Supplemental document accompanying submission to *Optica*

**Title:** Frequency-modulated comb in a VECSEL

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**Submitted:** 18.12.2020 09:25:54
Signatures of a frequency-modulated comb in a VECSEL: supplemental document

1. SAMPLE STRUCTURE

![Sample structure diagram]

Fig. S1. Sample structure and normalized standing electric field intensity at 303.65 THz (987.3 nm).

2. DISPERSION MEASUREMENT

![Dispersion measurement graphs]

Fig. S2. (a) Calculated (blue solid line) and measured (black dashed line) reflectivity spectrum of the studied VECSEL chip. Additionally, the calculated longitudinal confinement factor (LCF) of the chip’s microcavity is shown (red solid line). (b) Exemplary white-light interferogram of the probed VECSEL chip. (c) Measured (blue dots) and calculated (red line) group delay dispersion (GDD).

For the measurement of the group delay dispersion (GDD) of the unpumped VECSEL chip, a home-built white-light interferometer was used, similar to the ones in Refs. [1] and [2]. A He-Ne
laser was used as position reference of the moving mirror, mounted on a mechanical shaker, and spatially overlapped with the white-light from an halogen lamp. After the interferometer, the He-Ne laser and the white light are separated by appropriate dichroic mirrors and recorded each with a photodiode. A fast digitizer card records around 1000 interferograms while the shaker is oscillating. An exemplary interferogram is shown in Fig. S2(b). The mirror position has been obtained by taking the zero-crossings of the He-Ne laser interferogram as position reference. The GDD is then obtained by Fourier-transforming it with respect to the delay $\tau = \Delta x / c$, with $\Delta x$ the mirror position and $c$ the speed of light. The phase of the obtained spectrum is then twice numerically derived with respect to the angular frequency $\omega$ which provides the GDD. For averaging, multiple interferograms are interpolated over the same mirror position/delay grid before Fourier-transforming each of them and subsequently averaging the calculated GDD. Finally, the GDD of the balanced interferometer with two metal mirrors (and the sample removed) is recorded to account for the interferometer’s transfer function. Figure S2(c) shows the measured dispersion of the sample obtained by averaging over 997 interferograms (blue dots) and shows good correspondence to the calculated dispersion. The calculated dispersion has been obtained by calculating the complex reflectivity spectrum of the sample structure with the transfer matrix formalism detailed in [3] and obtaining the second derivative of the electric field’s phase with respect to $\omega$. The calculated and measured reflectivity spectra show good correspondence (see Fig. S2(a)). Also shown in Fig. S2(a) is the calculated longitudinal confinement factor (LCF) that characterizes the overlap of the intensity of the electric field with the quantum wells (see Ref. [3]). It is at its minimum in the spectral region where the laser operates, which yields the desired flat and low dispersion, but reduces the achievable maximum output power of the sample.

3. SWIFTS MEASUREMENT

To understand the working principle of the SWIFTS measurement (see also Refs. [4, 5]), we write the electric field of the laser emission as

$$E(t) = \sum_n E_n e^{i(\omega_n t + \phi_n)} + c.c., \quad (S1)$$

where $E_n$ is the amplitude, $\phi_n$ the phase and $\omega_n$ the oscillation frequency of an individual mode $n$. When detecting this field with a fast photodetector after a Michelson interferometer, which introduces a delay $\tau$ in one of its arms, the detected signal becomes

$$I(t, \tau) = \frac{1}{2} (E(t) + E(t - \tau))^2 = \frac{1}{2} \sum_{n,m} (E_n E_m e^{i(\omega_n - \omega_m) t + \phi_n - \phi_m})$$

$$+ E_n E_m e^{i((\omega_n - \omega_m)(t-\tau) + \phi_n - \phi_m)} + 2E_n E_m e^{i((\omega_n - \omega_m)t + \omega_m \tau + \phi_n - \phi_m)} + c.c., \quad (S2)$$

where we have omitted any terms with an $e^{i(\omega_n + \omega_m)t}$-dependency, which will not be detected by a photodiode. Equation S2 can be rewritten as

$$I(t, \tau) = \frac{1}{2} \sum_{n,m} E_n E_m e^{i((\omega_n - \omega_m) t + \phi_n - \phi_m)(1 + e^{i(\omega_n - \omega_m)\tau} + 2e^{i\omega_n \tau}) + c.c.. \quad (S3)$$

A slow photodiode will only detect terms with $n = m$, which leads to

$$I(\tau) = \sum_n E_n^2 (1 + e^{i\omega_n \tau}) + c.c.. \quad (S4)$$

This interferogram provides the optical intensity spectrum $I(\omega)$ of the laser when Fourier-transformed.

In SWIFTS, the detected signal in Eq. S3 is mixed with an in-phase ($\cos(\omega_0 t)$) and a quadrature signal ($\sin(\omega_0 t)$) at a frequency $\omega_0$. This operation, which is performed by the lock-in amplifier, leads to two signals,

$$X(t, \tau) = \frac{1}{4} \sum_{n,m} E_n E_m (e^{i((\omega_n - \omega_m - \omega_0) t + \phi_n - \phi_m)} + e^{i((\omega_n - \omega_m + \omega_0) t + \phi_n - \phi_m)}) + c.c., \quad (S5a)$$

$$Y(t, \tau) = \frac{i}{4} \sum_{n,m} E_n E_m (e^{i((\omega_n - \omega_m - \omega_0) t + \phi_n - \phi_m)} - e^{i((\omega_n - \omega_m + \omega_0) t + \phi_n - \phi_m)}) + c.c.. \quad (S5b)$$
Only variations in the order of the time constant of the lock-in amplifier will be detected, which means that only \( \omega_0 = \omega_n - \omega_m \) will result in a non-zero signal. For the measurement of the phase and coherence of two adjacent modes, \( \omega_0 = \omega_{n+1} - \omega_m \) is chosen. Note, however, that the time constant does not set the condition for the equidistance of two laser lines but just blocks lines other than nearest-neighbor in the lock-in detection. The precision of the equidistance assessment is instead determined by the inverse of the total measurement time of the interferogram (around 1 minute in our case) as pointed out in Ref. [5]. Remarkably, this means that the equidistance measurement exhibits sub-Hz precision. In the self-referenced scheme, as done in this work, the frequency \( \omega_0 \) of the reference signals is directly obtained from the fundamental beatnote of the laser, measured before the Michelson interferometer with another fast photodiode. Note that, as in our case the bandwidth of the lock-in amplifier is lower (200 MHz) than the fundamental repetition rate of the laser (1.6 GHz), we use a local oscillator and RF mixers to down-convert the signals from the fast photodiodes.

After the lock-in detection, Eqs. S5 thus become

\[
X(\tau) = \frac{1}{4} \sum_{m} E_{m+1} E_m e^{i(\varphi_{m+1} - \varphi_m)}(\ldots) + c.c., \quad \text{(S6a)}
\]

\[
Y(\tau) = \frac{i}{4} \sum_{m} E_{m+1} E_m e^{i(\varphi_{m+1} - \varphi_m)}(\ldots) + c.c., \quad \text{(S6b)}
\]

The two SWIFTS interferograms (in-phase and quadrature) can be combined in one complex interferogram,

\[
X(\tau) - iY(\tau) = \frac{1}{2} \sum_{m} E_{m+1} E_m e^{i(\varphi_{m+1} - \varphi_m)}(1 + e^{i\Delta\omega \tau} + 2e^{i\Delta\omega \tau}). \quad \text{(S7)}
\]

One can see that the product of two adjacent mode amplitudes and their phase difference will be resolved over the optical spectrum when the interferograms are Fourier-transformed with respect to the delay \( \tau \). Now it is obvious that \( \text{arg}(X(\omega) - iY(\omega)) \) will provide the intermode phase \( \varphi_{m+1} - \varphi_m \). However, the contribution of two adjacent modes to the SWIFTS spectrum \( |X(\omega) - iY(\omega)| \) will disappear when their phases fluctuate over time scales of the total measurement time. Thus, the comparison of \( |X(\omega) - iY(\omega)| \) with the intensity spectrum \( I(\omega) \) provides a measure of coherence (i.e. phase stability) of the laser.

Equations S4 and S7 (or S6) can be used to show that, if for a phase-locked two-color spectrum the intermode phase difference of both colors amounts to \( \pi \), respectively, with no phase-offset between both colors, the beating of the SWIFTS envelope will be shifted by 180° with respect to the beating envelope of the intensity interferogram. To show this, we assume an idealized spectrum of two lobes of the same width and amplitude, separated by a frequency difference \( \Delta\omega \).

The electric field of such a spectrum can then be written as

\[
E(t) = \sum_{n} E_0 e^{i(\omega_n t + \varphi_n)} + e^{i(\omega_{n+1} t + \varphi_{n+1})} + \ldots + c.c., \quad \text{(S8)}
\]

where \( N \) is the total number of modes (each lobe contains \( N/2 \) modes) and all mode amplitudes have been assumed constant (i.e. \( E_n = E_0 \)). With the above-mentioned assumption, the intermode phase is \( \Delta\varphi_n = \varphi_{n+1} - \varphi_n = \frac{2\pi}{\Delta\omega} n \), which means \( \Delta\varphi_{n+N/2} = \Delta\varphi_n + \pi \).

Consequently, the intensity interferogram in Eq. S4 becomes

\[
I(\tau) = \sum_{n} E_0^2(1 + e^{i\omega_n \tau} + 1 + e^{i(\omega_{n+1} + \Delta\omega) \tau}) + c.c. = \sum_{n} E_0^2(2 + e^{i\omega_n \tau} \left(1 + e^{i\Delta\omega \tau} \right)) + c.c., \quad \text{(S9)}
\]

where the term \( 1 + e^{i\Delta\omega \tau} \) describes the envelope beating of the interferogram.

Now we take the SWIFTS interferogram in Eq. S7 (we use the complex interferogram and complex electric field here for simpler expressions) and apply the assumptions for the electric...
field which leads to

\[ X(\tau) - iY(\tau) = \frac{1}{2} \sum_{n} E_0^2 (e^{i\Delta \phi_n (1 + e^{i\omega_0 \tau} + 2e^{i\omega_n \tau})} + e^{i\Delta \phi_n+N/2 (1 + e^{i\omega_0 \tau} + 2e^{i(\omega_n+\Delta \omega) \tau})}) = \]

\[ \frac{1}{2} \sum_{n} E_0^2 ((e^{i\Delta \phi_n} + e^{i(\Delta \phi_n+\pi)}) (1 + e^{i\omega_0 \tau} + 2e^{i(\omega_n+\Delta \omega) \tau)}) (1 + e^{i(\Delta \omega_0 \tau+\pi)}). \] (S10)

Here, the envelope beating term is \((1 + e^{i(\Delta \omega_0 \tau+\pi)})\), which is phase-shifted by \(\pi\) compared to the beating term of the intensity interferogram. This explains why the envelope beating that we observe in the measured intensity and SWIFTS interferograms is phase-shifted by \(180^\circ\).

4. LONG-SPAN RF SPECTRA AND HIGHER-ORDER TRANSVERSE MODE

![Image](image_url)

**Fig. S3.** Effect of spectral and spatial filtering on the long-span RF spectrum at a pump power of 3.2 W. The laser operation point is the same as in Fig. 4 of the main text, the power in front of the photodiode is adjusted to 2.2 mW in the filtered and unfiltered case. (a) RF spectrum (resolution bandwidth 100 kHz) of unfiltered laser emission for two positions of the photodiode with respect to the optical beam, indicated by the sketch in the upper right corner. The red ellipse visualizes the laser beam and the black circle the photodiode. (b) Intensity (blue solid line) and SWIFTS spectrum (orange dashed line) of unfiltered laser emission. (c) RF spectrum (resolution bandwidth 100 kHz) of filtered laser emission for two positions of the photodiode. (d) Intensity (blue solid line) and SWIFTS spectrum (orange dashed line) of filtered laser emission.

Here, we discuss the effect on the long-span RF spectra of filtering the lower lobe of the laser spectrum and show the high-resolution RF spectra of the beatnote of the transverse mode and its mixing product with the longitudinal mode (in the filtered and unfiltered case). These data are complementary to the data in Fig. 4 of the main text (i.e. taken at the same operation point of the laser at 3.2 W pump power). In Fig. S3(a), the long-span RF spectra of the unfiltered laser emission is shown, once for the case when the photodiode mainly captures the higher-order transverse mode (upper graph) and once when the photodiode is positioned in a way to only capture the contribution of the fundamental longitudinal mode (lower graph). The position of
Fig. S4. High-resolution RF beatnotes (1 kHz resolution bandwidth and 500 kHz span) of the transverse mode at frequency $f_T=0.67$ GHz ((a), (d)), the longitudinal mode at $f_L=1.59$ GHz ((c), (f)) and the mixing product of the two at $f_L-f_T=0.92$ GHz ((b), (d)). The 3-dB bandwidth $\Delta f$ is indicated in each figure with arrows. (a),(b),(c) correspond to the unfiltered case as shown in Fig. S3(a) and (d),(e),(f) correspond to the filtered case as shown in Fig. S3(c).

the photodiode with respect to the TEM$_{01}$ beamprofile is sketched in the corresponding graphs. In the former case, the mixing product at frequency $f_L-f_T=0.92$ GHz dominates the spectrum, in the latter case, no contribution from the higher-order modes can be seen. When the incoherent, low-frequency lobe is suppressed with the short-pass filter, one observes a significant increase of the beatnote, both, for the longitudinal mode and the transverse mode, while the mixing product maintains roughly the same magnitude. Here, the power in front of the photodiode has been adjusted to 2.2 mW in the filtered and unfiltered case to assure comparability. Now, the longitudinal mode exhibits the largest magnitude. Similarly to the unfiltered case, any sign of the higher-order mode disappears when the photodiode is moved to the center of the beam. It is somewhat surprising that also the transverse mode beating increases in magnitude when using the filter. This might be attributed to the fact that it also experiences some phase-locking and anti-phase synchronization (although not as strong as the longitudinal mode). When the spectrum is filtered, the anti-phase synchronization might be partially eliminated and thus the beatnote increases due to increased constructive interference of the intermode beatings. While the SWIFTS measurements did not work with the transverse mode, the width of its beatnote (shown in Fig. S4(a) and(d) for the unfiltered and filtered case, respectively) is 30-40 kHz and thus still narrower than the beatnote of the longitudinal mode when it becomes incoherent for high pump powers (as shown in Fig. 3 of the main text, with a width of hundreds of kHz). Such situation might indicate some weak phase-locking of the transverse mode. Thus, it is difficult to assess whether filtering the low-frequency lobe removes the relative contribution of the higher-order transverse mode from the laser emission just by looking at the relative difference in magnitude of the RF lines. However, the good correspondence of the SWIFTS and optical spectrum for the high-frequency lobe indicates that this is the case, as contributions from the higher-order transverse mode will not be accounted in the SWIFTS measurement that uses the longitudinal mode as reference.
REFERENCES