

# Experimental generalized contextuality with single-photon qubits: supplementary material

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This document provides supplementary information to "Experimental generalized contextuality with single-photon qubits," <https://doi.org/10.1364/optica.4.000966>. We provide the details of the experiment and data analysis. The raw probabilities to demonstrate the joint measurability of positive operator-valued measures (POVMs) are also provided. © 2017 Optical Society of America

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## 1. IMPLEMENTATION OF THE ELEMENTS OF THE JOINT POVM $G$

Each element of the POVM can be written as  $G_{\epsilon\epsilon}^{ij} = \lambda_{\epsilon\epsilon} \left| \xi_{\epsilon\epsilon}^{ij} \right\rangle \left\langle \xi_{\epsilon\epsilon}^{ij} \right|$ , with  $\epsilon, \epsilon \in \{+1, -1\}$ ,  $i \neq j \in \{1, 2, 3\}$ ,

$$\lambda_{++} = \lambda_{--} = \frac{(2 - \eta^2)}{4}, \lambda_{+-} = \lambda_{-+} = \frac{(2 + \eta^2)}{4}, \quad (\text{S1})$$

and

$$\left| \xi_{+-}^{12} \right\rangle = \begin{pmatrix} \sqrt{3}\eta + i\sqrt{4 - 8\eta^2 + \eta^4} \\ -2 + 3\eta - \eta^2 \end{pmatrix} / \sqrt{8 - 12\eta + 8\eta^2 - 6\eta^3 + 2\eta^4},$$

$$\left| \xi_{-+}^{12} \right\rangle = \begin{pmatrix} \sqrt{3}\eta - i\sqrt{4 - 8\eta^2 + \eta^4} \\ 2 + 3\eta + \eta^2 \end{pmatrix} / \sqrt{8 + 12\eta + 8\eta^2 + 6\eta^3 + 2\eta^4},$$

$$\left| \xi_{++}^{12} \right\rangle = \begin{pmatrix} \sqrt{3}\eta + i\sqrt{4 - 8\eta^2 + \eta^4} \\ 2 - \eta - \eta^2 \end{pmatrix} / \sqrt{8 - 4\eta - 8\eta^2 + 2\eta^3 + 2\eta^4},$$

$$\left| \xi_{--}^{12} \right\rangle = \begin{pmatrix} \sqrt{3}\eta - i\sqrt{4 - 8\eta^2 + \eta^4} \\ -2 - \eta + \eta^2 \end{pmatrix} / \sqrt{8 + 4\eta - 8\eta^2 - 2\eta^3 + 2\eta^4},$$

$$\left| \xi_{+-}^{13} \right\rangle = \begin{pmatrix} \sqrt{3}\eta - i\sqrt{4 - 8\eta^2 + \eta^4} \\ 2 - 3\eta + \eta^2 \end{pmatrix} / \sqrt{8 - 12\eta + 8\eta^2 - 6\eta^3 + 2\eta^4},$$

$$\left| \xi_{-+}^{13} \right\rangle = \begin{pmatrix} -\sqrt{3}\eta + i\sqrt{4 - 8\eta^2 + \eta^4} \\ 2 + 3\eta + \eta^2 \end{pmatrix} / \sqrt{8 + 12\eta + 8\eta^2 + 6\eta^3 + 2\eta^4},$$

$$\left| \xi_{++}^{13} \right\rangle = \begin{pmatrix} \sqrt{3}\eta - i\sqrt{4 - 8\eta^2 + \eta^4} \\ -2 + \eta + \eta^2 \end{pmatrix} / \sqrt{8 - 4\eta - 8\eta^2 + 2\eta^3 + 2\eta^4},$$

$$\begin{aligned}
|\xi_{+-}^{13}\rangle &= \begin{pmatrix} -\sqrt{3}\eta + i\sqrt{4-8\eta^2+\eta^4} \\ -2-\eta+\eta^2 \end{pmatrix} / \sqrt{8+4\eta-8\eta^2-2\eta^3+2\eta^4}, \\
|\xi_{+-}^{23}\rangle &= \begin{pmatrix} \sqrt{12}\eta - i\sqrt{4-8\eta^2+\eta^4} \\ 2+\eta^2 \end{pmatrix} / \sqrt{2}(2+\eta^2), \\
|\xi_{-+}^{23}\rangle &= \begin{pmatrix} -\sqrt{12}\eta - i\sqrt{4-8\eta^2+\eta^4} \\ 2+\eta^2 \end{pmatrix} / \sqrt{2}(2+\eta^2), \\
|\xi_{++}^{23}\rangle &= \begin{pmatrix} -i\sqrt{4-4\eta-4\eta^2+2\eta^3+\eta^4} \\ -\sqrt{4+4\eta-4\eta^2-2\eta^3+\eta^4} \end{pmatrix} / \sqrt{2}(-2+\eta^2), \\
|\xi_{--}^{23}\rangle &= \begin{pmatrix} -i\sqrt{4+4\eta-4\eta^2-2\eta^3+\eta^4} \\ -\sqrt{4-4\eta-4\eta^2+2\eta^3+\eta^4} \end{pmatrix} / \sqrt{2}(-2+\eta^2),
\end{aligned}$$

Each element can be implemented by a single-qubit rotation followed by a two-outcome measurement. In the first step, to realize the element  $G_{+-}^{ij}$  the single-qubit rotation is designed as

$$U_{+-} = |0\rangle\langle\phi'_{+-}| + |1\rangle\langle\phi_{+-}^\perp|, |\phi'_{+-}\rangle = |\xi_{+-}^{ij\perp}\rangle. \quad (\text{S2})$$

The POVM element  $\{P_{+-}^0, P_{+-}^1\}$  takes the form

$$P_{+-}^0 = |0\rangle\langle 0| + (1-\chi_{+-})|1\rangle\langle 1|, P_{+-}^1 = \chi_{+-}|1\rangle\langle 1|, \chi_{+-} = \lambda_{+-}. \quad (\text{S3})$$

The choice of  $|\phi'_{+-}\rangle$  guarantees if  $P_{+-}^1$  clicks the initial state is projected onto the eigenstate  $|\xi_{+-}^{ij}\rangle$  and let the component of the state  $|\xi_{+-}^{ij\perp}\rangle$  which is orthogonal to  $|\xi_{+-}^{ij}\rangle$  all pass through for the next measurement. Therefore the state after the first rotation is  $U_{+-}|\phi_0\rangle\langle\phi_0|U_{+-}^\dagger$  for any input  $|\phi_0\rangle$ . The first detector ( $P_{+-}^1$ ) clicks with the probability  $\text{Tr}(P_{+-}^1 U_{+-}|\phi_0\rangle\langle\phi_0|U_{+-}^\dagger) = \lambda_{+-} \langle\xi_{+-}^{ij}|\phi_0\rangle\langle\phi_0|\xi_{+-}^{ij}\rangle = \text{Tr}(G_{+-}^{ij}|\phi_0\rangle\langle\phi_0|)$ . Thus we implement the element of POVM  $G_{+-}^{ij}$ . The other state without click is

$$\sqrt{P_{+-}^0} U_{+-} |\phi_0\rangle = \langle\xi_{+-}^{ij\perp}|\phi_0\rangle |0\rangle + \sqrt{1-\chi_{+-}} \langle\xi_{+-}^{ij}|\phi_0\rangle |1\rangle. \quad (\text{S4})$$

In the second step, to implement the element  $G_{-+}^{ij}$ , the single-qubit rotation is designed as

$$U_{-+} = |0\rangle\langle\phi'_{-+}| + |1\rangle\langle\phi_{-+}^\perp|, \quad (\text{S5})$$

where

$$|\phi'_{-+}\rangle = \frac{1}{\mathcal{N}_{-+}} \left( \langle\xi_{-+}^{ij\perp}|\xi_{-+}^{ij\perp}\rangle |0\rangle + \sqrt{1-\chi_{-+}} \langle\xi_{-+}^{ij}|\xi_{-+}^{ij}\rangle |1\rangle \right) \quad (\text{S6})$$

with the normalization factor  $\mathcal{N}_{-+}$ . Compared Eqs. (S5) and (S7), one can see that we only replace  $|\phi_0\rangle$  in Eq. (S5) by  $|\xi_{-+}^{ij\perp}\rangle$  in Eq. (S7). The second POVM element takes the form  $\{P_{-+}^0, P_{-+}^1\}$ , where

$$P_{-+}^0 = |0\rangle\langle 0| + (1-\chi_{-+})|1\rangle\langle 1|, P_{-+}^1 = \chi_{-+}|1\rangle\langle 1|, \chi_{-+} = \frac{\lambda_{-+}\mathcal{N}_{-+}^2}{1-\lambda_{-+}}. \quad (\text{S7})$$

The parameter  $\chi_{-+}$  is chosen such that after the projector is performed, the probability of the click of  $P_{-+}^1$  is that of the measurement  $G_{-+}^{ij}$  performing on the initial state  $|\phi_0\rangle$ . The state after the second qubit rotation is  $U_{-+}P_{-+}^0 U_{-+}|\phi_0\rangle\langle\phi_0|U_{-+}^\dagger U_{-+}^\dagger$ . The second detector ( $P_{-+}^1$ ) clicks with the probability  $\text{Tr}(P_{-+}^1 U_{-+}P_{-+}^0 U_{-+}|\phi_0\rangle\langle\phi_0|U_{-+}^\dagger U_{-+}^\dagger) = \lambda_{-+} \langle\xi_{-+}^{ij}|\phi_0\rangle\langle\phi_0|\xi_{-+}^{ij}\rangle = \text{Tr}(G_{-+}^{ij}|\phi_0\rangle\langle\phi_0|)$ . Thus we implement the element  $G_{-+}^{ij}$ .

The other state without click is

$$\begin{aligned}
&\sqrt{P_{-+}^0} U_{-+} \sqrt{P_{-+}^0} U_{-+} |\phi_0\rangle \\
&= \left( \langle\phi'_{-+}|0\rangle \langle\xi_{-+}^{ij\perp}|\phi_0\rangle + \sqrt{1-\chi_{-+}} \langle\phi'_{-+}|1\rangle \langle\xi_{-+}^{ij}|\phi_0\rangle \right) |0\rangle \\
&+ \sqrt{1-\chi_{-+}} \left( \langle\phi'_{-+}|0\rangle \langle\xi_{-+}^{ij\perp}|\phi_0\rangle + \sqrt{1-\chi_{-+}} \langle\phi'_{-+}|1\rangle \langle\xi_{-+}^{ij}|\phi_0\rangle \right) |1\rangle.
\end{aligned} \quad (\text{S8})$$

In the third step, to implement  $G_{++}^{ij}$  the single-qubit rotation is designed as

$$U_{++} = |0\rangle \langle \phi'_{++}| + |1\rangle \langle \phi'^{\perp}_{++}|, \quad (\text{S9})$$

where

$$|\phi'_{++}\rangle = \frac{1}{\mathcal{N}_{++}} \left[ (\langle \phi'_{-+}|0\rangle \langle \xi_{+-}^{ij\perp} | \xi_{++}^{ij\perp} \rangle + \sqrt{1-\chi_{+-}} \langle \phi'_{-+}|1\rangle \langle \xi_{+-}^{ij} | \xi_{++}^{ij\perp} \rangle) |0\rangle \right. \\ \left. + \sqrt{1-\chi_{+-}} (\langle \phi'_{-+}|0\rangle \langle \xi_{+-}^{ij\perp} | \xi_{++}^{ij\perp} \rangle + \sqrt{1-\chi_{+-}} \langle \phi'_{-+}|1\rangle \langle \xi_{+-}^{ij} | \xi_{++}^{ij\perp} \rangle) |1\rangle \right] \quad (\text{S10})$$

with the normalization factor  $\mathcal{N}_{++}$ . Comparing Eqs. (S9) and (S11), one can find that we replace  $|\phi_0\rangle$  in Eq. (S9) by  $|\xi_{++}^{ij\perp}\rangle$  in Eq. (S11). The third POVM element takes the form  $\{P_{++}^0, P_{++}^1\}$ , where

$$P_{++}^0 = |0\rangle \langle 0|, P_{++}^1 = |1\rangle \langle 1|. \quad (\text{S11})$$

With this setup when the input state is  $|\xi_{++}^{ij\perp}\rangle$  the detector ( $P_{++}^1$ ) never clicks. Thus the measurement corresponding to click of  $P_{++}^1$  is proportional to  $|\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}|$ . We now prove the measurement we implement is exactly  $G_{++}^{ij}$ .

Assume the POVM elements corresponding to clicks of  $P_{++}^1$  and  $P_{++}^0$  are  $x |\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}|$  and  $y |\psi\rangle \langle \psi|$ . Due to the fact that we have realized  $\lambda_{+-} |\xi_{+-}^{ij}\rangle \langle \xi_{+-}^{ij}|$  and  $\lambda_{-+} |\xi_{-+}^{ij}\rangle \langle \xi_{-+}^{ij}|$ , we have

$$x |\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}| + y |\psi\rangle \langle \psi| = \mathbb{1} - \lambda_{+-} |\xi_{+-}^{ij}\rangle \langle \xi_{+-}^{ij}| - \lambda_{-+} |\xi_{-+}^{ij}\rangle \langle \xi_{-+}^{ij}| \\ = \lambda_{++} |\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}| - \lambda_{--} |\xi_{--}^{ij}\rangle \langle \xi_{--}^{ij}| \quad (\text{S12})$$

Tracing both sides of Eq. (S12) leads to

$$x + y = \lambda_{++} + \lambda_{--}, \quad (\text{S13})$$

and

$$\langle \xi_{++}^{ij\perp} | (x |\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}| + y |\psi\rangle \langle \psi|) |\xi_{++}^{ij\perp}\rangle = \langle \xi_{++}^{ij\perp} | (\lambda_{++} |\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}| - \lambda_{--} |\xi_{--}^{ij}\rangle \langle \xi_{--}^{ij}|) |\xi_{++}^{ij\perp}\rangle \quad (\text{S14})$$

leads to

$$y = \lambda_{--} \quad (\text{S15})$$

Then we also have

$$x = \lambda_{++} \quad (\text{S16})$$

Thus the POVM element corresponding to the click of  $P_{++}^1$  is  $G_{++}^{ij} = \lambda_{++} |\xi_{++}^{ij}\rangle \langle \xi_{++}^{ij}|$ . The POVM element corresponding to the click of  $P_{++}^0$  is  $G_{--}^{ij} = \lambda_{--} |\xi_{--}^{ij}\rangle \langle \xi_{--}^{ij}|$ .

## 2. THE MEASUREMENT STAGE OF REALIZING JOINT POVMS

In the measurement stage, the single-qubit rotations can be realized by a combination of quarter-wave plates (QWPs) and half-wave plates (HWPs), so-called a sandwich-type QWP-HWP-QWP set, with certain setting angles depending on the parameters of the joint positive operator-valued measure (POVM). The setting angles of the wave plates (WPs) used to realize the corresponding elements of joint POVMs are shown in Table S1. We employ partially polarizing beam splitters (PPBSs) with specific transmission probabilities for vertical polarization  $T_V$  and same transmission probability for horizontal polarization  $T_H = 1$ . This allows us to implement the measurement  $\langle V|$  on the reflected port of the PPBSs.

In the basis  $\{|H\rangle, |V\rangle\}$ , the single-qubit rotations realized by HWP and QWP are

$$R_{HWP}(\theta_H) = \begin{pmatrix} \cos 2\theta_H & \sin 2\theta_H \\ \sin 2\theta_H & -\cos 2\theta_H \end{pmatrix}, \quad (\text{S17})$$

$$R_{QWP}(\theta_Q) = \begin{pmatrix} \cos^2 \theta_Q + i \sin^2 \theta_Q & (1-i) \sin \theta_Q \cos \theta_Q \\ (1-i) \sin \theta_Q \cos \theta_Q & \sin^2 \theta_Q + i \cos^2 \theta_Q \end{pmatrix},$$

respectively, where  $\theta_H$  and  $\theta_Q$  are the angles between the optic axes of HWP and QWP and horizontal direction.

**Table S1.** The setting angles of WPs and the transmission probabilities for vertical polarization  $T_V^{1(2)}$  of the two PPBSs for realization of joint POVMs.

$G_{ij}$	$\theta_{Q1}/\theta_{Q2}$	$\theta_{H1}$	$\theta_{Q3}/\theta_{Q4}$	$\theta_{H2}$	$\theta_{Q5}/\theta_{Q6}$	$\theta_{H3}$	$T_V^1$	$T_V^2$
$G_{12}$	$-7.3^\circ$	$32.9^\circ$	$1.4^\circ$	$-29.5^\circ$	$22.4^\circ$	$69.2^\circ$	0.3904(45)	0.2897(50)
$G_{23}$	$21.7^\circ$	$60.2^\circ$	$-4.9^\circ$	$-36.5^\circ$	$21.7^\circ$	$60.2^\circ$	0.3904(45)	0.2897(50)
$G_{13}$	$7.3^\circ$	$47.5^\circ$	$-1.4^\circ$	$-32.2^\circ$	$-112.4^\circ$	$110.8^\circ$	0.3904(45)	0.2897(50)

### 3. RELATIVE DETECTION EFFICIENCIES OF DETECTORS

After applying joint POVM on the single-photon qubit, the photons are detected by single-photon avalanche photodiodes (APDs) on the reflected ports of the two PPBSs, and both reflected and transmitted ports of PBS, in coincidence with the trigger photons. The relative detection efficiencies of the detectors D1-D4 are measured and used to correct the coincidence counts.

To measure the relative efficiencies of the different detectors D1, D2, D3 and D4, we make a reasonable assumption that the total number of photons is fixed. We tune the setting angles of WPs to change the photon distribution. That is, for each time after we tune the setting angles of WPs, the normalized number of photons at each output port with respect to the total number of photons is changed, which can be read at the corresponding detector. The readout photon counts at each detector equal to the number of photons at each output port multiplied by the relative efficiency of the corresponding detector. After we tune the setting angles of WPs for four times, we have four linear equations with four variables (relative efficiencies of detectors) and then solve them to obtain the relative efficiencies of the detectors D1, D2, D3, and D4.

In our experiment, the relative efficiencies of the detectors D1, D2, D3, and D4 are  $1$ ,  $0.9499 \pm 0.0070$ ,  $0.9199 \pm 0.0069$  and  $0.9801 \pm 0.0063$  respectively calculated from the experimental data. Thus we can use the relative efficiencies of the detectors to correct the photon counts in the measurement stage. For example, after the correction the photon counts at D2 which are used to calculate the probability of the photons being measured at D2 should be the readout photon counts divided by the relative efficiency of D2  $0.9499 \pm 0.0070$ .

### 4. JOINT MEASURABILITY OF POVMS.

We test the joint measurability of the constructed joint POVM  $G_{ij}$ . For different qubit states, we analyze the experimental results of the joint POVM and test whether the marginal condition  $\sum_{X_j(i)} G_{X_i, X_j}^{ij} = E_{X_i(i)}^{i(j)}$  is satisfied. Without loss of generality, we choose the four states  $|H\rangle$ ,  $|V\rangle$ ,  $|R\rangle = (|H\rangle + i|V\rangle)/\sqrt{2}$ , and  $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$  as states being measured. The results are shown in Table S2.

The measured probabilities  $\text{Tr}(\sum_{X_j} G_{X_i, X_j}^{ij} \rho)$  are in agreement with the theoretical predictions of the probabilities  $\text{Tr}(E_{X_i}^i \rho)$  of the POVM element  $E_{X_i}^i$  on the state  $\rho \in \{|H\rangle\langle H|, |V\rangle\langle V|, |R\rangle\langle R|, |D\rangle\langle D|\}$ , which proves the marginal condition is satisfied. Thus the constructed joint POVM  $G_{ij}$  shows the joint measurability.

**Table S2.** Experimental results of  $\sum_{X_j} G_{X_i, X_j}^{ij}$  on the four different single-qubit states compared to the theoretical predictions of the probabilities of  $E_{X_i}^i$  on these states.

	$ H\rangle$	$ V\rangle$	$ R\rangle$	$ D\rangle$
$G_{++}^{12} + G_{+-}^{12}$	0.8357(13)	0.1607(13)	0.4990(21)	0.4958(20)
$G_{++}^{13} + G_{+-}^{13}$	0.8353(13)	0.1699(13)	0.5003(22)	0.5029(21)
$E_+^1$ (Theory)	0.8350	0.1650	0.5000	0.5000
$G_{-+}^{12} + G_{--}^{12}$	0.1642(13)	0.8393(13)	0.5009(21)	0.5042(20)
$G_{-+}^{13} + G_{--}^{13}$	0.1647(13)	0.8301(13)	0.4997(22)	0.4971(21)
$E_-^1$ (Theory)	0.1650	0.8350	0.5000	0.5000
$G_{++}^{12} + G_{-+}^{12}$	0.3155(20)	0.6747(19)	0.5013(22)	0.7818(17)
$G_{++}^{23} + G_{+-}^{23}$	0.3179(19)	0.6748(19)	0.5001(21)	0.7788(16)
$E_+^2$ (Theory)	0.3325	0.6675	0.5000	0.7901
$G_{+-}^{12} + G_{--}^{12}$	0.6845(20)	0.3253(19)	0.4987(22)	0.2182(17)
$G_{+-}^{23} + G_{--}^{23}$	0.6821(19)	0.3252(19)	0.4999(21)	0.2212(16)
$E_-^2$ (Theory)	0.6675	0.3325	0.5000	0.2099
$G_{++}^{13} + G_{-+}^{13}$	0.3454(21)	0.6652(19)	0.5008(22)	0.2083(16)
$G_{++}^{23} + G_{-+}^{23}$	0.3448(20)	0.6519(21)	0.5020(22)	0.2102(17)
$E_+^3$ (Theory)	0.3325	0.6675	0.5000	0.2099
$G_{+-}^{13} + G_{--}^{13}$	0.6546(21)	0.3348(19)	0.4992(22)	0.7917(16)
$G_{+-}^{23} + G_{--}^{23}$	0.6551(20)	0.3481(21)	0.4980(22)	0.7898(17)
$E_-^3$ (Theory)	0.6675	0.3325	0.5000	0.7901