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Geometrical acoustics of Lamb waves

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Geometrical acoustics of Lamb waves

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Abstract

In the present work, an overview of the developments of the geometrical acoustics (GA) theory of Lamb waves in plates of variable thickness is given, based mainly on the original results of the present author. The main attention is paid to the lowest order Lamb modes in plates of variable thickness, i.e. flexural and quasi-longitudinal plate waves. It is shown that the GA approach is an ideal tool to describe propagation of ultrasonic Lamb waves in complex plate-like and wedge-like structures. In particular, it is demonstrated that the developed GA theory involving both lowest order Lamb modes can be used for theoretical description of the classical problem of Rayleigh surface wave reflection from the tip of an elastic wedge of arbitrary angle, both at normal and at oblique incidence. The GA approach operating with flexural waves alone can be used for the development of the theory of localised waves propagating along sharp edges of different wedge-like structures. Another important application of GA is the development of the theory of ‘acoustic black holes’ for flexural waves that can absorb almost all of the incident wave energy. The obtained theoretical results are illustrated by recent experiments.

1. Introduction

Geometrical acoustics (GA) theory or geometrical acoustics approximation is an asymptotic high-frequency solution (sometimes called ray-tracing solution) to the differential equations and boundary conditions describing wave propagation in any particular elastic medium or structure. Similarly to geometrical optics (GO) theory for electromagnetic waves, it can be used if elastic properties of media change slowly with the distance of propagation. Although geometrical acoustics is used widely to describe propagation of underwater or atmospheric sound^(1, 2), its use in the acoustics of solids is not so common, which can be partly explained by the complexity of real inhomogeneous solid structures. In the same time, the use of GA to describe wave

propagation in such inhomogeneous solids is very efficient, and it can assist in obtaining explicit analytical solutions to many important practical problems ⁽³⁾.

In this paper, an overview of the developments of the geometrical acoustics theory of Lamb waves in inhomogeneous plates is given, based largely on the original results of the present author. The main attention is paid to the lowest order Lamb modes in plates of variable thickness, i.e. flexural and quasi-longitudinal plate waves. Although the developed GA approach can be used for general description of acoustic and ultrasonic wave propagation in complex solid structures, in this paper only three major topics will be discussed where GA was instrumental in obtaining fundamentally new theoretical results. Initially, it is demonstrated that the developed GA theory involving both lowest order Lamb modes can be used for theoretical description of the classical problem of Rayleigh surface wave reflection from the tip of an elastic wedge of arbitrary wedge angle, both at normal and at oblique incidence. Then, propagation of guided flexural waves in plates of variable thickness is considered. The obtained results are used for the development of the theory of localised flexural waves propagating along the tips of slender elastic wedges, also called ‘wedge elastic waves’. The use of GA in this case permits to consider wedges of arbitrary shape, curved wedges and truncated wedges. Another important application of geometrical acoustics to inhomogeneous plates is the development of the theory of ‘acoustic black holes’ for flexural waves. Acoustic black holes are one-dimensional wedges or two-dimensional indentations of power-law profile that can absorb almost all of the incident wave energy. They can be used for efficient damping of vibrations in light-weight structures.

2. Propagation of Lamb waves in plates of variable thickness

Using geometrical acoustics approach, it is possible to predict propagation of Lamb waves in real life complex solid structures, for example those that are used in ultrasonic non-destructive testing applications. The most important modes of Lamb waves are lowest order symmetric (longitudinal) and antisymmetric (flexural) modes, or compression and flexural waves respectively. In what follows, a particular type of plates of variable thickness will be discussed – plates with linearly variable thickness, i.e. slender elastic wedges (Figure 1).

To develop a geometrical acoustics theory of longitudinal and flexural wave propagation in wedges having a small wedge angle θ one can refer to the well-known thin plate equations ^(4, 5) (see also the monograph ⁽³⁾) for longitudinal and compression waves in plates of variable thickness $h(x) = 2x \tan(\theta/2) \approx \theta x$. Considering, for example, wave propagation in x - direction (one-dimensional case), the GA solutions to these equations can be sought in the forms (the time factor $\exp(-i\omega t)$ is assumed):

$$u(x) = A_1(x) \exp\left[\int ik_s(x) dx\right], \quad (1)$$

$$w(x) = A_2(x) \exp\left[\int ik_a(x) dx\right]. \quad (2)$$

Here $u(x)$ is the longitudinal displacement, $w(x)$ is the flexural (normal) displacement of a plate (this displacement is linked to $u(x)$), $A_1(x)$ and $A_2(x)$ are slowly varying functions describing the change of wave amplitudes with the distance of propagation, $k_s(x)$ is the local wavenumber of a symmetric (longitudinal) wave in a thin plate (for

very thin plates $k_s(x) = k_p = \omega/c_p$, where $c_p = 2c_t(1-c_t^2/c_l^2)^{1/2}$ is the so-called plate wave velocity, c_l and c_t are the velocities of longitudinal and shear acoustic waves in the plate material, $k_a(x) = 12^{1/4} k_p^{1/2}(\theta x)^{-1/2}$ is the local wavenumber of a flexural wave in a plate of linearly variable thickness. It can be shown that equations (1) and (2) satisfy the thin plate equations asymptotically at relatively high frequencies ^(4, 5). In practice though these frequencies are well within the range of practically used frequencies.

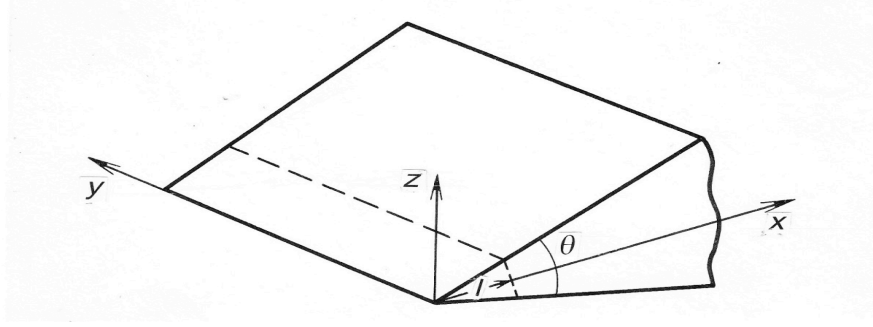


Figure 1. Geometry of a slender solid wedge

Functions $A_1(x)$ and $A_2(x)$ can be obtained from the so-called transport equations following from the geometrical acoustics theory ^(4, 5). In addition to the transport equations, they can be also defined from the energy conservation law.

3. Rayleigh wave reflection and transmission in a wedge

One of the important applications of the above-mentioned GA approach to Lamb wave propagation is the development of the analytical theory of Rayleigh wave reflection and transmission in a wedge undertaken by the present author and his co-workers ^(6, 7). As it is well known, the displacements in the Rayleigh waves on the upper and lower surfaces of a plate (or a wedge in the case under consideration), u_{upp} and u_{low} , can be represented as the sum or difference respectively of symmetric and antisymmetric Lamb modes of the lowest order (or longitudinal and flexural waves):

$$u_{upp} = u_s + u_a, \quad (3)$$

$$u_{low} = u_s - u_a. \quad (4)$$

Considering propagation of the incident Rayleigh wave in the normal direction towards the edge of a wedge (see Figure 1), one can represent u_s and u_a in line with the geometrical acoustic expressions (1) and (2) ^(6, 7):

$$u_s = \frac{u_0}{2} \exp \left[-i \int_{x_0}^x k_s(x') dx' \right], \quad (5)$$

$$u_a = \frac{u_0}{2} \exp \left[-i \int_{x_0}^x k_a(x') dx' \right]. \quad (6)$$

Here u_0 is the amplitude of the incident Rayleigh wave (either normal or longitudinal), x_0 is a certain initial coordinate located sufficiently far from the edge (so that the local thickness $h(x_0)$ is larger than the Rayleigh wavelength), $k_s(x)$ is the local wavenumber of the lowest order symmetric Lamb mode (longitudinal plate wave) that tends to the thickness independent (nondispersive) value $k_p = \omega/c_p$ for very thin plates, $k_a(x)$ is the above-mentioned local wavenumber of the lowest order antisymmetric Lamb mode (flexural wave), which is dispersive. It is obvious from equations (5), (6) and (3), (4) that, if a Rayleigh wave is generated at the point $x = x_0$ on the upper surface of a plate or a wedge, its amplitude on the lower surface at $x = x_0$ is zero, as expected.

It is important to note that for values of the plate thickness being larger than the Rayleigh wavelength at given frequency (i.e. at relatively high frequencies) the velocities of both symmetric and antisymmetric Lamb modes tend to the Rayleigh wave velocity c_R . However, when these Lamb modes approach the wedge tip, their phase velocities begin to differ from each other, resulting in phase difference between these modes accumulated during their propagation to the tip and back. This may cause oscillations in the amplitudes of the reflected and transmitted Rayleigh waves with changing the wedge angle θ .

It should be also noted that amplitude dependence on the propagation distance (described by the functions $A_1(x)$ and $A_2(x)$ in equations (1) and (2)) is ignored in equations (5) and (6) because, if to consider the final coordinate as $x = x_0$, i.e. to assume that the reflected signal is observed at the same point where the incident signal was generated, the amplitudes of both symmetric and antisymmetric modes will be restored to their initial levels after the direct reflection of each of them from the wedge tip and their propagation back to the initial position $x = x_0$. The direct reflection of each of these modes from the wedge tip is considered as reflection from a free edge of a plate, i.e. the modules of the corresponding reflection coefficients are assumed to be equal to one, whereas their phases are either zero or $\pi/2$ for symmetric or antisymmetric modes respectively.

The resulting expressions for the absolute values of Rayleigh wave reflection and transmission coefficients in a wedge defined at the position $x = x_0$ have the form ⁽⁶⁾:

$$|R| = \left| \sin \left[\frac{\delta}{\tan(\theta/2)} - \frac{(\pi - \theta)}{4} \right] \right|, \quad (7)$$

$$|T| = \left| \cos \left[\frac{\delta}{\tan(\theta/2)} - \frac{(\pi - \theta)}{4} \right] \right|, \quad (8)$$

where

$$\delta = \frac{1}{2} \int_0^\infty [k_a(h) - k_s(h)] dh \quad (9)$$

is a dimensionless constant depending on the elastic properties of the wedge material (integration in (9) is taken over the local thickness h rather than over the distance x). Functions $k_a(h)$ and $k_s(h)$ in equation (9) should represent the complete dependences valid for the whole range of h . The approximate analytical expressions for such functions have been constructed ⁽⁶⁾, so that for $h \rightarrow 0$ they coincide with the corresponding functions for flexural and longitudinal waves in thin plates, whereas for $h \rightarrow \infty$ they go to the dispersion relation for Rayleigh waves. Calculations show that, for example, for duraluminium $\delta = 2.75$.

Calculated values of $|R|$ and $|T|$ as functions of the wedge angle θ are shown in Figure 2 for $\delta = 2.75$ by solid lines. Dots indicate experimental values earlier obtained by Viktorov ⁽⁸⁾. It can be seen that the developed theory based on GA explains the experiment well, correctly describing the nature of the observed decreases in the periods of oscillations of both $|R|$ and $|T|$ as the angle θ decreases.

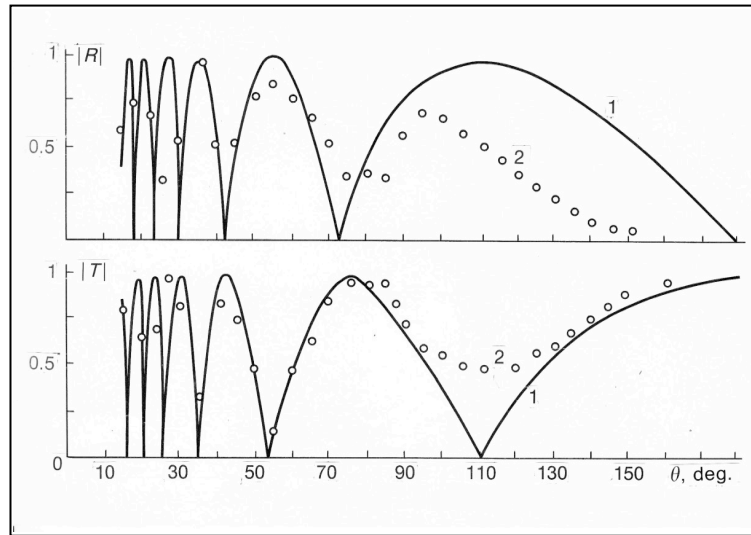


Figure 2. Reflection and transmission of Rayleigh waves in a solid wedge as functions of the wedge angle θ ⁽⁶⁾

It is evident from equations (7) and (8) that the theoretical values of $|R|$ and $|T|$ satisfy the condition $|R|^2 + |T|^2 = 1$. This condition describes energy conservation law for reflection and transmission of Rayleigh waves in a wedge, meaning that there is no energy conversion into other plate modes. Comparison with the measurements ⁽⁸⁾ shows that for measured values of $|R|$ and $|T|$ the sum $|R|^2 + |T|^2$ is less than one, but it is very close to it for most values of the angle θ , except for the range of θ between 100° and 150° , where GA theory is not expected to be valid.

In addition to the problem of reflection and transmission of Rayleigh wave in a wedge at normal incidence, which was discussed above, it is important to consider a more general case of oblique incidence ⁽⁷⁾ (see Figure 3). In this case, even for a fixed wedge angle θ , the total accumulated phases for propagating flexural and longitudinal waves depend on the angle of incidence α . Therefore, according to the GA theory, oscillations of the reflection and transmission coefficients of Rayleigh waves can take place as a result of changing the angle of incidence α . Naturally, the developed theory

of Rayleigh wave reflection and transmission is more complex in the case of oblique incidence.

First of all, one should take into account the different characteristics of refraction of the symmetric and antisymmetric Lamb modes during their propagation to the wedge tip and back as their velocities vary according to different laws near the tip. The velocity of the antisymmetric (flexural) mode tends to zero near the edge. This is why it approaches the tip almost at normal angle (see Figure 3, trajectory 1). On the other hand, the velocity of the symmetric (longitudinal) mode increases near the edge from the Rayleigh wave velocity c_R to the plate wave velocity c_p . As the result of this, the ray trajectory of the symmetric mode is more oblique.

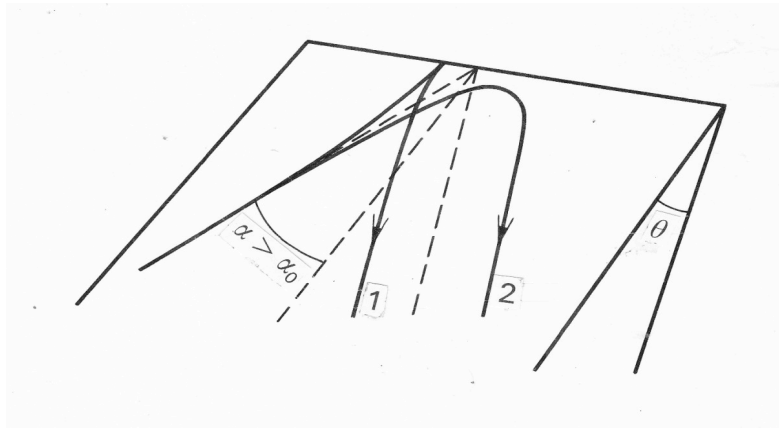


Figure 3. Geometry of Rayleigh wave propagation in a wedge at oblique incidence; a Rayleigh wave is represented as a sum of two lowest order Lamb modes ⁽⁷⁾

Especially interesting situation happens for angles of incidence α that are larger than a certain angle α_0 . For such angles, the symmetric mode turns back before reaching the edge, thus forming a simple caustic ⁽⁷⁾. It is well known from the general geometrical acoustics (optics) theory ⁽²⁾ that, upon reflection from a simple caustic (ray congestion), the reflected wave obtains an additional phase shift of $-\pi/2$ called the 'caustic phase shift'. This phase shift must be added to the total accumulated phase of the symmetric plate mode in the process of calculations of Rayleigh wave reflection and transmission coefficients. It can be shown ⁽⁷⁾ that the caustic phase shift results in the additional spike around the angle $\alpha = \alpha_0$. This prediction has been confirmed by the experiments ⁽⁷⁾.

4. Geometrical acoustics theory of wedge elastic waves

One of the important applications of geometrical acoustics of antisymmetric (flexural) Lamb waves is the development of the analytical theory of localised elastic waves propagating along the tip of an elastic wedge, also known as wedge elastic (or acoustic) waves. The GA theory of localised elastic waves in slender solid wedges has been developed by the present author ^(4, 5) (see also the monograph 3), and it is based on the geometrical acoustics approach considering a slender wedge as a plate with a local variable thickness $d \approx x\theta$, where θ is the wedge apex angle and x is the distance from the wedge tip measured in the middle plane (see Figure 1). The velocities c of the

localised wedge modes propagating in y -direction can be determined in the geometrical acoustics approximation as the solutions of the Bohr - Sommerfeld type equation ^(4,5):

$$\int_0^{x_t} [k_a^2(x) - \beta^2]^{1/2} dx = \pi n, \quad (10)$$

where $\beta = \omega/c$ is yet unknown wavenumber of a wedge mode, $k_a(x)$ is a local wavenumber of a flexural wave in a plate of variable thickness, $n = 1, 2, 3, \dots$ is the mode number, and x_t is the so called ray turning point being determined from the equation $k_a^2(x) - \beta^2 = 0$.

As was mentioned in the previous section, for a slender wedge in contact with vacuum $k_a(x) = 12^{1/4} k_p^{1/2} (\theta x)^{-1/2}$, so that $x_t = 2\sqrt{3} k_p / \theta \beta^2$. Here $k_p = \omega/c_p$ and $c_p = 2c_l(1 - c_t^2/c_l^2)^{1/2}$ is the plate wave velocity, c_l and c_t are the velocities of longitudinal and shear acoustic waves in the plate material. Taking the integral in equation (10) analytically and solving the resulting algebraic equation yields the extremely simple analytical expression for wedge wave velocities ^(4,5):

$$c = \frac{c_p n \theta}{\sqrt{3}}. \quad (11)$$

The expression (11) for wedge wave velocities agrees well with the other (numerically based) theories and with the available experimental results. The analytical expressions for amplitude distributions of wedge modes are rather cumbersome ⁽⁴⁾, and they are not displayed here for shortness. Figure 4 illustrates the first three modes of wedge elastic waves calculated in geometrical acoustics approximation ⁽⁴⁾.

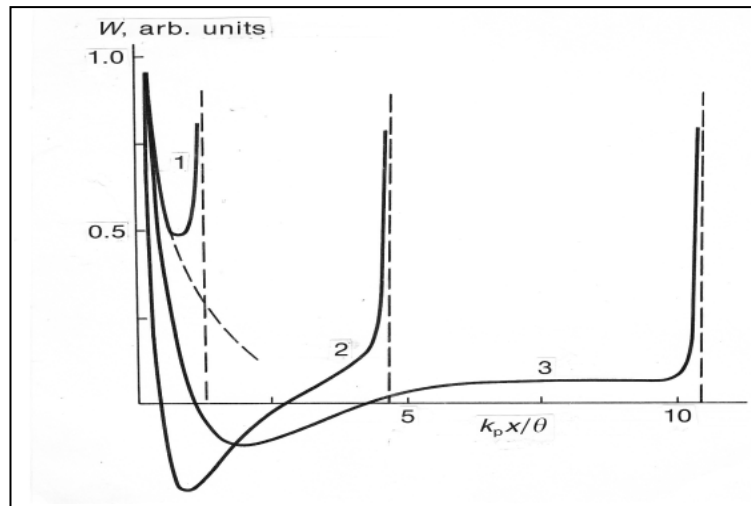


Figure 4. First three modes of wedge elastic waves calculated using geometrical acoustics approach ⁽⁴⁾

The structure of the wedge modes shown in Figure 4 agrees well with the results of the numerical calculations, with the exception of the clearly seen singularities in the areas marked by dashed vertical lines. Beyond these lines, which show the locations of caustics (ray congestions), the modes do not penetrate into the depth of a wedge. The above singularities manifest the well-known limitation of all geometrical acoustics (optics) theories that become invalid near caustics ^(1, 2).

The above-mentioned geometrical acoustics theory of wedge elastic waves can be generalised to consider localised modes of quadratically-shaped elastic wedges ⁽⁹⁾, wedges immersed in liquids ⁽¹⁰⁾, cylindrical and conical wedge-like structures (curved wedges) ⁽¹¹⁾, wedges of general power-law shape ⁽¹²⁾, and wedges made of anisotropic materials ⁽¹³⁾.

5. GA theory of acoustic black holes for flexural waves

Another important application of geometrical acoustics of Lamb waves in plates of variable thickness is the theory of the so-called ‘acoustic black holes’ for flexural waves that can absorb almost 100% of the incident wave energy. This effect can be used as a new method of damping structural vibrations. It should be noted that vibration damping represents a vitally important topic of general vibration engineering ⁽¹⁴⁾.

To understand the phenomenon of acoustic black holes, or the acoustic black hole effect, one can start with the simplest one-dimensional case of plane flexural wave propagation in the normal direction towards the edge of a free wedge having the local thickness $h(x)$ described by a power-law relationship $h(x) = \varepsilon x^m$, where m is a positive rational number and ε is a constant (see Figure 5).

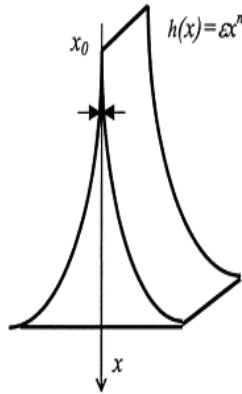


Figure 5. Solid wedge of power-law profile

Since flexural wave propagation in such wedges can be described using the geometrical acoustics approach ^(4, 5), the total accumulated phase Φ representing part of the geometrical acoustics solution in the form described by equations (1) or (2) and resulting from the wave propagation from an arbitrary point x located in the wedge medium plane to the wedge tip ($x = 0$) can be written as

$$\Phi = -\int_x^0 k_a(x)dx = \int_0^x k_a(x)dx. \quad (12)$$

Here $k_a(x)$ is a local wavenumber of a flexural wave for a wedge in contact with vacuum. For a slender wedge, $k_a(x) = 12^{1/4} k_p^{1/2} (\epsilon x^m)^{-1/2}$, where $k_p = \omega/c_p$ is the wavenumber of a symmetrical plate wave, $c_p = 2c_l(1-c_t^2/c_l^2)^{1/2}$ is its phase velocity, and c_l and c_t are longitudinal and shear wave velocities in the wedge material.

It is easy to see that the integral in equation (12) diverges for $m \geq 2$. This indicates that the total accumulated phase Φ becomes infinite under these circumstances, which means that the wave never reaches the edge. Therefore, it never reflects back either, i.e. the wave becomes trapped at the edge, thus indicating that the above mentioned ideal wedges of power-law profile for $m \geq 2$ materialise what can be called 'acoustic black holes' (ABH) for flexural waves^(15, 16). The above-mentioned effect of zero reflection of flexural waves from the edge of a wedge of power-law profile has been first predicted by Mironov⁽¹⁷⁾ who gave a slightly different interpretation of this phenomenon.

Real fabricated wedges, however, are not ideally sharp, and they always have truncated edges, which adversely affects their performance as 'black holes'. If for ideal wedges of power-law shape (with $m \geq 2$) it follows from equation (12) that even an infinitely small material attenuation, described by the imaginary part of $k_a(x)$, would be sufficient for the whole wave energy to be absorbed, this is not so for truncated wedges. Indeed, for truncated wedges the lower integration limit in equation (12) must be changed from 0 to a certain value x_0 describing the length of truncation, thus resulting in the amplitude of the total reflection coefficient R_0 being expressed in the form⁽¹⁷⁾

$$R_0 = \exp(-2 \int_{x_0}^x \text{Im} k_a(x) dx). \quad (13)$$

According to equation (13), for low values of wave attenuation in such typical wedge materials as steel, even very small truncations x_0 result in the reflection coefficients R_0 becoming as large as 50-70 %, which makes it impossible to use such wedges in practice.

In order to improve the situation for real wedges (with truncations), it has been proposed by the present author to cover wedge surfaces near the edges by thin absorbing layers (films) of thickness δ_l , e.g. by polymeric films^(15, 16). To analyse the effect of thin absorbing films on flexural wave propagation in a wedge using geometrical acoustics approximation one should consider first the effect of such films on flexural wave propagation in plates of constant thickness. The simplest way of approaching this problem is to use the already known solutions for plates covered by absorbing layers of arbitrary thickness obtained by different authors with regard to the description of damped vibrations in sandwich plates⁽¹⁴⁾. Using this approach, one can derive the analytical expressions for the reflection coefficients of flexural waves from the edges of truncated wedges covered by thin absorbing layers^(15, 16).

Typical calculated values of the reflection coefficient R_0 for flexural waves in real quadratic wedges are as follows: in the presence of the damping film $R_0 = 2 - 4$ %, whereas in the absence of the damping film $R_0 = 50 - 70$ %. Thus, in the presence of the damping film the values of the reflection coefficient are much smaller than for a

wedge with the same value of truncation, but without a film. Note that almost all absorption of the incident wave energy takes place in the vicinity of the sharp edge of a wedge. The above-mentioned theoretical predictions have been confirmed experimentally and numerically ^(18, 19).

Figure 6 shows the measured point mobility (particle velocity divided by the value of an applied concentrated force) for a free quadratic wedge made of steel and for the same wedge covered by a thin strip of an absorbing polymeric layer ⁽¹⁸⁾. It can be seen from Figure 6 that the addition of a thin absorbing layer leads to a dramatic increase in damping (up to 20 dB) of resonant peaks in the initial mobility function.

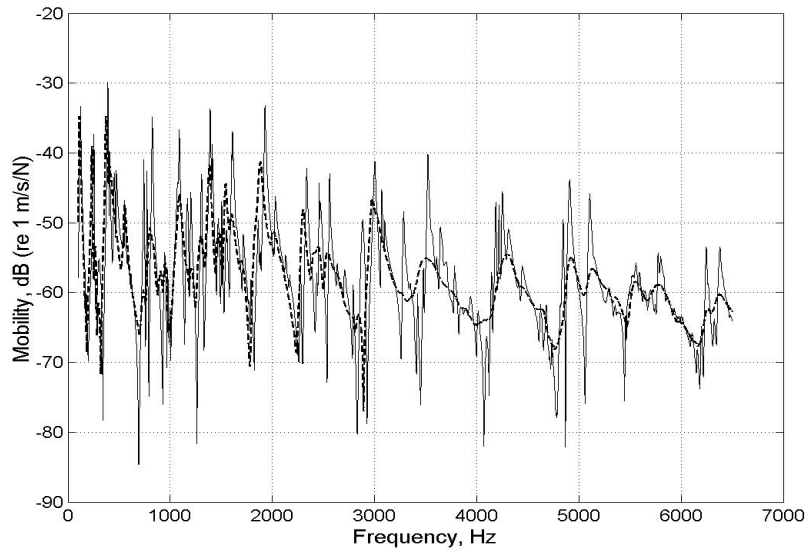


Figure 6. Measured point mobility of a quadratic steel wedge: free wedge (solid curve) and wedge covered by a thin adhesive strip of thickness $\delta_t = 0.2$ mm (dashed curve) ⁽¹⁸⁾

The logical developments of acoustic black holes based on wedges of power-law profile, which can be considered as one-dimensional ABH, are two-dimensional acoustic black holes formed by nearly or slightly protruding cylindrically symmetrical indentations (pits) of power-law profile drilled in a regular thin plate of constant thickness ⁽²⁰⁾. To consider flexural wave propagation over two-dimensional pits of power-law profile it is convenient to use the geometrical acoustics approach in Hamiltonian formulation ⁽²⁰⁾. The analysis shows that the presence of a pit influences the rays of flexural waves that propagate through it. In particular, in the case of symmetrical pits of power-law profile with $m \geq 2$, all rays incident at angles α_0 less than a certain critical value α_{cr} (that depends on the distance r_0 from the source to the centre of the pit), including a direct ray for which $\alpha_0 = 0$, will deflect towards the centre of the pit. If a piece of polymeric film is attached to the central area of a pit, then all such rays will be efficiently absorbed by the pit, which thus materialises a two-dimensional acoustic black hole ⁽²⁰⁾.

For practical purposes, the above-mentioned two-dimensional acoustic black holes (power-law pits) can be placed at any point of a plate- or shell-like structure. The effect of such black holes will be in eliminating the flexural wave rays intersecting with them

from contributing to the overall frequency response function of a structure, which will result in substantial damping of its resonant vibrations. To amplify the effect of two-dimensional acoustic black holes one can use combinations or arrays of several black holes distributed over the structure, if this does not compromise the main functions of the latter ⁽²⁰⁾.

One of the most important advantages of the acoustic black holes as dampers of structural vibrations is that they are efficient even for relatively thin and narrow strips of attached absorbing layers ^(15, 16, 18, 19). This is in contrast with the traditional techniques requiring covering the entire surfaces of structures by relatively thick layers of absorbing materials ⁽¹⁴⁾. This important feature of acoustic black holes can be very attractive for many practical applications, e.g in transport engineering, where the reduction of mass of a vehicle is highly desirable for fuel economy and environmental reasons. Because of these attractive features, the topic of acoustic black holes has become popular within the research community, which is reflected in numerous recent publications, for example ⁽²¹⁻²⁴⁾.

6. Conclusions

It has been demonstrated in this work that geometrical acoustics theory of Lamb waves, especially flexural waves, is a powerful tool that provides researchers with the possibility to develop theoretical descriptions of wave propagation in a variety of important complex solid structures.

In many practical cases, including those described in this work, geometrical acoustics also gives clear and physically explicit explanations of the physical mechanisms behind the wave phenomena in question.

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