp \Delta = \{b_1, \dots, b_n\}.$$

Here the degree of a homogeneous polynomial vector field $X = \xi_i(z) \frac{\partial}{\partial z_i}$ is defined as the degree of any of its non-zero components: $\deg X = \deg \xi_i$.

Saito's criterion [37] says that Δ is free if and only if there are n homogeneous linearly independent over P_n logarithmic vector fields X_1, \dots, X_n such that the sum of the

degrees equals the number of hyperplanes $|\Delta|$:

$$\sum_{i=1}^n \deg X_i = |\Delta|. \tag{17}$$

Such fields can be chosen as free generators of the module $Der(\log \Delta)$.

However, a satisfactory characterisation of all free arrangements is still an open problem. There is a conjecture due to Terao, that the freeness property is combinatorial (see [37, p. 154]), but there is still not much evidence in its favour.

Probably the most remarkable result in this area is the following *Factorisation Theorem* proved by Terao [49]: Poincaré polynomial of the complement $\Sigma = \mathbb{C}^n \setminus \Delta$ for a free arrangement Δ has the form

$$P_{\Sigma}(t) = \prod_{i=1}^n (1 + b_i t) \tag{18}$$

with b_1, \dots, b_n being the exponents of Δ . This is a far-going generalisation of Arnold's formula

$$P_{\Sigma_{n+1}}(t) = (1 + t)(1 + 2t) \dots (1 + nt)$$

for the Poincaré polynomial of the configuration space of $n + 1$ distinct points on the plane, corresponding to A_n -type arrangement, see [4].

It is well-known (Arnold, Saito) that all Coxeter arrangements are free with the exponents $b_i = m_i$ being the exponents of the corresponding Coxeter (finite reflection) group G [18]. The corresponding generators $X_i = grad f_i$, $i = 1, \dots, n$, where f_1, \dots, f_n are some free generators of the corresponding algebra of polynomial G -invariants $\mathbb{C}[z_1, \dots, z_n]^G$, which exist by Chevalley theorem. Indeed, it is easy to see that the corresponding fields are logarithmic and, by Saito's criterion, generate $Der(\log \Delta)$ because the sum of the exponents of a Coxeter group is known to be the number of the reflection hyperplanes:

$$m_1 + \dots + m_n = |\Delta|,$$

see, for example, [18].

It is known also that any linear arrangement in \mathbb{C}^2 is free and that a generic arrangement in \mathbb{C}^n with $n > 2$ is not free [37].

The arrangement Δ is called *hereditarily free* if it is free and all restriction arrangements to the hyperplanes of Δ and their intersections are also free [37]. The property of Δ being free is not hereditary [37], but it is known that all Coxeter arrangements are hereditarily free [38].

Conjecture. For any \vee -system \mathcal{A} the associated arrangement $\Delta_{\mathcal{A}}$ is hereditarily free. \square

We have shown that the class of \vee -systems is closed under the restriction [17], so to prove the conjecture it is enough to show that $\Delta_{\mathcal{A}}$ is free. In particular, this would imply by Terao's theorem that the corresponding Poincaré polynomial $P_{\Sigma_{\mathcal{A}}}(t)$ is factorisable in the form (18), which would be already a strong topological restriction of the arrangement.

Theorem 2. For all known \vee -systems [16, 17] the conjecture is true. \square

Proof. In dimension 2 this follows from the fact that in that case any system is a \vee -system and any line arrangement is free [37].

For the classical series of \vee -systems $A_n(c)$ the corresponding arrangements (10) coincide with the Coxeter arrangement of type A_n , and thus are free with the exponents $1, 2, \dots, n$.

In the $B_n(c)$ case the corresponding arrangements have Coxeter type B_n unless $c_0 + c_i = 0$ for some i . In that case we have Zaslavsky arrangements D_n^k consisting of hyperplanes $x_i = 0$, $i = 1, \dots, k$, $1 \leq k \leq n$ and $x_i \pm x_j = 0$, $1 \leq i < j \leq n$. These arrangements appear also as some restrictions of the systems of type D_N , see [37, 54], and thus are free with the exponents $1, 3, 5, \dots, 2n - 3, k + n - 1$ [36].

In the F_4 case we have the \vee -system consisting of the covectors

$$e_i \pm e_j, 2te_i, t(e_1 \pm e_2 \pm e_3 \pm e_4).$$

If $t \neq 0$ this gives the Coxeter arrangement of type F_4 with the exponents $1, 5, 7, 11$. If $t = 0$ we have the Coxeter D_4 arrangement with the exponents $1, 3, 3, 5$.

For the Coxeter restrictions we will use the notations from [16] as pairs (G, H) , where G is Coxeter group and H is the corresponding parabolic subgroup (see also [36]).

There are two restrictions of F_4 , but they are equivalent (see [16]) to

$$F_3(t) = \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, \sqrt{a}e_1, \sqrt{a}e_2, \sqrt{a}e_3, t\sqrt{2}(e_1 \pm e_2 \pm e_3)\}, \quad a = 4t^2 + 2,$$

where $t^2 \neq -1/2$. If $t \neq 0$ we have the arrangement of 13 planes equivalent to the Coxeter restriction (E_8, D_5) , which is free with the exponents 1, 5, 7 [36]. If $t = 0$ we have the B_3 arrangement with the exponents 1, 3, 5.

Consider now the generalised systems related to exceptional basic classical Lie superalgebras and their deformations [17, 46].

In the AB_4 case related to Lie superalgebra of type $AB(1, 3)$ we have generally 18 covectors

$$\sqrt{a}e_i, i = 1, 2, 3, \sqrt{b}(e_i \pm e_j), 1 \leq i < j \leq 3, \sqrt{c}e_4, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4),$$

where

$$a = \frac{3k + 1}{2}, b = \frac{3k - 1}{4}, c = \frac{1 - k}{2k},$$

$k \neq 0, -1/3$ is an arbitrary parameter (see [46]). When $k = -1/3$ the canonical form is zero, so we have complex Euclidean ∇ -system but not a ∇ -system.

If all a, b, c are not zero then we have the free arrangement equivalent to the Coxeter restriction (E_7, A_3) of E_7 -type system, which is free with the exponents 1, 5, 5, 7 (see [36]). If $c = 0, k = 1$ we have the hyperplane $x_4 = 0$ disappeared, so we can apply Terao's Addition-Deletion theorem (see Theorems 4.49 and 4.51 in [37]) to claim that the corresponding arrangement of 17 hyperplanes is free with the exponents 1, 4, 5, 7. Alternatively, one can check that the corresponding ∇ -system is just the Coxeter restriction $(E_6, A_1 \times A_1)$ (see [16]) and use the results from [36, 38].

Finally, if $b = 0, k = 1/3$, we have the ∇ -system consisting of 12 covectors

$$e_i, i = 1, 2, 3, 4, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4),$$

which is equivalent to Coxeter system of type D_4 and thus is free with exponents 1, 3, 3, 5.

We should consider also the 3D restrictions. There are two types of these restrictions consisting of the following covectors [17]:

$$(AB_4(t), A_1)_1 = \{\sqrt{2(2t^2 + 1)}e_1, 2\sqrt{2(t^2 + 1)}e_2, t\sqrt{\frac{2(2t^2 - 1)}{t^2 + 1}}e_3, \\ \sqrt{2}(e_1 \pm e_2), t\sqrt{2}(e_1 \pm e_3), t(e_1 \pm 2e_2 \pm e_3)\}$$

with $t^2 \neq -1, -1/2$, and

$$(AB_4(t), A_1)_2 = \{e_1 + e_2, e_1 + e_3, e_2 + e_3, \sqrt{2}e_1, \sqrt{2}e_2, \sqrt{2}e_3, \frac{t\sqrt{2}}{\sqrt{t^2+1}}(e_1 + e_2 + e_3), \\ \frac{1}{\sqrt{4t^2+1}}(e_1 - e_2), \frac{1}{\sqrt{4t^2+1}}(e_1 - e_3), \frac{1}{\sqrt{4t^2+1}}(e_2 - e_3)\}$$

with $t^2 \neq -1, -1/2, -1/4$. If $t^2 \neq 0, 1/2$ then the 3D arrangement $(AB_4(t), A_1)_1$ consists of 11 planes, is equivalent to the Coxeter restriction $(E_7, A_1 \times A_3)_2$, and thus is free with the exponents 1, 5, 5 [36]. When $t = 0$ we have the reducible arrangement of five planes $B_2 \times A_1$, which is free with exponents 1, 1, 3 (see Proposition 4.28 in [37]). When $t^2 = 1/2$ we have arrangement of ten planes equivalent to Coxeter restriction (E_6, A_1^3) which is free with exponents 1, 4, 5 [36]. Similarly, if $t \neq 0$ the 3D arrangement $(AB_4(t), A_1)_2$ consists of ten planes and is equivalent to the Coxeter restriction $(E_6, A_1 \times A_2)$, which is free with the exponents 1, 4, 5 [36]. When $t = 0$ we have nine planes, forming B_3 arrangement with the exponents 1, 3, 5.

The \vee -systems of type G_3 related to the Lie superalgebra $G(2, 1)$ consist of covectors [46]

$$G_3(t) = \{\sqrt{a}e_1, \sqrt{a}e_2, \sqrt{a}(e_1 + e_2), \sqrt{b}(e_1 - e_2), \sqrt{b}(2e_1 + e_2), \sqrt{b}(e_1 + 2e_2), \sqrt{c}e_3, \\ e_1 \pm e_3, e_2 \pm e_3, e_1 + e_2 \pm e_3\}, \quad a = 2t + 1, \quad b = \frac{2t - 1}{3}, \quad c = \frac{3}{t},$$

depending on the parameter $t \neq 0, -1/2$. In the case $t \neq 1/2$ we have the arrangement of 13 planes equivalent to the Coxeter restriction (E_7, A_2^2) (or, equivalently, (E_8, A_5)), which is free with the exponents 1, 5, 7 (see [16, 36]). When $t = 1/2$ we have the arrangement of ten planes, which can be shown to be equivalent to the Coxeter restriction (E_6, A_1^3) , and thus is free with the exponents 1, 4, 5 (see [36]).

Finally the \vee -systems of type D_3 related to the exceptional Lie superalgebra $D(2, 1, \lambda)$ consist of covectors [17]

$$D_3(t, s) = \{e_1 \pm e_2 \pm e_3, \sqrt{2(s+t-1)}e_1, \sqrt{\frac{2(s-t+1)}{t}}e_2, \sqrt{\frac{2(t-s+1)}{s}}e_3\},$$

where s, t are non-zero parameters, such that $s + t + 1 \neq 0$. For generic values of parameters the corresponding arrangement is equivalent to the Zaslavsky configuration $D_3^1 = (D_6, A_3)$ with the exponents 1, 3, 3. If one of the coefficients of e_1, e_2, e_3 vanishes, then we have type A_3 arrangement with the exponents 1, 2, 3. This analysis together with all other Coxeter restrictions exhausts all the cases and completes the proof. \blacksquare

Remark. For the complex Euclidean ∇-systems the conjecture is not true. A counterexample is given by the ∇-system of type $F_3(t)$ with $t^2 = -\frac{1}{2}$, consisting of the following ten vectors in \mathbb{C}^3

$$\mathcal{A} = \{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, i(e_1 \pm e_2 \pm e_3)\}.$$

The corresponding Poincaré polynomial

$$P_{\mathcal{A}}(t) = (1 + t)(1 + 9t + 26t^2)$$

is not factorisable, so the arrangement is not free and is not combinatorially equivalent to any Coxeter restriction. The corresponding canonical form $G_{\mathcal{A}} = 0$ in this case. \square

We are going to show now that at least for a subclass of ∇-systems we can find the corresponding generating logarithmic vector fields X_1, \dots, X_n as polynomial ∇-parallel vector fields (9) for special values of κ being the exponents b_1, \dots, b_n .

Theorem 3. The polynomial solutions ψ of (9) are gradient logarithmic vector fields for the corresponding arrangement $\Delta_{\mathcal{A}}$ with the degrees

$$\deg \psi = \kappa. \tag{19}$$

\square

Proof. To prove this it is convenient to choose an orthonormal basis in V , so that the canonical form $G_{\mathcal{A}}$ becomes standard. Then we can identify V and V^* with \mathbb{C}^n , so that $\alpha^\vee = \alpha$ and

$$\sum_{\alpha \in \mathcal{A}} \alpha_i \alpha_j = \delta_{ij}, \quad i, j = 1, \dots, n,$$

where α_i is the i th coordinate of α . The system (9) takes the form

$$\partial_i \psi_j = \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha_i \alpha_j}{(\alpha, z)} (\alpha, \psi), \quad z, \psi(z) \in \mathbb{C}^n, \quad i, j = 1, \dots, n. \tag{20}$$

Now from (20) it is immediate that $\partial_i \psi_j = \partial_j \psi_i$, so

$$\psi_i = \partial_i F, \quad i = 1, \dots, n \tag{21}$$

for some polynomial potential $F(z)$. The fact that ψ is logarithmic follows from the regularity of the left-hand side on the hyperplane $(\alpha, z) = 0$, which implies that $(\alpha, \psi) = 0$ on this hyperplane, so that ψ is tangent. To find the degree of ψ multiply the relations (20) by $z_i \beta_j$ and add over all i, j to have

$$E(\beta, \psi) = \kappa \sum_{\alpha \in \mathcal{A}} (\alpha, \beta)(\alpha, \psi) = \kappa(\beta, \psi),$$

where $E = \sum_{i=1}^n z_i \partial_i$ is the Euler vector field. ■

The potential F of a \vee -parallel vector field ψ can be defined in coordinate-free way by the relation

$$\alpha(\psi) = G_{\mathcal{A}}(\alpha^\vee, \psi) = dF(\alpha^\vee) = \partial_{\alpha^\vee} F \quad (22)$$

for any $\alpha \in V^*$. The parallel transport condition

$$\partial_{\xi} \psi = \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi) \alpha(\psi)}{\alpha(z)} \alpha^\vee = \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi) \partial_{\alpha^\vee} F}{\alpha(z)} \alpha^\vee \quad (23)$$

implies that the potential F satisfies compatible system of the Euler–Poisson–Darboux type equations

$$\partial_{\xi} \partial_{\eta} F = \kappa \sum_{\alpha \in \mathcal{A}} \frac{\alpha(\xi) \alpha(\eta)}{\alpha(z)} \partial_{\alpha^\vee} F, \quad \xi, \eta \in V. \quad (24)$$

So the question is for which integer values of parameter κ do the polynomial solutions of (9) exist, and whether we can find enough such solutions to generate all logarithmic vector fields over polynomial algebra. For $\kappa = 1$ we always have the solution $\psi_i = z_i$, $i = 1, \dots, n$ corresponding to the Euler vector field $\psi = E$.

To understand the situation better let us consider the case of rank-2 systems \mathcal{A} . In this case the gradient generators of logarithmic vector fields may not exist. It is well-known [37] that any such arrangement is free and $Der(\log \Delta_{\mathcal{A}})$ is generated by Euler vector field E and

$$X = (\partial_2 Q) \partial_1 - (\partial_1 Q) \partial_2,$$

where $Q = \prod_{\alpha \in \mathcal{A}} (\alpha, z)$. The last vector field is gradient if and only if Q is harmonic:

$$\Delta Q = 0, \quad \Delta = \partial_1^2 + \partial_2^2$$

which in general is not the case. Indeed, consider a particular case of four lines with

$$Q = x_1x_2(x_1 - x_2)(x_1 - ax_2).$$

Then $\Delta Q = 2(1 + a)(-x_1^2 + 3x_1x_2 - x_2^2)$, which vanishes only when $a = -1$, so the lines form a harmonic set (which means that their cross-ratio is -1), projectively equivalent to B_2 case. Adding to X a multiple of Euler field E also would not make it gradient. Indeed if $X + f(x)E = \text{grad}F$ then

$$\Delta Q = x_2\partial_1f - x_1\partial_2f = \partial_\xi f,$$

where $\xi = (x_2, -x_1)$. Since vector field ξ has closed circular orbits, the necessary condition for the existence of polynomial f is $\int_\gamma \Delta Q dt = 0$, where γ is the circle $x_1 = \cos t, x_2 = \sin t$. In our case

$$\int_\gamma \Delta Q dt = 2(1 + a) \int_0^{2\pi} (3 \cos t \sin t - 1) dt = -4(1 + a)\pi,$$

which is zero only if $a = -1$. One can also check that in general the corresponding systems (20) do not have polynomial solutions for $\kappa = 3$ if $a \neq -1$.

This means that we are dealing with a special subclass of both free arrangements and ∇ -systems. This motivates the following definition.

We say that ∇ -system \mathcal{A} of rank n is *harmonic* if the corresponding system (9) has n linearly independent (at generic point) polynomial solutions for $\kappa = \kappa_1, \dots, \kappa_n$ such that

$$\kappa_1 + \dots + \kappa_n = |\mathcal{A}|, \tag{25}$$

where $|\mathcal{A}|$ is the number of covectors in \mathcal{A} .

Theorem 4. The arrangement Δ of any harmonic ∇ -system is free with exponents $b_i = \kappa_i, i = 1, \dots, n$ and the Poincaré polynomial of $\Sigma = V \setminus \Delta$ has the form

$$P_\Sigma(t) = \prod_{i=1}^n (1 + \kappa_i t). \tag{26}$$

□

The proof follows immediately from Theorem 3 and the Saito criterion.

As one can see from the Euler–Poisson–Darboux type equations (24) the corresponding potentials F_1, \dots, F_n belong to the algebra of *quasi-invariants* of \mathcal{A}

$$\mathcal{Q}_{\mathcal{A}} = \{p(z) \in \mathbb{C}[z_1, \dots, z_n] : \partial_{\alpha^\vee}|_{\alpha(z)=0} p(z) = 0, \quad \alpha \in \mathcal{A}\}. \quad (27)$$

It would be interesting to understand their role for these algebras (cf. [15]).

As we will see now for the classical series the corresponding potentials turn out to be certain deformations of Saito’s generators of the algebra of invariants.

4 Analysis of the Classical Series

Consider first \vee -systems of type A_n from [8]:

$$A_n(c) = \{\sqrt{c_i c_j}(e_i - e_j), 0 \leq i < j \leq n\},$$

where all c_i are assumed to be non-zero. One can check that the corresponding canonical form is non-degenerate if

$$\sigma = c_0 + c_1 + \dots + c_n \neq 0,$$

and the vector α^\vee for $\alpha = \sqrt{c_i c_j}(e_i - e_j)$ has the form

$$\alpha^\vee = \sigma^{-1} \sqrt{c_i c_j}(c_i^{-1} e_i - c_j^{-1} e_j)$$

(see [8]). The corresponding KZ equations $\nabla_\xi^\vee \psi = 0$ with ∇_ξ^\vee given by (13) and $\psi = (\psi_0, \dots, \psi_n) \in V^*$ have the form

$$\partial_i \psi_j = -\kappa \sigma^{-1} \frac{c_j \psi_i - c_i \psi_j}{x_i - x_j}, \quad i \neq j \quad (28)$$

with $\partial_i \psi_i$ determined from the relation $\psi_0 + \dots + \psi_n = 0$:

$$\partial_i \psi_i = \kappa \sigma^{-1} \sum_{j \neq i} \frac{c_j \psi_i - c_i \psi_j}{x_i - x_j}. \quad (29)$$

These equations are nothing but the Jordan–Pochhammer linear system for the integrals of the hypergeometric type

$$I_\lambda(x_0, \dots, x_n) = \int_\gamma \prod_{j=0}^n (x - x_j)^{\lambda_j} dx$$

(see, e.g., [3], [9, Section 2.3], [29], [39, formula (1) on p. 71], [40]). More precisely, we have the following:

Theorem 5. The ∇-systems $A_n(c)$ are harmonic with the potentials given by the Pochhammer type integrals

$$F_\kappa(x_0, \dots, x_n) = \frac{1}{2\pi i} \int_\gamma \prod_{j=0}^n (x - x_j)^{\lambda_j} dx, \quad \kappa = 1, 2, \dots, n, \quad (30)$$

where $\lambda_j = \kappa \frac{c_j}{\sigma}$ and contour γ is a large circle surrounding all x_0, \dots, x_n . □

Proof. Let $\Phi_\lambda(x; x_0, \dots, x_n) = \prod_{j=0}^n (x - x_j)^{\lambda_j}$. Then $I_\lambda(x_0, \dots, x_n) = \int_\gamma \Phi_\lambda(x; x_0, \dots, x_n) dx$ and

$$\psi_j = \partial_j I_\lambda(x_0, \dots, x_n) = -\lambda_j \int_\gamma \frac{\Phi_\lambda(x; x_0, \dots, x_n)}{x - x_j} dx. \quad (31)$$

For the chosen contour γ the integral (30) is well-defined if and only if $\lambda_0 + \lambda_1 + \dots + \lambda_n$ is an integer, which we will assume to be the case. Then we have

$$\psi_0 + \dots + \psi_n = - \int_\gamma d\Phi_\lambda = 0.$$

Consider the derivative

$$\partial_i \psi_j = \lambda_i \lambda_j \int_\gamma \frac{\Phi_\lambda}{(x - x_i)(x - x_j)} dx = \frac{\lambda_i \lambda_j}{x_i - x_j} \int_\gamma \left(\frac{\Phi_\lambda}{x - x_i} - \frac{\Phi_\lambda}{x - x_j} \right) dx = \frac{\lambda_i \psi_j - \lambda_j \psi_i}{x_i - x_j}$$

if $i \neq j$ and

$$\partial_i \psi_i = - \sum_{j \neq i} \frac{\lambda_i \psi_j - \lambda_j \psi_i}{x_i - x_j},$$

which coincides with the equations (28), (29) with $\lambda_j = \kappa \frac{c_j}{\sigma}$.

Since $\lambda_0 + \dots + \lambda_n = \kappa$, we need κ to be integer. We claim that if we choose simply the smallest $\kappa = 1, 2, \dots, n$ then we will have the basic gradient logarithmic vector fields X with components

$$\xi_i = \lambda_i^{-1} \psi_i, \quad i = 0, \dots, n$$

(note that the canonical form is not standard in this case). Indeed, Φ_λ is meromorphic in x at infinity with the expansion

$$\Phi_\lambda = x^\kappa \prod_{i=0}^n \left(1 - \frac{x_i}{x}\right)^{\lambda_i} = x^\kappa \prod_{i=0}^n \left(1 - \lambda_i \frac{x_i}{x} + \frac{\lambda_i(\lambda_i - 1)}{2} \frac{x_i^2}{x^2} + \dots\right). \quad (32)$$

The contour integral (30) is simply the coefficient at x^{-1} in this expansion, so F_κ is polynomial in both x_0, \dots, x_n and $\lambda_0, \dots, \lambda_n$. Simple algebraic arguments show that the determinant of the matrix of the partial derivatives $\|\partial F_i / \partial x_j\|, i, j = 0, \dots, n$ is equal to

$$\det \|\partial F_i / \partial x_j\| = \lambda_0 \dots \lambda_n \prod_{i < j} (x_i - x_j).$$

This implies their independence for all non-zero λ_i , or equivalently, for all non-zero c_i with $\sum c_i = \sigma \neq 0$. Since the degrees of these polynomials in x are the same as in the non-deformed case: $2, 3, \dots, n + 1$, by Saito's criterion we have the claim. \blacksquare

There is a more explicit way to represent potentials F_κ from Theorem 5. Let us introduce deformed Newton sums by

$$p_s^\lambda = \sum_{i=0}^n \lambda_i x_i^s, \quad s \in \mathbb{N}.$$

Further, for a partition μ define the corresponding deformed power sum by

$$p_\mu^\lambda = (p_1^\lambda)^{m_1} (p_2^\lambda)^{m_2} \dots,$$

where m_i is the number of parts i in μ . Let $l(\mu)$ be the number of non-zero parts in the partition μ and define

$$z_\mu = \prod_{j \geq 1} (j^{m_j} m_j!).$$

The \vee -system $A_n(c)$ is considered in the subspace where $p_1^\lambda \equiv 0$, but it is convenient to keep this polynomial in the statement and proof of the next theorem.

Theorem 6. Potentials (30) have the form

$$F_\kappa = \frac{(-1)^{\kappa+1}}{(\kappa+1)!} \det \begin{pmatrix} p_1^\lambda & 1 & 0 & 0 \dots & 0 \\ p_2^\lambda & p_1^\lambda & 2 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_\kappa^\lambda & p_{\kappa-1}^\lambda & p_{\kappa-2}^\lambda & \dots & \kappa \\ p_{\kappa+1}^\lambda & p_\kappa^\lambda & p_{\kappa-1}^\lambda & \dots & p_1^\lambda \end{pmatrix} = \sum_{\mu:|\mu|=\kappa+1} (-1)^{l(\mu)} z_\mu^{-1} p_\mu^\lambda, \quad (33)$$

where $\kappa = 1, \dots, n$ and the sum is taken over all partitions μ of $\kappa + 1$. □

Proof. We know that $F_\kappa(x_0, \dots, x_n)$ equals the coefficient at $x^{-\kappa-1}$ in the expansion of the function $\widehat{\Phi}_\lambda = x^{-\kappa} \Phi_\lambda$, where Φ_λ is given by (32). On the other hand

$$\widehat{\Phi}_\lambda = e^{\log \prod_{i=0}^n (1 - \frac{x_i}{x})^{\lambda_i}} = e^{\sum_{i=0}^n \lambda_i \log(1 - \frac{x_i}{x})} = e^{-p_1^\lambda x^{-1} - \frac{1}{2} p_2^\lambda x^{-2} - \frac{1}{3} p_3^\lambda x^{-3} - \dots}.$$

The required coefficient is a polynomial in p_j^λ with $1 \leq j \leq \kappa + 1$. All these polynomials p_j^λ are algebraically independent. Therefore the required coefficient is a polynomial in variables p_j^λ whose coefficients do not depend on λ , and it is sufficient to establish the statement when $\lambda_i = 1$ for all i , which is well known (see pages 25, 28 in [30]). (Note also that the statement for $\lambda_0 = \lambda_1 = \dots = \lambda_n$ is contained in [33], and that a related analysis of functions F_κ is contained in the recent paper [21, Remark 6.6].) ■

Consider now the ∇ -systems of B_n -type [8]

$$B_n(c) = \left\{ \sqrt{c_i c_j} (e_i \pm e_j), 1 \leq i < j \leq n; \quad \sqrt{2c_i(c_i + c_0)} e_i, 1 \leq i \leq n \right\},$$

where all c_i with $i \geq 1$ are assumed to be non-zero. Let us also assume for the beginning that $c_i + c_0 \neq 0$ for all $i = 1, \dots, n$, so the corresponding arrangement is of type B_n .

The canonical form has the matrix $G = 2\sigma C$, $C = \text{diag}(c_1, \dots, c_n)$ with

$$\sigma = c_0 + c_1 + \dots + c_n,$$

so for $\alpha = \sqrt{c_i c_j} (e_i \pm e_j)$ we have

$$\alpha^\vee = 2^{-1} \sigma^{-1} \sqrt{c_i c_j} (c_i^{-1} e_i \pm c_j^{-1} e_j)$$

and for $\alpha = \sqrt{2c_i(c_i + c_0)} e_i$ we have

$$\alpha^\vee = (2\sigma c_i)^{-1} \sqrt{2c_i(c_i + c_0)} e_i.$$

The corresponding equations (13) for $\psi = (\psi_1, \dots, \psi_n) \in V^*$ have the form

$$2\sigma\kappa^{-1}\partial_i\psi_j = -\frac{c_j\psi_i - c_i\psi_j}{x_i - x_j} + \frac{c_j\psi_i + c_i\psi_j}{x_i + x_j}, \quad i \neq j, \quad (34)$$

$$2\sigma\kappa^{-1}\partial_i\psi_i = \sum_{j \neq i} \left(\frac{c_j\psi_i - c_i\psi_j}{x_i - x_j} + \frac{c_j\psi_i + c_i\psi_j}{x_i + x_j} \right) + \frac{2(c_i + c_0)\psi_i}{x_i}. \quad (35)$$

Consider the product

$$\Phi_\lambda = \prod_{j=1}^n (x^2 - x_j^2)^{\lambda_j} x^{2\lambda_0}$$

and the corresponding integral

$$J_\lambda(x_1, \dots, x_n) = \int_\gamma \Phi_\lambda dx = \int_\gamma \prod_{j=1}^n (x^2 - x_j^2)^{\lambda_j} x^{2\lambda_0} dx, \quad (36)$$

where γ as before is a large circle. The integral is well-defined if the sum

$$2(\lambda_0 + \lambda_1 + \dots + \lambda_n) \in \mathbb{Z}.$$

Theorem 7. \vee -systems $B_n(c)$ with $c_j + c_0 \neq 0$ for all $j = 1, \dots, n$ are harmonic with the corresponding potentials $F_k = \frac{1}{2\pi i} J_\lambda$, where J_λ are contour integrals (36) with $\lambda_i = (2k - 1)\frac{c_i}{2\sigma}$ and $k = 1, \dots, n$. The corresponding value of κ is $2k - 1$. \square

Proof. We have

$$\psi_j = \partial_j J_\lambda(x_1, \dots, x_n) = -2\lambda_j \int \frac{x_j \Phi_\lambda}{x^2 - x_j^2} dx. \quad (37)$$

One can easily check that

$$\partial_i \psi_j = 4\lambda_i \lambda_j \int_\gamma \frac{x_i x_j \Phi_\lambda}{(x^2 - x_i^2)(x^2 - x_j^2)} dx = -\frac{\lambda_j \psi_i - \lambda_i \psi_j}{x_i - x_j} + \frac{\lambda_j \psi_i + \lambda_i \psi_j}{x_i + x_j}$$

when $i \neq j$. When $i = j$ we have

$$\partial_i \psi_i = -2 \int_\gamma \left(\frac{\lambda_i}{x^2 - x_i^2} - \frac{2\lambda_i(\lambda_i - 1)x_i^2}{(x^2 - x_i^2)^2} \right) \Phi_\lambda dx = -2 \int_\gamma \left(\frac{\lambda_i(2\lambda_i - 1)}{x^2 - x_i^2} - \frac{2\lambda_i(\lambda_i - 1)x^2}{(x^2 - x_i^2)^2} \right) \Phi_\lambda dx.$$

On the other hand

$$\begin{aligned} & \sum_{j \neq i} \left(\frac{\lambda_j \psi_i - \lambda_i \psi_j}{x_i - x_j} + \frac{\lambda_j \psi_i + \lambda_i \psi_j}{x_i + x_j} \right) + \frac{2(\lambda_i + \lambda_0) \psi_i}{x_i} \\ &= -2 \int_{\gamma} \left(\sum_{j \neq i} \frac{2\lambda_i \lambda_j x^2 \Phi_\lambda}{(x^2 - x_i^2)(x^2 - x_j^2)} + \frac{2\lambda_i(\lambda_i + \lambda_0) \Phi_\lambda}{x^2 - x_i^2} \right) dx. \end{aligned}$$

Since the difference of the right-hand sides of the last two formulas is the integral of the total derivative $\int_{\gamma} d \frac{x \Phi_\lambda}{x^2 - x_i^2}$, we see that the integrals (37) satisfy equations (34),(35) with $\lambda_j = \kappa \frac{c_j}{2\sigma}$.

Note that $2(\lambda_0 + \lambda_1 + \dots + \lambda_n) = \kappa$, so κ must be an integer. It is easy to see that the integral (36) vanishes for even κ , so the minimal values of κ are $1, 3, 5, \dots, 2n - 1$, which are the exponents of the Weyl group B_n . Similarly to the A_n case, one can show that the corresponding logarithmic vector fields are independent for all non-zero c_i , so the claim follows again from Saito's criterion. ■

As in the A_n case, the potential F_k can be given by formula (33) with $\kappa = k - 1$, where one has to replace p_s^λ with

$$q_s^\lambda = \sum_{i=1}^n \lambda_i x_i^{2s}.$$

If $c_1 = c_2 = \dots = c_k = -c_0$ for some $k = 1, \dots, n - 1$ then associated arrangement Δ is not of Coxeter type. It was studied first by Zaslavsky [54] and is usually denoted as D_n^k [36]. It is known to be free with the exponents of $1, 3, \dots, 2n - 3, 2n - k - 1$ (see [36, 37]). The first $n - 1$ generating potentials F can be found by the same integrals (36) with $\lambda_i = \kappa \frac{c_i}{2\sigma}$ and $\kappa = 1, 3, \dots, 2n - 3$, but the last one of the required degree $2n - k$ appears not to exist for general values of the remaining parameters c_{k+1}, \dots, c_n (see below the example with $k = 2, n = 3, c_3 = 3$).

For special c this is however possible. Let

$$c_1 = c_2 = \dots = c_k = -c_0 = 1, c_{k+1} = \dots = c_n = 2,$$

then $\sigma = 2n - k - 1$ and the integral (36) becomes

$$J_\lambda(x_1, \dots, x_n) = \int_{\gamma} \prod_{i=1}^k (x^2 - x_i^2)^{1/2} \prod_{j=k+1}^n (x^2 - x_j^2) x^{-1} dx.$$

Taking now small contour γ surrounding $x = 0$ we have up to a non-essential multiple

$$J_\lambda(x_1, \dots, x_n) = x_1 \dots x_k (x_{k+1} \dots x_n)^2,$$

which is the remaining potential for the arrangement D_n^k (cf. [37]).

This case corresponds to the restriction of the Coxeter arrangement of type D_{k+2n} to the subspace $x_{k+1} = x_{k+n+1}$, $x_{k+2} = x_{k+n+2}$, $x_{k+n} = x_{k+2n}$. So one might expect that the restrictions of Coxeter systems are always harmonic. This however is not true as the following example shows.

Consider the restriction of the Coxeter system D_5 to the subspace $x_3 = x_4 = x_5$. The corresponding \vee -system $B_3(-1; 1, 1, 3)$ is of type $D_3(3/2, 3/2)$ in the notations of [17] and belongs to the deformation family of the roots of the exceptional Lie superalgebra $D(2, 1, \lambda)$.

Theorem 8. The restricted Coxeter \vee -system $B_3(-1; 1, 1, 3)$ is not harmonic. □

Proof. We have seven hyperplanes in the corresponding arrangements. Assume that there are polynomial solutions for the corresponding system (20) for $\kappa_1 \leq \kappa_2 \leq \kappa_3$ with

$$\kappa_1 + \kappa_2 + \kappa_3 = 7.$$

Direct check shows that there are no quasi-invariants of degree 3 and the space of quasi-invariants of degree 4 is two-dimensional. We have $\kappa_1 \geq 1, \kappa_2 \geq 3$, so $\kappa_3 \leq 3$ and the only possible choice is $\kappa_1 = 1, \kappa_2 = \kappa_3 = 3$. As the space of quasi-invariants of degree 4 contains the square of the quasi-invariant of degree 2 one cannot have three independent solutions of the system (20) at the specified κ_i . ■

The corresponding arrangement can be given by

$$x_3(x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2) = 0$$

and has Poincaré polynomial

$$P_\Sigma(t) = (1 + t)(1 + 3t)^2.$$

It is free with a basis of logarithmic vector fields

$$X_1 = \sum_{i=1}^3 x_i \partial_i, X_2 = \sum_{i=1}^3 x_i^3 \partial_i, X_3 = x_1 x_2 x_3^2 \sum_{i=1}^3 x_i^{-1} \partial_i$$

(see [37, p. 251]). The restriction of the D_5 invariant $x_1 \dots x_5$ gives the polynomial $x_1 x_2 x_3^3$ of degree 5.

5 Coxeter Arrangements and Saito Flat Coordinates

Let G be an irreducible finite Coxeter group generated by reflections in a real Euclidean space V of dimension n and Δ be the set of all corresponding reflection hyperplanes. Define the corresponding Coxeter root system \mathcal{R} as a set of normals chosen in a G -invariant way. We have either one or two different orbits of G on \mathcal{R} , so such a system in general depends on the additional parameter $q = |\alpha|/|\beta|$, which is a ratio of the lengths of the roots from two different orbits.

The positive part $\mathcal{A} = \mathcal{R}_+$ of Coxeter root system is known to be a ∇ -system ([50], see also [32]), which we call *Coxeter ∇ -system*. We are going to show that it is harmonic and that the corresponding potentials of the gradient logarithmic vector fields are given by Saito flat coordinates [42].

Recall briefly the definition of these remarkable coordinates, which can be considered as a canonical choice of generators in the algebra of G -invariant polynomials $S^G(V)$. Let y_1, \dots, y_n be any set of homogeneous generators in $S^G(V)$ of degrees $d_1 > d_2 \geq d_3 \geq \dots > d_n = 2$. The image of the Euclidean contravariant metric on V is degenerate on the orbit space V/G , but its Lie derivative along well-defined vector field $\frac{\partial}{\partial y_1}$ gives flat metric η (called *Saito metric*), which is non-degenerate everywhere [12, 41].

The corresponding flat coordinates $t_1, \dots, t_n \in S^G(V)$ are called *Saito flat coordinates*. They were found explicitly by Saito *et al.* in [44] for all the cases except E_7, E_8 (for the latter cases see [2, 35, 48]). These coordinates play an important role in 2D topological field theory [10] and related theory of Frobenius manifolds developed by Dubrovin [11–13]. In the A_n case they appear in the theory of the dispersionless KP hierarchy [34].

For the classical Coxeter groups of types A_n and B_n the Saito coordinates can be written as the residues at infinity [10, 19]:

$$t_k = \text{Res}_\infty \prod_{i=1}^{n+1} (x - x_i)^{\frac{k}{n+1}}, \quad \sum_{i=1}^{n+1} x_i = 0$$

in type A_n and

$$t_k = \text{Res}_\infty \prod_{i=1}^n (x^2 - x_i^2)^{\frac{2k-1}{2n}}$$

in type B_n . Comparing this with the formulas (30), (36) we see that they coincide with the potentials of the $A_n(c)$ -type \vee -systems with $c_0 = c_1 = \dots = c_n$ and of $B_n(c)$ -type \vee -systems with $c_0 = 0, c_1 = \dots = c_n$, respectively.

We also note that in the A_n case the Saito coordinate t_k is proportional to the Jack polynomial $J_{[k+1]}^\alpha$, corresponding to a single-row Young diagram with $k+1$ boxes and the parameter $\alpha = -\frac{n+1}{k}$ (see [47]).

It turns out that this link with harmonic \vee -systems is not accidental.

Theorem 9. The Coxeter \vee -system \mathcal{R}_+ is harmonic. In the case when all the normals have the same length the potentials of the corresponding gradient logarithmic vector fields are the Saito flat coordinates t_1, \dots, t_n . \square

Proof. In the case when all the vectors are normalised to have the same length this follows from the results of [14], where it was shown that the Saito polynomials satisfy the corresponding system (24) with $\kappa = \deg t_i - 1$ being the corresponding exponents of the Coxeter group.

This covers completely one-orbit cases: simply laced ADE as well as H_3, H_4 and odd dihedral groups $I_2(2k+1)$. The B_n case follows from Theorem 7: for a general choice of normals

$$\mathcal{B}_n = \{e_i \pm e_j, \sqrt{2(1+c_0)}e_i, 1 \leq i < j \leq n\}$$

the potentials are given by

$$F_k = \text{Res}_\infty \prod_{i=1}^n [x^{2c_0}(x^2 - x_i^2)]^{\frac{2k-1}{2n+2c_0}}, k = 1, \dots, n$$

(the case of equal lengths corresponds to $c_0 = 0$). Thus it remains to consider only the case F_4 and even dihedral groups $I_2(2p)$.

The Coxeter \vee -system of type F_4 consists of the following covectors:

$$e_i \pm e_j, t\sqrt{2}e_i, \frac{t\sqrt{2}}{2}(e_1 \pm e_2 \pm e_3 \pm e_4), 1 \leq i < j \leq 4.$$

In the case $t = 1$ all the roots have equal length, the case $t = 1/\sqrt{2}$ corresponds to the root system F_4 . In the complex case we have to add that $t^2 \neq -1$ for the non-degeneracy of the corresponding bilinear form.

Consider the polynomials

$$I_n = \sum_{i < j}^4 (x_i - x_j)^n + (x_i + x_j)^n.$$

The polynomials I_2, I_6, I_8, I_{12} are basic invariants for the Weyl group of type F_4 (see, e.g., [44]). The Mathematica calculations lead to the following potentials:

$$\begin{aligned} & I_2, 648(1 + t^2)I_6 - 5(5 + 4t^2)I_2^3, \\ & 69984(1 + t^2)^2I_8 - 9072(7 + 2t^2)(1 + t^2)I_2I_6 + 35(49 + 46t^2 + 4t^4)I_2^4, \\ & 10077696(1 + t^2)^3I_{12} - 384912(11 + 8t^2)(1 + t^2)^2I_8I_2^2 + 769824(4t^2 - 11)(1 + t^2)^2I_6^2 \\ & \quad + 7128(319 + 376t^2 + 112t^4)(1 + t^2)I_6I_2^3 - 11(3641 + 7032t^2 + 4560t^4 + 1048t^6)I_2^6. \end{aligned}$$

At $t = 1$ the above potentials are proportional to the corresponding Saito flat coordinates [44] (there seem to be typos in [44] in the expressions for the 6th- and 12th-order polynomials).

Consider now even dihedral case $I_2(2p)$ with $p > 1$. Let us fix the corresponding vectors as $\alpha_k = a(\cos \varphi_k, \sin \varphi_k)$, $\beta_k = b(\cos \psi_k, \sin \psi_k)$, where

$$\varphi_k = \pi/2 + \pi k/p, \quad \psi_k = \pi/2 + \pi/2p + \pi k/p, \quad k = 0, 1, \dots, p-1.$$

One can show that in the complex coordinate $z = x_1 + ix_2$ the potentials of the corresponding ∇-system are

$$F_1 = z\bar{z}, \quad F_2 = z^{2p} + \bar{z}^{2p} + \frac{2(2p-1)}{p-1} \frac{a^2 - b^2}{a^2 + b^2} (z\bar{z})^p.$$

When $a = b$ we have the basic invariants $z\bar{z}, z^{2p} + \bar{z}^{2p}$, known to be Saito flat coordinates in this case [44]. ■

6 Concluding Remarks

We have shown that the theory of ∇-systems has natural links both with the representation theory of holonomy Lie algebras and with Saito's theory of logarithmic vector fields, which could be used in both ways. This led us to the notion of the harmonic ∇-system, which seems to be important in its own right.

The homological representations of braid groups were first studied by Lawrence [27] in relation with Hecke algebras and later by Bigelow and Krammer, who showed

that they provide faithful representations of braid groups [6, 26]. The relationship to the monodromy representations of the KZ connections was investigated by Kohno [23]. Our results can be used to extend these important developments to a larger class of the hyperplane arrangements.

We would like to mention a few more very interesting and relevant questions, which were left aside. The first one is related to the so called “almost duality” discovered by Dubrovin [13]. More precisely, Dubrovin found a remarkable connection between the polynomial Frobenius structure on the orbit space of a Coxeter group and the logarithmic Frobenius structure with the prepotential

$$\mathcal{F} = \sum_{\alpha \in \mathcal{R}_+} (\alpha, x)^2 \log(\alpha, x)^2,$$

where \mathcal{R} is the corresponding Coxeter root system with all the roots of the same length [13]. A natural question is what is the dual structure in the case when \mathcal{R} is a general Coxeter \vee -system with roots of different lengths, or more generally, if \mathcal{R} is any harmonic \vee -system.

The second question is about possible differential-geometric interpretation of the corresponding potentials. As we have seen above these potentials are certain deformations of Saito flat coordinates. Finally, it would be interesting to investigate the role of these potentials for the representations of rational Cherednik algebras similarly to Saito coordinates which are shown to be related to special singular vectors in the polynomial representations [14]. We hope to address these questions elsewhere.

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