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Extended Wiener-Khinchin theorem for quantum spectral analysis: supplementary material

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1. THE CONVENTIONAL WIENER-KHINCHIN THEOREM BASED ON MULTI-MODE MACH-ZEHNDER INTERFERENCE

In Section 1, we provide a multi-mode theory for Mach-Zehnder (MZ) interference. Based on the equations of this interferometry, we can construct the conventional Wiener-Khinchin theorem (WKT), which is the foundation for the classical interferometric spectroscopy. The setup of the MZ interference is shown in Fig. S1(a). Assume there is a single photon state $|\psi\rangle$, which has a frequency distribution (i.e., one-photon spectral amplitude) of $f(\omega_s)$

$$|\psi\rangle = \int_0^\infty d\omega_s f(\omega_s)\hat{a}_s^\dagger(\omega_s)\,|0\rangle\,, \tag{S1}$$

where $\hat{a}_s^\dagger$ is the creation operator and ω_s is the angular frequency.

The photons from the single photon source are split by the first 50/50 beam splitter (BS1) and then pass through path 1 and 2. Then, after an optical delay τ, the photons combine at the second 50/50 beam splitter (BS2). The photons at the output port 3 of BS2 are detected by a single photon detector D. The detection field operator of detector (D) is $\hat{E}^{(+)}(t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega \hat{a}(\omega)e^{-i\omega t}$, where $\hat{a}(\omega)$ is the annihilation operator for the frequency ω in the detection filed. By considering the relation of $\hat{a}(\omega) = \frac{1}{\sqrt{2}}[\hat{a}_1(\omega)e^{-i\omega\tau} + \hat{a}_2(\omega)] = \frac{1}{2}\hat{a}_s(\omega)(e^{-i\omega\tau}+1)$, where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for path 1 and path 2 respectively, the detection filed can be rewritten as

$$\hat{E}^{(+)}(t) = \frac{1}{2\sqrt{2\pi}}\int_0^\infty d\omega \hat{a}_s(\omega)(e^{-i\omega\tau}+1)e^{-i\omega t}. \tag{S2}$$

The one-photon detection probability $P(\tau)$ is determined by

$$P(\tau) = \int dt \left\langle \psi \left| \hat{E}^{(-)}\hat{E}^{(+)} \right| \psi \right\rangle. \tag{S3}$$

Consider $\hat{E}^{(+)}|\psi\rangle$,

$$\frac{1}{2\sqrt{2\pi}}\int_0^\infty d\omega \hat{a}_s(\omega)(e^{-i\omega\tau}+1)e^{-i\omega t} \times \int_0^\infty d\omega_s f(\omega_s)\hat{a}_s^\dagger(\omega_s)\,|0\rangle = \frac{1}{2\sqrt{2\pi}}\int_0^\infty d\omega e^{-i\omega t} f(\omega)(e^{-i\omega\tau}+1), \tag{S4}$$

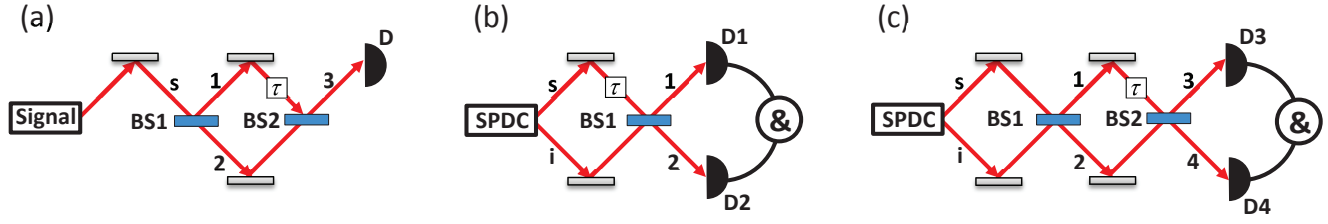

Fig. S1. The setups. (a)Mach-Zehnder (MZ) interference, (b) Hong-Ou-Mandel (HOM) interference, (c) NOON-state interference.

where the relation of $\hat{a}_s(\omega)\hat{a}_s^\dagger(\omega_s) - \hat{a}_s^\dagger(\omega_s)\hat{a}_s(\omega) = \delta(\omega - \omega_s)$ is used. So,

$$\begin{aligned} \left\langle \psi \left| \hat{E}^{(-)}\hat{E}^{(+)} \right| \psi \right\rangle &= \tfrac{1}{8\pi}\int_0^\infty d\omega e^{-i\omega t} f(\omega)(e^{-i\omega\tau}+1) \times \int_0^\infty d\omega' e^{i\omega' t} f^*(\omega')(e^{i\omega'\tau}+1) \\ &= \tfrac{1}{8\pi}\int_0^\infty \int_0^\infty d\omega d\omega' f(\omega) f^*(\omega')(e^{-i\omega\tau}+1)(e^{i\omega'\tau}+1)e^{-i(\omega-\omega')t}. \end{aligned} \tag{S5}$$

where f^* is the complex conjugate of f . Finally,

$$\begin{aligned} P(\tau) &= \int dt \left\langle \psi \left| \hat{E}^{(-)}\hat{E}^{(+)} \right| \psi \right\rangle \\ &= \tfrac{1}{4}\int_0^\infty \int_0^\infty d\omega d\omega' f(\omega) f^*(\omega')(e^{-i\omega\tau}+1)(e^{i\omega'\tau}+1)\delta(\omega-\omega') \\ &= \tfrac{1}{4}\int_0^\infty d\omega f(\omega) f^*(\omega) \left|(e^{-i\omega\tau}+1)\right|^2 \\ &= \tfrac{1}{2}\int_0^\infty d\omega \left|f(\omega)\right|^2 [1+\cos(\omega\tau)]. \end{aligned} \tag{S6}$$

In this calculation, the relation of $\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt$ is used. For a normalized $f(\omega)$, i.e. $\int_0^\infty d\omega \left|f(\omega)\right|^2 = 1$,

$$P(\tau) = \frac{1}{2}[1 + \int_0^\infty d\omega \left|f(\omega)\right|^2 \cos(\omega\tau)]. \tag{S7}$$

After omitting the constant component ("direct current" component) and the coefficients, we can define the first-order correlation function as

$$G_1(\tau) \equiv \int_0^\infty d\omega \left|f(\omega)\right|^2 e^{-i\omega\tau}, \tag{S8}$$

where $P(\tau) = \frac{1}{2}[1 + Re\{G_1(\tau)\}]$. This definition is consistent with the definition in Eq. (3.3.9) on Page 94 of Book by R. Loudon [*The Quantum Theory of Light*, 3ed, *Oxford*, (2000)]. The inverse Fourier transform of $G_1(\tau)$ is

$$F_1(\omega) \equiv \left|f(\omega)\right|^2 = \frac{1}{2\pi}\int_0^\infty d\tau G_1(\tau) e^{i\omega\tau} \tag{S9}$$

This is the traditional WKT, which express power spectrum in terms of autocorrelation function by Fourier transform. Therefore, we can extract the frequency information of the photon source from the time-domain MZ interference pattern.

2. THE EXTENDED WIENER-KHINCHIN THEOREM FOR HONG-OU-MANDEL INTERFERENCE

In Section 2, we deduce the equations for the Hong-Ou-Mandel (HOM) interference using multi-mode theory. Based on this theory, we can construct the extend WKT (e-WKT) for differential frequency. The setup of the HOM interference is shown in Fig. S1(b). The two-photon state from a spontaneous parametric down-conversion (SPDC) process can be described as

$$\left|\psi\right\rangle = \int_0^\infty \int_0^\infty d\omega_s d\omega_i f(\omega_s, \omega_i)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\left|00\right\rangle, \tag{S10}$$

where ω is the angular frequency; $\hat{a}^\dagger$ is the creation operator and the subscripts s and i denote the signal and idler photons from SPDC, respectively; $f(\omega_s, \omega_i)$ is the two-photon spectral amplitude (also called joint spectral amplitude) of the signal and idler photons.

The detection field operators of detector 1 (D1) and detector 2 (D2) are $\hat{E}_1^{(+)}(t_1) = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega_1 \hat{a}_1(\omega_1)e^{-i\omega_1 t_1}$ and $\hat{E}_2^{(+)}(t_2) = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega_2 \hat{a}_2(\omega_2)e^{-i\omega_2 t_2}$, where the subscripts 1 and 2 denote the photons detected by D1 and D2 respectively. The transformation rule of the 50/50 beamsplitter (BS) after a delay time τ is $\hat{a}_1(\omega_1) = \frac{1}{\sqrt{2}}[\hat{a}_s(\omega_1) + \hat{a}_i(\omega_1)e^{-i\omega_1\tau}]$ and $\hat{a}_2(\omega_2) = \frac{1}{\sqrt{2}}[\hat{a}_s(\omega_2) - \hat{a}_i(\omega_2)e^{-i\omega_2\tau}]$. So, we can rewrite the field operators as

$$\hat{E}_1^{(+)}(t_1) = \tfrac{1}{\sqrt{4\pi}}\int_0^\infty d\omega_1[\hat{a}_s(\omega_1)e^{-i\omega_1 t_1} + \hat{a}_i(\omega_1)e^{-i\omega_1(t_1+\tau)}], \tag{S11}$$

and

$$\hat{E}_2^{(+)}(t_2) = \tfrac{1}{\sqrt{4\pi}}\int_0^\infty d\omega_2[\hat{a}_s(\omega_2)e^{-i\omega_2 t_2} - \hat{a}_i(\omega_2)e^{-i\omega_2(t_2+\tau)}]. \tag{S12}$$

The two-photon detection probability $P(\tau)$ can be expressed as

$$P(\tau) = \int\int dt_1 dt_2 \left\langle \psi \left| \hat{E}_1^{(-)}\hat{E}_2^{(-)}\hat{E}_2^{(+)}\hat{E}_1^{(+)} \right| \psi \right\rangle. \tag{S13}$$

Consider $\hat{E}_2^{(+)}\hat{E}_1^{(+)}\left|\psi\right\rangle$, only 2 out of 4 terms exist. The first term is

$$\begin{aligned} &-\tfrac{1}{4\pi}\int_0^\infty \int_0^\infty d\omega_1 d\omega_2 \hat{a}_s(\omega_1)\hat{a}_i(\omega_2)e^{-i\omega_1 t_1}e^{-i\omega_2(t_2+\tau)} \int_0^\infty \int_0^\infty d\omega_s d\omega_i f(\omega_s,\omega_i)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\left|00\right\rangle \\ &= -\tfrac{1}{4\pi}\int_0^\infty \int_0^\infty d\omega_1 d\omega_2 f(\omega_1,\omega_2)e^{-i\omega_1 t_1}e^{-i\omega_2(t_2+\tau)}\left|00\right\rangle. \end{aligned} \tag{S14}$$

In the above calculation, the equations of $\hat{a}_s(\omega_1)\hat{a}_s^\dagger(\omega_s)\left|0\right\rangle = \delta(\omega_1-\omega_s)\left|0\right\rangle$ and $\hat{a}_i(\omega_2)\hat{a}_i^\dagger(\omega_i)\left|0\right\rangle = \delta(\omega_2-\omega_i)\left|0\right\rangle$ are used.

The second term is

$$\begin{aligned}&\tfrac{1}{4\pi}\int_0^\infty\int_0^\infty d\omega_1 d\omega_2 \hat{a}_i(\omega_1)\hat{a}_s(\omega_2)e^{-i\omega_1(t_1+\tau)}e^{-i\omega_2 t_2}\int_0^\infty\int_0^\infty d\omega_s d\omega_i f(\omega_s,\omega_i)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\left|00\right\rangle\\&=\tfrac{1}{4\pi}\int_0^\infty\int_0^\infty d\omega_1 d\omega_2 f(\omega_2,\omega_1)e^{-i\omega_1(t_1+\tau)}e^{-i\omega_2 t_2}\left|00\right\rangle.\end{aligned}\tag{S15}$$

Combine these two terms:

$$\hat{E}_2^{(+)}\hat{E}_1^{(+)}\left|\psi\right\rangle = \tfrac{1}{4\pi}\int_0^\infty\int_0^\infty d\omega_1 d\omega_2 e^{-i\omega_1 t_1}e^{-i\omega_2 t_2}[f(\omega_2,\omega_1)e^{-i\omega_1\tau}-f(\omega_1,\omega_2)e^{-i\omega_2\tau}]\left|00\right\rangle.\tag{S16}$$

Then,

$$\begin{aligned}\left\langle\psi\left|\hat{E}_1^{(-)}\hat{E}_2^{(-)}\hat{E}_2^{(+)}\hat{E}_1^{(+)}\right|\psi\right\rangle &= (\tfrac{1}{4\pi})^2\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2' e^{-i(\omega_1-\omega_1')t_1}e^{-i(\omega_2-\omega_2')t_2}\\&\quad\times[f^*(\omega_2',\omega_1')e^{i\omega_1'\tau}-f^*(\omega_1',\omega_2')e^{i\omega_2'\tau}][f(\omega_2,\omega_1)e^{-i\omega_1\tau}-f(\omega_1,\omega_2)e^{-i\omega_2\tau}].\end{aligned}\tag{S17}$$

Finally,

$$\begin{aligned}P(\tau) &= \int\int dt_1 dt_2\left\langle\psi\left|\hat{E}_1^{(-)}\hat{E}_2^{(-)}\hat{E}_2^{(+)}\hat{E}_1^{(+)}\right|\psi\right\rangle\\&=\tfrac{1}{4}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2'\delta(\omega_1-\omega_1')\delta(\omega_2-\omega_2')\\&\quad\times[f^*(\omega_2',\omega_1')e^{i\omega_1'\tau}-f^*(\omega_1',\omega_2')e^{i\omega_2'\tau}][f(\omega_2,\omega_1)e^{-i\omega_1\tau}-f(\omega_1,\omega_2)e^{-i\omega_2\tau}]\\&=\tfrac{1}{4}\int_0^\infty\int_0^\infty d\omega_1 d\omega_2|[f(\omega_1,\omega_2)-f(\omega_2,\omega_1)e^{-i(\omega_1-\omega_2)\tau}]|^2.\end{aligned}\tag{S18}$$

If we assume $f^*=f$, i.e. f is real, we can further simplify the equation to be

$$P(\tau) = \tfrac{1}{4}\int_0^\infty\int_0^\infty d\omega_1 d\omega_2[|f(\omega_1,\omega_2)|^2+|f(\omega_2,\omega_1)|^2-2f(\omega_1,\omega_2)f(\omega_2,\omega_1)\cos(\omega_1-\omega_2)\tau].\tag{S19}$$

For a normalized $f(\omega_1,\omega_2)$, i.e. $\int_0^\infty\int_0^\infty d\omega_1 d\omega_2|f(\omega_1,\omega_2)|^2=1$,

$$P(\tau) = \tfrac{1}{2}[1-\int_0^\infty\int_0^\infty d\omega_1 d\omega_2 f(\omega_1,\omega_2)f(\omega_2,\omega_1)\cos(\omega_1-\omega_2)\tau].\tag{S20}$$

If we assume $f(\omega_1,\omega_2)$ has the exchange symmetry of $f(\omega_1,\omega_2)=f(\omega_2,\omega_1)$, we can further simplify the equation as

$$P(\tau) = \tfrac{1}{2}[1-\int_0^\infty\int_0^\infty d\omega_1 d\omega_2|f(\omega_1,\omega_2)|^2\cos(\omega_1-\omega_2)\tau].\tag{S21}$$

In order to introduce less variables, Eq. (S21) can be rewritten as

$$P(\tau) = \tfrac{1}{2}[1-\int_0^\infty\int_0^\infty d\omega_s d\omega_i|f(\omega_s,\omega_i)|^2\cos(\omega_s-\omega_i)\tau].\tag{S22}$$

Next, we introduce new parameters $\omega_+=(\omega_s+\omega_i)$ and $\omega_-=(\omega_s-\omega_i)$. So, $\omega_s=\frac{1}{2}(\omega_++\omega_-)$ and $\omega_i=\frac{1}{2}(\omega_+-\omega_-)$, and $P(\tau)$ can be rewritten as

$$P(\tau) = \tfrac{1}{2}[1-\tfrac{1}{2}\int_0^\infty\int_{-\infty}^\infty d\omega_+ d\omega_-|f(\omega_s,\omega_i)|^2\cos(\omega_-\tau)] = \tfrac{1}{2}[1-\int_{-\infty}^\infty d\omega_- F_2(\omega_-)\cos(\omega_-\tau)],\tag{S23}$$

where

$$F_2(\omega_-)\equiv\frac{1}{2}\int_0^\infty d\omega_+|f(\omega_s,\omega_i)|^2\tag{S24}$$

is the projection of $|f(\omega_s,\omega_i)|^2$ onto diagonal axis. For a normalized $f(\omega_s,\omega_i)$, $F_2(\omega_-)$ is also normalized, i.e. $\int_{-\infty}^\infty F_2(\omega_-)d\omega_-=1$, Note, the following rule is used in the change of variables in the double integral,

$$\int\int f(\omega_s,\omega_i)d\omega_s d\omega_i=\int\int f(\omega_s(\omega_+,\omega_-),\omega_i(\omega_+,\omega_-))\left|\frac{\partial(\omega_s,\omega_i)}{\partial(\omega_+,\omega_-)}\right|d\omega_+ d\omega_-,\tag{S25}$$

which can be further simplified as $\int\int f(\omega_s,\omega_i)d\omega_s d\omega_i=\frac{1}{2}\int\int f(\omega_s,\omega_i)d\omega_+ d\omega_-$, because

$$\frac{\partial(\omega_s,\omega_i)}{\partial(\omega_+,\omega_-)}=\begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{1}{2}\tag{S26}$$

So,

$$P(\tau) = \tfrac{1}{2}[1-\int_{-\infty}^\infty d\omega_- F_2(\omega_-)\cos(\omega_-\tau)].\tag{S27}$$

After omitting the constant component ("direct current" component) and the coefficients, we can define the second order correlation function $G_2(\tau)$ in the HOM interference.

$$G_2(\tau)\equiv\int_{-\infty}^\infty d\omega_- F_2(\omega_-)e^{-i\omega_-\tau},\tag{S28}$$

where, $P(\tau)=\frac{1}{2}[1-Re\{G_2(\tau)\}$. The inverse Fourier transform of $G_2(\tau)$ is

$$F_2(\omega_-) = \tfrac{1}{2\pi}\int_{-\infty}^\infty d\tau G_2(\tau)e^{i\omega_-\tau}\tag{S29}$$

This is the extended Wiener-Khinchin theorem (e-WKT) for the HOM interference, which can provide the differential frequency information of the photon source from the time-domain HOM interference patterns.

3. THE EXTENDED WIENER-KHINCHIN THEOREM FOR NOON-STATE INTERFERENCE

In Section 3, we deduce the equations for the NOON-state interference using multi-mode theory. Based on this theory, we can construct the extended Wiener-Khinchin theorem (e-WKT) for sum frequency interference. The setup of the NOON-state interference is shown in Fig. S1(c).

Assume we have the same input state as described in Section 2. The two-photon state from a spontaneous parametric down-conversion (SPDC) process can be described as

$$|\psi\rangle = \int_0^\infty \int_0^\infty d\omega_s d\omega_i f(\omega_s,\omega_i)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\,|00\rangle\,, \tag{S30}$$

where ω is the angular frequency; $\hat{a}^\dagger$ is the creation operator and the subscripts s and i denote the signal and idler photons from SPDC, respectively; $f(\omega_s,\omega_i)$ is two-photon spectral amplitude (also called the join spectral amplitude) of the signal and idler photons.

The detection field operators of detector 3 (D3) and detector 4 (D4) are $\hat{E}_3^{(+)}(t_3) = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega_3\hat{a}_3(\omega_3)e^{-i\omega_3 t_3}$ and $\hat{E}_4^{(+)}(t_4) = \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega_4\hat{a}_4(\omega_4)e^{-i\omega_4 t_4}$, where the subscripts 3 and 4 denote the photons detected by D3 and D4 respectively. The transformation rule of the second 50/50 beamsplitter (BS2) after a delay time τ is $\hat{a}_3(\omega_3) = \frac{1}{\sqrt{2}}[\hat{a}_1(\omega_3)e^{-i\omega_3\tau} + \hat{a}_2(\omega_3)]$ and $\hat{a}_4(\omega_4) = \frac{1}{\sqrt{2}}[\hat{a}_1(\omega_4)e^{-i\omega_4\tau} - \hat{a}_2(\omega_4)]$. The transformation rule of the first 50/50 beamsplitter (BS1) is $\hat{a}_1(\omega_3) = \frac{1}{\sqrt{2}}[\hat{a}_s(\omega_3) + \hat{a}_i(\omega_3)]$, $\hat{a}_2(\omega_3) = \frac{1}{\sqrt{2}}[\hat{a}_s(\omega_3) - \hat{a}_i(\omega_3)]$, $\hat{a}_1(\omega_4) = \frac{1}{\sqrt{2}}[\hat{a}_s(\omega_4) + \hat{a}_i(\omega_4)]$ and $\hat{a}_2(\omega_4) = \frac{1}{\sqrt{2}}[\hat{a}_s(\omega_4) - \hat{a}_i(\omega_4)]$. So, we can rewrite the detection field operators as

$$\hat{E}_3^{(+)}(t_3) = \tfrac{1}{2\sqrt{2\pi}}\int_0^\infty d\omega_3[\hat{a}_s(\omega_3)(e^{-i\omega_3\tau}+1)e^{-i\omega_3 t_3} + \hat{a}_i(\omega_3)(e^{-i\omega_3\tau}-1)e^{-i\omega_3 t_3}] \tag{S31}$$

and

$$\hat{E}_4^{(+)}(t_4) = \tfrac{1}{2\sqrt{2\pi}}\int_0^\infty d\omega_4[\hat{a}_s(\omega_4)(e^{-i\omega_4\tau}-1)e^{-i\omega_4 t_4} + \hat{a}_i(\omega_4)(e^{-i\omega_4\tau}+1)e^{-i\omega_4 t_4}]. \tag{S32}$$

The two-photon detection probability $P(\tau)$ can be expressed as

$$P(\tau) = \int\int dt_3 dt_4 \left\langle \psi \left| \hat{E}_3^{(-)}\hat{E}_4^{(-)}\hat{E}_4^{(+)}\hat{E}_3^{(+)} \right| \psi \right\rangle. \tag{S33}$$

Consider $\hat{E}_4^{(+)}\hat{E}_3^{(+)}|\psi\rangle$, only 2 out of 4 terms exist. The first term is

$$\begin{aligned} &\tfrac{1}{8\pi}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 \hat{a}_s(\omega_3)(e^{-i\omega_3\tau}+1)e^{-i\omega_3 t_3}\times\hat{a}_i(\omega_4)(e^{-i\omega_4\tau}+1)e^{-i\omega_4 t_4}\times\int_0^\infty\int_0^\infty d\omega_s d\omega_i f(\omega_s,\omega_i)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\,|00\rangle \\ &= \tfrac{1}{8\pi}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 e^{-i\omega_3 t_3}e^{-i\omega_4 t_4}f(\omega_3,\omega_4)(e^{-i\omega_3\tau}+1)(e^{-i\omega_4\tau}+1)\,|00\rangle\,. \end{aligned} \tag{S34}$$

In the above calculation, the equation of $\hat{a}(\omega)\hat{a}^\dagger(\omega')|0\rangle = \delta(\omega-\omega')|0\rangle$ is used. The second term is

$$\begin{aligned} &\tfrac{1}{8\pi}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 \hat{a}_i(\omega_3)(e^{-i\omega_3\tau}-1)e^{-i\omega_3 t_3}\hat{a}_s(\omega_4)(e^{-i\omega_4\tau}-1)e^{-i\omega_3 t_3}\times\int_0^\infty\int_0^\infty d\omega_s d\omega_i f(\omega_s,\omega_i)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\,|00\rangle \\ &= \tfrac{1}{8\pi}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 f(\omega_4,\omega_3)(e^{-i\omega_3\tau}-1)(e^{-i\omega_4\tau}-1)e^{-i\omega_3 t_3}e^{-i\omega_3 t_3}\,|00\rangle\,. \end{aligned} \tag{S35}$$

Combine these two terms:

$$\begin{aligned} &\hat{E}_4^{(+)}\hat{E}_3^{(+)}|\psi\rangle \\ &= \tfrac{1}{4}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 e^{-i\omega_3 t_3}e^{-i\omega_4 t_4}[f(\omega_3,\omega_4)(e^{-i\omega_3\tau}+1)(e^{-i\omega_4\tau}+1) + f(\omega_4,\omega_3)(e^{-i\omega_3\tau}-1)(e^{-i\omega_4\tau}-1)]\,|00\rangle\,. \end{aligned} \tag{S36}$$

Then,

$$\begin{aligned} &\left\langle \psi \left| \hat{E}_3^{(-)}\hat{E}_4^{(-)}\hat{E}_4^{(+)}\hat{E}_3^{(+)} \right| \psi \right\rangle \\ &= \tfrac{1}{8\pi}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 e^{-i\omega_3 t_3}e^{-i\omega_4 t_4}[f(\omega_3,\omega_4)(e^{-i\omega_3\tau}+1)(e^{-i\omega_4\tau}+1) + f(\omega_4,\omega_3)(e^{-i\omega_3\tau}-1)(e^{-i\omega_4\tau}-1)] \\ &\times\tfrac{1}{8\pi}\int_0^\infty\int_0^\infty d\omega_3' d\omega_4' e^{i\omega_3' t_3}e^{i\omega_4' t_4}[f^*(\omega_3',\omega_4')(e^{i\omega_3'\tau}+1)(e^{i\omega_4'\tau}+1) + f^*(\omega_4',\omega_3')(e^{i\omega_3'\tau}-1)(e^{i\omega_4'\tau}-1)]. \end{aligned} \tag{S37}$$

Finally,

$$\begin{aligned} P(\tau) &= \int\int dt_3 dt_4 \left\langle \psi \left| \hat{E}_3^{(-)}\hat{E}_4^{(-)}\hat{E}_4^{(+)}\hat{E}_3^{(+)} \right| \psi \right\rangle \\ &= \tfrac{1}{16}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 d\omega_3' d\omega_4'\delta(\omega_3-\omega_3')\delta(\omega_4-\omega_4') \\ &\times [f(\omega_3,\omega_4)(e^{-i\omega_3\tau}+1)(e^{-i\omega_4\tau}+1) + f(\omega_4,\omega_3)(e^{-i\omega_3\tau}-1)(e^{-i\omega_4\tau}-1)] \\ &\times [f^*(\omega_3',\omega_4')(e^{i\omega_3'\tau}+1)(e^{i\omega_4'\tau}+1) + f^*(\omega_4',\omega_3')(e^{i\omega_3'\tau}-1)(e^{i\omega_4'\tau}-1)] \\ &= \tfrac{1}{16}\int_0^\infty\int_0^\infty d\omega_3 d\omega_4 \left|[f(\omega_3,\omega_4)(e^{-i\omega_3\tau}+1)(e^{-i\omega_4\tau}+1) + f(\omega_4,\omega_3)(e^{-i\omega_3\tau}-1)(e^{-i\omega_4\tau}-1)]\right|^2. \end{aligned} \tag{S38}$$

In the above calculation, the relation of $\delta(\omega-\omega')=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\omega-\omega')t}dt$ is used; f^* is the complex conjugate of f. Assuming f has the symmetry of $f(\omega_3,\omega_4)$=$f(\omega_4,\omega_3)$, $P(\tau)$ can be further simplified as

$$P(\tau)=\frac{1}{2}\int_0^{\infty}\int_0^{\infty}d\omega_3 d\omega_4|f(\omega_3,\omega_4)|^2[\cos(\omega_3+\omega_4)\tau+1]. \tag{S39}$$

For a normalized $f(\omega_3,\omega_4)$, i.e. $\int_0^{\infty}\int_0^{\infty}d\omega_3 d\omega_4|f(\omega_3,\omega_4)|^2=1$,

$$P(\tau)=\frac{1}{2}[1+\int_0^{\infty}\int_0^{\infty}d\omega_3 d\omega_4|f(\omega_3,\omega_4)|^2\cos(\omega_3+\omega_4)\tau]. \tag{S40}$$

In order to introduce less variables, Eq. (S40) can be rewritten as

$$P(\tau)=\frac{1}{2}[1+\int_0^{\infty}\int_0^{\infty}d\omega_s d\omega_i|f(\omega_s,\omega_i)|^2\cos(\omega_s+\omega_i)\tau] \tag{S41}$$

Next, we use the parameters $\omega_+=(\omega_s+\omega_i)$ and $\omega_-=(\omega_s-\omega_i)$, which are similar as in the case of HOM interference. So, $\omega_s=\frac{1}{2}(\omega_++\omega_-)$ and $\omega_i=\frac{1}{2}(\omega_+-\omega_-)$, and $P(\tau)$ can be rewritten as

$$P(\tau)=\frac{1}{2}[1+\frac{1}{2}\int_0^{\infty}\int_{-\infty}^{\infty}d\omega_+ d\omega_-|f(\omega_s,\omega_i)|^2\cos(\omega_+\tau)]=\frac{1}{2}[1+\int_{-\infty}^{\infty}d\omega_+ F_2(\omega_+)\cos(\omega_+\tau)]. \tag{S42}$$

where,

$$F_2(\omega_+)\equiv\frac{1}{2}\int_{-\infty}^{\infty}d\omega_-|f(\omega_s,\omega_i)|^2 \tag{S43}$$

is the projection of $|f(\omega_s,\omega_i)|^2$ on to the anti-diagonal axis. So,

$$P(\tau)=\tfrac{1}{2}[1+\textstyle\int_0^{\infty}d\omega_+ F_2(\omega_+)\cos(\omega_+\tau)]. \tag{S44}$$

Omitting the constant component ("direct current" component) and the coefficients, we can define the second-order correlation function $G_2(\tau)$ in the NOON-state interference

$$G_2(\tau)\equiv\textstyle\int_0^{\infty}d\omega_+ F_2(\omega_+)e^{-i\omega_+\tau}. \tag{S45}$$

where, $P_2(\tau)=\frac{1}{2}[1+Re\{G_2(\tau)\}]$. The inverse Fourier transform of $G_2(\tau)$ is

$$F_2(\omega_+)=\tfrac{1}{2\pi}\textstyle\int_0^{\infty}d\tau G_2(\tau)e^{i\omega_+\tau}. \tag{S46}$$

This is the extended Wiener-Khinchin theorem for the NOON-state interference, which can provide the sum frequency information of the photon source from the time-domain NOON-state interference patterns.

4. SUMMARY OF THE WKT AND THE E-WKT

For simplicity in the deduction of these equations in Sections 1-3, the notifications in each section were "local variables", which is valid in each section. However, for comparison, here we slightly revise the notifications and summarize these equations using "global variables" as follow.

A. The WKT

The input state in MZ interference is

$$|\psi_1\rangle=\int_{-\infty}^{\infty}d\omega f_1(\omega)\hat{a}^{\dagger}(\omega)\,|0\rangle\,, \tag{S47}$$

where $f_1(\omega)$ is one-photon spectral amplitude. In order to keep the uniformity, here the integration range is enlarged from $[0,\infty]$ to $[-\infty,\infty]$.

The one-photon detection probability is

$$P_1(\tau)=\frac{1}{2}[1+\int_{-\infty}^{\infty}d\omega\,|f_1(\omega)|^2\cos(\omega\tau)]. \tag{S48}$$

The first order correlation function is

$$G_1(\tau)\equiv\int_{-\infty}^{\infty}d\omega\,|f_1(\omega)|^2\,e^{-i\omega\tau}. \tag{S49}$$

The connection between $P_1(\tau)$ and $G_1(\tau)$ is

$$P(\tau)=\frac{1}{2}[1+Re\{G_1(\tau)\}]. \tag{S50}$$

The conventional WKT is

$$F_1(\omega)\equiv|f_1(\omega)|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau G_1(\tau)e^{i\omega\tau}. \tag{S51}$$

B. The e-WKT

The input state in HOM interference and NOON-state interference is

$$|\psi_2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_s d\omega_i f_2(\omega_s,\omega_i)\hat{a}_s^{\dagger}(\omega_s)\hat{a}_i^{\dagger}(\omega_i)\,|00\rangle\,, \tag{S52}$$

where $f_2(\omega_s,\omega_i)$ is the two-photon spectral amplitude.

The two-photon detection probability $P_2^{\pm}(\tau)$ is

$$P_2^{\pm}(\tau) \quad = \quad \tfrac{1}{2}[1 \pm \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_s d\omega_i |f_2(\omega_s,\omega_i)|^2 \cos(\omega_s \pm \omega_i)\tau], \tag{S53}$$

where P_2^{+} is for NOON-state interference, while P_2^{-} is for HOM interference.

After the transformation of variables: $\omega_{\pm} = \omega_s \pm \omega_i$,

$$P_2^{\pm}(\tau) \quad = \quad \tfrac{1}{2}[1 \pm \tfrac{1}{2}\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_+ d\omega_- |f_2(\omega_s,\omega_i)|^2 \cos(\omega_{\pm}\tau)]. \tag{S54}$$

Using the definition of the sum- or difference-frequency spectrum of the two-photon state, i.e.,

$$F_2^{\pm}(\omega_{\pm}) \equiv \frac{1}{2}\int_{-\infty}^{\infty} d\omega_{\mp} |f_2(\omega_s,\omega_i)|^2 \tag{S55}$$

$P_2^{\pm}(\tau)$ can be further simplified as

$$P_2^{\pm}(\tau) \quad = \quad \tfrac{1}{2}[1 \pm \textstyle\int_{-\infty}^{\infty} d\omega_{\pm} F_2^{\pm}(\omega_{\pm}) \cos(\omega_{\pm}\tau)]. \tag{S56}$$

Omitting the constant component ("direct current" component) and the coefficients, we can define the second-order correlation function $G_2^{\pm}(\tau)$ as

$$G_2^{\pm}(\tau) \equiv \int_{-\infty}^{\infty} d\omega_{\pm} F_2(\omega_{\pm}) e^{-i\omega_{\pm}\tau}. \tag{S57}$$

The connection between $P_2^{\pm}$ and $G_2^{\pm}$ is,

$$P_2^{\pm}(\tau) = \frac{1}{2}[1 \pm Re\{G_2^{\pm}(\tau)\}]. \tag{S58}$$

G_2^{+} is for NOON-state interference, while G_2^{-} is for HOM interference. The inverse Fourier transform of $G_2^{\pm}(\tau)$ is

$$F_2^{\pm}(\omega_{\pm}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau G_2^{\pm}(\tau) e^{i\omega_{\pm}\tau}. \tag{S59}$$

This is the unified form of e-WKT.