A-HYPERGEOMETRIC SYSTEMS AND D-MODULE FUNCTORS

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Avi Steiner

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THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. Uli Walther, Chair
Department of Mathematics
Dr. Donu Arapura
Department of Mathematics
Dr. Linquan Ma
Department of Mathematics
Dr. Jaroslaw Wlodarczyk
Department of Mathematics

Approved by:
Dr. David Goldberg
Head of the School Graduate Program

To Papa, Grammy, Grandma Sue, Grandma Rose, and Grandpa Joe

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ABSTRACT

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Let A be a d by n integer matrix. Gel'fand et al. proved that most A-hypergeometric systems have an interpretation as a Fourier–Laplace transform of a direct image. The set of parameters for which this happens was later identified by Schulze and Walther as the set of not strongly resonant parameters of A. A similar statement relating A-hypergeometric systems to exceptional direct images was proved by Reichelt. In the first part of this thesis, we consider a hybrid approach involving neighborhoods U of the torus of A and consider compositions of direct and exceptional direct images. Our main results characterize for which parameters the associated A-hypergeometric system is the inverse Fourier–Laplace transform of such a "mixed Gauss-Manin" system.

If the semigroup ring of A is normal, we show that every A-hypergeometric system is "mixed Gauss–Manin".

In the second part of this thesis, we use our notion of mixed Gauss–Manin systems to show that the projection and restriction of a normal A-hypergeometric system to the coordinate subspace corresponding to a face are isomorphic up to cohomological shift; moreover, they are essentially hypergeometric. We also show that, if A is in addition homogeneous, the holonomic dual of an A-hypergeometric system is itself A-hypergeometric. This extends a result of Uli Walther, proving a conjecture of Nobuki Takayama in the normal homogeneous case.

1. INTRODUCTION

Let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ such that $\mathbb{Z}A = \mathbb{Z}^d$; we abuse notation and also use A to denote the set of its columns. Assume that $\mathbb{N}A$ is pointed, i.e. that $\mathbb{N}A \cap -\mathbb{N}A = 0$. Associated to this data, Gel′fand, Graev, Kapranov, and Zelevinskiĭ defined in [2, 3] a family of modules over the sheaf $\mathcal{D}_{\mathbb{C}^n}$ of algebraic linear partial differential operators on $\mathbb{C}^n$ today referred to either as *GKZ-* or *A-hypergeometric* systems. These systems are defined as follows:

The *Euler operators* of A are the operators $E_i \coloneqq a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n$ ($i = 1, \dots, d$), and the *toric ideal* of A is the $\mathbb{C}[\partial_1, \dots, \partial_n]$-ideal

$$I_A \coloneqq \langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^n \rangle .$$

The *A-hypergeometric system* corresponding to the parameter $\beta \in \mathbb{C}^d$ is then defined to be

$$\mathcal{M}_A(\beta) \coloneqq \mathcal{D}_{\mathbb{C}^n} / \left(\mathcal{D}_{\mathbb{C}^n} I_A + \mathcal{D}_{\mathbb{C}^n}\{E_1 - \beta_1, \dots, E_d - \beta_d\}\right). \tag{1.0.1}$$

(In Chapter 2, this is denoted by $M_A(\beta)$ and is thought of as a module rather than a sheaf). If the condition that $\mathbb{Z}A = \mathbb{Z}^d$ is relaxed, $\mathcal{M}_A(\beta)$ may still be defined as above by first choosing a $\mathbb{Z}$-basis of $\mathbb{Z}A$; the resulting $\mathcal{D}_{\mathbb{C}^n}$-module is independent of this choice.

Solutions to A-hypergeometric systems have found applications in a wide variety of areas of both mathematics and physics including in the study of Aomoto–Gel′fand functions, toric residues, Picard–Fuchs equations for the variation of Hodge structure of Calabi–Yau toric hypersurfaces, and generating functions for intersection numbers on moduli spaces of curves (see [4–6]). In fact, most functions one comes across “in the wild” are solutions to such a system.

1.1 Torus Embeddings and Direct Images

The torus embedding

$$\begin{aligned} \varphi\colon T_A &\hookrightarrow \widehat{\mathbb{C}^n} := \operatorname{Spec}\mathbb{C}[\partial_1,\dots,\partial_n] \\ t &\mapsto (t^{\mathbf{a}_1},\dots,t^{\mathbf{a}_n}) \end{aligned} \tag{1.1.1}$$

induces a closed immersion of X_A into $\widehat{\mathbb{C}^n}$. On the torus, the data A and β give a D-module

$$\mathcal{O}^{\beta}_{T_A} := \mathcal{O}_{T_A} t^{-\beta}.$$

A natural question is then whether and how this $\mathcal{D}_{T_A}$-module is related to (the inverse Fourier–Laplace transform (see §2.2.3) of) the A-hypergeometric system $\mathcal{M}_A(\beta)$. A foundational result in this direction was given by Gelf'and, et al. in [7, Theorem 4.6]: For non-resonant β, the Fourier–Laplace transform of the D-module direct image $\varphi_+\mathcal{O}^{\beta}_{T_A}$ is isomorphic to $\mathcal{M}_A(\beta)$. This result was strengthened in [8, Corollary 3.7] to: the Fourier–Laplace transform of $\varphi_+\mathcal{O}^{\beta}_{T_A}$ is isomorphic to $\mathcal{M}_A(\beta)$ if and only if β is not in the set

$$\operatorname{sRes}(A) := \bigcup_{j=1}^{n} \operatorname{qdeg} H^1_{\langle t^{\mathbf{a}_j}\rangle}(S_A) \tag{1.1.2}$$

of *strongly resonant* parameters. Here, qdeg denotes the set of quasidegrees of a $\mathbb{Z}^d$-graded module and is defined in Definition 2.4.1. The $\mathbb{Z}^d$-grading on S_A is defined in §2.2.1.

It was then shown in [9, Proposition 1.14] that for certain other β, the inverse Fourier–Laplace transform of $\mathcal{M}_A(\beta)$ may be related to the D-module exceptional direct image $\varphi_\dagger\mathcal{O}^{\beta}_{T_A}$. Namely, $\varphi_\dagger\mathcal{O}^{\beta}_{T_A} \cong \mathrm{FL}^{-1}(\mathcal{M}_A(\beta))$ if A is homogeneous (i.e. the vector $(1,\dots,1)$ is in the row span of A), $\beta \in \mathbb{Q}^d$, and β is not in the set

$$\bigcup_{F \text{ face of } A} \left[(\mathbb{Z}^d \cap \mathbb{R}_{\geq 0}A) + \mathbb{C}F\right]. \tag{1.1.3}$$

However, in a certain sense, neither the theorem of Gelf'and, et al. nor the theorem of Reichelt hold for "most" β. A natural question is therefore whether there is a similar description which works for these remaining β. We discuss this question in

Chapter 2 using the author's notion of mixed Gauss–Manin systems and parameters, giving a complete answer in the normal case.

Remark 1.1.1 *Both* $\mathrm{sRes}(A)$ *and the set* (1.1.3) *are contained in the set*

$$\mathrm{Res}(A) := \bigcup_{F \text{ face of } A} (\mathbb{Z}^d + \mathbb{C}F)$$

of resonant *parameters defined in [7, Section 2.9] (the containment for* $\mathrm{sRes}(A)$ *follows from the definitions of* qdeg *and local cohomology; the containment for the set* (1.1.3) *is immediate). The complement* $\mathbb{C}^d \setminus \mathrm{Res}(A)$ *of* $\mathrm{Res}(A)$ *is open dense in the analytic topology. Therefore, if* $\beta \in \mathbb{C}^d$ *is generic, then* $\mathrm{FL}^{-1}(M_A(\beta))$ *is isomorphic to both* $\varphi_+\mathcal{O}_{T_A}^\beta$ *and to* $\varphi_\dagger\mathcal{O}_{T_A}^\beta$.

1.2 Applying D-module functors to A-hypergeometric systems

1.2.1 Projection and restriction

Explicit formulas for restriction (i.e. pullback via the D-module inverse image) to a coordinate subspace were computed in [10, Th. 4.4] and [11, Th. 4.2] for certain classes of GKZ systems. These formulas were generalized in [12, Th. 2.2] under certain hypotheses about the genericity of the parameter β and the size of the coordinate subspace. We focus on a different situation, and explicitly compute, when the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is normal, the restriction of $\mathcal{M}_A(\beta)$ to the coordinate subspace $\mathbb{C}^F$ corresponding to a face $F \preceq A$ (see (3.2.6) for the notation $\mathbb{C}^F$). We also compute the projection (i.e. the pushforward via the D-module direct image) of $\mathcal{M}_A(\beta)$ to $\mathbb{C}^F$. Both computations appear in Theorem 3.5.4. Note that, unless $F = A$, the subspace $\mathbb{C}^F$ does not satisfy the size requirements of [12, Th. 2.2], hence there is no nontrivial overlap between this paper and [12]. We also show that at most one of them can be nonzero (Corollary 3.5.9).

1.2.2 Duality

N. Takayama conjectured that the holonomic dual of an A-hypergeometric system is itself a GKZ system (after applying the coordinate transformation $x \mapsto -x$ if A is non-homogeneous, i.e. if the columns of A do not all lie in a hyperplane). U. Walther, in [13], provided a class of counterexamples to this conjecture. However, each of these counterexamples is rank-jumping (i.e. the holonomic rank is higher than expected), and in the same paper, Walther shows that for generic parameters, Takayama's conjecture does indeed hold. In particular, when the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is normal, he proves ([13, Prop. 4.4]) that the set of all parameters β for which the holonomic dual of $\mathcal{M}_A(\beta)$ is not a GKZ system has codimension at least three. We show in Chapter 3 using the notion of mixed and dual mixed Gauss–Manin systems that if A is homogeneous, this set is in fact empty.

2. A-HYPERGEOMETRIC MODULES AND GAUSS–MANIN SYSTEMS[1]

2.1 Introduction

Let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ such that $\mathbb{Z}A = \mathbb{Z}^d$. Assume that $\mathbb{N}A$ is pointed, i.e. that $\mathbb{N}A \cap -\mathbb{N}A = 0$. Define the following objects:

$$
\begin{aligned}
S_A &= \mathbb{C}[\mathbb{N}A], \text{ the semigroup ring of } A\\
X_A &= \operatorname{Spec} S_A, \text{ the toric variety of } A\\
T_A &= \operatorname{Spec} \mathbb{C}[\mathbb{Z}^d], \text{ the torus of } A\\
D_A &= \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n], \text{ the } n\text{th Weyl algebra}
\end{aligned}
$$

Let $\beta \in \mathbb{C}^d$. The *Euler operators* of A are the operators

$$E_i := a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n, \quad i = 1, \dots, d. \tag{2.1.1}$$

The A-hypergeometric system corresponding to β is then defined to be

$$M_A(\beta) := \frac{D_A}{\langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^n\rangle + \langle E_1 - \beta_1, \dots, E_d - \beta_d\rangle},$$

where the brackets (here and throughout this paper) denote a left ideal.

In Theorem 2.8.17 and Theorem 2.8.19, we give simultaneous generalizations of both [8, Corollary 3.3] and [9, Proposition 1.14]. These generalizations allow (the inverse Fourier–Laplace transform of) more A-hypergeometric systems to be equipped with a mixed Hodge module structure. In Chapter 3, we will use the normal case of these generalizations (Theorem 2.9.3) to compute for normal A the projection and restriction of $M_A(\beta)$ to coordinate subspaces of the form $\mathbb{C}^F$, where F is a face of A; and, if A is in addition homogeneous, to show that the holonomic dual of $M_A(\beta)$ is itself A-hypergeometric.

[1] A version of this chapter has been published in *Journal of Algebra* as [14].

2.1.1 Main Idea

Given a Zariski open subset $U \subseteq \widehat{\mathbb{C}^n}$ containing T_A, write

$$\iota_U \colon T_A \hookrightarrow U$$

for the embedding of T_A into U and

$$\varpi_U \colon U \hookrightarrow \widehat{\mathbb{C}^n}$$

for the inclusion of U into $\widehat{\mathbb{C}^n}$. The first main result in this paper, Theorem 2.8.17, provides an equivalent condition (in terms of the various local cohomology complexes $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)$ with supports in the orbit $\mathrm{O}(F)$; see §2.2.1 and §2.2.4) for

$$K^A_\bullet(S_A; E_A - \beta) \cong \mathrm{FL}(\varpi_{U+}\iota_{U\dagger}\mathcal{O}^\beta_{T_A})$$

for some such U, while the second main result, Theorem 2.8.19, does the same (this time in terms of the various localizations $S_A[\partial^{-F}]$) for

$$K^A_\bullet(S_A; E_A - \beta) \cong \mathrm{FL}(\varpi_{U\dagger}\iota_{U+}\mathcal{O}^\beta_{T_A}).$$

The condition for the first main result has two parts: First is a requirement that β not be rank-jumping. Second is a requirement about certain sets akin to Saito's $E_F(\beta)$ sets (see Definition 2.8.11 and Definition 2.8.15). Those parameters β for which both these conditions hold are called *dual mixed Gauss–Manin* (see Definition 2.8.15).

On the other hand, the condition for the second main result can be expressed as a requirement about Saito's $E_F(\beta)$ sets themselves. Those parameters β for which this condition holds are called *mixed Gauss–Manin* (see Definition 2.8.15).

The proof of Theorem 2.8.17 is accomplished as follows: First, we restate in terms of local cohomology via Lemma 2.8.1. Then, using the relationship between fiber support (Definition 2.3.1) and local cohomology in Proposition 2.3.7, we focus in on the restriction to torus orbits. These restrictions are computed for general inverse-Fourier–Laplace-transformed Euler–Koszul complexes in Theorem 2.7.2.

We also use in the proof that $\varphi_\dagger \mathcal{O}_{T_A}^\beta$ can be expressed in two ways as an Euler–Koszul complex (see Theorem 2.2.2): As an Euler–Koszul complex of the dualizing complex of S_A (Corollary 2.5.5), and as an Euler–Koszul complex of S_A itself (Proposition 2.6.2).

The proof of Theorem 2.8.19 follows a similar route.

2.2 Notation and Conventions

Subsection 2.2.1 defines various symbols related to the affine semigroup $\mathbb{N}A$. Subsection 2.2.2 recalls some common notions and facts about (multi-)graded rings and modules. Local cohomology with supports in a locally closed subset is recalled in subsection 2.2.4. Conventions and notation relating to varieties, D-modules, sheaves, and derived categories are given in subsection 2.2.3 along with the definition of the Fourier–Laplace transform. Finally, in subsection 2.2.5, we recall the notion of Euler–Koszul complexes.

2.2.1 Toric and GKZ Conventions/Notation

Let R_A be the polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_n]$, and set

$$\widehat{\mathbb{C}^n} := \operatorname{Spec} R_A. \tag{2.2.1}$$

This space is to be (loosely) interpreted as the "Fourier–Laplace-transformed version" of $\mathbb{C}^n$, hence the $\widehat{\ }$ (cf. §2.2.3).

Let $I_A \subseteq R_A$ be the toric ideal corresponding to the embedding φ from (1.1.1)—we identify S_A with the quotient R_A/I_A. The torus embedding also induces an action of T_A on $\widehat{\mathbb{C}^n}$, which in turn induces an action (the *contragredient* action) of T_A on R_A via

$$(t \cdot f)(\partial_1, \ldots, \partial_n) = f(t^{-\mathbf{a}_1}\partial_1, \ldots, t^{-\mathbf{a}_n}\partial_n).$$

An element $f \in R_A$ is *homogeneous of degree* $\alpha \in \mathbb{Z}^d$ if $t \cdot f = t^{-\alpha} f$ for all points $t \in (\mathbb{C}^*)^n$; it is *homogeneous* if it is homogeneous for some α. In particular, $\deg(\partial_i) = \mathbf{a}_i$, and S_A is a $\mathbb{Z}^d$-graded R_A-module.

Set

$$\varepsilon_A \coloneqq \mathbf{a}_1 + \cdots + \mathbf{a}_n. \tag{2.2.2}$$

Write $\hat{M}_A(\beta)$ for the inverse Fourier–Laplace transform (see §2.2.3) of the GKZ system $M_A(\beta)$.

Faces

A submatrix F of A is called a *face* of A, written $F \preceq A$, if F has d rows and $\mathbb{R}_{\geq 0} F$ is a face of $\mathbb{R}_{\geq 0} A$. Given $F \preceq A$, we make the following definitions:

$$T_F \coloneqq \operatorname{Spec} \mathbb{C}[\mathbb{Z}F] \tag{2.2.3}$$

is the torus of F. The monomial in $\mathbb{C}[\mathbb{Z}F]$ corresponding to $\alpha \in \mathbb{Z}F$ is written t^α. Denote by

$$\mathrm{O}(F) \coloneqq T_A \cdot \mathbb{1}_F \subseteq \widehat{\mathbb{C}^n} \tag{2.2.4}$$

the orbit in $\widehat{\mathbb{C}^n}$ corresponding to F (where the ith coordinate of $\mathbb{1}_F$ is 1 if $\mathbf{a}_i \in F$ and 0 otherwise). Note that the inclusion $\mathbb{Z}F \hookrightarrow \mathbb{Z}^d$ induces an isomorphism $\mathrm{O}(F) \cong T_F$. The rank of F is denoted by d_F, and if $G \preceq A$ with $G \succeq F$, we set

$$d_{G/F} \coloneqq d_G - d_F. \tag{2.2.5}$$

Define the ideal

$$I_F^A \coloneqq I_A + \langle \partial_i \mid \mathbf{a}_i \notin F \rangle \tag{2.2.6}$$

of R_A, and set

$$\partial^{kF} \coloneqq \prod_{\mathbf{a}_i \in F} \partial_i^k \qquad (k \in \mathbb{Z}). \tag{2.2.7}$$

Given $u \in (\mathbb{C}F)^* \coloneqq \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}F, \mathbb{C})$, define ϑ_u to be the invariant vector field on T_F defined by

$$\vartheta_u(t^\alpha) \coloneqq \langle \alpha, u \rangle\, t^\alpha \quad (\alpha \in \mathbb{Z}F), \tag{2.2.8}$$

where $\langle , \rangle$ denotes the standard pairing of dual spaces. These vector fields span the Lie algebra of T_F; therefore, D_{T_F} is generated as a $\mathbb{C}$-algebra by $\mathcal{O}_{T_F}$ and the vector fields $\{ \vartheta_u \mid u \in (\mathbb{C}F)^* \}$ (both of these claims may be proven in a straightforward manner, e.g. by choosing coordinates).

For $\lambda \in \mathbb{C}F$, define the D_{T_F}-module

$$\mathcal{O}_{T_F}^{\lambda} \coloneqq \mathcal{O}_{T_F} t^{-\lambda}, \tag{2.2.9}$$

where $t^{-\lambda}$ is a formal symbol subject to the D_{T_F}-action

$$\vartheta_u(ft^{-\lambda}) \coloneqq [\vartheta_u(f) - \langle \lambda, u \rangle\, f]t^{-\lambda} \qquad (u \in (\mathbb{C}F)^*).$$

This module is isomorphic to $\mathcal{O}_{T_F}$ as an $\mathcal{O}_{T_F}$-module and so is in particular an integrable connection. Moreover, it is a simple D_{T_F}-module.

2.2.2 Graded Rings and Modules

For more details about (multi-)graded rings and modules than are given here, refer to [15–17].

Twists

Let M be a graded module over a $\mathbb{Z}^k$-graded ring R. Given an $\alpha \in \mathbb{Z}^k$, define the graded module $M(\alpha)$ to be M as an ungraded R-module and to have degree γ component

$$M(\alpha)_\gamma \coloneqq M_{\alpha+\gamma}.$$

*- Properties

A **-simple ring* is a graded ring with no homogeneous (two-sided) ideals. A graded module over a graded ring S is **-free* if it is a direct sum of graded twists of S. A graded module over a graded ring S is **-injective* if it is an injective object in the category of graded S-modules.

(Weakly) $\mathbb{N}A$-Closed Subsets

As in [18, p143], we make the following definitions: A subset E of $\mathbb{Z}^d$ is $\mathbb{N}A$-*closed* if $E + \mathbb{N}A \subseteq E$. If E is $\mathbb{N}A$-closed, define $\mathbb{C}\{E\}$ to be the graded S_A-submodule of $\mathbb{C}[\mathbb{Z}^d] \coloneqq \mathbb{C}[t_1^{\pm}, \dots, t_d^{\pm}]$

$$\mathbb{C}\{E\} \coloneqq \mathbb{C}\,\{\, t^\alpha \mid \alpha \in E \,\} \tag{2.2.10}$$

given as the vector space spanned by $\{\, t^\alpha \mid \alpha \in E \,\}$.

A subset E of $\mathbb{Z}^d$ is *weakly* $\mathbb{N}A$-*closed* if $(E+\mathbb{N}A) \setminus E$ is $\mathbb{N}A$-closed. If E is weakly $\mathbb{N}A$-closed, define

$$\mathbb{C}\{E\} \coloneqq \mathbb{C}\{E + \mathbb{N}A\}/\mathbb{C}\{(E + \mathbb{N}A) \setminus E\}. \tag{2.2.11}$$

*-Injective Modules

By [17, Prop. 11.24], every indecomposable *-injective S_A-module is a $\mathbb{Z}^d$-graded twist of $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ for some face $F \preceq A$. Note that by [17, Lem. 11.12 together with Prop. 11.24], $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ is the injective envelope of S_F in the category of graded S_A-modules.

Graded Hom

Given graded modules M and N over a $\mathbb{Z}^k$-graded ring R, define for each $\alpha \in \mathbb{Z}^k$ the vector space

$$\underline{\mathrm{Hom}}_R(M, N)_\alpha \coloneqq \left\{\, f \in \mathrm{Hom}_R(M, N) \;\middle|\; f(M_\gamma) \subseteq N_{\gamma+\alpha} \text{ for all } \gamma \in \mathbb{Z}^k \,\right\} \tag{2.2.12}$$

of degree-α homomorphisms from M to N. Define $\underline{\mathrm{Hom}}_R(M, N)$ to be the graded R-module

$$\underline{\mathrm{Hom}}_R(M, N) \coloneqq \bigoplus_{\alpha \in \mathbb{Z}^k} \underline{\mathrm{Hom}}_R(M, N)_\alpha, \tag{2.2.13}$$

where the direct sum is taken inside $\mathrm{Hom}_R(M, N)$.

2.2.3 Other Conventions/Notation

Varieties

Varieties (smooth or otherwise) are not required to be irreducible. A subvariety of a variety X is a locally closed subset. The inclusion morphism of a subvariety $Z \subseteq X$ is usually denoted by i_Z, unless $Z = \{x\}$ is a point, in which case we write i_x instead of $i_{\{x\}}$.

Sheaves

The support of a sheaf M is

$$\operatorname{Supp} M \coloneqq \{\, x \in X \mid M_x \neq 0 \,\}\,. \tag{2.2.14}$$

The support of a complex $M^\bullet$ of sheaves is

$$\operatorname{Supp} M^\bullet \coloneqq \bigcup_i \operatorname{Supp} H^i(M^\bullet).$$

Complexes and Derived Categories

If $M^\bullet$ is a (cochain) complex with differential $d_M^i \colon M^i \to M^{i+1}$ and $k \in \mathbb{Z}$, define the complex $M^\bullet[k]$ to have ith component

$$M^\bullet[k]^i \coloneqq M^{k+i}$$

and differential

$$d^i_{M[k]} \coloneqq (-1)^k d_M^{k+i}.$$

The bounded derived category of D_X-modules is denoted by $\mathrm{D}^{\mathrm{b}}(D_X)$. The full subcategories of $\mathrm{D}^{\mathrm{b}}(D_X)$ generated by complexes with D_X-coherent and $\mathcal{O}_X$-quasicoherent cohomology are denoted by $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(D_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{qc}}(D_X)$, respectively. If $Z \subseteq X$ is a closed subvariety of X and $\sharp \in \{\mathrm{c}, \mathrm{qc}\}$, then $\mathrm{D}^{\mathrm{b},Z}(D_X)$ (respectively $\mathrm{D}^{\mathrm{b},Z}_\sharp(D_X)$) denotes the full subcategory of $\mathrm{D}^{\mathrm{b}}(D_X)$ (respectively $\mathrm{D}^{\mathrm{b}}_\sharp(D_X)$) of complexes supported in Z.

D-Modules

Given a morphism $f\colon X \to Y$ of smooth varieties, we write f_+ for the D-module direct image functor,

$$f^+ = \mathrm{L}f^*[\dim X - \dim Y] \quad \text{and} \quad f^\dagger = \mathbb{D}_X f^+ \mathbb{D}_Y$$

for the (shifted) D-module inverse image functor and the D-module exceptional inverse image functor, respectively, and

$$f_\dagger = \mathbb{D}_Y f_+ \mathbb{D}_X$$

for the D-module exceptional direct image functor.

Fourier–Laplace Transform and $\widehat{\mathbb{C}^n}$

Recall from (2.2.1) that $\widehat{\mathbb{C}^n} \coloneqq \operatorname{Spec} R_A$ with $R_A \coloneqq \mathbb{C}[\partial_1, \ldots, \partial_n]$. We identify $D_{\widehat{\mathbb{C}^n}}$ with $D_{\mathbb{C}^n}$ via the $\mathbb{C}$-algebra isomorphism

$$\partial_i \mapsto \partial_i \qquad \text{and} \qquad \partial_{\partial_i} \mapsto -x_i. \tag{2.2.15}$$

The *Fourier–Laplace transform* $\mathrm{FL}(N)$ of a $D_{\widehat{\mathbb{C}^n}}$-module N is N viewed as a $D_{\mathbb{C}^n}$-module via the isomorphism (2.2.15). This functor is an exact equivalence of categories. Its inverse functor is called the *inverse Fourier–Laplace transform*.

For a description of FL in terms of D-module direct and inverse image functors, see [19].

2.2.4 Local Cohomology

We recall the notion of local cohomology with supports in a locally closed set. As we will only need this notion for (complexes of) modules on an affine variety, we will only discuss local cohomology in this case. The reader is referred to [20] for more detail.

Let Z be a locally closed subset of an affine variety $X = \operatorname{Spec} R$, and let M be an R-module. Choose an open subset $U \subseteq X$ which contains Z as a closed subset. Then

$$\Gamma_Z(M) := \ker(\Gamma_U(M) \to \Gamma_{U \setminus Z}(M)),$$

independent of U. This defines a left-exact functor Γ_Z taking R-modules to R-modules. If Z' is another locally closed subset of X, then $\mathrm{R}\Gamma_{Z'}\, \mathrm{R}\Gamma_Z \cong \mathrm{R}\Gamma_{Z' \cap Z}$. In particular, if $Z = Y \cap U$ with Y closed in X and U open in X, then

$$\mathrm{R}\Gamma_U\, \mathrm{R}\Gamma_Y \cong \mathrm{R}\Gamma_Z \cong \mathrm{R}\Gamma_Y\, \mathrm{R}\Gamma_U\,. \tag{2.2.16}$$

Now, assume that X is smooth. Then Γ_Z takes D_X-modules to D_X-modules. The right derived functor of $\Gamma_Z \colon \operatorname{Mod}_{qc}(D_X) \to \operatorname{Mod}_{qc}(D_X)$ agrees with the derived functor of $\Gamma_Z \colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ (here $\operatorname{Mod}_{qc}(D_X)$ is the category of quasi-coherent left D_X-modules, and $\operatorname{Mod}(R)$ is the category of R-modules).

Example 2.2.1 *Let M be an R_A-module, $F \preceq A$ a face. The orbit $\mathrm{O}(F)$ is the intersection of the closed subset $V(\langle \partial_i \mid \mathbf{a}_i \notin F \rangle) \subseteq \widehat{\mathbb{C}^n}$ and the principal open subset $U = \widehat{\mathbb{C}^n} \setminus V(\prod_{\mathbf{a}_i \in F} \partial_i)$. So, by (2.2.16),*

$$\mathrm{R}\Gamma_{\mathrm{O}(F)}(M) \cong \mathrm{R}\Gamma_{\langle \partial_i | \mathbf{a}_i \notin F \rangle}(R_A[\partial^{-F}] \otimes_{R_A} M),$$

where $\partial^{-F} := \prod_{\mathbf{a}_i \in F} \partial_i^{-1}$. If M is in addition a graded R_A-module, then $\mathrm{R}\Gamma_{\mathrm{O}(F)}(M)$ is a complex of graded R_A-modules.

2.2.5 Euler–Koszul Complex

In this section, we recall the notion of Euler–Koszul complexes given in [21] and prove an elementary lemma (Lemma 2.2.3) relating Euler–Koszul complexes and local cohomology.

Define the vector

$$E_A = [E_1, \ldots, E_d]^\top$$

whose components are the Euler operators E_i from (2.1.1). Given a $\mathbb{Z}^d$-graded D_A-module N and a vector $\beta \in \mathbb{C}^d$, we define an action $\circ$ of $E_i - \beta_i$ on N by

$$(E_i - \beta_i) \circ m := (E_i - \beta_i + \deg_i(m)) \cdot m \qquad (m \neq 0 \text{ homogeneous})$$

and extending by $\mathbb{C}$-linearity. The maps $(E_i - \beta_i)\circ \colon N \to N$ are D_A-linear and pairwise commuting.

Definition 2.2.2 ([21, Definition 4.2]) *The* Euler–Koszul complex *of a $\mathbb{Z}^d$-graded R_A-module M with respect to A and β is*

$$K_\bullet^A(M; E_A - \beta) := K_\bullet\big((E_A - \beta)\circ; D_A \otimes_{R_A} M\big);$$

i.e. it is the Koszul complex of left D_A-modules defined by the sequence $(E_A - \beta)\circ$ of commuting endomorphisms on the left D_A-module $D_A \otimes_{R_A} M$. The complex is concentrated in homological degrees d to 0. The ith Euler–Koszul homology *is $H_i^A(M; E_A - \beta) := H_i(K_\bullet^A(M; E_A - \beta))$.*

The inverse Fourier–Laplace transform of the complex $K_\bullet^A(M; E_A - \beta)$ and the modules $H_i^A(M; E_A - \beta)$ will be denoted by $\hat{K}_\bullet^A(M; E_A - \beta)$ and $\hat{H}_i^A(M; E_A - \beta)$, respectively.

A standard computation shows that for $\alpha \in \mathbb{Z}^d$,

$$K_\bullet^A(M(\alpha); E_A - \beta) = K_\bullet^A(M; E - A - \beta - \alpha)(\alpha). \tag{2.2.17}$$

The $\mathbb{Z}^d$-grading on the Euler–Koszul complex will usually be ignored throughout this article, so the twist by α on the right-hand side will usually be left out.

Lemma 2.2.3 *Let $M^\bullet$ be a bounded complex of graded R_A-modules, and let $\beta \in \mathbb{C}^d$. Then for all faces $F \preceq A$, there is a canonical isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{O}(F)}\, \hat{K}_\bullet^A(M^\bullet; E_A - \beta) \cong \hat{K}_\bullet^A(\mathrm{R}\Gamma_{\mathrm{O}(F)}(M^\bullet); E_A - \beta).$$

Proof Use Example 2.2.1 together with the fact that localization at a monomial of R_A commutes with $\hat{K}_\bullet^A(-; E_A - \beta)$. ■

2.3 Fiber Support and Local Cohomology

We now establish a relationship between fiber support, defined below, and local cohomology. The main result of this section, Proposition 2.3.7, describes how for a sufficiently nice bounded complex $M^\bullet$ of D-modules (e.g. one with holonomic cohomology), the local cohomology of $M^\bullet$ with supports in a subvariety Z vanishes if and only if the fiber support of $M^\bullet$ is disjoint from Z. We also introduce cofiber support, which will be used later in the statement of Theorem 2.8.19.

Definition 2.3.1 *Let $M^\bullet \in \mathrm{D}^b(\mathcal{O}_X)$.*

1. *The* fiber support *of $M^\bullet$, denoted* $\operatorname{fSupp} M^\bullet$*, is defined to be the set*
$$\operatorname{fSupp} M^\bullet := \left\{ x \in X \;\middle|\; k(x) \otimes^{\mathrm{L}}_{\mathcal{O}_{X,x}} M^\bullet_x \neq 0 \right\}.$$
2. *If $M^\bullet \in \mathrm{D}^b_c(D_X)$, the* cofiber support *of $M^\bullet$, denoted* $\operatorname{cofSupp} M^\bullet$*, is defined to be the set*
$$\operatorname{cofSupp} M^\bullet := \left\{ x \in X \mid i_x^\dagger M^\bullet \neq 0 \right\} = \operatorname{fSupp} \mathbb{D} M^\bullet.$$

If $M^\bullet \in \mathrm{D}^b(D_X)$ has regular holonomic cohomology, then its fiber support is exactly the support (recall the definition of support in (2.2.14)) of the analytic solution complex $\mathrm{R}\mathscr{H}\!om_{D_{X^{\mathrm{an}}}}((M^\bullet)^{\mathrm{an}}, O_{X^{\mathrm{an}}})$, and its cofiber support is exactly the support of the analytic de Rham complex $\Omega_{X^{\mathrm{an}}} \otimes^{\mathrm{L}}_{D_{X^{\mathrm{an}}}} (M^\bullet)^{\mathrm{an}}$, where $(-)^{\mathrm{an}}$ denotes analytification.

The following two elementary lemmas are included for convenience:

Lemma 2.3.2 *Let X be a smooth variety, Z a smooth subvariety, $\bar{Z}$ its closure. Then i_Z^+ takes $\mathrm{D}^{b,\bar{Z}}_c(D_X)$ to $\mathrm{D}^b_c(D_Z)$.*

Proof Let $M^\bullet \in \mathrm{D}^b_c(D_X)$. Let U be an open subset of X containing Z in which Z is closed. Then $i_U^+ M^\bullet \cong i_U^{-1} M^\bullet$ is in $\mathrm{D}^b_c(D_X)$ by definition of coherence. Because $i_U^+ M^\bullet$ is supported on $\bar{Z} \cap U = Z$, Kashiwara's Equivalence (or more specifically [22, Corollary 1.6.2]) then tells us that the restriction of $i_U^+ M^\bullet$ to Z is in $\mathrm{D}^b_c(D_Z)$. This restriction is just $i_Z^+ M^\bullet$. ∎

Lemma 2.3.3 *Let Y, Z be smooth subvarieties of a smooth variety X, and let i_Y, i_Z be their inclusions into X. If $Y \cap Z = \emptyset$, then $i_Y^+ i_{Z+} = 0$ and $i_Y^\dagger i_{Z\dagger} = 0$ on $\mathrm{D}_c^b(D_Z)$.*

Proof Let $U = X \setminus Y$, and let $j\colon U \to X$ be inclusion. Write i'_Z for the inclusion $Z \to U$. Then $i_Y^+ i_{Z+} \cong i_Y^+ j_+ i'_{Z+} = 0$, where the isomorphism is because $i_Z = j \circ i'_Z$, and the equality is by [22, Proposition 1.7.1(ii)]. This proves the first statement. The second statement follows by duality. ■

Proposition 2.3.4 *Let X be a smooth variety, and let $M^\bullet \in \mathrm{D}_c^b(D_X)$. Then* $\operatorname{fSupp} M^\bullet$ *is a dense subset of* $\operatorname{Supp} M^\bullet$.

Proof We first show that $\operatorname{fSupp} M^\bullet \subseteq \operatorname{Supp} M^\bullet$. Let $x \in X$. If $x \notin \operatorname{Supp} M^\bullet$, then $M_x^\bullet \overset{\text{qi}}{=} 0$, and therefore $k(x) \otimes^L_{\mathcal{O}_{X,x}} M_x^\bullet$ vanishes. Hence, $x \notin \operatorname{fSupp} M^\bullet$, proving the claim.

Next, let $Y = \operatorname{Supp} M^\bullet$ (note that this is closed by [23, Proposition 2.3]). We show that the fiber support of $M^\bullet$ contains an open dense subset of Y; the result follows. This is accomplished in two steps: First, we show that there exists a smooth open dense subset $V \subseteq Y$ such that $i_V^+ M^\bullet$ is non-zero with $\mathcal{O}_V$-projective cohomology. Second, we show that for locally projective quasi-coherent $\mathcal{O}_X$-modules, the support agrees with the fiber support.

Choose a smooth dense open subset V of Y. By Lemma 2.3.2, $i_V^+ M^\bullet \in \mathrm{D}_c^b(D_V)$, and therefore by [22, Proposition 3.3.2], there exists a dense open subset V' of V such that all cohomology modules of $i_{V'}^+ M^\bullet$ are $\mathcal{O}_{V'}$-projective. Replace V with V'.

Suppose that $i_V^+ M^\bullet$ vanishes. Since V is smooth, $\mathrm{R}\Gamma_V(M^\bullet) \cong i_{V+} i_V^+ M^\bullet$, which by assumption is zero. So, $M^\bullet \cong \mathrm{R}\Gamma_{X \setminus V}(M^\bullet)$. But $M^\bullet$ is supported in Y, so

$$\mathrm{R}\Gamma_{X\setminus V}(M^\bullet) \cong \mathrm{R}\Gamma_{Y \cap (X \setminus V)}(M^\bullet) \cong \mathrm{R}\Gamma_{Y \setminus V}(M^\bullet).$$

Hence, $M^\bullet \cong \mathrm{R}\Gamma_{Y\setminus V}(M^\bullet)$ and therefore, using that $Y \setminus V$ is closed in X, is supported in $Y \setminus V$. This contradicts the fact that V is dense in the non-empty set $Y = \operatorname{Supp} M^\bullet$. Thus, $i_V^+ M^\bullet \neq 0$, proving the first claim.

To prove the second claim, let P be a locally projective quasi-coherent $\mathcal{O}_X$-module. By [24, Tag 058Z], each stalk of P is free, hence faithfully flat. Thus, $k(x) \otimes^L_{\mathcal{O}_{X,x}} P_x \cong k(x) \otimes_{\mathcal{O}_{X,x}} P_x$, and this vanishes if and only if P_x vanishes. ■

Example 2.3.5 *Although Proposition 2.3.4 tells us that the fiber support of a D-module is always contained in its support, this containment is in general strict:*

Consider the $D_{\mathbb{C}}$-module $M = \mathcal{O}_{\mathbb{C}}[x^{-1}]$, where x is the coordinate function on $\mathbb{C}$. The restriction of M to $\mathbb{C}^*$ is a (non-0) integrable connection, so the support and fiber support of M both contain $\mathbb{C}^*$. By [23, Proposition 2.3], $\operatorname{Supp} M$ is closed and therefore equal to $\mathbb{C}$. On the other hand, x acts invertibly on the stalk M_0, so the (total) fiber $k(0) \otimes^{\mathrm{L}}_{\mathcal{O}_{\mathbb{C},0}} M_0 = 0$. Hence, $\operatorname{fSupp} M = \mathbb{C}^*$.

Corollary 2.3.6 *Let X be a smooth variety, and let $M^\bullet \in \mathrm{D}^b_c(D_X)$. Then $\operatorname{fSupp} M^\bullet$ is empty if and only if $M^\bullet \cong 0$.* ■

Proposition 2.3.7 *Let X be a smooth variety, $Z \subseteq X$ be a subvariety, and $M^\bullet \in \mathrm{D}^b_{qc}(D_X)$. If $\mathrm{R}\Gamma_Z(M^\bullet) \cong 0$, then $Z \cap \operatorname{fSupp} M^\bullet = \emptyset$. The converse holds if both $M^\bullet$ and $\mathrm{R}\Gamma_Z(M^\bullet)$ are in $\mathrm{D}^b_c(D_X)$ (e.g. if $M^\bullet \in \mathrm{D}^b_h(D_X)$).*

Proof By Kashiwara's Equivalence, $i_x^+ \cong i_x^+ i_{x+} i_x^+$ (on $\mathrm{D}^b_{qc}(D_X)$), which in turn is isomorphic to $i_x^+ \mathrm{R}\Gamma_{\{x\}}$. On the other hand, if $x \in Z$, then $\mathrm{R}\Gamma_{\{x\}} \mathrm{R}\Gamma_Z \cong \mathrm{R}\Gamma_{\{x\}}$. Combining these, we get that $i_x^+ \mathrm{R}\Gamma_Z(M^\bullet) \cong i_x^+ M^\bullet$ for all $x \in Z$. Hence, if $\mathrm{R}\Gamma_Z(M^\bullet)$ vanishes, the same applies to $i_x^+ M^\bullet$ for every $x \in Z$. This proves the first statement.

To prove the second statement, let $M^\bullet \in \mathrm{D}^b_c(D_X)$, and assume that $Z \cap \operatorname{fSupp} M^\bullet = \emptyset$. We show that $\operatorname{fSupp} \mathrm{R}\Gamma_Z(M^\bullet) = \emptyset$, so that $\mathrm{R}\Gamma_Z(M^\bullet)$ vanishes by Corollary 2.3.6 (note that Corollary 2.3.6 applies by the coherence assumption on $\mathrm{R}\Gamma_Z(M^\bullet)$).

By the first part of this proof, if $x \in Z$, then $i_x^+ \mathrm{R}\Gamma_Z(M^\bullet) \cong i_x^+ M^\bullet$, which vanishes by assumption. To see that $i_x^+ \mathrm{R}\Gamma_Z(M^\bullet)$ also vanishes for $x \notin Z$, let $U \subseteq X$ be an open neighborhood of Z in which Z is closed, $j\colon U \to X$ inclusion. Then

$$i_x^+ \mathrm{R}\Gamma_Z(M^\bullet) \cong i_x^+ \mathrm{R}\Gamma_U \mathrm{R}\Gamma_Z(M^\bullet) \cong i_x^+ j_+ j^+ \mathrm{R}\Gamma_Z(M^\bullet) \cong i_x^+ j_+ \mathrm{R}\Gamma_Z(M_\bullet|_U). \quad (2.3.1)$$

There are two cases: If $x \notin U$, then the right-hand side of (2.3.1) vanishes by Lemma 2.3.3. On the other hand, suppose $x \in U \setminus Z$. Then $i_x^+ j_+ \cong (i_x')^+ j^+ j_+ \cong (i_x')^+$, where $i_x' \colon \{x\} \to U$ is inclusion. Combined with (2.3.1), this gives

$$i_x^+ \,\mathrm{R}\Gamma_Z(M^\bullet) \cong (i_x')^+ \,\mathrm{R}\Gamma_Z(M^\bullet|_U).$$

But Z is closed in U, so $\operatorname{Supp} \mathrm{R}\Gamma_Z(M^\bullet|_U) \subseteq Z$, which by assumption doesn't contain x. Hence, by Corollary 2.3.6, $x \notin \operatorname{fSupp} \mathrm{R}\Gamma_Z(M^\bullet)$. ∎

2.4 Quasidegrees

In this section we prove some lemmas on quasidegrees (Definition 2.4.1). These lemmas will be needed later to establish quasi-isomorphisms of certain Euler–Koszul complexes, and in Proposition 2.8.6 to establish $\mathcal{E}_A$ as the union of certain other related quasidegree sets. Lemma 2.4.2 provides a sufficient condition on a graded R_A-module M for there to be a face $F \preceq A$ such that $\operatorname{qdeg} M$ is a union of translates of $\mathbb{C}F$. Lemma 2.4.3 states that for a finitely-generated graded S_A-module M, the quasidegree set of M has the same dimension as the support of M.

We begin by generalizing the definition of quasidegrees from that given in [8, Definition 5.3] (which is itself a generalization of [21, Proposition 5.3], where the notion originated).

Definition 2.4.1 *The* true degree set *of a $\mathbb{Z}^d$-graded R_A-module M, denoted* $\operatorname{tdeg} M$*, is defined to be the set of $\alpha \in \mathbb{Z}^d$ such that $M_\alpha \neq 0$.*

The quasidegree set *of a finitely-generated $\mathbb{Z}^d$-graded R_A-module M, denoted* $\operatorname{qdeg} M$*, is defined to be the Zariski closure (in $\mathbb{C}^d$) of* $\operatorname{tdeg} M$*. We extend the definition of* qdeg *to arbitrary $\mathbb{Z}^d$-graded R_A-modules by*

$$\operatorname{qdeg} M := \bigcup_{M'} \operatorname{qdeg} M',$$

where the union is over all finitely-generated graded submodules $M' \subseteq M$. If $M^\bullet$ is a complex of such modules, we define

$$\operatorname{qdeg} M^\bullet := \bigcup_i \operatorname{qdeg} H^i(M^\bullet).$$

Before continuing, recall from (2.2.6) and (2.2.7) that $I_F^A = I_A + \langle \partial_i \mid \mathbf{a}_i \notin F \rangle$ and $\partial^{kF} = \prod_{\mathbf{a}_i \in F} \partial_i^k$. Recall also the definitions of *-simple and *-free given in §2.2.2.

Lemma 2.4.2 *Let M be a $\mathbb{Z}^d$-graded R_A-module, $F \preceq A$ a face. If M is both an $R_A[\partial^{-F}]$-module and I_F^A-torsion, then every irreducible component of* $\operatorname{qdeg} M$ *is a translate of $\mathbb{C}F$. Hence,*

$$\operatorname{qdeg} M = \left\{ \beta \in \mathbb{C}^d \;\middle|\; M_{\beta + \mathbb{C}F} \neq 0 \right\}.$$

Proof Consider the exhaustive filtration $M_k = 0 :_M (I_F^A)^k$ of M (this is exhaustive because M is I_F^A-torsion). Since M is an $R_A[\partial^{-F}]$-module, each M_k is an $R_A[\partial^{-F}]$-submodule. Moreover, each factor module M_k/M_{k-1} is by construction killed by I_F^A. Thus, M_k/M_{k-1} is an $S_F[\partial^{-F}]$-module for all k. But $S_F[\partial^{-F}]$ is a *-simple ring, so each M_k/M_{k-1} is a *-free $S_F[\partial^{-F}]$-module. Now, every finitely generated graded submodule of a direct sum is contained in a finite sub-sum, and the quasidegree set of a finite direct sum is the union of the quasidegree sets of its summands; so, the same is true for an infinite direct sum. Thus, $\operatorname{qdeg}(M_k/M_{k-1})$ is a union of translates of $\operatorname{qdeg}(S_F[\partial^{-F}]) = \mathbb{C}F$, proving the first claim.

For the second claim, $\beta \in \operatorname{qdeg} M$ if and only if it is contained in an irreducible component of $\operatorname{qdeg} M$. But by the first claim, every irreducible component of $\operatorname{qdeg} M$ is a translate of $\mathbb{C}F$. The only such translate containing β is $\beta + \mathbb{C}F$, and in the present situation this is an irreducible component of $\operatorname{qdeg} M$ if and only if it intersects $\operatorname{tdeg} M$, i.e. if and only if $M_{\beta+\mathbb{C}F} \neq 0$. ■

Lemma 2.4.3 *Let M be a finitely generated $\mathbb{Z}^d$-graded S_A-module. Then*

$$\dim \operatorname{Supp} M = \dim \operatorname{qdeg} M.$$

Proof Choose a filtration $0 = M_0 \subseteq \cdots \subseteq M_s = M$ of M by graded submodules such that each M_i/M_{i-1} is of the form $S_F(-\alpha)$ for some face $F \preceq A$ and some $\alpha \in \mathbb{Z}^d$ (in the terminology of [21, Definition 4.5], $\{M_i\}$ is a *toric filtration*). Then

$$\operatorname{Supp} M = \bigcup_{i=1}^{s} \operatorname{Supp} M_i/M_{i-1}.$$

So, $\dim \operatorname{Supp} M$ is equal to the maximum of the dimensions $\dim \operatorname{Supp}(M_i/M_{i-1})$.

On the other hand, each of the sets $\operatorname{qdeg}(M_i/M_{i-1})$ is a translate of the span of one of the finitely many faces of $\mathbb{N}A$. So, the dimension of $\operatorname{qdeg} M$ is equal to the maximum of the dimensions $\dim \operatorname{qdeg}(M_i/M_{i-1})$.

Thus, we are reduced to the case $M = S_F(-\alpha)$ for some $F \preceq A$, $\alpha \in \mathbb{Z}^d$. Then $\operatorname{qdeg}(S_F(-\alpha)) = \mathbb{C}F + \alpha$ and $\operatorname{Supp}(S_F(-\alpha)) = V(I_F^A)$. Since both of these have dimension d_F, we arrive at the result. ■

Lemma 2.4.4 *Let M be a finitely generated graded S_A-module, $F \preceq G \preceq A$. Then a subset $Z \subseteq \operatorname{qdeg} M[\partial^{-G}]$ is an irreducible component of $\operatorname{qdeg} M[\partial^{-G}]$ if and only if it is an irreducible component of $\operatorname{qdeg} M[\partial^{-F}]$. In particular,*

$$\operatorname{qdeg} M[\partial^{-G}] \subseteq \operatorname{qdeg} M[\partial^{-F}].$$

Proof Let H be any face of A. Choose a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_r = M$$

of M as in Lemma 2.4.3. Write $M_i/M_{i-1} \cong S_{F_i}(-\alpha_i)$. Then $\{M_i[\partial^{-F}]\}$ is a filtration of $M[\partial^{-H}]$, and its ith factor module is isomorphic to $S_{F_i}[\partial^{-F}](-\alpha_i)$, which is nonzero if and only if $F_i \succeq H$. Therefore,

$$\begin{aligned}
\operatorname{qdeg} M[\partial^{-H}] &= \bigcup_{i=1}^{r} \operatorname{qdeg}\big((M_i/M_{i-1})[\partial^{-H}]\big) \\
&= \bigcup_{i=1}^{r} \operatorname{qdeg}\big(S_{F_i}[\partial^{-H}](-\alpha_i)\big) \\
&= \bigcup_{F_i \succeq H} \operatorname{qdeg}\big(S_{F_i}[\partial^{-H}](-\alpha_i)\big)
\end{aligned}$$

$$= \bigcup_{F_i \succeq H} \left[\operatorname{qdeg}\left(S_{F_i}[\partial^{-H}]\right) + \alpha_i\right]$$

If $F_i \succeq H$, then every finitely generated submodule of $S_{F_i}[\partial^{-H}]$ is contained in $S_{F_i}\partial^{-kH}$ for some k, and $\operatorname{qdeg}(S_{F_i}\partial^{-kH}) = \mathbb{C}F_i + k \deg \partial^H = \mathbb{C}F_i$. Therefore, $\operatorname{qdeg} S_{F_i}[\partial^{-H}] = \mathbb{C}F_i$. Hence,

$$\operatorname{qdeg} M[\partial^{-H}] = \bigcup_{F_i \succeq H} (\mathbb{C}F_i + \alpha_i). \tag{2.4.1}$$

Set $\mathcal{Z}_H \coloneqq \{\, \mathbb{C}F_i + \alpha_i \mid F_i \succeq H \,\}$. Each $\mathbb{C}F_i + \alpha_i$ is irreducible, so by (2.4.1), the irreducible components of $\operatorname{qdeg} M[\partial^{-H}]$ are exactly the maximal elements of $\mathcal{Z}_H$. We show that $\mathcal{Z}_H$ is an upper subset of $\mathcal{Z} \coloneqq \mathcal{Z}_\emptyset$ (recall that a subset Y of an ordered set $(X, \leq)$ is *upper* if for all $y \in Y$, we have $\{\, x \in X \mid y \leq x \,\} \subseteq Y$). It follows that $\mathcal{Z}_G$ is an upper subset of $\mathcal{Z}_F$, and therefore that an element of $\mathcal{Z}_G$ is maximal in $\mathcal{Z}_G$ if and only if it is maximal in $\mathcal{Z}_F$, proving the lemma.

Let $\mathbb{C}F_i + \alpha_i \in \mathcal{Z}_H$, and suppose $\mathbb{C}F_j + \alpha_j$ contains $\mathbb{C}F_i + \alpha_i$. Then $F_j \succeq F_i \succeq H$, so $\mathbb{C}F_j + \alpha_j \in \mathcal{Z}_H$. Thus, $\mathcal{Z}_H$ is an upper subset of $\mathcal{Z}$, as claimed. ■

Remark 2.4.5 *The proofs of Lemmas 2.4.3 and 2.4.4 work also for toric R_A-modules as defined in [21, Definition 4.5]. With minor adjustments, they can even be made to work for weakly toric modules (see [8, Definition 5.1]).*

2.5 The Holonomic Dual of Euler–Koszul Complexes

The following theorem, Theorem 2.5.2, will be used in Corollary 2.5.5 to give a first description of $\varphi_\dagger \mathcal{O}_{T_A}^\beta$, and then in the next section to describe $\mathrm{FL}(\varphi_\dagger \mathcal{O}_{T_A}^\beta)$ as an A-hypergeometric system.

Before stating the theorem, we need a definition. Also recall from (2.2.13) that $\underline{\operatorname{Hom}}_{R_A}(M, N)$ denotes the graded R_A-module whose degree-α component is the vector space of R_A-module homomorphisms from M to N of degree α.

Definition 2.5.1 *Given a bounded complex $M^\bullet$ of finitely generated graded R_A-modules, define*

$$\mathbf{D}M^\bullet := \mathrm{R}\underline{\mathrm{Hom}}_{R_A}(M^\bullet, \omega_{R_A})[n-d],$$

where the shift is cohomological and $\omega_{R_A} := R_A(-\sum_i \mathbf{a}_i)$.

Theorem 2.5.2 below is proved in essentially the same way as is [21, Theorem 6.3]—no problems occur translating from statements about modules and spectral sequences to statements in the derived category. We therefore omit the proof of Theorem 2.5.2. Note that the reason that Theorem 2.5.2 does not need the auto-equivalence $N \mapsto N^-$ as does [21, Theorem 6.3] is that we work with the inverse-Fourier–Laplace-transformed Euler–Koszul complex, whereas [21] works with the Euler–Koszul complex itself.

Recall from (2.2.2) that $\varepsilon_A := \mathbf{a}_1 + \cdots + \mathbf{a}_n$.

Theorem 2.5.2 *Let $M^\bullet$ be a bounded complex of finitely-generated graded R_A-modules, $\beta \in \mathbb{C}^d$. Then*

$$\mathbb{D}\hat{K}^A_\bullet(M^\bullet; E_A - \beta) \cong \hat{K}^A_\bullet(\mathbf{D}M^\bullet; E_A + \beta).$$

If the $\mathbb{Z}^d$-grading is taken into account, the right-hand side must be twisted by $-\varepsilon_A$.

■

Definition 2.5.3 *Define*

$$\omega^\bullet_{S_A} := \bigoplus_{F \preceq A} \mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}, \tag{2.5.1}$$

*where the summand $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ sits in cohomological degree $d_{A/F}$, and the coboundary maps are the natural projections with signs chosen appropriately (for details, see [17, Def. 12.7] or [18, §2]). This is a complex of *-injective modules (see §2.2.2).*

By [18, Theorem 3.2], $\omega^\bullet_{S_A}$ is a dualizing complex in the ungraded category; the arguments there show that $\omega^\bullet_{S_A}$ is also a dualizing complex in the $\mathbb{Z}^d$-graded category. With minor changes to its proof, [25, Theorem V.3.1] implies that $\omega^\bullet_{S_A}$ is unique (in the $\mathbb{Z}^d$-graded derived category) up to cohomological shift. Its cohomological degrees are chosen such that $\underline{\mathrm{Hom}}_{S_A}(\mathbb{C}, \omega^\bullet_{S_A})$ is quasi-isomorphic to the complex $\mathbb{C}[-d]$; this choice implies that

$$\mathbf{D}(-) \cong \underline{\mathrm{Hom}}_{S_A}(-, \omega^\bullet_{S_A}) \tag{2.5.2}$$

and

$$\omega^\bullet_{S_A} \cong \mathbf{D}(S_A) \tag{2.5.3}$$

in the derived category of graded S_A-modules.

Remark 2.5.4 *Let $F \preceq A$ be a face. Since $\omega^\bullet_{S_A}$ is a complex of *-injective modules, $\mathrm{R}\Gamma_{\mathrm{O}(F)}(\omega^\bullet_{S_A}) \cong \Gamma_{\mathrm{O}(F)}(\omega^\bullet_{S_A})$. By Example 2.2.1, we have*

$$\Gamma_{\mathrm{O}(F)}(\omega^\bullet_{S_A}) \cong \Gamma_{\langle \partial_i | \mathbf{a}_i \notin F\rangle}(R_A[\partial^{-F}] \otimes_{R_A} \omega^\bullet_{S_A}).$$

If a face $F' \preceq A$ does not contain F, then $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ is ∂^F-torsion and therefore vanishes upon tensoring with $R_A[\partial^{-F}]\otimes_{R_A}$. Then because $\langle \partial_i \mid \mathbf{a}_i \notin F\rangle\, S_A$ is the homogeneous prime ideal corresponding to F, the only module $\mathbb{C}\{\mathbb{N}F' - \mathbb{N}A\}$ with $F' \succeq F$ which is not killed by $\Gamma_{\langle \partial_i | \mathbf{a}_i \notin F\rangle}$ is $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$. Hence,

$$\mathrm{R}\Gamma_{\mathrm{O}(F)}(\omega^\bullet_{S_A}) \cong \mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}[-d_{A/F}]. \tag{2.5.4}$$

Recall that $\mathrm{sRes}(A)$ was defined in (1.1.2).

Corollary 2.5.5 *If $-\beta \in \mathbb{C}^d \setminus \mathrm{sRes}(A)$, then $\varphi_\dagger \mathcal{O}^\beta_{T_A} \cong \hat{K}^A_\bullet(\omega^\bullet_{S_A}; E_A - \beta)$.*

Proof The holonomic dual of $\mathcal{O}^\beta_{T_A}$ is $\mathcal{O}^{-\beta}_{T_A}$, and by [8, Corollary 3.7], applying φ_+ to this gives $\hat{K}^A_\bullet(S_A; E_A + \beta)$. So, by Theorem 2.5.2,

$$\varphi_\dagger \mathcal{O}^\beta_{T_A} \cong \mathbb{D}\varphi_+ \mathbb{D}\mathcal{O}^\beta_{T_A} \cong \hat{K}^A_\bullet(\mathbf{D}S_A; E_A - \beta).$$

Now use (2.5.3). ■

2.6 The Exceptional Direct Image of $\mathcal{O}_{T_A}^{\beta}$

Reichelt proves in [9, Proposition 1.14] that $\mathrm{FL}(\varphi_{\dagger}\mathcal{O}_{T_A}^{\beta})$ is isomorphic to a GKZ system for homogeneous A and $\beta \in \mathbb{Q}^d$. We now generalize this to arbitrary A, β. This generalization, or rather Proposition 2.6.2, will be used later in the proof of Theorem 2.8.17.

Lemma 2.6.1 *For all $i \in \mathbb{N}$, $\dim \operatorname{qdeg} H^i(\omega_{S_A}^{\bullet}) \leq d - i$.*

Proof By definition, $\omega_{S_A}^i$ is the direct sum of $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ for faces F with $d_{A/F} = i$. Each $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ has support equal to $V(I_F^A)$, which has dimension $d_F = d - i$. So, $\dim \operatorname{Supp} \omega_{S_A}^i = d - i$. Hence, because $H^i(\omega_{S_A}^{\bullet})$ is a subquotient of $\omega_{S_A}^i$, its support must have dimension at most $d - i$. Now apply Lemma 2.4.3. ∎

Proposition 2.6.2 *Let $\beta \in \mathbb{C}^d$. Then for all $k \gg 0$,*

$$\varphi_{\dagger}\mathcal{O}_{T_A}^{\beta} \cong \hat{K}_{\bullet}^{A}(S_A; E_A - \beta + k\varepsilon_A).$$

Proof First, notice that by [8, Corollaries 3.1 and 3.7], $-\beta + k\varepsilon_A \notin \mathrm{sRes}(A)$ for all $k \gg 0$. Also notice that $\mathcal{O}_{T_A}^{\beta} \cong \mathcal{O}_{T_A}^{\beta'}$ for all $\beta' \equiv \beta \pmod{\mathbb{Z}^d}$. Hence, in light of Corollary 2.5.5, we may replace β with $\beta - k\varepsilon_A$ to assume that $-\beta \notin \mathrm{sRes}(A)$.

Step 1: We show that $\beta - k\varepsilon_A \notin \operatorname{qdeg} \operatorname{cone}(H^0(\omega_{S_A}^{\bullet}) \to \omega_{S_A}^{\bullet})$ for all $k \gg 0$. Then, applying [8, Theorem 5.4(3)] along with a basic spectral sequence argument, it follows that $\hat{K}_{\bullet}^{A}(-; E_A - \beta + k\varepsilon_A)$ applied to the morphism $H^0(\omega_{S_A}^{\bullet}) \to \omega_{S_A}^{\bullet}$ is a quasi-isomorphism for all $k \gg 0$.

By Lemma 2.6.1, the union $\bigcup_{i>0} \operatorname{qdeg} H^i(\omega_{S_A}^{\bullet})$ has codimension at least 1, and because each cohomology module of $\omega_{S_A}^{\bullet}$ is finitely generated, this union has finitely many irreducible components. Therefore, since ε_A is in the relative interior of $\mathbb{N}A$, we see that $\beta - k\varepsilon_A \notin \bigcup_{i>0} \operatorname{qdeg} H^i(\omega_{S_A}^{\bullet})$ for all $k \gg 0$. But the ith cohomology of $\operatorname{cone}(H^0(\omega_{S_A}^{\bullet}) \to \omega_{S_A}^{\bullet})$ is 0 if $i \leq 0$ and is $H^i(\omega_{S_A}^{\bullet})$ if $i > 0$. Hence, $\beta - k\varepsilon_A \notin \operatorname{cone}(H^0(\omega_{S_A}^{\bullet}) \to \omega_{S_A}^{\bullet})$ for all $k \gg 0$, as promised.

Step 2: We construct a quasi-isomorphism

$$\hat{K}^A_\bullet(S_A; E_A - \beta + k\varepsilon_A) \xrightarrow{\text{qi}} \hat{K}^A_\bullet(H^0(\omega^\bullet_{S_A}); E_A - \beta + k\varepsilon_A)$$

for $k \gg 0$. Let $0 \neq m \in H^0(\omega^\bullet_{S_A})$ be homogeneous. Since $H^0(\omega^\bullet_{S_A}) \subseteq \mathbb{C}[\mathbb{Z}^d]$ and is non-zero (it contains m), the zero ideal is one of its associated primes (in fact the only one). Therefore, $H^0(\omega^\bullet_{S_A})$ must contain a twist of S_A; in particular, it must contain $k_0\varepsilon_A$ for some $k_0 \in \mathbb{N}$. Hence, m may be chosen to have degree $k_0\varepsilon_A$.

Now, consider the quotient $H^0(\omega^\bullet_{S_A})/S_A m$. The quasidegree set of this quotient has codimension at least 1, so as before, $\beta - k\varepsilon_A \notin \text{qdeg}(H^0(\omega^\bullet_{S_A})/S_A m)$ for all $k \gg 0$. Hence, the morphism

$$\hat{K}^A_\bullet(S_A(-k_0\varepsilon_A); E_A - \beta + k\varepsilon_A) \to \hat{K}^A_\bullet(H^0(\omega^\bullet_{S_A}); E_A - \beta + k\varepsilon_A)$$

induced by right-multiplication by m is a quasi-isomorphism for $k \gg 0$. Applying (2.2.17) gives the result. ■

The promised generalization is given by the following corollary:

Corollary 2.6.3 *Let $\beta \in \mathbb{C}^d$. Then for all $k \gg 0$,*

$$\text{FL}(\varphi_\dagger \mathcal{O}^\beta_{T_A}) \cong M_A(\beta - k\varepsilon_A).$$

Proof By Proposition 2.6.2, it suffices to show that $\varphi_\dagger \mathcal{O}^\beta_{T_A}$ has cohomology only in degree 0. The holonomic dual of $\varphi_\dagger \mathcal{O}^\beta_{T_A}$ is $\varphi_+ \mathcal{O}^{-\beta}_{T_A}$, which by [8, Proposition 2.1] has cohomology only in degree 0. Then because $\mathbb{D}$ is exact, the same applies to $\varphi_\dagger \mathcal{O}^\beta_{T_A}$. ■

2.7 Restricting Euler–Koszul Complexes to Orbits

We now compute the restriction and exceptional restriction to an orbit (as defined in (2.2.4)) of an (inverse Fourier–transformed) Euler–Koszul complex in terms of local cohomology and localizations, respectively. Recall from §2.2.2 that a *-injective S_A-module is an injective object in the category of $\mathbb{Z}^d$-graded S_A-modules, and every indecomposable *-injective S_A-module is a $\mathbb{Z}^d$-graded twist of $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$ for some face $F \preceq A$.

Proposition 2.7.1 *Let $J^\bullet$ be a bounded below complex of *-injective S_A-modules, $\beta \in \mathbb{C}^d$, and $F \preceq A$. Assume that each J^i is a direct sum of twists of $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}$. Then there exists a quasi-isomorphism of double complexes*[2]

$$\{K_{-p}(J^q; E_A - \beta)\} \overset{qi}{=} \left\{ \bigoplus_{\lambda+\mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F} K_{-p}(S_F[\partial^{-F}]; E_A - \lambda) \otimes_{\mathbb{C}} J^q_{\beta-\lambda} \right\}.$$

Proof Consider the subcomplexes of $J^\bullet$ given by $M^\bullet = S_A[\partial^{-F}]J^\bullet_{\beta+\mathbb{C}F}$ and $M'^\bullet = \langle \partial_i \mid \mathbf{a}_i \notin F\rangle\, M^\bullet$ (note that $J^\bullet$ is a complex of $S_A[\partial^{-F}]$-modules, so $M^\bullet$ is in fact a subcomplex of $J^\bullet$). We claim for all q that $\beta \notin \mathrm{qdeg}(M'^q)$ and $\beta \notin \mathrm{qdeg}(J^q/M^q)$. To see this, notice that the intersections of $\beta+\mathbb{C}F$ with $\mathrm{tdeg}(M'^q)$ and with $\mathrm{tdeg}(J^q/M^q)$ are both empty by construction. But both M'^q and J^q/M^q are I^A_F-torsion (because J^q is). Hence, by Lemma 2.4.2, β is a quasidegree of neither, proving the claim.

From the claim, we get that for all q, the morphisms

$$K_\bullet(M^q; E_A - \beta) \to K_\bullet(J^q; E_A - \beta)$$

and

$$K_\bullet(M^q; E_A - \beta) \to K_\bullet(M^q/M'^q; E_A - \beta)$$

are both quasi-isomorphisms, and therefore we get a quasi-isomorphism of double complexes

$$\{K_{-p}(J^q; E_A - \beta)\} \overset{\mathrm{qi}}{=} \{K_{-p}(M^q/M'^q; E_A - \beta)\}. \tag{2.7.1}$$

Next, notice that $M^\bullet/M'^\bullet$ is a complex of graded modules over $\mathbb{C}[\mathbb{Z}F]$ (which we identify with $S_F[\partial^{-F}]$). So, since $\mathbb{C}[\mathbb{Z}F]$ is a *-simple ring, each M^q/M'^q is a direct sum of $\mathbb{Z}^d$-graded twists of $\mathbb{C}[\mathbb{Z}F]$. Therefore, by gradedness,

$$M^\bullet/M'^\bullet = \bigoplus_{\alpha+\mathbb{Z}F \in \mathbb{Z}^d/\mathbb{Z}F} (M^\bullet/M'^\bullet)_\alpha \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}F](-\alpha). \tag{2.7.2}$$

[2] By a *quasi-isomorphism of double complexes* between $M^{\bullet\bullet}$ and $M'^{\bullet\bullet}$, we mean a pair of morphisms

$$M^{\bullet\bullet} \xleftarrow{f} N^{\bullet\bullet} \xrightarrow{g} M'^{\bullet\bullet}$$

such that $\mathrm{Tot}(f)$ and $\mathrm{Tot}(g)$ are quasi-isomorphisms of complexes.

Now, as complexes of vector spaces, $(M^\bullet/M'^\bullet)_\alpha$ is isomorphic to $J^\bullet_\alpha$ if $\alpha \in \beta + \mathbb{C}F$ and is zero otherwise. So, combining this with eqs. (2.7.1) and (2.7.2), we get

$$\begin{aligned}\{K_{-p}(J^q; E_A - \beta)\} &\overset{\text{qi}}{=} \left\{ \bigoplus_{\substack{\alpha+\mathbb{Z}F\in\mathbb{Z}^d/\mathbb{Z}F \\ \alpha\in\beta+\mathbb{C}F}} K_{-p}(\mathbb{C}[\mathbb{Z}F](-\alpha)\otimes_{\mathbb{C}} J^q_\alpha; E_A-\beta) \right\} \\ &= \left\{ \bigoplus_{\substack{\alpha+\mathbb{Z}F\in\mathbb{Z}^d/\mathbb{Z}F \\ \alpha\in\beta+\mathbb{C}F}} K_{-p}(\mathbb{C}[\mathbb{Z}F](-\alpha); E_A-\beta)\otimes_{\mathbb{C}} J^q_\alpha \right\}. \end{aligned} \tag{2.7.3}$$

But $K_\bullet(\mathbb{C}[\mathbb{Z}F](-\alpha); E_A - \beta) = K_\bullet(\mathbb{C}[\mathbb{Z}F]; E_A - \beta + \alpha)$. So, re-indexing the sum, we are done. ■

Let $M^\bullet$ be a bounded complex of $\mathbb{Z}^d$-graded S_A-modules, let $\beta \in \mathbb{C}^d$, and let $F \preceq A$ be a face. For $\lambda \in \mathbb{C}F$, we give $\mathrm{R}\Gamma_{\mathrm{O}(F)}(M^\bullet)_{\beta-\lambda+\mathbb{Z}F}$ the structure of a complex of D_{T_F}-modules as follows: Let m be a homogeneous element of $H^i_{\mathrm{O}(F)}(S_A)_{\beta-\lambda+\mathbb{Z}F}$ for some i. Recalling the definition of ϑ_u from (2.2.8), we set

$$\vartheta_u \cdot m := \langle \deg(m) - \beta, u\rangle\, m \qquad (u \in (\mathbb{C}F)^*). \tag{2.7.4}$$

Observing that (2.7.4) makes no reference to λ, we get an isomorphism

$$\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F} \mathrm{R}\Gamma_{\mathrm{O}(F)}(M^\bullet)_{\beta-\lambda+\mathbb{Z}F} \cong \mathrm{R}\Gamma_{\mathrm{O}(F)}(M^\bullet)_{\beta+\mathbb{C}F}. \tag{2.7.5}$$

In the theorem below, we use the convention that $\bigwedge\mathbb{C}^k$ lives in cohomological degrees $-k$ through 0.

Theorem 2.7.2 *Let $M^\bullet$ be a bounded complex of $\mathbb{Z}^d$-graded S_A-modules, $\beta \in \mathbb{C}^d$. Then for all faces $F \preceq A$,*

$$i^+_{\mathrm{O}(F)}\hat{K}_\bullet(M^\bullet; E_A-\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F} \mathcal{O}^\lambda_{T_F}\otimes_{\mathbb{C}} \mathrm{R}\Gamma_{\mathrm{O}(F)}(M^\bullet)_{\beta-\lambda}\otimes_{\mathbb{C}} \bigwedge\mathbb{C}^{d_{A/F}}.$$

This isomorphism is functorial in $M^\bullet$.

An equivalent presentation, absorbing the $\mathcal{O}^\lambda_{T_F}$ into the local cohomology, is

$$i^+_{\mathrm{O}(F)}\hat{K}_\bullet(M^\bullet; E_A-\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F} \mathrm{R}\Gamma_{\mathrm{O}(F)}(M^\bullet)_{\beta-\lambda+\mathbb{Z}F}\otimes_{\mathbb{C}} \bigwedge\mathbb{C}^{d_{A/F}}.$$

This can be further compacted using (2.7.5) to give

$$i^{+}_{\mathrm{O}(F)}\hat{K}_{\bullet}(M^{\bullet};E_A-\beta)\cong \mathrm{R}\Gamma_{\mathrm{O}(F)}(M^{\bullet})_{\beta+\mathbb{C}F}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}. \tag{2.7.6}$$

Proof Let $J^{\bullet}$ be a (bounded below) *-injective S_A-module resolution of $M^{\bullet}$. Then $\mathrm{R}\Gamma_{\mathrm{O}(F)}(M^{\bullet})\cong\Gamma_{\mathrm{O}(F)}(J^{\bullet})$, which is a complex of *-injective S_A-modules each of which is either 0 or has I^A_F as its only associated prime; that is, each $\Gamma_{\mathrm{O}(F)}(J^i)$ is a direct sum of twists of $\mathbb{C}\{\mathbb{N}F-\mathbb{N}A\}$. Thus, noting that $i^{+}_{\mathrm{O}(F)}\cong i^{+}_{\mathrm{O}(F)}\,\mathrm{R}\Gamma_{\mathrm{O}(F)}$, Proposition 2.7.1 and Lemma 2.2.3 give

$$\begin{aligned} i^{+}_{\mathrm{O}(F)}\hat{K}_{\bullet}(M^{\bullet};E_A-\beta)&\cong i^{+}_{\mathrm{O}(F)}\hat{K}_{\bullet}(\mathrm{R}\Gamma_{\mathrm{O}(F)}(M^{\bullet});E_A-\beta)\\ &\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F} i^{+}_{\mathrm{O}(F)}\hat{K}_{\bullet}(S_F[\partial^{-F}];E_A-\lambda)\otimes_{\mathbb{C}}\mathrm{R}\Gamma_{\mathrm{O}(F)}(M^{\bullet})_{\beta-\lambda}. \end{aligned} \tag{2.7.7}$$

But for $\lambda\in\mathbb{C}F$,

$$i^{+}_{\mathrm{O}(F)}\hat{K}_{\bullet}(S_F[\partial^{-F}];E_A-\lambda)\cong i^{+}_{\mathrm{O}(F)}\hat{K}^{F}_{\bullet}(S_F[\partial^{-F}];E_F-\lambda)\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}. \tag{2.7.8}$$

Now by [8, Prop. 2.1], $\hat{K}^{F}_{\bullet}(S_F[\partial^{-F}];E_F-\lambda)$ is isomorphic to the direct image $\varphi_{F+}\mathcal{O}^{\lambda}_{T_F}$, where φ_F is the torus embedding of T_F into $V(\partial_i\mid\mathbf{a}_i\notin F)$. Then because $i^{+}_{\mathrm{O}(F)}\varphi_{F+}\cong\mathrm{id}$, we get that

$$i^{+}_{\mathrm{O}(F)}\hat{K}_{\bullet}(S_F[\partial^{-F}];E_A-\lambda)\cong\mathcal{O}^{\lambda}_{T_F}.$$

Combining this with (2.7.7) and (2.7.8) gives the result. ∎

Before stating Theorem 2.7.4, we recall the notion of ($\mathbb{Z}^d$-graded) Matlis duality:

Definition 2.7.3 *Let Q be an affine semigroup. The* Matlis dual *of the graded $\mathbb{C}[Q]$-module M is the graded $\mathbb{C}[Q]$-module $M^{\vee}:=\underline{\mathrm{Hom}}_{\mathbb{C}}(M,\mathbb{C})$.*

Theorem 2.7.4 *Let $\beta\in\mathbb{C}^d$, and let M be a finitely generated graded S_A-module. Then for all faces $F\preceq A$,*

$$i^{\dagger}_{\mathrm{O}(F)}\hat{K}^{A}_{\bullet}(M;E_A-\beta)\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F}\mathcal{O}^{\lambda}_{T_F}\otimes_{\mathbb{C}}M[\partial^{-F}]_{\beta-\lambda}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}.$$

This isomorphism is functorial in M.

As in Theorem 2.7.2, this isomorphism may also be written as

$$i^{\dagger}_{\mathrm{O}(F)}\hat{K}^{A}_{\bullet}(M;E_A-\beta)\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F}M[\partial^{-F}]_{\beta-\lambda+\mathbb{Z}F}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}$$

and as

$$i^{\dagger}_{\mathrm{O}(F)}\hat{K}^{A}_{\bullet}(M;E_A-\beta)\cong M[\partial^{-F}]_{\beta+\mathbb{C}F}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}.$$

Proof By Theorems 2.5.2 and 2.7.2,

$$i^{+}_{\mathrm{O}(F)}\mathbb{D}\hat{K}^{A}_{\bullet}(M;E_A-\beta)\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F}\mathcal{O}^{\lambda}_{T_F}\otimes_{\mathbb{C}}\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)_{-\beta-\lambda}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}.$$

Dualizing, we get

$$\begin{aligned}
i^{\dagger}_{\mathrm{O}(F)}\hat{K}^{A}_{\bullet}(M;E_A-\beta)&\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F}\mathcal{O}^{-\lambda}_{T_F}\otimes_{\mathbb{C}}[\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)_{-\beta-\lambda}]^{*}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}[-d_{A/F}]\\
&\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F}\mathcal{O}^{-\lambda}_{T_F}\otimes_{\mathbb{C}}[\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)^{\vee}]_{\beta+\lambda}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}[-d_{A/F}]\\
&\cong\bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F}\mathcal{O}^{\lambda}_{T_F}\otimes_{\mathbb{C}}[\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)^{\vee}]_{\beta-\lambda}\otimes_{\mathbb{C}}\bigwedge\mathbb{C}^{d_{A/F}}[-d_{A/F}],
\end{aligned}$$

where $(-)^{*}$ is the vector space duality functor. In the notation of Example 2.2.1, we have

$$\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)\cong\mathrm{R}\Gamma_{\langle\partial_i|\mathbf{a}_i\notin F\rangle}\left(\mathbf{D}(M)[\partial^{-F}]\right).$$

So, by (2.5.2) and because M is finitely generated, we get

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)&\cong\mathrm{R}\Gamma_{\langle\partial_i|\mathbf{a}_i\notin F\rangle}(S_A)\otimes^{\mathrm{L}}_{S_A}\mathrm{R}\underline{\mathrm{Hom}}_{S_A}(M,\omega^{\bullet}_{S_A})[\partial^{-F}]\\
&\cong\mathrm{R}\Gamma_{\langle\partial_i|\mathbf{a}_i\notin F\rangle}(S_A)\otimes^{\mathrm{L}}_{S_A}\mathrm{R}\underline{\mathrm{Hom}}_{S_A}\big(M,\omega^{\bullet}_{S_A}[\partial^{-F}]\big)\\
&\cong\mathrm{R}\underline{\mathrm{Hom}}_{S_A[\partial^{-F}]}\big(M[\partial^{-F}],\mathrm{R}\Gamma_{\langle\partial_i|\mathbf{a}_i\notin F\rangle}(S_A)\otimes^{\mathrm{L}}_{S_A}\omega^{\bullet}_{S_A}[\partial^{-F}]\big)\\
&\cong\mathrm{R}\underline{\mathrm{Hom}}_{S_A[\partial^{-F}]}\big(M[\partial^{-F}],\mathrm{R}\Gamma_{\langle\partial_i|\mathbf{a}_i\notin F\rangle}\big(\omega^{\bullet}_{S_A}[\partial^{-F}]\big)\big)\\
&\cong\mathrm{R}\underline{\mathrm{Hom}}_{S_A[\partial^{-F}]}\big(M[\partial^{-F}],\mathrm{R}\Gamma_{\mathrm{O}(F)}(\omega^{\bullet}_{S_A})\big)\\
&\cong\underline{\mathrm{Hom}}_{S_A[\partial^{-F}]}\big(M[\partial^{-F}],\Gamma_{\mathrm{O}(F)}(\omega^{\bullet}_{S_A})\big)\,.
\end{aligned}$$

But $\Gamma_{\mathrm{O}(F)}(\omega^\bullet_{S_A}) \cong \mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}[-d_{A/F}]$ by (2.5.4), and $\mathbb{C}\{\mathbb{N}F - \mathbb{N}A\} \cong S_A[\partial^{-F}]^\vee$. So, applying [17, Lem. 11.16], we see that

$$\mathrm{R}\Gamma_{\mathrm{O}(F)}(\mathbf{D}M)^\vee \cong (M[\partial^{-F}]^\vee[-d_{A/F}])^\vee \cong M[\partial^{-F}][d_{A/F}].$$

Therefore,

$$i^\dagger_{\mathrm{O}(F)}\hat{K}^A_\bullet(M; E_A - \beta) \cong \bigoplus_{\lambda+\mathbb{Z}F\in\mathbb{C}F/\mathbb{Z}F} \mathcal{O}^\lambda_{T_F} \otimes_{\mathbb{C}} M[\partial^{-F}]_{\beta-\lambda} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}},$$

as hoped. ∎

2.8 A-Hypergeometric Systems via Direct Images

In subsection 2.8.1, we introduce the notion of strongly (A, F)-exceptional quasidegrees and prove some related lemmas. In subsection 2.8.2, we study an effect of contiguity on Euler–Koszul complexes. We then state and prove the main theorems, Theorems 2.8.17 and 2.8.19, in subsection 2.8.3.

Given an open subset $U \subseteq \widehat{\mathbb{C}^n}$ containing (the image of) T_A, consider the inclusion maps in the commutative diagram below:

$$\begin{tikzcd}
& (\mathbb{C}^*)^n \arrow[dd, hook, "j_U"] \arrow[dr, hook, "\varpi"] & \\
T_A \arrow[ur, hook, "\iota"] \arrow[dr, hook, "\iota_U"'] & & \widehat{\mathbb{C}^n} \\
& U \arrow[ur, hook, "\varpi_U"'] &
\end{tikzcd}$$

The morphisms ι and ι_U are the torus embedding φ with codomain restricted to $(\mathbb{C}^*)^n$ and U, respectively. The remaining morphisms are the inclusions.

Lemma 2.8.1 *With notation as above, there are, for every $M^\bullet \in \mathrm{D}^b_c(D_{T_A})$, natural isomorphisms $\varpi_{U+}\iota_{U\dagger}M^\bullet \cong \mathrm{R}\Gamma_U\, \varphi_\dagger M^\bullet$ and $\varpi_{U\dagger}\iota_{U+}M^\bullet \cong \varpi_{U\dagger}\varpi_U^{-1}\varphi_+ M^\bullet$.*

Proof The map j_U is an affine open immersion, so $j_{U+} \cong \mathrm{R}j_{U*} \cong j_{U*}$. Now, for any open subset $V \subseteq U$ and any sheaf F on $(\mathbb{C}^*)^n$, one has

$$\Gamma(V, \varpi_U^*\varpi_*F) = \Gamma(V, \varpi_*F) = \Gamma(V \cap (\mathbb{C}^*)^n, F) = \Gamma(V, j_{U*}F),$$

so $j_{U*} = \varpi_U^* \varpi_*$, which is isomorphic to $\varpi_U^+ \varpi_+$ because ϖ_U is an open immersion and ϖ is an affine open immersion. So, $j_{U+} \cong \varpi_U^+ \varpi_+$. Therefore,

$$\iota_{U\dagger} = \mathbb{D}_U \iota_{U+} \mathbb{D}_{T_A} \cong \mathbb{D}_U j_{U+} \iota_+ \mathbb{D}_{T_A} \cong \mathbb{D}_U \varpi_U^+ \varpi_+ \iota_+ \mathbb{D}_{T_A} \cong \varpi_U^+ \mathbb{D}_{\widehat{\mathbb{C}^n}} \varpi_+ \iota_+ \mathbb{D}_{T_A}.$$

Since $\mathbb{D}_{\widehat{\mathbb{C}^n}} \varpi_+ \iota_+ \mathbb{D}_{T_A} \cong \varphi_\dagger$ and $\varpi_{U+} \varpi_U^+ \cong \mathrm{R}\Gamma_U$, we get the first isomorphism. The second isomorphism follows via duality. ■

2.8.1 Exceptional and Strongly Exceptional Quasidegrees

In this section we introduce the notion of strongly (A, F)-exceptional quasidegrees for $F \preceq A$. These are then related in Proposition 2.8.6 to the set

$$\mathcal{E}_A := \bigcup_{i>0} \operatorname{qdeg} H^i(\omega_{S_A}^\bullet)$$

of A-exceptional quasidegrees. In Lemma 2.8.8, we prove that $\hat{K}_\bullet^A(S_A; E_A - \beta)$ has relatively open fiber support if $\beta \notin \mathcal{E}_A$.

Definition 2.8.2 *Given a face F, we define the set of* strongly (A, F)-exceptional quasidegrees *to be*

$$\mathcal{E}_{A,F}^{\text{strong}} := \bigcup_{i<d_{A/F}} \operatorname{qdeg} H^i_{\mathrm{O}(F)}(S_A).$$

When $F = \emptyset$, this is just the set of strongly A-exceptional quasidegrees defined in [26, Definition 2.9]. More generally, if M is a graded R_A-modules, we define the set of strongly (A, F)-exceptional quasidegrees for M *to be*

$$\mathcal{E}_{A,F}^{\text{strong}}(M) := \bigcup_{i<d_{A/F}} \operatorname{qdeg} H^i_{\mathrm{O}(F)}(M).$$

Remark 2.8.3 *From Example 2.2.1, we know that*

$$H^i_{\mathrm{O}(F)}(S_A) \cong H^i_{\langle \partial_i | \mathbf{a}_i \notin F \rangle}(S_A[\partial^{-F}]).$$

The ideal $\langle \partial_i \mid \mathbf{a}_i \notin F \rangle\, S_A[\partial^{-F}]$ is the maximal homogeneous ideal of $S_A[\partial^{-F}]$, so $\mathcal{E}_{A,F}^{\text{strong}} = \emptyset$ if and only if the affine semigroup ring $S_A[\partial^{-F}]$ is Cohen–Macaulay.

Example 2.8.4 *If $d \leq 2$, then the localization $S_A[\partial^{-F}]$ is Cohen–Macaulay for all faces $F \neq \emptyset$. Therefore, $\mathcal{E}_{A,F}^{\text{strong}} = \emptyset$ for all faces $F \neq \emptyset$.*

Example 2.8.5 *Let*

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

The semigroup $\mathbb{N}A$ is equal to $\mathbb{N}^3 \backslash \{\, (0,0,c) \mid c \text{ is odd} \,\}$. Let $F \preceq A$ be a face. If $F = A$ or $\mathbb{C}F$ does not contain the z-axis, then the semigroup ring $S_A[\partial^{-F}]$ is normal, hence Cohen–Macaulay, and therefore $\mathcal{E}_{A,F}^{\text{strong}} = \emptyset$ by Remark 2.8.3. If $\mathbb{C}F$ equals the z-axis, then $H^0_{\mathrm{O}(F)}(S_A)$ is zero, so $\mathcal{E}_{A,F}^{\text{strong}} = \operatorname{qdeg} H^1_{\mathrm{O}(F)}(S_A) = \mathbb{C}F$. If $F = \emptyset$, then $H^i_{\mathrm{O}(F)}(S_A)$ is zero if $i = 0$ or 1, so $\mathcal{E}_{A,F}^{\text{strong}} = \operatorname{qdeg} H^2_{\mathrm{O}(F)}(S_A) = \{\, (0,0,c) \mid c \in \mathbb{Z}_{<0} \text{ and } c \text{ is odd} \,\}$.

Proposition 2.8.6 $\mathcal{E}_A = \bigcup_{F \preceq A} \mathcal{E}_{A,F}^{\text{strong}}$.

Proof It suffices to show that

$$\operatorname{qdeg} H^{>0}(\omega^\bullet_{S_A}) = \bigcup_{F \preceq A} \left\{ \beta \in \mathbb{C}^d \;\middle|\; H^{>0}(\omega^\bullet_{S_A[\partial^{-F}]})_{\beta + \mathbb{C}F} \neq 0 \right\}.$$

($\subseteq$) Let $Z = \mathbb{C}F + \beta$ be an irreducible component of $\operatorname{qdeg} H^{>0}(\omega^\bullet_{S_A})$. Then by Lemma 2.4.4, Z is also an irreducible component of $\operatorname{qdeg} H^{>0}(\omega^\bullet_{S_A})[\partial^{-F}]$, and therefore $H^{>0}(\omega^\bullet_{S_A})[\partial^{-F}]_{\beta+\mathbb{C}F} \neq 0$. Now use that

$$H^{>0}(\omega^\bullet_{S_A})[\partial^{-F}] \cong H^{>0}(\omega^\bullet_{S_A[\partial^{-F}]}). \tag{2.8.1}$$

($\supseteq$) Suppose $H^{>0}(\omega^\bullet_{S_A[\partial^{-F}]})_{\beta+\mathbb{C}F} \neq 0$. Then by (2.8.1) and Lemma 2.4.4, the irreducible component of $\operatorname{qdeg} H^{>0}(\omega^\bullet_{S_A[\partial^{-F}]})$ containing β is also an irreducible component of $\operatorname{qdeg} H^{>0}(\omega^\bullet_{S_A})$. ∎

The proof of Lemma 2.8.8 requires the Ishida complex of an affine semigroup ring, which we now recall.

Definition 2.8.7 *Let Q be an affine semigroup, $\pi := \mathbb{R}_{\geq 0}Q$ its cone. The* Ishida complex *of $\mathbb{C}[Q]$ is the complex*

$$\mho^{\bullet}_{\mathbb{C}[Q]} := \bigoplus_{\sigma \text{ a face of } \pi} \mathbb{C}[Q]_{\sigma}, \tag{2.8.2}$$

where $\mathbb{C}[Q]_{\sigma}$ sits in cohomological degree $\dim(\sigma) - \dim(\pi \cap -\pi)$ and denotes the localization of $\mathbb{C}[Q]$ with respect to the multiplicative system $\{ t^{\alpha} \mid \alpha \in \sigma \cap Q \}$. The coboundary maps are the natural localization maps with signs chosen appropriately (for details, see [17, Def. 13.21] or [18, §2]).

Lemma 2.8.8 *If $\beta \notin \mathcal{E}_A$, then* $\operatorname{fSupp} \hat{K}^A_{\bullet}(S_A; E_A - \beta)$ *is open in X_A.*

Proof By Theorem 2.7.2, the orbit-cone correspondence, and Proposition 2.8.6, it suffices to prove the following: For all faces $F \preceq G \preceq A$,

$$\operatorname{qdeg} H^{d_{A/F}}_{\mathrm{O}(F)}(S_A) \subseteq \operatorname{qdeg} H^{d_{A/F}}_{\mathrm{O}(G)}(S_A).$$

To prove this, consider the short exact sequence of complexes

$$0 \to \mho^{\bullet}_{S_A[\partial^{-F}]}[d_{G/F}] \to \mho^{\bullet}_{S_A[\partial^{-G}]} \to C^{\bullet} \to 0, \tag{2.8.3}$$

where the first two complexes are the Ishida complexes of $S_A[\partial^{-F}]$ and $S_A[\partial^{-G}]$, respectively, the first map is the natural inclusion, and the third complex is the cokernel. Since the Ishida complex of $S_F[\partial^{-F}]$ represents $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A[\partial^{-F}]) = \mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)$ (and similarly for G), the long exact sequence in cohomology gives an exact sequence

$$H^{d_{A/F}}_{\mathrm{O}(F)}(S_A[\partial^{-F}]) \to H^{d_{A/G}}_{\mathrm{O}(G)}(S_A[\partial^{-G}]) \to H^{d_{A/G}}(C^{\bullet}) \to 0.$$

But the first two complexes in (2.8.3) are both equal to $S_A[\partial^{-A}]$ in cohomological degree $d_{A/G}$, so $H^{d_{A/G}}(C^{\bullet}) = 0$. Now use that if M is a graded quotient of a graded module N, then $\operatorname{qdeg} M \subseteq \operatorname{qdeg} N$. ■

2.8.2 Contiguity

In this subsection we discuss how right multiplication by a monomial of R_A (a "contiguity" operator) affects the restrictions and exceptional restrictions, respectively, of an Euler–Koszul complex to orbits.

Lemma 2.8.9 *Let $F \preceq A$ be a face, and let M be a finitely-generated $\mathbb{Z}^d$-graded S_A-submodule of $\mathbb{C}[\mathbb{Z}^d]$. Let $\beta \in \mathbb{C}^d$ and $\alpha \in \mathbb{N}A$. Assume that $\beta, \beta - \alpha \notin \mathcal{E}_{A,F}^{\text{strong}}(M)$. Then the following are equivalent:*

(a) *The morphism $i^+_{\mathrm{O}(F)} \hat{K}^A_\bullet(M; E_A - \beta + \alpha) \to i^+_{\mathrm{O}(F)} \hat{K}^A_\bullet(M; E_A - \beta)$ induced by right-multiplication by ∂^α is an isomorphism.*

(b) *For all $\lambda \in \mathbb{C}F$,*

$$\mathrm{R}\Gamma_{\mathrm{O}(F)}(M)_{\beta-\alpha-\lambda} \neq 0 \textit{ if and only if } \mathrm{R}\Gamma_{\mathrm{O}(F)}(M)_{\beta-\lambda} \neq 0.$$

Proof By Theorem 2.7.2 and because neither β nor $\beta - \alpha$ are strongly (A, F)-exceptional for M, it suffices to show that the morphism $f_\lambda \colon H^{d_{A/F}}_{\mathrm{O}(F)}(M(-\alpha))_{\beta-\lambda} \to H^{d_{A/F}}_{\mathrm{O}(F)}(M)_{\beta-\lambda}$ induced by multiplication by ∂^α is an isomorphism for all $\lambda \in \mathbb{C}F$ if and only if (b).

The "only if" direction is immediate. For the "if" direction, the long exact sequence of local cohomology gives an exact sequence

$$H^{d_{A/F}}_{\mathrm{O}(F)}(M(-\alpha))_{\beta-\lambda} \xrightarrow{f_\lambda} H^{d_{A/F}}_{\mathrm{O}(F)}(M)_{\beta-\lambda} \to H^{d_{A/F}}_{\mathrm{O}(F)}(M/\partial^\alpha M)_{\beta-\lambda} \to 0.$$

But $\dim(M/\partial^\alpha M) < d_{A/F}$ because M is finitely generated and ∂^α is M-regular. So, $H^{d_{A/F}}_{\mathrm{O}(F)}(M/\partial^\alpha M)_{\beta-\lambda} = 0$, and therefore f_λ is always surjective. Moreover, because the Hilbert function of $H^{d_{A/F}}_{\mathrm{O}(F)}(M)$ takes values in $\{0, 1\}$, the hypothesis (b) implies that both the domain and codomain of f_λ have dimension 1. Therefore, f_λ is an isomorphism for all λ. ■

Lemma 2.8.10 *Let $F \preceq A$ be a face, and let M be a finitely-generated $\mathbb{Z}^d$-graded S_A-submodule of $\mathbb{C}[\mathbb{Z}^d]$. The following are equivalent for $\beta \in \mathbb{C}^d$ and $\alpha \in \mathbb{N}A$:*

(a) *The morphism $i^\dagger_{\mathrm{O}(F)} \hat{K}^A_\bullet(M; E_A - \beta) \to i^\dagger_{\mathrm{O}(F)} \hat{K}^A_\bullet(M; E_A - \beta - \alpha)$ induced by right-multiplication by ∂^α is an isomorphism.*

(b) *For all $\lambda \in \mathbb{C}F$,*

$$M[\partial^{-F}]_{\beta-\lambda} \neq 0 \textit{ if and only if } M[\partial^{-F}]_{\beta+\alpha-\lambda} \neq 0.$$

Proof By Theorem 2.7.4, it suffices to show that, as with Lemma 2.8.9, the morphism $f_\lambda\colon M[\partial^{-F}]_{\beta-\lambda} \to M(\alpha)[\partial^{-F}]_{\beta-\lambda}$ induced by multiplication by ∂^α is an isomorphism for all $\lambda \in \mathbb{C}F$ if and only if (b).

As before, the "only if" direction is immediate. For the "if" direction, ∂^α is M- (and therefore $M[\partial^{-F}]$-) regular, so f_λ is always injective. Now proceed as in Lemma 2.8.9 using the fact that the Hilbert function of $M[\partial^{-F}]$ takes values in $\{0,1\}$. ■

2.8.3 Main Theorems

Definition 2.8.11 *Given a face F and a parameter $\beta \in \mathbb{C}^d$, define the sets*

$$\mathrm{E}^*_F(\beta) := \left\{ \lambda \in \mathbb{C}F/\mathbb{Z}F \;\middle|\; \mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta-\lambda} \neq 0 \right\}$$

and

$$\mathrm{E}_F(\beta) := \left\{ \lambda \in \mathbb{C}F/\mathbb{Z}F \;\middle|\; S_A[\partial^{-F}]_{\beta-\lambda} \neq 0 \right\}.$$

Because $S_A[\partial^{-F}] \cong \mathbb{C}\{\mathbb{N}A - \mathbb{N}F\}$, the second set is the set $E_F(\beta)$ defined by Saito in [27].

Remark 2.8.12 *The definitions of $\mathrm{E}_F(\beta)$ and $\mathrm{E}^*_F(\beta)$ along with Theorems 2.7.2 and 2.7.4 show that for all $\beta \in \mathbb{C}^d$,*

$$\mathrm{fSupp}\,\hat{K}^A_\bullet(S_A; E_A - \beta) = \bigsqcup_{\mathrm{E}^*_F(\beta)\neq\emptyset} \mathrm{O}(F)$$

and

$$\mathrm{cofSupp}\,\hat{K}^A_\bullet(S_A; E_A - \beta) = \bigsqcup_{\mathrm{E}_F(\beta)\neq\emptyset} \mathrm{O}(F).$$

Remark 2.8.13 *Let $\beta \in \mathbb{C}^d$ and $F \precneqq A$. Suppose $\lambda + \mathbb{Z}F \in \mathrm{E}_F(\beta)$, so that $\beta - \lambda \in \mathbb{N}A - \mathbb{N}F$. Then $H^i_{\mathrm{O}(F)}(S_A)_{\beta-\lambda}$ is isomorphic to the reduced cohomology $\tilde{H}^i(\mathcal{P};\mathbb{C})$ of a (nonempty) convex polytope $\mathcal{P}$ (cf. [17, Rmk. 13.25 and Cor. 13.26]). As convex polytopes are contractible, this cohomology vanishes, and therefore $\lambda + \mathbb{Z}F \notin \mathrm{E}^*_F(\beta)$. In other words,*

$$\mathrm{E}_F(\beta) \cap \mathrm{E}^*_F(\beta) = \emptyset.$$

Before continuing, we state a small lemma about $\mathrm{E}^*_F(\beta)$ and $\mathrm{E}_F(\beta)$. Parts (a) and (b) follow from Lemma 2.8.9 and Lemma 2.8.10, respectively. Note that (b) is also [27, Prop. 2.2 (5)].

Lemma 2.8.14 *Let $\beta \in \mathbb{C}^d$ and $\alpha \in \mathbb{N}A$.*

(a) *If $\beta, \beta - \alpha \notin \mathcal{E}^{\mathrm{strong}}_{A,F}$, then $\mathrm{E}^*_F(\beta) \subseteq \mathrm{E}^*_F(\beta - \alpha)$.*

(b) $\mathrm{E}_F(\beta) \subseteq \mathrm{E}_F(\beta + \alpha)$.

Definition 2.8.15 *1. A parameter $\beta \in \mathbb{C}^d$ is* mixed Gauss–Manin along the face *$F \preceq A$ if either $\mathrm{E}_F(\beta) = \emptyset$ or there exists a $\beta' \in \mathbb{C}^d \setminus \mathrm{sRes}(A)$ with $\beta - \beta' \in \mathbb{Z}^d$ such that $\mathrm{E}_F(\beta) = \mathrm{E}_F(\beta')$. A parameter $\beta \in \mathbb{C}^d$ is* mixed Gauss–Manin *if it is mixed Gauss–Manin along every face.*

2. A parameter $\beta \in \mathbb{C}^d$ is dual mixed Gauss–Manin along the face *$F \preceq A$ if $\beta \notin \mathcal{E}_A$ and if either $\mathrm{E}^*_F(\beta) = \emptyset$ or there exists a $-\beta' \in \mathbb{C}^d \setminus \mathrm{sRes}(A)$ with $\beta - \beta' \in \mathbb{Z}^d$ such that $\mathrm{E}^*_F(\beta) = -\mathrm{E}_F(-\beta')$. A parameter $\beta \in \mathbb{C}^d$ is* dual mixed Gauss–Manin *if it is dual mixed Gauss–Manin along every face.*

Remark 2.8.16 *The proof of Lemma 2.8.9 shows that, at least if $\mathcal{E}^{\mathrm{strong}}_{A,F} = \emptyset$, the condition of being dual mixed Gauss–Manin along F is partially stable in the following sense: If β is dual mixed Gauss–Manin along F with $\mathrm{E}^*_F(\beta) \neq \emptyset$, then $\beta - \alpha$ is also dual mixed Gauss–Manin along F for every $\alpha \in \mathbb{N}A$. Similarly, the proof of Lemma 2.8.10 shows that if β is mixed Gauss–Manin along F with $\mathrm{E}_F(\beta) \neq \emptyset$, then $\beta + \alpha$ is also mixed Gauss–Manin along F for every $\alpha \in \mathbb{N}A$.*

Before stating Theorems 2.8.17 and 2.8.19, we recall the following notation and definitions:

- fSupp and cofSupp denote fiber support and cofiber support, respectively, and were defined in (2.3.1).
- $\widehat{\mathbb{C}^n} := \operatorname{Spec} \mathbb{C}[\partial_1, \dots, \partial_n]$.

- For an open subset $U \subseteq \widehat{\mathbb{C}^n}$ containing T_A, the embeddings $T_A \hookrightarrow U$ and $U \hookrightarrow \widehat{\mathbb{C}^n}$ are denoted by ι_U and ϖ_U, respectively—these were discussed at the start of §2.8.

Theorem 2.8.17 *The following are equivalent for $\beta \in \mathbb{C}^d$:*

(a) *β is dual mixed Gauss–Manin.*

(b) $K_\bullet^A(S_A; E_A - \beta) \overset{qi}{=} \mathrm{FL}(\varpi_{U+}\iota_{U\dagger}\mathcal{O}_{T_A}^\beta)$ *for some open subset $U \subseteq \widehat{\mathbb{C}^n}$ containing T_A.*

(c) $K_\bullet^A(S_A; E_A - \beta) \overset{qi}{=} \mathrm{FL}(\varpi_{U+}\iota_{U\dagger}\mathcal{O}_{T_A}^\beta)$ *for any open subset $U \subseteq \widehat{\mathbb{C}^n}$ satisfying $U \cap X_A = \mathrm{fSupp}\, \hat{K}_\bullet^A(S_A; E_A - \beta)$.*

Proof ((b) $\Longrightarrow$ (a)) Let $F \preceq A$ be a face. If $\mathrm{O}(F)$ is not contained in (hence is disjoint from) U, then by Lemma 2.3.3, the restriction to $\mathrm{O}(F)$ of $\varpi_{U+}\iota_{U\dagger}\mathcal{O}_{T_A}^\beta$ vanishes. Therefore, by the hypothesis, the same applies to the restriction to $\mathrm{O}(F)$ of $\hat{K}_\bullet^A(S_A; E_A - \beta)$. Hence, (2.7.6) from Theorem 2.7.2 implies that $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta+\mathbb{C}F} = 0$. In particular, $\beta \notin \mathcal{E}_{A,F}^{\mathrm{strong}}$ and $\mathrm{E}_F^*(\beta) = \emptyset$.

Next, suppose $\mathrm{O}(F) \subseteq U$. By Proposition 2.6.2, there exists a β', which may be chosen such that $-\beta'$ is not strongly resonant (cf. [8, the discussion preceding Cor. 3.9]), with $\beta - \beta' \in \mathbb{N}A$ and such that $\varphi_\dagger\mathcal{O}_{T_A}^\beta$ is isomorphic to $\hat{K}_\bullet^A(S_A; E_A - \beta')$. We fix such a β'. By Theorem 2.7.2,

$$i_{\mathrm{O}(F)}^+ \hat{K}_\bullet^A(S_A; E_A - \beta) \cong \bigoplus_{\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F} \mathcal{O}_{T_F}^\lambda \otimes_{\mathbb{C}} \mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta-\lambda} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}, \tag{2.8.4}$$

By Theorem 2.7.2 together with Lemma 2.8.1,

$$\begin{aligned} i_{\mathrm{O}(F)}^+ \varpi_{U+}\iota_{U\dagger}\mathcal{O}_{T_A}^\beta &\cong i_{\mathrm{O}(F)}^+ \mathrm{R}\Gamma_U(\varphi_\dagger \mathcal{O}_{T_A}^\beta) \\ &\cong i_{\mathrm{O}(F)}^+ \varphi_\dagger \mathcal{O}_{T_A}^\beta \\ &\cong i_{\mathrm{O}(F)}^+ \hat{K}_\bullet^A(S_A; E_A - \beta') \\ &\cong \bigoplus_{\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F} \mathcal{O}_{T_F}^\lambda \otimes_{\mathbb{C}} \mathrm{R}\Gamma_{\mathrm{O}(F)}(\omega_{S_A}^\bullet)_{\beta'-\lambda} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}. \end{aligned} \tag{2.8.5}$$

The left hand sides of (2.8.4) and (2.8.5) are quasi-isomorphic by hypothesis. Hence, the same is true of the right hand sides of (2.8.4) and (2.8.5)—call this isomorphism ψ. Now, the modules $\mathcal{O}^{\lambda}_{T_A}$ are simple of different weights, and the differentials of $\bigwedge \mathbb{C}^{d_{A/F}}$ are all 0. Therefore, ψ induces a quasi-isomorphism between $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta-\lambda}$ and $\mathrm{R}\Gamma_{\mathrm{O}(F)}(\omega^{\bullet}_{S_A})_{\beta'-\lambda}$ for all $\lambda \in \mathbb{C}F$. But by (2.5.4), we know that $\mathrm{R}\Gamma_{\mathrm{O}(F)}(\omega^{\bullet}_{S_A}) \cong \mathbb{C}\{\mathbb{N}F - \mathbb{N}A\}[-d_{A/F}]$. Hence, $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta-\lambda}$ can have cohomology only in cohomological degree $d_{A/F}$ and is nonzero if and only if $\lambda + \mathbb{Z}F \in -\mathrm{E}_F(-\beta')$. Thus, β is not strongly (A, F)-resonant, and $\mathrm{E}^*_F(\beta) = -\mathrm{E}_F(-\beta')$. Now use Proposition 2.8.6 and Definition 2.8.15.

((a) $\implies$ (c)) Let β' be as above. Consider the morphism

$$\eta = \cdot\partial^{\beta-\beta'} : \hat{K}^A_{\bullet}(S_A; E_A - \beta') \to \hat{K}^A_{\bullet}(S_A; E_A - \beta).$$

Let U be an open subset of $\widehat{\mathbb{C}^n}$ with $U \cap X_A = \mathrm{fSupp}\, \hat{K}^A_{\bullet}(S_A; E_A - \beta)$; such a U exists by Lemma 2.8.8. Then $\mathrm{R}\Gamma_{\widehat{\mathbb{C}^n}\setminus U}\, \hat{K}^A_{\bullet}(S_A; E_A - \beta)$ vanishes by Proposition 2.3.7. So, from the distinguished triangle relating $\mathrm{R}\Gamma_U$ and $\mathrm{R}\Gamma_{\widehat{\mathbb{C}^n}\setminus U}$, we get that $\mathrm{R}\Gamma_U\, \hat{K}^A_{\bullet}(S_A; E_A - \beta)$ is isomorphic to $\hat{K}^A_{\bullet}(S_A; E_A - \beta)$. Thus, it remains to show that $\mathrm{R}\Gamma_U(\eta)$ is an isomorphism.

Now, $\mathrm{R}\Gamma_U(\eta)$ is an isomorphism if and only if its cone vanishes, and cones commute with $\mathrm{R}\Gamma_U$, so we need to show that $\mathrm{R}\Gamma_U(\operatorname{cone}\eta) = 0$. By Proposition 2.3.7, this is true if and only if the fiber support of $\operatorname{cone}\eta$ is disjoint from U. So, we just need to show that $i^+_{\mathrm{O}(F)} \operatorname{cone}\eta = 0$ for all $\mathrm{O}(F) \subseteq U$. Pulling out the cone, we just need to show that $\operatorname{cone}(i^+_{\mathrm{O}(F)}\eta) = 0$ for all $\mathrm{O}(F) \subseteq U$, i.e. that $i^+_{\mathrm{O}(F)}\eta$ is an isomorphism for all $\mathrm{O}(F) \subseteq U$. This is true by Lemma 2.8.9.

((c) $\implies$ (b)) Immediate. ∎

Remark 2.8.18 *Let $\beta \in \mathbb{C}^d$ with $\varphi_{\dagger}\mathcal{O}_{T_A} \cong \hat{K}^A_{\bullet}(S_A; E_A - \beta)$. Then the proof of (a $\implies$ b) in Theorem 2.8.17 shows that $\beta \notin \mathcal{E}_A$, and $\mathrm{E}^*_F(\beta) = -\mathrm{E}_F(-\beta)$ for all $F \preceq A$.*

Theorem 2.8.19 *The following are equivalent for $\beta \in \mathbb{C}^d$:*

(a) β *is mixed Gauss–Manin.*

(b) $K_\bullet^A(S_A; E_A - \beta) \overset{qi}{=} \mathrm{FL}(\varpi_{U\dagger}\iota_{U+}\mathcal{O}_{T_A}^\beta)$ *for some open subset* $U \subseteq \widehat{\mathbb{C}^n}$ *containing* T_A.

(c) $K_\bullet^A(S_A; E_A - \beta) \overset{qi}{=} \mathrm{FL}(\varpi_{U\dagger}\iota_{U+}\mathcal{O}_{T_A}^\beta)$ *for any open subset* $U \subseteq \widehat{\mathbb{C}^n}$ *satisfying* $U \cap X_A = \operatorname{cofSupp} \hat{K}_\bullet^A(S_A; E_A - \beta)$.

Proof ((b) $\Longrightarrow$ (a)) Let $F \preceq A$ be a face. If $\mathrm{O}(F)$ is not contained in (hence disjoint from) U, then $i^\dagger_{\mathrm{O}(F)}\varpi_{U\dagger}\iota_U^+\mathcal{O}_{T_A}^\beta = 0$ by Lemma 2.3.3. So, $\mathrm{E}_F(\beta) = \emptyset$ by the hypothesis and Theorem 2.7.4.

Next, suppose $\mathrm{O}(F)$ is contained in U. Choose a $\beta' \in \beta + \mathbb{Z}^d$ which is not strongly resonant. Then by Theorem 2.7.4,

$$i^\dagger_{\mathrm{O}(F)}\hat{K}_\bullet^A(S_A; E_A - \beta) \cong \bigoplus_{\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F} \mathcal{O}_{T_F}^\lambda \otimes_{\mathbb{C}} S_A[\partial^{-F}]_{\beta-\lambda} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}, \tag{2.8.6}$$

and by Theorem 2.7.4 together with [8, Cor. 3.7] and Lemma 2.8.1,

$$\begin{aligned} i^\dagger_{\mathrm{O}(F)}\varpi_{U\dagger}\iota_{U+}\mathcal{O}_{T_A}^\beta &\cong i^\dagger_{\mathrm{O}(F)}\varpi_{U\dagger}\varpi_U^{-1}\varphi_+\mathcal{O}_{T_A}^\beta \\ &\cong i^\dagger_{\mathrm{O}(F)}\varphi_+\mathcal{O}_{T_A}^\beta \\ &\cong i^\dagger_{\mathrm{O}(F)}\hat{K}_\bullet^A(S_A; E_A - \beta') \\ &\cong \bigoplus_{\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F} \mathcal{O}_{T_F}^\lambda \otimes_{\mathbb{C}} S_A[\partial^{-F}]_{\beta'-\lambda} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}. \end{aligned} \tag{2.8.7}$$

The left hand sides of (2.8.6) and (2.8.7) are isomorphic by hypothesis. Hence, the same is true of the right hand sides—call this isomorphism ψ. As in the proof of Theorem 2.8.17, the modules $\mathcal{O}_{T_A}^\lambda$ are simple of different weights, and the differentials of $\bigwedge \mathbb{C}^{d_{A/F}}$ are all 0. Therefore, ψ induces an isomorphism between $S_A[\partial^{-F}]_{\beta-\lambda}$ and $S_A[\partial^{-F}]_{\beta'-\lambda}$. Now use the definition of E_F.

((a) $\Longrightarrow$ (c)) Let β' be as above. Consider the morphism

$$\eta = \cdot\partial^{\beta'-\beta} \colon \hat{K}_\bullet^A(S_A; E_A - \beta') \to \hat{K}_\bullet^A(S_A; E_A - \beta).$$

Let U be a Zariski open subset of $\widehat{\mathbb{C}^n}$ with $U \cap X_A = \operatorname{cofSupp} \hat{K}_\bullet^A(S_A; E_A - \beta)$; such a U exists by [27, Prop. 2.2 (4)] and the orbit-cone correspondence. Now use the same argument as in the proof of Theorem 2.8.17 with $\mathbb{D}\hat{K}^A_\bullet(S_A; E_A - \beta)$, $\mathbb{D}\eta$, and Lemma 2.8.10 in place of $\hat{K}^A_\bullet(S_A; E_A - \beta)$, η, and Lemma 2.8.9, respectively.

((c) $\Longrightarrow$ (b)) Immediate. ■

The following example shows that in general, not every β is mixed or dual mixed Gauss–Manin even if S_A is Cohen–Macaulay.

Example 2.8.20 *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

The associated semigroup ring S_A is Cohen–Macaulay but not normal. For simplicity, we only discuss $\beta \in \mathbb{Z}^2$. There are 8 isomorphism classes—these are pictured in Figure 2.1 and were computed using [27, Th. 2.1] together with direct calculation (note that although [27] assumes homogeneity, [28, Th. 3.4.4] shows that this assumption may be removed). Of these, only the first four (numbered from left to right then top to bottom) are mixed Gauss–Manin, and only these first four are dual mixed Gauss–Manin. The fiber supports of the 8 classes are, in order,

$$\mathrm{O}(A), \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3]), \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_1]), \quad X_A,$$
$$\mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3]), \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3]), \quad X_A, \quad X_A.$$

The cofiber supports of the 8 classes are, in order,

$$X_A, \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_1]), \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3]), \quad \mathrm{O}(A)$$
$$\mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_1]) \cup \mathrm{O}([\mathbf{a}_3]), \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_1]) \cup \mathrm{O}([\mathbf{a}_3])$$
$$\mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3]), \quad \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3]).$$

The fiber supports were computed using Macaulay2 ([29]) by restricting the various modules $\hat{M}_A(\beta)$ to the various distinguished points $\mathbb{1}_F$ and asking whether or not the result vanished. To compute the cofiber supports, we implemented [28, Algorithms 3.4.2 and 3.4.3] in Macaulay2.

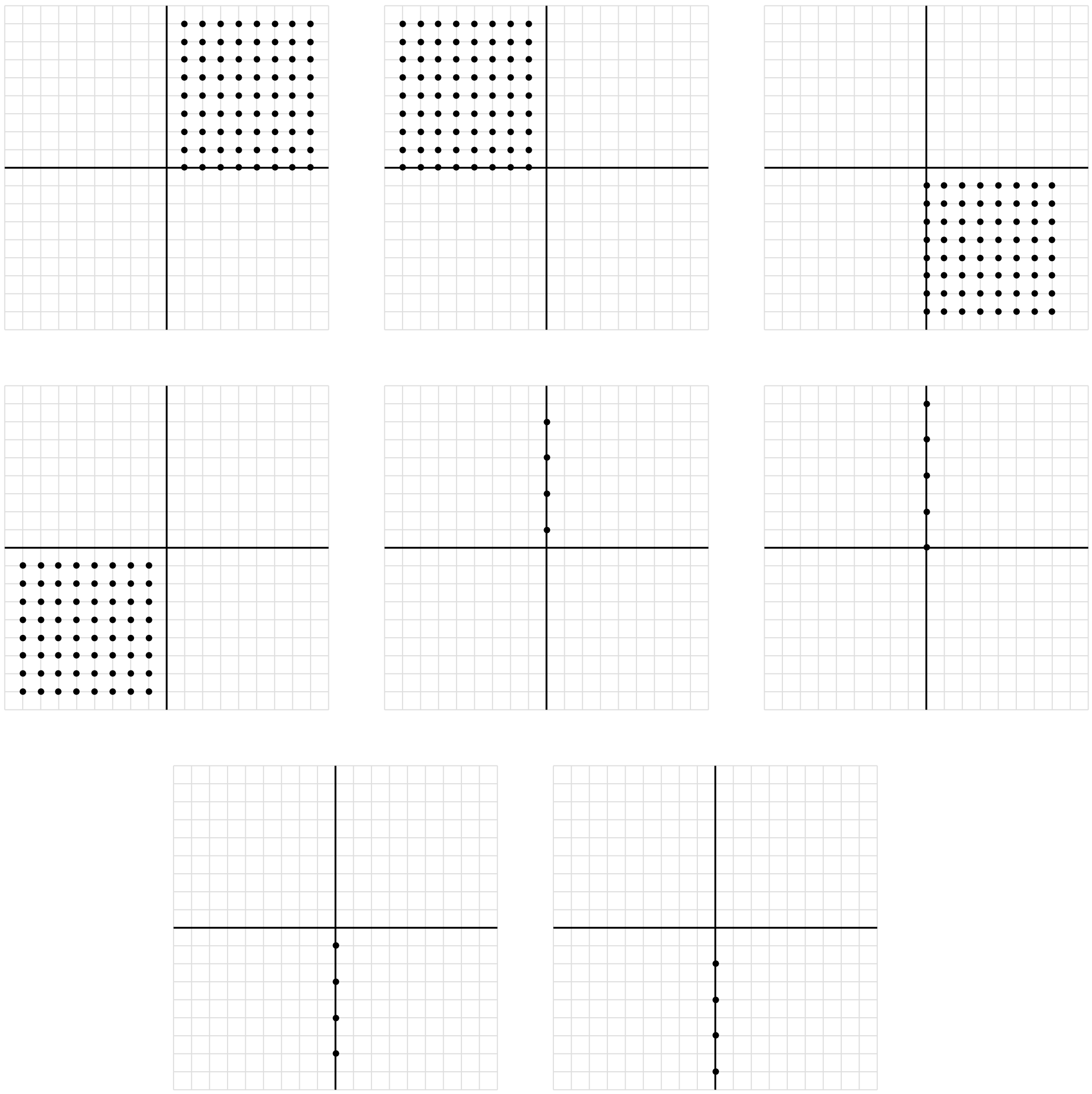


Figure 2.1. The eight integral isomorphism classes from Example 2.8.20.

2.9 Normal Case

In this section, we prove (Theorem 2.9.3) that if S_A is normal, then every parameter β is both mixed Gauss–Manin and dual mixed Gauss–Manin. Lemma 2.9.1 provides an explicit description of the fiber and cofiber supports of $\hat{M}_A(\beta)$ and computes the restrictions of $\hat{M}_A(\beta)$ to the various orbits. In a future paper, we will apply Theorem 2.9.3 to compute for such A the projection and restriction of $M_A(\beta)$ to coordinate subspaces of the form $\mathbb{C}^F$, where F is a face of A; and, if A is in addition homogeneous, to show that the holonomic dual of $M_A(\beta)$ is itself A-hypergeometric.

Recall that for a facet $G \preceq \mathbb{N}A$, there is a unique linear form $h_G \colon \mathbb{Z}^d \to \mathbb{Z}$, called the *primitive integral support function* of G, satisfying the following conditions:

1. $h_G(\mathbb{Z}^d) = \mathbb{Z}$.
2. $h_G(\mathbf{a}_i) \geq 0$ for all i.
3. $h_G(\mathbf{a}_i) = 0$ for all $\mathbf{a}_i \in G$.

Lemma 2.9.1 *Assume S_A is normal. Let $\beta \in \mathbb{C}^d$ and $F \preceq A$.*

(a) $i^+_{\mathrm{O}(F)}\hat{M}_A(\beta)$ *is either zero or isomorphic to* $\mathcal{O}^\lambda_{T_F} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[-d_{A/F}]$ *for some (equiv. any)* $\lambda \in \mathbb{C}F$ *with* $\beta - \lambda \in \mathbb{Z}^d$.

(b) $i^\dagger_{\mathrm{O}(F)}\hat{M}_A(\beta)$ *is either zero or isomorphic to* $\mathcal{O}^\lambda_{T_F} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}$ *for some (equiv. any)* $\lambda \in \mathbb{C}F$ *with* $\beta - \lambda \in \mathbb{Z}^d$.

(c) $\mathrm{O}(F) \subseteq \mathrm{fSupp}\,\hat{M}_A(\beta)$ *if and only if* $(\beta + \mathbb{C}F) \cap \mathbb{Z}^d \neq \emptyset$ *and* $h_G(\beta) \in \mathbb{Z}_{<0}$ *for every facet* $G \succeq F$.

(d) $\mathrm{O}(F) \subseteq \mathrm{cofSupp}\,\hat{M}_A(\beta)$ *if and only if* $(\beta + \mathbb{C}F) \cap \mathbb{Z}^d \neq \emptyset$ *and* $h_G(\beta) \in \mathbb{N}$ *for every facet* $G \succeq F$.

Proof Before proving the statements, notice that because S_A is normal, it is Cohen–Macaulay by [30, Theorem 1]. Therefore, $\hat{M}_A(\beta) \overset{\mathrm{qi}}{=} \hat{K}^A_\bullet(S_A; E_A - \beta)$ by [21, Th. 6.6].

(a) Since S_A is Cohen–Macaulay, the complex $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)$ has cohomology only in cohomological degree $d_{A/F}$, so that

$$\mathrm{E}^*_F(\beta) = \left\{ \lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F \;\middle|\; H^{d_{A/F}}_{\mathrm{O}(F)}(S_A)_{\beta-\lambda} \neq 0 \right\}. \tag{2.9.1}$$

Suppose $\lambda + \mathbb{Z}F, \lambda' + \mathbb{Z}F \in \mathrm{E}^*_F(\beta)$. Then λ and λ' differ by an element $\mathbb{C}F \cap \mathbb{Z}^d$. But by normality, $\mathbb{C}F \cap \mathbb{Z}^d = \mathbb{Z}F$. Hence, $\lambda + \mathbb{Z}F = \lambda' + \mathbb{Z}F$. Now apply Theorem 2.7.2, and use (2.9.1) along with the fact that the Hilbert function of $H^{d_{A/F}}_{\mathrm{O}(F)}(S_A)$ takes values in $\{0, 1\}$.

(b) As in (a), normality implies that $\mathrm{E}_F(\beta)$ has at most one element (this also follows from [27, Prop. 2.3 (1)]). Now apply Theorem 2.7.4 along with the fact that the Hilbert function of $S_A[\partial^{-F}]$ takes values in $\{0, 1\}$.

(c) By Theorem 2.7.2, we need to show that $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta+\mathbb{C}F} \neq 0$ if and only if $h_G(\beta) \in \mathbb{Z}_{<0}$ for all facets $G \succeq F$. As in (a), $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)$ is concentrated in cohomological degree $d_{A/F}$. Since S_A is normal, $H^{d_{A/F}}_{\mathrm{O}(F)}(S_A) = \mathbb{C}\{-\operatorname{relint}(\mathbb{N}A - \mathbb{N}F)\}$, where relint denotes the relative interior of an affine semigroup (i.e. the set of points in the affine semigroup which are not on any of its facets). In terms of the primitive integral support functions, $-\operatorname{relint}(\mathbb{N}A - \mathbb{N}F)$ consists of those points $\alpha \in \mathbb{Z}^d$ such that $h_G(\alpha) < 0$ for all facets of A which contain F. Thus, $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta+\mathbb{C}F} \neq 0$ if and only if there exists a $\lambda \in \mathbb{C}F$ with $\beta - \lambda \in \mathbb{Z}^d$ such that $h_G(\beta - \lambda) \in \mathbb{Z}_{<0}$ for all facets $G \succeq F$. But h_G kills $\mathbb{C}F$ by definition, and $\beta + \mathbb{C}^d$ intersects $\mathbb{Z}^d$ by assumption. So, $\mathrm{R}\Gamma_{\mathrm{O}(F)}(S_A)_{\beta+\mathbb{C}F} \neq 0$ if and only if $h_G(\beta) \in \mathbb{Z}_{<0}$ for all facets $G \succeq F$.

(d) The proof of [27, Th. 5.2] shows that $\mathrm{E}_F(\beta)$ is non-empty if and only if $(\beta + \mathbb{C}F) \cap \mathbb{Z}^d \neq \emptyset$ and $h_G(\beta) \in \mathbb{N}$ for every facet $G \succeq F$. Now use Theorem 2.7.4. ■

The condition $(\beta + \mathbb{C}F) \cap \mathbb{Z}^d \neq \emptyset$ in Lemma 2.9.1(c) and (d) is necessary, as the following example shows:

Example 2.9.2 *Choose a matrix A generating the affine semigroup pictured in Figure 2.2. As in the figure, denote by G_1 and G_2, respectively, the facets $\begin{bmatrix} 2 & 3 \end{bmatrix}^\top$ and $\begin{bmatrix} 2 & -1 \end{bmatrix}^\top$ of A. Then $h_{G_1} = \begin{bmatrix} 3 & -2 \end{bmatrix}$ and $h_{G_2} = \begin{bmatrix} 1 & 2 \end{bmatrix}$.*

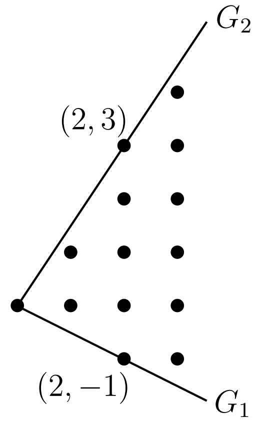


Figure 2.2. The affine semigroup from Example 2.9.2.

Consider the parameter $\beta = (-1, -1/2)$. *Then* $h_{G_1}(\beta) = -3 - 2(-1/2) = -2 \in \mathbb{Z}_{<0}$ *and* $h_{G_2}(\beta) = -1 + 2(-1/2) = -2 \in \mathbb{Z}_{<0}$, *so by Lemma 2.9.1(c), the fiber support of* $\hat{M}_A(\beta)$ *contains both* $\mathrm{O}(G_1)$ *and* $\mathrm{O}(G_2)$. *But* $\beta + \mathbb{C}\emptyset = \beta \notin \mathbb{Z}^2$, *so by Lemma 2.9.1(c), the fiber support of* $\hat{M}_A(\beta)$ *does not contain* $\mathrm{O}(\emptyset)$.

To see the necessity of the condition for Lemma 2.9.1(d), use a similar argument with the same A *for* $\beta = (1, 1/2)$.

Theorem 2.9.3 *Assume* S_A *is normal. Let* $\beta \in \mathbb{C}^d$, *let* $U \subseteq \widehat{\mathbb{C}^n}$ *be an open subset with* $U \cap X_A = \operatorname{fSupp} \hat{M}_A(\beta)$, *and let* $V \subseteq \widehat{\mathbb{C}^n}$ *be an open subset with* $V \cap X_A = \operatorname{cofSupp} \hat{M}_A(\beta)$. *Then*

$$\mathrm{FL}(\varpi_{V\dagger}\iota_{V+}\mathcal{O}_{T_A}^{\beta}) \cong M_A(\beta) \cong \mathrm{FL}(\varpi_{U+}\iota_{U\dagger}\mathcal{O}_{T_A}^{\beta}).$$

Proof As in Lemma 2.9.1, $\hat{M}_A(\beta) \overset{\mathrm{qi}}{=} \hat{K}_\bullet^A(S_A; E_A - \beta)$. By [21, Th. 6.6], this implies that $\mathcal{E}_A = \emptyset$.

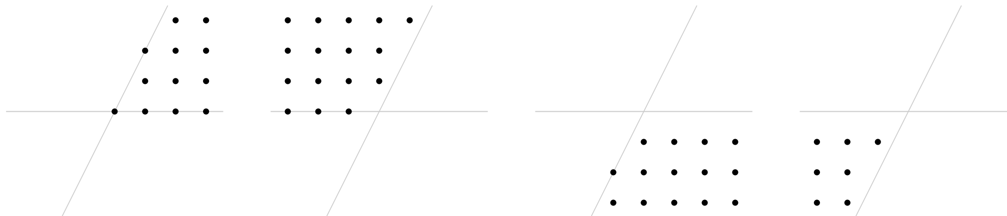
To prove the first isomorphism, choose a $\beta' \in \mathbb{C}^d \setminus \mathrm{sRes}(A)$ with $\beta' - \beta \in \mathbb{N}A$ (this is always possible—see [8, the discussion preceding Cor. 3.9]). Let $F \preceq A$ be a face. By Lemma 2.8.14(b), we have $\mathrm{E}_F(\beta) \subseteq \mathrm{E}_F(\beta')$, and by Lemma 2.9.1(b), both $\mathrm{E}_F(\beta)$ and $\mathrm{E}_F(\beta')$ consist of at most one element. Therefore, if $\mathrm{E}_F(\beta)$ is non-empty, then it equals $\mathrm{E}_F(\beta')$. Hence, β is mixed Gauss–Manin along F. Thus, β is mixed Gauss–Manin.

We now prove the second isomorphism. As in the proof of ((b) $\Longrightarrow$ (a)) in Theorem 2.8.17, choose a $-\beta' \in \mathbb{C}^d \setminus \mathrm{sRes}(A)$ with $\beta - \beta' \in \mathbb{N}A$ such that $\varphi_\dagger \mathcal{O}_{T_A}^{\beta}$ is isomorphic to $\hat{M}_A(\beta')$. Now proceed as for the first isomorphism, using Lemma 2.8.14(a), E_F^*, and Lemma 2.9.1(a) in place of Lemma 2.8.14(b), E_F, and Lemma 2.9.1(b), respectively. ■

Example 2.9.4 *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The associated semigroup ring S_A *is a normal. For simplicity, we only discuss* $\beta \in \mathbb{Z}^2$.



$U = (\mathbb{C}^*)^3$	$U = \widehat{\mathbb{C}^3} \setminus \{\partial_3\text{-axis}\}$	$U = \widehat{\mathbb{C}^3} \setminus \{\partial_1\text{-axis}\}$	$U = \widehat{\mathbb{C}^3}$
$V = \widehat{\mathbb{C}^3}$	$V = \widehat{\mathbb{C}^3} \setminus \{\partial_1\text{-axis}\}$	$V = \widehat{\mathbb{C}^3} \setminus \{\partial_3\text{-axis}\}$	$V = (\mathbb{C}^*)^3$

Figure 2.3. The four isomorphism classes from Example 2.9.4. The lines are the spans of the two facets of $\mathbb{R}_{\geq 0}A$.

There are four isomorphism classes of A-hypergeometric systems with $\beta \in \mathbb{Z}^2$; these are pictured in Figure 2.3 along with a U and a V as in Theorem 2.9.3, and they were computed using [27, Th. 5.2]. We now explain why these U and V work by computing the fiber and cofiber supports, using Lemma 2.9.1, for each of the four isomorphism classes.

The primitive integral support function corresponding to the facets $[\mathbf{a}_1]$ and $[\mathbf{a}_3]$ are $h_1(t_1, t_2) = t_2$ and $h_2(t_1, t_2) = 2t_1 - t_2$, respectively. If $M_A(\beta)$ is in the first (counted from left to right in Figure 2.3) isomorphism class, then $h_1(\beta)$ and $h_2(\beta)$ are both in $\mathbb{N}$, so $\operatorname{fSupp} \hat{M}_A(\beta) = \mathrm{O}(A)$ and $\operatorname{cofSupp} \hat{M}_A(\beta) = X_A$. If $M_A(\beta)$ is in the second class, then $h_1(\beta) \in \mathbb{N}$ and $h_2(\beta) \in \mathbb{Z}_{<0}$, so $\operatorname{fSupp} \hat{M}_A(\beta) = \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3])$ and $\operatorname{cofSupp} \hat{M}_A(\beta) = \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_1])$. If $M_A(\beta)$ is in the third class, then $h_1(\beta) \in \mathbb{Z}_{<0}$ and $h_2(\beta) \in \mathbb{N}$, so $\operatorname{fSupp} \hat{M}_A(\beta) = \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_1])$ and $\operatorname{cofSupp} \hat{M}_A(\beta) = \mathrm{O}(A) \cup \mathrm{O}([\mathbf{a}_3])$. If $M_A(\beta)$ is in the fourth class, then $h_1(\beta)$ and $h_2(\beta)$ are both in $\mathbb{Z}_{<0}$, so $\operatorname{fSupp} \hat{M}_A(\beta) = X_A$ and $\operatorname{cofSupp} \hat{M}_A(\beta) = \mathrm{O}(A)$.

3. DUALIZING, PROJECTING, AND RESTRICTING GKZ SYSTEMS[1]

3.1 Introduction

Our approach to studying the projection, restriction, and holonomic dual of A-hypergeometric systems is to use the notion of mixed and dual mixed Gauss–Manin systems (see §3.2.3) from Chapter 2.

We first study these in slightly more generality in §3.3. In §3.4, we generalize the notion of quasi-equivariant D-module (introduced by T. Reichelt and U. Walther in [26]) to what we are calling twistedly quasi-equivariant D-modules (Definition 3.4.2). We then follow a similar process to that in [26] to relate the restriction and projection of such modules (Lemma 3.4.4) and to show that mixed and dual mixed Gauss–Manin systems are twistedly quasi-equivariant (Proposition 3.4.5). These results are combined in §3.5 first to compute the restriction and projection to $\mathbb{C}^F$ of dual mixed Gauss–Manin and mixed Gauss–Manin systems, respectively (Theorem 3.5.4), and then to do the same for normal A-hypergeometric systems (Theorem 3.5.8).

We then show in Theorem 3.6.3, again using the notion of mixed and dual mixed Gauss–Manin systems, that if A is homogeneous and normal, the holonomic dual of every A-hypergeometric system is itself A-hypergeometric.

3.2 Notation and conventions

In §3.2.1, we define various notations and conventions related to varieties, derived categories, D-modules, and mixed Hodge modules. §3.2.4 recalls the notions of fiber and cofiber support. §3.2.2 defines various notations related to the semigroup $\mathbb{N}A$,

[1]A version of this chapter will be appearing in *Journal of Pure and Applied Algebra* as [1].

and in §3.2.3 we recall and discuss the notions of mixed and dual mixed Gauss–Manin parameters and systems.

3.2.1 General geometric conventions/notation

Varieties, smooth or otherwise, are not required to be irreducible, are defined over $\mathbb{C}$, and are always considered with the Zariski topology. The closure of a subset Z of a topological space X is written $\overline{Z}$. If X is a smooth variety, denote by $\mathcal{D}_X$ its sheaf of algebraic linear partial differential operators. A subset Z of a topological space X is *relatively open* if it is an open subset of its closure.

Derived categories

The category of mixed Hodge modules on a variety X is denoted $\mathrm{MHM}(X)$. The bounded derived category of $\mathrm{MHM}(X)$ is denoted $\mathrm{D^b\,MHM}(X)$. If X is smooth, the bounded derived category of $\mathcal{D}_X$-modules with coherent and holonomic cohomology are denoted by $\mathrm{D^b_c}(\mathcal{D}_X)$ and $\mathrm{D^b_h}(\mathcal{D}_X)$, respectively. If Z is a closed subvariety, a superscript Z in the notation for any of these categories denotes the full subcategory of objects whose cohomology is supported in Z.

D-module functors

(cf. [22]) The holonomic duality functor ([22, Def. 2.6.1]) is denoted $\mathbb{D}$. Let $f\colon X \to Y$ be a morphism of smooth varieties. We write f_+ for the D-module direct image, $f_\dagger \coloneqq \mathbb{D} f_+ \mathbb{D}$ for the D-module exceptional direct image, $f^+ \coloneqq \mathrm{L}f^*[\dim X - \dim Y]$ for the D-module inverse image, and $f^\dagger \coloneqq \mathbb{D} f^+ \mathbb{D}$ for the D-module exceptional direct image. If X_1 and X_2 are smooth varieties and $\mathcal{M}_i^\bullet \in \mathrm{D}^b(\mathcal{D}_{X_i})$ $(i = 1, 2)$, the exterior tensor product (see [22, p38]) of $\mathcal{M}_1^\bullet$ and $\mathcal{M}_2^\bullet$ is

$$\mathcal{M}_1^\bullet \boxtimes \mathcal{M}_2^\bullet \coloneqq \mathcal{D}_{X_1 \times X_2} \otimes_{p_1^{-1}\mathcal{D}_{X_1} \otimes_{\mathbb{C}} p_2^{-1}\mathcal{D}_{X_2}} (p_1^{-1}\mathcal{M}_1^\bullet \otimes_{\mathbb{C}} p_2^{-1}\mathcal{M}_2^\bullet).$$

Note that [22] denotes the functors f_+, f^+, $f_\dagger$, and $f^\dagger$ by $\int_f$, $f^\dagger$, $\int_{f!}$, and $f^\bigstar$, respectively. They define the first two on pages 33 and 40, respectively, while they define the second two in Def. 3.2.13 on page 91.

Fourier–Laplace transform

(cf. [31, pp85-102]) The Fourier–Laplace transform is denoted by FL. By definition, $\mathrm{FL}(\mathcal{M}^\bullet)$ is the pullback of $\mathcal{M}^\bullet \in \mathrm{D}^b(\mathcal{D}_{\mathbb{C}^n})$ by the $\mathbb{C}$-algebra automorphism of $\mathcal{D}_{\mathbb{C}^n}$ taking $x_i \mapsto \partial/\partial x_i$ and $\partial/\partial x_i$ to $-x_i$. The inverse Fourier transform is denoted by FL^{-1} and is defined similarly.

For a description of FL in terms of D-module direct and inverse image functors, see [19].

Mixed Hodge modules

Let $\mathcal{M}^\bullet$ be a complex of mixed Hodge modules, and let F be a functor of D-modules. If the mixed Hodge module structure on $\mathcal{M}^\bullet$ induces a mixed Hodge module structure on $F(\mathcal{M}^\bullet)$, we will always take $F(\mathcal{M}^\bullet)$ to be this induced mixed Hodge module unless otherwise specified.

3.2.2 Toric and GKZ conventions/notation

The semigroup ring of A is $S_A \coloneqq \mathbb{C}[\mathbb{N}A] = \mathbb{C}[\partial_1, \ldots, \partial_n]/I_A$. The toric variey of A is $X_A \coloneqq \mathrm{Var}(I_A)$, and the torus of A is $T_A \coloneqq \operatorname{Spec}\mathbb{C}[\mathbb{Z}A]$. Given $\beta \in \mathbb{C}^d$, define the $\mathcal{D}_{T_A}$-module

$$\mathcal{O}^\beta_{T_A} \coloneqq \mathcal{D}_{T_A}/\mathcal{D}_{T_A}\{\, t_i\partial_{t_i} + \beta_i \mid i = 1, \ldots, d\,\} = \mathcal{O}_{T_A} t^{-\beta}. \tag{3.2.1}$$

Note that $\mathcal{O}^\beta_{T_A}$ can be defined in a coordinate-free manner (see Equation (2.2.9)). Set

$$\hat{\mathcal{M}}_A(\beta) \coloneqq \mathrm{FL}^{-1}(\mathcal{M}_A(\beta)). \tag{3.2.2}$$

Definition 3.2.1 *A submatrix F of A is called a* face *of A, written $F \preceq A$, if F has d rows and $\mathbb{R}_{\geq 0}F$ is a face of $\mathbb{R}_{\geq 0}A$. A* facet *of A is a face of rank $d-1$.*

Given a face $F \preceq A$, set

$$d_{A/F} \coloneqq d - \operatorname{rank} F = d - \dim \mathbb{R}_{\geq 0}F, \tag{3.2.3}$$

and

$$n_{A/F} \coloneqq n - \#(\text{columns of } F) = n - \dim \mathbb{C}^F. \tag{3.2.4}$$

The torus embedding $t \mapsto (t^{\mathbf{a}_1}, \ldots, t^{\mathbf{a}_n})$ of T_A into $\mathbb{C}^n$ defined by A induces an action of T_A on $\mathbb{C}^n$ which makes T_A-equivariant the inclusion $X_A \subseteq \mathbb{C}^n$. If $F \preceq A$ is a face, the T_A-orbit of X_A corresponding to F is

$$\mathrm{O}_A(F) \coloneqq T_A \cdot \mathbb{1}_F, \tag{3.2.5}$$

where the ith coordinate of $\mathbb{1}_F$ is 1 if $\mathbf{a}_i \in F$ and 0 otherwise. Set

$$\mathbb{C}^F \coloneqq \{\, x \in \mathbb{C}^n \mid x_i = 0 \text{ for all } \mathbf{a}_i \notin F \,\}. \tag{3.2.6}$$

Definition 3.2.2 *For a facet $G \preceq \mathbb{N}A$, there is a unique linear form $h_G = h_{G,A} \colon \mathbb{Z}^d \to \mathbb{Z}$, called the* primitive integral support function *of G, satisfying the following conditions:*

1. $h_G(\mathbb{Z}^d) = \mathbb{Z}$.

2. $h_G(\mathbf{a}_i) \geq 0$ *for all* i.

3. $h_G(\mathbf{a}_i) = 0$ *for all* $\mathbf{a}_i \in G$.

Euler–Koszul complex

We recall the definition of Euler–Koszul complex $\mathcal{K}^A_\bullet(S_A; E_A - \beta)$ from [21]:

$$\mathcal{K}^A_\bullet(S_A; E_A - \beta) \coloneqq K_\bullet\big(\cdot(E_A - \beta); \mathcal{D}_{\mathbb{C}^n}/\mathcal{D}_{\mathbb{C}^n} I_A\big);$$

i.e. it is the Koszul complex of left $\mathcal{D}_{\mathbb{C}^n}$-modules defined by the (right) action of the sequence $E_A - \beta = E_1 - \beta_1, \ldots, E_d - \beta_d$ on the left $\mathcal{D}_{\mathbb{C}^n}$-module $\mathcal{D}_{\mathbb{C}^n}/\mathcal{D}_{\mathbb{C}^n} I_A$. The more general Euler–Koszul complexes defined in [21] will not be needed. The inverse Fourier–Laplace transform of $\mathcal{K}_\bullet^A(S_A; E_A - \beta)$ is denoted by $\hat{\mathcal{K}}_\bullet^A(S_A; E_A - \beta)$. Recall also that the zeroth homology sheaf of the Euler–Koszul complex $\mathcal{K}_\bullet^A(S_A; E_A - \beta)$ is exactly the A-hypergeometric system $\mathcal{M}_A(\beta)$. Moreover, if S_A is Cohen–Macaulay (in particular, by Hochster's Theorem, if S_A is normal), then $\mathcal{K}_\bullet^A(S_A; E_A - \beta)$ is actually a resolution of $\mathcal{M}_A(\beta)$ [21, Th. 6.6].

3.2.3 Mixed and dual mixed Gauss–Manin systems

Given a T_A-stable open neighborhood $U \subseteq \mathbb{C}^n$ of T_A and a $\beta \in \mathbb{C}^d$, set

$$\mathrm{MGM}(U, \beta) \coloneqq \varpi_\dagger \iota_+ \mathcal{O}_{T_A}^\beta \quad \text{and} \quad \mathrm{MGM}^*(U, \beta) \coloneqq \varpi_+ \iota_\dagger \mathcal{O}_{T_A}^\beta, \tag{3.2.7}$$

where $\iota \colon T_A \hookrightarrow U$ is the torus embedding and $\varpi \colon U \hookrightarrow \mathbb{C}^n$ is inclusion.

Definition 3.2.3 *A complex $\mathcal{M}^\bullet \in \mathrm{D}_\mathrm{h}^\mathrm{b}(\mathcal{D}_{\mathbb{C}^n})$ is* mixed Gauss–Manin *(resp.* dual mixed Gauss–Manin*) if it is isomorphic to* $\mathrm{MGM}(U, \beta)$ *(resp.* $\mathrm{MGM}^*(U, \beta)$ *for some U and β.*

Definition 3.2.4 *A parameter $\beta \in \mathbb{C}^d$ is* mixed Gauss–Manin *(resp.* dual mixed Gauss–Manin*) if $\hat{\mathcal{K}}_A(S_A; E_A - \beta)$ is mixed Gauss–Manin (resp. dual mixed Gauss–Manin).*

Note that the definitions of mixed and dual mixed Gauss–Manin parameters in Definition 2.8.15 is different than that in Definition 3.2.4. However, the two definitions are equivalent by Theorems 2.8.17 and 2.8.19.

3.2.4 Fiber and cofiber support

We recall from Definition 2.3.1 the notions of fiber and cofiber support—refer there for main properties along with additional examples. The *fiber support* of a (bounded) complex $\mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules is

$$\operatorname{fSupp} \mathcal{M}^\bullet := \left\{ x \in X \;\middle|\; k(x) \otimes^{\mathrm{L}}_{\mathcal{O}_{X,x}} \mathcal{M}^\bullet_x \neq 0 \right\}, \tag{3.2.8}$$

where $k(x)$ denotes the residue field of the point $x \in X$. If $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$, its *cofiber support* is

$$\operatorname{cofSupp} \mathcal{M}^\bullet := \operatorname{fSupp} \mathbb{D}\mathcal{M}^\bullet. \tag{3.2.9}$$

Note that both the fiber support and cofiber support are independent of the complex representing the object $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)$.

Example 3.2.5 *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ *and* $\beta = (-1,1)^\top$. *We describe the fiber and cofiber support of* $\hat{\mathcal{M}}_A(\beta)$ *using Lemma 2.9.1. The facets of* A *are* $F_1 = [\mathbf{a}_1]$ *and* $F_2 = [\mathbf{a}_4]$, *and their primitive integral support functions (see Definition 3.2.2) are* $h_1(x,y) = y$ *and* $h_2(x,y) = 3x - y$, *resp. Applying these to the vector* β, *we get* $h_1(\beta) = 1 \in \mathbb{N}$ *and* $h_2(\beta) = -4 \in \mathbb{Z}_{<0}$. *Therefore,* $\mathrm{O}_A(F_1)$ *is in the cofiber support but not the fiber support,* $\mathrm{O}_A(F_2)$ *is in the fiber support but not in the cofiber support,* $\mathrm{O}_A(\emptyset)$ *is in neither, and* $\mathrm{O}_A(A)$ *is in both. In summary,*

$$\operatorname{fSupp} \hat{\mathcal{M}}_A(\beta) = \mathrm{O}_A(F_2) \cup \mathrm{O}_A(A)$$

and

$$\operatorname{cofSupp} \hat{\mathcal{M}}_A(\beta) = \mathrm{O}_A(F_1) \cup \mathrm{O}_A(A).$$

3.3 Alternating direct images

In this section we discuss a generalization of mixed and dual mixed Gauss–Manin systems which we will refer to by the name “alternating direct images”.

In §3.3.1, we characterize in terms of fiber and cofiber support when a D-module or mixed Hodge module is isomorphic to a given alternating direct image.

In §3.3.2, we use the results of §3.3.1 to characterize, under a certain openness condition, when a D-module or mixed Hodge module is isomorphic to *some* alternating direct image.

In §3.3.3, we specialize Corollaries 3.3.6 and 3.3.7 to the GKZ case (Theorem 3.3.8). As a consequence, we obtain Corollary 3.3.9, which states that for GKZ systems, being dual mixed Gauss–Manin is the same as being mixed Gauss–Manin and not rank-jumping.

3.3.1 Characterizing alternating direct images passing through a fixed U

Let

$$Z \xrightarrow{\iota} U \xrightarrow{\varpi} X$$

be inclusions of smooth (locally closed) subvarieties, where U is open in X, and set $\varphi := \varpi \circ \iota$. We associate to this situation the *alternating direct image* functors $\varpi_+\iota_\dagger$ and $\varpi_\dagger\iota_+$.

Remark 3.3.1 *Note that if $\mathcal{N}^\bullet$ is in $\mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ or $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$, then $\varphi^+\mathcal{N}^\bullet$ is canonically isomorphic to $\varphi^\dagger\mathcal{N}^\bullet$. To see this, notice that because ϖ is an open embedding, $\varpi^\dagger = \varpi^+$; now shrink U so that ι is a closed immersion, then apply Kashiwara's equivalence.*

Lemma 3.3.2 *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$). Then $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ is the unique object in $\mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$) such that*

1. *the restriction to Z is isomorphic to $\mathcal{M}^\bullet$;*

2. *the fiber support is contained in U; and*

3. *the cofiber support intersected with U is contained in Z.*

Proof We first show that $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ satisfies the required properties. Because both ι and ϖ are inclusions of (locally closed) subvarieties, $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ is supported on $\overline{Z}$. Applying φ^+ to $\varpi_+\iota_\dagger\mathcal{M}^\bullet$, we get

$$\varphi^+\varpi_+\iota_\dagger\mathcal{M}^\bullet = \iota^+\iota_\dagger\mathcal{M}^\bullet = \iota^\dagger\iota_\dagger\mathcal{M}^\bullet = \mathcal{M}^\bullet,$$

where the second equality follows for the same reason as in Theorem 3.3.1. So the restriction to Z is $\mathcal{M}^\bullet$. Let i_x denote inclusion of a point $x \in X$. If $x \notin U$, then $i_x^+\varpi_+\iota_\dagger\mathcal{M}^\bullet$ vanishes by Lemma 2.3.3, so the fiber support is contained in U. If $x \in U \setminus Z$, then also by Lemma 2.3.3,

$$i_x^\dagger\varpi_+\iota_\dagger\mathcal{M}^\bullet = i_x^\dagger\iota_\dagger\mathcal{M}^\bullet = 0.$$

So, the cofiber support intersected with U is contained in Z.

We now prove uniqueness. Suppose $\mathcal{N}^\bullet$ also satisfies the properties. Then the equality of $\varphi^+\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ induces a morphism $f\colon \iota_\dagger\mathcal{M}^\bullet \to \varpi^+\mathcal{N}^\bullet$. By property 3, $i_x^\dagger f = 0$ for all $x \in U \setminus Z$, while by property 1, the restriction $\iota^+ f$ is an equality. Hence, cone(f) has empty fiber support, and therefore it vanishes by Corollary 2.3.6. Thus, f is an isomorphism. By duality, the same argument applied to the case $Z = U$ and $\mathcal{M}^\bullet = \varpi^+\mathcal{N}^\bullet$ gives an isomorphism $\mathcal{N}^\bullet \to \varpi_+\iota_\dagger\mathcal{M}^\bullet$. ■

Lemma 3.3.3 *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$). Then $\varpi_\dagger\iota_+\mathcal{M}^\bullet$ is the unique object in $\mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$) such that*

1. *the restriction to Z equals $\mathcal{M}^\bullet$;*
2. *the cofiber support is contained in U; and*
3. *the fiber support intersected with U is contained in Z.*

Proof This follows from Lemma 3.3.2 by duality. ■

Remark 3.3.4 *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$. Lemmas 3.3.2 and 3.3.3 imply that if there are open neighborhoods U and U' of Z such that $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ and $\varpi'_\dagger\iota'_+\mathcal{M}^\bullet$ are isomorphic as $\mathcal{D}_X$-modules, then they are also isomorphic as mixed Hodge modules.*

Finally, we relate the fiber (resp. cofiber) supports of $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ and $\varphi_\dagger\mathcal{M}^\bullet$ (resp. of $\varpi_\dagger\iota_+\mathcal{M}^\bullet$ and $\varphi_+\mathcal{M}^\bullet$). Part (1) of the following lemma generalizes Lemma 2.8.1. Recall that a set is relatively open if it is an open subset of its closure.

Lemma 3.3.5 *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$).*

1. *There are natural isomorphisms*

$$\varpi_+\iota_\dagger\mathcal{M}^\bullet \cong \varpi_+\varpi^+\varphi_\dagger\mathcal{M}^\bullet \quad \textit{and} \quad \varpi_\dagger\iota_+\mathcal{M}^\bullet \cong \varpi_\dagger\varpi^\dagger\varphi_+\mathcal{M}^\bullet.$$

2. *If* $\operatorname{fSupp}\varphi_\dagger\mathcal{M}^\bullet$ *(resp.* $\operatorname{cofSupp}\varphi_+\mathcal{M}^\bullet$*) is relatively open, then so is* $\operatorname{fSupp}\varpi_+\iota_\dagger\mathcal{M}^\bullet$ *(resp.* $\operatorname{cofSupp}\varpi_\dagger\iota_+\mathcal{M}^\bullet$*).*

Proof (1) We prove the first isomorphism. The second follows via duality.

It suffices to show that $\varpi_+\varpi^+\varphi_\dagger\mathcal{M}^\bullet$ satisfies the three conditions of Lemma 3.3.2. Conditions 1 and 2 are straightforward from the definitions. To prove condition 3, observe that

$$\varpi^+\varpi_+\varpi^+\varphi_\dagger\mathcal{M}^\bullet \cong \varpi^+\varphi_\dagger\mathcal{M}^\bullet \cong \varpi^+\varpi_+\iota_\dagger\mathcal{M}^\bullet \cong \iota_\dagger\mathcal{M}^\bullet.$$

Now apply Lemma 2.3.3.

(2) Use part (1) along with Lemma 2.3.3. ■

3.3.2 The relatively open (co)fiber support case

If the fiber support of $\varphi_\dagger\mathcal{M}^\bullet$ is relatively open, then the same is true of $\varpi_+\iota_\dagger\mathcal{M}^\bullet$ by Lemma 3.3.5(2). We may therefore shrink U so that $U\cap\overline{Z} = \operatorname{fSupp}\varpi_+\iota_\dagger\mathcal{M}^\bullet$ without changing $\varpi_+\iota_\dagger\mathcal{M}^\bullet$. Similarly, if the cofiber support of $\varphi_+\mathcal{M}^\bullet$ is relatively open, then we may shrink U so that $U\cap\overline{Z} = \operatorname{cofSupp}\varpi_\dagger\iota_+\mathcal{M}^\bullet$ without changing $\varpi_\dagger\iota_+\mathcal{M}^\bullet$. As an immediate consequence, we get the following corollaries of Lemmas 3.3.2 and 3.3.3:

Corollary 3.3.6 *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$), and assume that the fiber support of $\varphi_\dagger\mathcal{M}^\bullet$ is relatively open. Let $\mathcal{N}^\bullet \in \mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$). Then there exists an open neighborhood $U \subseteq X$ of Z such that (in the notation of §3.3.1) $\varpi_+\iota_\dagger\mathcal{M}^\bullet \cong \mathcal{N}^\bullet$ if and only if all of the following conditions hold:*

1. $\varphi^{+}\mathcal{N}^{\bullet} \cong \mathcal{M}^{\bullet}$;

2. $\operatorname{fSupp}\mathcal{N}^{\bullet} \cap \operatorname{cofSupp}\mathcal{N}^{\bullet} \subseteq Z$; *and*

3. $\operatorname{fSupp}\mathcal{N}^{\bullet}$ *is relatively open.*

Corollary 3.3.7 *Let $\mathcal{M}^{\bullet} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_Z)$ (resp. $\mathcal{M}^{\bullet} \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Z))$), and assume that the cofiber support of $\varphi_{+}\mathcal{M}^{\bullet}$ is relatively open. Let $\mathcal{N}^{\bullet} \in \mathrm{D}^{\mathrm{b},\overline{Z}}_{\mathrm{c}}(\mathcal{D}_X)$ (resp. in $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}^{\overline{Z}}(X))$). Then there exists an open neighborhood $U \subseteq X$ of Z such that (in the notation of §3.3.1) $\varpi_{\dagger}\iota_{+}\mathcal{M}^{\bullet} \cong \mathcal{N}^{\bullet}$ if and only if all of the following conditions hold:*

1. $\varphi^{+}\mathcal{N}^{\bullet} \cong \mathcal{M}^{\bullet}$;

2. $\operatorname{fSupp}\mathcal{N}^{\bullet} \cap \operatorname{cofSupp}\mathcal{N}^{\bullet} \subseteq Z$; *and*

3. $\operatorname{cofSupp}\mathcal{N}^{\bullet}$ *is relatively open.*

3.3.3 A different characterization of mixed and dual mixed Gauss–Manin parameters

Specializing Corollaries 3.3.6 and 3.3.7 to the GKZ case, we get Theorem 3.3.8 below. Before stating it, we recall the definition of the set of *A-exceptional parameters*. This is the set $\mathcal{E}_A$ of parameters β for which the holonomic rank of $\mathcal{M}_A(\beta)$ is larger than for a generic parameter. Note that $\mathcal{E}_A$ also has a description in terms of local cohomology (see [21]).

Theorem 3.3.8 *Let $\beta \in \mathbb{C}^d$.*

1. *β is dual mixed Gauss–Manin for A if and only if*

$$\beta \notin \mathcal{E}_A \quad \textit{and} \quad \operatorname{fSupp}\hat{\mathcal{M}}_A(\beta) \cap \operatorname{cofSupp}\hat{\mathcal{M}}_A(\beta) = T_A.$$

2. *β is mixed Gauss–Manin for A if and only if*

$$\operatorname{fSupp}\hat{\mathcal{K}}^{A}_{\bullet}(S_A; E_A - \beta) \cap \operatorname{cofSupp}\hat{\mathcal{K}}^{A}_{\bullet}(S_A; E_A - \beta) = T_A.$$

Proof (1) By Theorem 2.8.17, a dual mixed Gauss–Manin parameter is not A-exceptional. By Lemma 2.8.8, if $\beta \notin \mathcal{E}_A$, then the fiber support of $\hat{\mathcal{M}}_A(\beta)$ is relatively open; in particular, as $\varphi_\dagger \mathcal{O}_{T_A}^\beta$ is isomorphic to $\hat{\mathcal{M}}_A(\beta')$ for some $\beta' \notin \mathcal{E}_A$ (Remark 2.8.16), the fiber support of $\varphi_\dagger \mathcal{O}_{T_A}^\beta$ is relatively open. Now use Corollary 3.3.6.

(2) By [27, Prop. 2.2 (4)], the orbit-cone correspondence, and Theorem 2.7.4, the cofiber support of $\hat{\mathcal{K}}_\bullet^A(S_A; E_A - \beta)$ is relatively open for all β. In particular, as $\varphi_+ \mathcal{O}_{T_A}^\beta$ is isomorphic to $\hat{\mathcal{K}}_\bullet^A(S_A; E_A - \beta')$ for some β' ([8, Cor. 3.7]), the cofiber support of $\varphi_+ \mathcal{O}_{T_A}^\beta$ is relatively open. Now use Corollary 3.3.7. ■

Corollary 3.3.9 *A parameter is dual mixed Gauss–Manin for A if and only if it is mixed Gauss–Manin for A and not A-exceptional.*

3.4 Twisted quasi-equivariance

Reichelt and Walther introduced in [26, Def. 3.2] the notion of a quasi-equivariant $\mathcal{D}_E$ module. For the purposes of this paper, we need to generalize this notion slightly (Definition 3.4.2) to incorporate a “twist” by a rank one integrable connection on $\mathbb{C}^*$ à la [32]. In Lemma 3.4.4, this generalization is used to relate certain projections and restrictions of twistedly equivariant D-modules. Proposition 3.4.5 shows that, when properly interpreted, every mixed and dual mixed Gauss–Manin module is twistedly equivariant. Note that Lemma 3.4.4 and proposition 3.4.5 are generalizations of [26, Lem. 3.3 and 3.4].

We begin by recalling the notion of a fibered $\mathbb{C}^*$-action on a trivial vector bundle. Let $\pi\colon E \to X$ be a trivial vector bundle on a smooth affine variety X, and denote by

$$i\colon X \hookrightarrow E$$

the zero section. Set

$$E^* \coloneqq E \setminus i(X).$$

Definition 3.4.1 ([26, Def. 3.1]) *A $\mathbb{C}^*$ action $\mu\colon \mathbb{C}^* \times E \to E$ is* fibered *if*

1. *μ preserves fibers;*

2. *μ extends under the inclusion $\mathbb{C}^* \hookrightarrow \mathbb{C}$ to a morphism (also denoted μ) $\mathbb{C} \times E \to E$;*

3. *$0 \in \mathbb{C}$ multiplies into the zero section, i.e. $\mu\colon \{0\} \times E \to i(X)$; and*

4. *$\mathbb{C}$ fixes the zero section.*

Definition 3.4.2 *Let $\mu\colon \mathbb{C}^* \times E \to E$ be a fibered action on E, let μ' be the restriction of this action to E^*, and let $\lambda \in \mathbb{C}$. A complex $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_E)$ is* λ-twistedly $\mathbb{C}^*$-quasi-equivariant *if*

$$\mu'^* \mathcal{M}^\bullet_{|E^*} \cong \mathcal{O}^\lambda_{\mathbb{C}^*} \boxtimes \mathcal{M}^\bullet_{|E^*}. \tag{3.4.1}$$

A complex $\mathcal{M}^\bullet$ is twistedly $\mathbb{C}^*$-quasi-equivariant *if it is λ-twistedly $\mathbb{C}^*$-quasi-equivariant for some λ.*

Remark 3.4.3 *Note that because μ' is smooth of relative dimension* 1*, (3.4.1) is equivalent to*

$$\mu'^+ \mathcal{M}^\bullet_{|E^*} \cong \mathcal{O}^\lambda_{\mathbb{C}^*}[1] \boxtimes \mathcal{M}^\bullet_{|E^*} \tag{3.4.2}$$

and also to

$$\mu'^\dagger \mathcal{M}^\bullet_{|E^*} \cong \mathcal{O}^\lambda_{\mathbb{C}^*}[-1] \boxtimes \mathcal{M}^\bullet_{|E^*}. \tag{3.4.3}$$

The following lemma is proved in exactly the same way as is [26, Lem. 3.3]. The only change to the proof is that "$\mathcal{O}_{\mathbb{G}_\mathrm{m}}$" must be replaced throughout with "$\mathcal{O}^\lambda_{\mathbb{C}^*}$". No issues occur with doing so, and no issues occur with the passage to the derived category as opposed to modules.

Lemma 3.4.4 *If $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_E)$ is λ-twistedly $\mathbb{C}^*$-quasi-equivariant, then $\pi_+\mathcal{M}^\bullet \cong i^\dagger\mathcal{M}^\bullet$ and $\pi_\dagger\mathcal{M}^\bullet \cong i^+\mathcal{M}^\bullet$.*

We now generalize [26, Lem. 3.4]. The basic idea of the proof is the same. However, sufficiently many technical details need to be modified that we feel it necessary to provide the proof in full.

Proposition 3.4.5 *Let $F \preceq A$ be a face, and view $\mathbb{C}^n$ as a vector bundle over $\mathbb{C}^F$ via the coordinate projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^F$. Let $\beta \in \mathbb{C}^d$. Then there exists a fibered $\mathbb{C}^*$-action on $\mathbb{C}^n$ such that for all T_A-stable open neighborhoods $U \subseteq \mathbb{C}^n$ of T_A, both* $\mathrm{MGM}(U,\beta)$ *and* $\mathrm{MGM}^*(U,\beta)$ *are twistedly quasi-equivariant.*

Proof Write E for $\mathbb{C}^n$ viewed as vector bundle over $\mathbb{C}^F$. Since $\mathbb{N}A$ is pointed and F is a face, there exists a $\mathbf{u} \in \mathbb{Z}^d$ such that $\langle \mathbf{a}_i, \mathbf{u}\rangle = 0$ for $\mathbf{a}_i \in F$ and $\langle \mathbf{a}_i, \mathbf{u}\rangle > 0$ for $\mathbf{a}_i \notin F$. We show that the monomial action $\mu\colon \mathbb{C}^* \times E \to E$ induced by $\mathbf{v} := A^\top \mathbf{u}$, i.e. $t \cdot (x_1, \dots, x_n) = (t^{v_1}x_1, \dots, t^{v_n}x_n)$, satisfies the requirements of the proposition.

Step 1: μ is a fibered action.

Proof of Step 1. Condition (1) of Theorem 3.4.1 holds because $v_i = 0$ for all $\mathbf{a}_i \in F$. Because in addition $v_i > 0$ for all $\mathbf{a}_i \notin F$, the action extends to $\mathbb{C}$; so, condition (2) holds. Conditions (3) and (4) follow immediately from the definition of this extension. This finishes the proof of Step 1.

Step 2: $\tilde{\mu}^*\mathcal{O}_{T_A}^\beta \cong \mathcal{O}_{\mathbb{C}^*}^{\langle \mathbf{u},\beta\rangle} \boxtimes \mathcal{O}_{T_A}^\beta$, where $\tilde{\mu}$ denotes the monomial action on T_A induced by $\mathbf{u}$.

Proof of Step 2. Let $f\colon \tilde{\mu}^*\mathcal{O}_{T_A}^\beta \to \mathcal{O}_{\mathbb{C}^*}^{\langle \mathbf{u},\beta\rangle} \boxtimes \mathcal{O}_{T_A}^\beta$ be the $\mathcal{O}_{\mathbb{C}^* \times T_A}$-module isomorphism taking the generator $1 \otimes t^{-\beta}$ to the generator $s^{-\langle \mathbf{u},\beta\rangle} \otimes t^{-\beta}$, where s denotes the coordinate on $\mathbb{C}^*$. The action of $1 \otimes t_i\partial_{t_i}$ on both generators is multiplication by $-\beta_i$, while the action of $s\partial_s$ on both generators is multiplication by $-\langle \mathbf{u}, \beta\rangle$. Therefore, f is an isomorphism of $\mathcal{D}_{\mathbb{C}^* \times T_A}$-modules. This finishes the proof of Step 2.

Step 3: Both $\mathrm{MGM}(U,\beta)$ and $\mathrm{MGM}^*(U,\beta)$ are $\langle \mathbf{u}, \beta\rangle$-twistedly quasi-equivariant.

Proof of Step 3. Since the two statements are equivalent via duality, we only prove the first. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^* \times T_A \arrow[r, "\mathrm{id}\times\iota'"] \arrow[d, "\tilde{\mu}"] & \mathbb{C}^* \times (U \cap E^*) \arrow[r, "\mathrm{id}\times\varpi'"] \arrow[d, "\mu''"] & \mathbb{C}^* \times E^* \arrow[d, "\mu'"] \\
T_A \arrow[r, "\iota'"] & U \cap E^* \arrow[r, "\varpi'"] & E^*
\end{tikzcd} \tag{3.4.4}$$

Here, ι' is the torus embedding, ϖ' is inclusion, μ' is the restriction of μ to E^*, and μ'' is the restriction of μ to $U \cap E^*$. By construction, the action μ factors through the action of T_A. So, because U is T_A-stable, it is also $\mathbb{C}^*$-stable, and therefore both squares in (3.4.4) are Cartesian. Then

$$\begin{aligned}
\mu'^{\dagger}\,\mathrm{MGM}(U,\beta)_{|E^*} &\cong \mu'^{\dagger}\varpi'_{\dagger}\iota'_{+}\mathcal{O}^{\beta}_{T_A}\\
&\cong (\mathrm{id}\times\varpi')_{\dagger}\mu''^{\dagger}\iota'_{+}\mathcal{O}^{\beta}_{T_A}\\
&\cong (\mathrm{id}\times\varpi')_{\dagger}(\mathrm{id}\times\iota')_{+}\tilde{\mu}^{\dagger}\mathcal{O}^{\beta}_{T_A}\\
&\cong (\mathrm{id}\times\varpi')_{\dagger}(\mathrm{id}\times\iota')_{+}(\mathcal{O}^{\langle\mathbf{u},\beta\rangle}_{\mathbb{C}^*}[-1]\boxtimes\mathcal{O}^{\beta}_{T_A})\\
&\cong \mathcal{O}^{\langle\mathbf{u},\beta\rangle}_{\mathbb{C}^*}[-1]\boxtimes\varpi'_{\dagger}\iota'_{+}\mathcal{O}^{\beta}_{T_A}\\
&\cong \mathcal{O}^{\langle\mathbf{u},\beta\rangle}_{\mathbb{C}^*}[-1]\boxtimes\mathrm{MGM}(U,\beta)_{|E^*},
\end{aligned}$$

where the second isomorphism is by base change, the third is by base change together with the fact that μ'' and $\tilde{\mu}$ are smooth of the same relative dimension, and the fourth is by Step 2 and the smoothness of $\tilde{\mu}$. Now use Theorem 3.4.3. This finishes the proof of Step 3 and thereby the proposition.□

■

3.5 Projections and restrictions

In §3.5.1, we use the framework of a $\mathbb{C}^*$-fibered vector bundle to show that the projection and restriction of alternating direct images are also alternating direct images. We apply this in §3.5.2 to mixed and dual mixed Gauss–Manin systems.

In §3.5.3, we specialize these results to the case of normal S_A, culminating in Theorem 3.5.8, where we compute the restriction and projection of $\mathcal{M}_A(\beta)$ to the coordinate subspace corresponding to a face of A, and Corollary 3.5.9, which says that at most one of the restriction and projection can be nonzero.

3.5.1 Restricting and projecting twistedly quasi-equivariant alternating direct images

Let X be a smooth affine variety, $\pi\colon E \to X$ a $\mathbb{C}^*$-fibered vector bundle, and as before, denote by $i\colon X \hookrightarrow E$ the zero section. Consider the following diagrams:

$$Z \xrightarrow{\ \iota\ } U \xrightarrow{\ \varpi\ } E \quad \text{and} \quad i^{-1}(U) \cap \pi(Z) \xrightarrow{\ \iota'\ } i^{-1}(U) \xrightarrow{\ \varpi'\ } X.$$

Here, Z is smooth and locally closed in E, U is an open subset of E containing Z, and the morphisms are inclusion. (Note that the role of X has changed from what it was in Section 3.3). Set $\varphi := \varpi \circ \iota$ and $\varphi' := \varpi' \circ \iota'$.

Proposition 3.5.1 *Let $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_Z)$. Assume that $U \supseteq \pi^{-1}(i^{-1}(U))$ and $\pi(Z)$ is locally closed.*

1. *If $\mathcal{N}^\bullet := \varpi_+\iota_\dagger\mathcal{M}^\bullet$ is twistedly $\mathbb{C}^*$-quasi-equivariant, then*

$$i^+\mathcal{N}^\bullet \cong \varpi'_+\iota'_\dagger(i \circ \varphi')^+\mathcal{N}^\bullet.$$

2. *If $\mathcal{N}^\bullet := \varpi_\dagger\iota_+\mathcal{M}^\bullet$ is twistedly $\mathbb{C}^*$-quasi-equivariant, then*

$$\pi_+\mathcal{N}^\bullet \cong \varpi'_\dagger\iota'_+(i \circ \varphi')^\dagger\mathcal{N}^\bullet.$$

Proof (1) By Lemma 3.3.2, the fiber support of $i^+\mathcal{N}^\bullet$ is contained in $i^{-1}(U)$. Suppose $x \in i^{-1}(U) \cap \operatorname{cofSupp} i^+\mathcal{N}^\bullet$. Then by Lemma 3.4.4 and the base change formula, $(\pi|_{E_x})_\dagger i^\dagger_{E_x}\mathcal{N}^\bullet \neq 0$, where $E_x := \pi^{-1}(x)$ is the fiber of E over x, and $i_{E_x}\colon E_x \hookrightarrow E$ is inclusion. So, $i^\dagger_{E_x}\mathcal{N}^\bullet \neq 0$, and therefore $E_x \cap \operatorname{cofSupp}\mathcal{N}^\bullet \neq \emptyset$. On the other hand, $x \in i^{-1}(U)$, so because $U \supseteq \pi^{-1}(i^{-1}(U))$, we have that $E_x \subseteq U$. Hence, $E_x \cap \operatorname{cofSupp}\mathcal{N}^\bullet$ is a nonempty subset of Z by Lemma 3.3.2, and therefore $\pi(x) \in \pi(Z) \cap i^{-1}(U)$. Thus,

$$i^+\mathcal{N}^\bullet \cong \varpi'_+\iota'_\dagger\varphi'^+ i^+\mathcal{N}^\bullet \cong \varpi'_+\iota'_\dagger(i \circ \varphi')^+\mathcal{N}^\bullet.$$

(2) This follows from (1) by duality together with Lemma 3.4.4. ∎

It may appear at first that the assumption that $U \supseteq \pi^{-1}(i^{-1}(U))$ in Proposition 3.5.1 is too restrictive to apply in the situation of Proposition 3.4.5. However, as we will see in Lemma 3.5.3, U can always be enlarged to satisfy this assumption without changing $\mathrm{MGM}(U,\beta)$ or $\mathrm{MGM}^*(U,\beta)$.

3.5.2 Restricting and projecting GKZ systems

Before stating Theorem 3.5.4, we recall the below facts about mixed and dual mixed Gauss–Manin systems. Also recall from (3.2.5) that $\mathrm{O}_A(F)$ is the T_A-orbit of the toric variety X_A which corresponds to F, and from (3.2.3) that $d_{A/F} = d - \operatorname{rank} F$.

Here and in the rest of this article, we follow that convention that $\bigwedge \mathbb{C}^k$ lives in cohomological degrees $-k$ through 0.

Fact 3.5.2 *Let $\beta \in \mathbb{C}^d$, and let $U \subseteq \mathbb{C}^n$ be a T_A-stable open neighborhood of T_A. Write $i_{\mathrm{O}_A(F)}$ for the inclusion $\mathrm{O}_A(F) \hookrightarrow \mathbb{C}^n$.*

1. *If $\mathrm{O}_A(F) \subseteq \operatorname{cofSupp} \mathrm{MGM}(U,\beta)$, then*

$$i^{\dagger}_{\mathrm{O}_A(F)} \mathrm{MGM}(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \mathcal{O}^{\lambda}_{T_F} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}},$$

   where the direct sum is over those $\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F$ for which $\beta - \lambda \in \mathbb{Z}^d$. This follows from Lemma 2.8.14, remark 2.8.16, and eq. (2.8.6).

2. *If $\mathrm{O}_A(F) \subseteq \operatorname{fSupp} \mathrm{MGM}^*(U,\beta)$, then*

$$i^{+}_{\mathrm{O}_A(F)} \mathrm{MGM}^*(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \mathcal{O}^{\lambda}_{T_F} \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[-d_{A/F}],$$

   where the direct sum is over those $\lambda + \mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F$ for which $\beta - \lambda \in \mathbb{Z}^d$. This follows from Fact 3.5.2(1) and Remark 2.8.18.

Let $F \preceq A$ be a face, and let $\pi_F \colon \mathbb{C}^n \to \mathbb{C}^F$ and $i_F \colon \mathbb{C}^F \hookrightarrow \mathbb{C}^n$ be coordinate projection and inclusion, respectively.

Lemma 3.5.3 *Let $\beta \in \mathbb{C}^n$ and $\mathcal{M}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_{\mathbb{C}^n})$. Let $U \subseteq \mathbb{C}^n$ be a T_A-stable open neighborhood of T_A, and let $U' = U \cup \pi_F^{-1}(i_F^{-1}(U))$. Then*

$$\mathrm{MGM}^*(U,\beta) \cong \mathrm{MGM}^*(U',\beta) \quad \text{and} \quad \mathrm{MGM}(U,\beta) \cong \mathrm{MGM}(U',\beta).$$

Proof It suffices to show that $U' \cap X_A = U \cap X_A$. The containment $U' \cap X_A \supseteq U \cap X_A$ is immediate. For the other containment, let G be a face of A such that $\mathrm{O}_A(G) \subseteq U'$, and suppose $\mathrm{O}_A(G) \subseteq \pi_F^{-1}(i_F^{-1}(U))$. Then $i_F(\pi_F(\mathrm{O}_A(G))) \subseteq U$. But $i_F(\pi_F(\mathrm{O}_A(G))) = i_F(\mathrm{O}_F(G \cap F)) = \mathrm{O}_A(G \cap F)$, so $\mathrm{O}_A(G \cap F) \subseteq U$. Therefore, because U is open, the orbit-cone correspondence implies that $\mathrm{O}_A(G) \subseteq U$. Thus, $U' \cap X_A = U \cap X_A$. ■

Theorem 3.5.4 *Let $\beta \in \mathbb{C}^n$, and let $U \subseteq \mathbb{C}^n$ be a T_A-stable open neighborhood of T_A.*

1. *If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$ or $U \not\supseteq \mathrm{O}_A(F)$, then*

$$\pi_{F+} \mathrm{MGM}_A(U,\beta) = i_F^+ \mathrm{MGM}^*_A(U,\beta) = 0.$$

2. *If $U \supseteq \mathrm{O}_A(F)$, then*

$$i_F^+ \mathrm{MGM}^*_A(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \mathrm{MGM}^*_F\big(i_F^{-1}(U),\lambda\big) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[-d_{A/F}]$$

and

$$\pi_{F+} \mathrm{MGM}_A(U,\beta) \cong \bigoplus_{\lambda+\mathbb{Z}F} \mathrm{MGM}_F\big(i_F^{-1}(U),\lambda\big) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}},$$

where the direct sums are over those $\lambda+\mathbb{Z}F \in \mathbb{C}F/\mathbb{Z}F$ for which $\beta - \lambda \in \mathbb{Z}^d$. If in addition, $\beta \in \mathbb{C}F + \mathbb{Z}^d$, then neither $i_F^+ \mathrm{MGM}^*_A(U,\beta)$ nor $\pi_{F+} \mathrm{MGM}_A(U,\beta)$ vanish.

Proof We only prove the dual MGM case. The MGM case follows by duality together with Lemma 3.4.4. For ease of notation, set $\pi = \pi_F$ and $i = i_F$.

(1) If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$ or $U \not\supseteq \mathrm{O}_A(F)$, then $\mathrm{O}_A(F)$ does not intersect the fiber support of $\mathrm{MGM}^*(U,\beta)$—the former by Fact 3.5.2(2) and the latter by Lemma 3.3.2(2).

Therefore, no orbit corresponding to a face of F intersects the fiber support of $\mathrm{MGM}^*(U,\beta)$. Hence, $\operatorname{fSupp} i^+ \mathrm{MGM}^*(U,\beta) = X_F \cap i^{-1}(\operatorname{fSupp} \mathrm{MGM}^*(U,\beta)) = \emptyset$, and therefore $i^+ \mathrm{MGM}^*(U,\beta) = 0$.

(2) Assume $U \supseteq \mathrm{O}_A(F)$. By Lemma 3.5.3, we may replace U with $U \cup \pi^{-1}(i^{-1}(U))$ (note that this leaves $i^{-1}(U)$ unchanged) to assume that $U \supseteq \pi^{-1}(i^{-1}(U))$. In addition, $\pi(T_A) = T_F$, which is locally closed in $\mathbb{C}^F$. Therefore, Proposition 3.5.1(1) applies to give

$$i^+ \mathrm{MGM}^*_A(U,\beta) \cong \varpi'_+ \iota'_\dagger (i \circ \varphi_F)^+ \mathrm{MGM}^*_A(U,\beta),$$

where $\varphi_F \colon T_F \hookrightarrow \mathbb{C}^F$ is the torus embedding induced by F, $\iota' \colon i^{-1}(U) \cap T_F \hookrightarrow i^{-1}(U)$ is the restriction of φ_F, and ϖ' is the inclusion $i^{-1}(U) \hookrightarrow \mathbb{C}^F$.

By assumption, $i^{-1}(U) \cap T_F = T_F$, and therefore $i \circ \varphi'$ is just the inclusion $T_F = \mathrm{O}_A(F) \hookrightarrow \mathbb{C}^n$. Now use Fact 3.5.2 together with the additivity of the D-module functors. This proves that $i_F^+ \mathrm{MGM}^*_A(U,\beta)$ is isomorphic to the requisite direct sum. That this does not vanish if $\beta \in \mathbb{C}F + \mathbb{Z}^d$ is because in such a case the direct sum is over a nonempty set. ■

3.5.3 Normal case

Throughout this section, S_A is assumed to be normal. Lemma 3.5.5 is a technical lemma which we will use (both in this section and in §3.6) to move a parameter β within the class of those parameters whose A-hypergeometric system is isomorphic to that of β. Lemma 3.5.6 will be needed in the proof of Theorem 3.5.8. Recall from Definition 3.2.2 the definition of the primitive integral support functions h_G.

Lemma 3.5.5 *Let $\beta \in \mathbb{C}^d$. Then there exists a $\gamma \in \mathbb{Z}^d$ such that for all facets $G \preceq A$,*

1. *$h_G(\gamma) \neq 0$ if $h_G(\beta) \notin \mathbb{Z}$;*

2. *$h_G(\gamma) > 0$ if $h_G(\beta) \in \mathbb{N}$; and*

3. *$h_G(\gamma) < 0$ if $h_G(\beta) \in \mathbb{Z}_{<0}$.*

Proof Consider the system of equations

$$\{\, h_G(x) = h_G(\beta) \mid G \preceq A \text{ is a facet with } h_G(\beta) \in \mathbb{Z} \,\}\,.$$

This has a solution in $\mathbb{C}^d$, namely β, and therefore has a solution in $\mathbb{R}^d$. Let α be one such solution. Then α describes a hyperplane

$$H_\alpha = \{\, f \in (\mathbb{R}^d)^* \mid f(\alpha) = 0 \,\}\,.$$

Denote by $H_\alpha^{\geq 0}$ the set of $f \in (\mathbb{R}^d)^*$ such that $f(\alpha) \geq 0$, and similarly for $H_\alpha^{>0}$, $H_\alpha^{\leq 0}$, and $H_\alpha^{<0}$.

Let us now consider the sets $P_\alpha = \{\, h_G \mid h_G(\alpha) \geq 0 \,\}$ and $N_\alpha = \{\, h_G \mid h_G(\alpha) < 0 \,\}$. By construction, $\mathbb{R}_{\geq 0}P_\alpha \cap \mathbb{R}_{\geq 0}N_\alpha = \{0\}$. Let Z be an affine hyperplane in $(\mathbb{R}^d)^*$ transverse to the dual cone $(\mathbb{R}_{\geq 0}A)^\vee$, and assume that the intersection $Z \cap (\mathbb{R}_{\geq 0}A)^\vee$ is nonempty. Then $Z \cap \mathbb{R}_{\geq 0}P_\alpha$ and $Z \cap \mathbb{R}_{\geq 0}N_\alpha$ are convex, compact, and disjoint. Hence, there exists a hyperplane L in Z separating $Z \cap \mathbb{R}_{\geq 0}P_\alpha$ and $Z \cap \mathbb{R}_{\geq 0}N_\alpha$. Choose a $\gamma \in \mathbb{R}^d$ such that $H_\gamma \cap Z = L$ and $H_\gamma^{>0} \supseteq Z \cap \mathbb{R}_{\geq 0}P_\alpha$. Then $H_\gamma^{<0} \supseteq Z \cap \mathbb{R}_{\geq 0}N_\alpha$. Then by convexity, $H_\gamma^{>0} \supseteq \mathbb{R}_{\geq 0}P_\alpha$ and $H_\gamma^{<0} \supseteq \mathbb{R}_{\geq 0}N_\alpha$. In particular, $H_\gamma^{>0} \supseteq P_\alpha$ and $H_\gamma^{<0} \supseteq N_\alpha$. Because $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, we may modify γ so that it is in $\mathbb{Q}^d$. Clearing denominators, we may take γ to be in $\mathbb{Z}^d$. ■

Note that because we are in the normal case, we may define

$$\operatorname{sRes}(A) = \mathbb{C}^d \setminus \left\{\, \beta \in \mathbb{C}^d \;\middle|\; h_G(\beta) \geq 0 \text{ whenever } h_G(\beta) \in \mathbb{Z} \,\right\}. \tag{3.5.1}$$

We will take this as the definition of $\operatorname{sRes}(A)$ since we are only dealing with normal A. However, (3.5.1) follows from the general definition given in [8] by applying Theorem 2.9.3 and lemma 2.9.1 along with [8, Cor. 3.8]

Lemma 3.5.6 *Let $\beta \in \mathbb{C}F + \mathbb{Z}^d$, and let $F \preceq A$ be a face. Then there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that for all facets F' of F,*

1. *$h_{F'}(\lambda) \in \mathbb{N}$ implies that $h_G(\beta) \in \mathbb{N}$ for all facets G of A with $G \cap F = F'$; and*

2. *$h_{F'}(\lambda) \in \mathbb{Z}_{<0}$ implies that $h_G(\beta) \in \mathbb{Z}_{<0}$ for all facets G of A with $G \cap F = F'$.*

Proof Step 1: The lemma holds for $\beta \in \mathbb{Z}^d$.

Proof of Step 1. By induction on the rank of F, we may assume that F is a facet of A. Let $F_1, \ldots, F_\ell$ be the facets of F. For each i, let G_i be the facet of A whose intersection with F is F_i. For each $I \subseteq \{1, \ldots, \ell\}$, consider the sets

$$X_I := \{\, x \in \mathbb{R}F \mid h_{F_i}(x) \geq 0 \text{ for all } i \in I \,\}$$
$$Y_I := \left\{\, x \in \mathbb{R}^d \;\middle|\; h_{G_i}(x) \geq 0 \text{ for all } i \in I \,\right\}.$$

When X_I is nonempty, neither is Y_I, and X_I and Y_I are chambers of the arrangments $\{\mathbb{R}F_1, \ldots, \mathbb{R}F_\ell\}$ and $\{\mathbb{R}G_1, \ldots, \mathbb{R}G_\ell\}$, respectively. But these two arrangements are combinatorially equivalent by construction, so they have the same number of chambers. Hence, X_I is nonempty if and only if Y_I is nonempty. Since both arrangements are central, $X_I \cap \mathbb{Z}F$ is nonempty if and only if $Y_I \cap \mathbb{Z}^d$ is nonempty. Therefore, if $\beta \in Y_I$, then any $\lambda \in X_I \cap \mathbb{Z}F$ has the required properties. This finishes the proof of Step 1.

Step 2: The lemma holds for general β.

Proof of Step 2. Apply Lemma 3.5.5 to β to get a $\gamma \in \mathbb{Z}^d$. Apply Step 1 to γ to get an $\alpha \in \mathbb{Z}F$. Let $\lambda_0 \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d) \setminus \mathrm{sRes}(A)$. By adding sufficiently many copies of $\sum_{\mathbf{a}_i \in F} \mathbf{a}_i$ to λ_0, we may assume that

$$h_{F'}(\lambda_0) \geq |h_{F'}(\alpha)| \tag{3.5.2}$$

for all facets F' of F with $h_{F'}(\lambda_0) \in \mathbb{Z}$. Set $\lambda = \lambda_0 + \alpha$. Let F' be a facet of F, and let G be a facet of A with $G \cap F = F'$.

Suppose $h_{F'}(\lambda) \in \mathbb{N}$. Then because $h_{F'}(\alpha) \in \mathbb{Z}$, $h_{F'}(\lambda_0)$ must be an integer and therefore a non-negative integer. Then by (3.5.2), $h_{F'}(\alpha) \geq 0$. Hence, $h_G(\gamma) \geq 0$, which by construction of γ means that $h_G(\beta) \in \mathbb{N}$.

Next, suppose $h_{F'}(\lambda) \in \mathbb{Z}_{<0}$. As before, this implies that $h_{F'}(\lambda_0)$ is a non-negative integer. But then $h_{F'}(\alpha)$ must be negative. Hence, $h_G(\gamma) \leq 0$, which by construction of γ means that $h_G(\beta) \in \mathbb{Z}_{<0}$. This finishes the proof of Step 2 and thereby the lemma.

■

The following example shows that even if $h_G(\beta) \in \mathbb{Z}$ for every facet G of A with $G \cap F = F'$, it is still possible that $h_{F'}(\lambda) \notin \mathbb{Z}$.

Example 3.5.7 *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The only facet of F is $\emptyset$, and the only facet of A whose intersection with F is $\emptyset$ is the facet $G = [1,0]^\top$. The primitive integral support functions of these facets are $h_{\emptyset,F}(c, 2c) = c$ and $h_{G,A}(a, b) = b$. Then $h_{G,A}(c, 2c) = 2c$, so $h_{G,A}|_{\mathbb{C}F} = 2h_{\emptyset,F}$.

Consider the parameter $\beta = (1/2, 1)$. This parameter is already in $\mathbb{C}F$. Since $h_{\emptyset,F}(\beta) = 1/2$ is not in $\mathbb{Z}$, the same is true of $h_{\emptyset,F}(\lambda)$ for every $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^2)$. However, $h_{G,A}(\beta) = 2 \in \mathbb{Z}$.

Recall that $n_{A/F}$ is the number of columns of A which are not in F; equivalently, $n_{A/F} = n - \dim \mathbb{C}^F$.

Theorem 3.5.8 *Assume S_A is normal, let $F \preceq A$ be a face, and let $\beta \in \mathbb{C}^d$.*

1. *If $\beta \in \mathbb{C}F + \mathbb{Z}^d$ and $h_G(\beta) \in \mathbb{Z}_{<0}$ for every facet $G \succeq F$, then there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that*

   $$\pi_{F+}\mathcal{M}_A(\beta) \cong \mathcal{M}_F(\lambda) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[n_{A/F} - d_{A/F}];$$

   otherwise, $\pi_{F+}\mathcal{M}_A(\beta) = 0$.

2. *If $\beta \in \mathbb{C}F + \mathbb{Z}^d$ and $h_G(\beta) \in \mathbb{Z}_{<0}$ for every facet $G \succeq F$, then there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that*

   $$i_F^+\mathcal{M}_A(\beta) \cong \mathcal{M}_F(\lambda) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[-n_{A/F}];$$

   otherwise, $i_F^+\mathcal{M}_A(\beta) = 0$.

Proof We prove (1); statement (2) is proved similarly.

Recall that the Fourier–Laplace transform interchanges π_{F+} and $i_F^+[n_{A/F}]$. Therefore, (1) is equivalent to the following statement (where we recall from (3.2.2) that $\hat{\mathcal{M}}_A(\beta) \coloneqq \mathrm{FL}^{-1}(\mathcal{M}_A(\beta))$):

(∗) If $\beta \in \mathbb{C}F + \mathbb{Z}^d$ and $h_G(\beta) \in \mathbb{Z}_{<0}$ for every facet $G \succeq F$, then there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that

$$i_F^+ \hat{\mathcal{M}}_A(\beta) \cong \hat{\mathcal{M}}_F(\lambda) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[-d_{A/F}];$$

otherwise, $i_F^+ \hat{\mathcal{M}}_A(\beta) = 0$.

We prove this Fourier–Laplace transformed statement.

Choose an open subset U of $\mathbb{C}^n$ such that $U \cap X_A = \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta)$. Theorem 2.9.3 establishes that

$$\hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}_A^*(U, \beta). \tag{3.5.3}$$

If $\beta \notin \mathbb{C}F + \mathbb{Z}^d$ or $h_G(\beta) \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ for some facet $G \succeq F$, then $\mathrm{O}_A(F)$ is not contained in U by Lemma 2.9.1(c). Theorem 3.5.4(1) then applies to give that $i_F^+ \hat{\mathcal{M}}_A(\beta) = 0$.

Suppose $\beta \in \mathbb{C}F + \mathbb{Z}^d$ and $h_G(\beta) \in \mathbb{Z}_{<0}$ for every facet $G \succeq F$. By normality, the direct sums in Theorem 3.5.4(2) collapse to a single summand, giving

$$i_F^+ \hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}_F^*\big(i_F^{-1}(U), \lambda_0\big) \otimes_{\mathbb{C}} \bigwedge \mathbb{C}^{d_{A/F}}[-d_{A/F}],$$

where $\lambda_0 \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ is arbitrary. Therefore, taking into account (3.5.3), it remains to show that there exists a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ such that

$$\mathrm{MGM}_F^*\big(i_F^{-1}(U), \lambda_0\big) \cong \hat{\mathcal{M}}_F(\lambda).$$

Choose a $\lambda \in \mathbb{C}F \cap (\beta + \mathbb{Z}^d)$ as in Lemma 3.5.6. By Theorem 2.9.3 together with Lemma 2.9.1(c), we need to show for all facets F' of F,

$$h_{F'}(\lambda) \in \mathbb{Z}_{<0} \text{ if and only if } i_F^{-1}(U) \supseteq \mathrm{O}_F(F').$$

Let F' be a facet of F.

If $h_{F'}(\lambda) \in \mathbb{Z}_{<0}$, then $h_G(\beta) \in \mathbb{Z}_{<0}$ for every facet G of A containing F' by assumption on β and by Lemma 3.5.6, and $\beta = \lambda + (\beta - \lambda) \in \mathbb{C}F' + \mathbb{Z}^d$; hence, $i_F^{-1}(U) \supseteq \mathrm{O}_F(F')$ by Lemma 2.9.1(c).

If $h_{F'}(\lambda) \in \mathbb{N}$, then $h_G(\beta) \in \mathbb{N}$ for some facet G of A containing F', and therefore $i_F^{-1}(U) \not\supseteq \mathrm{O}_F(F')$ by Lemma 2.9.1(c).

Finally, if $\beta \in \mathbb{C}F' + \mathbb{Z}^d$, then $\lambda = \beta + (\lambda - \beta) \in (\mathbb{C}F' + \mathbb{Z}^d) \cap \mathbb{C}F = \mathbb{C}F' + \mathbb{Z}F$ (because F is saturated), and therefore $h_{F'}(\lambda) \in \mathbb{Z}$. Hence, if $h_{F'}(\lambda) \notin \mathbb{Z}$, then $\beta \notin \mathbb{C}F' + \mathbb{Z}^d$. Thus, $i_F^{-1}(U) \not\supseteq \mathrm{O}_F(F')$ by Lemma 2.9.1(c). ■

Note that Theorem 3.5.8 only claims the existence of λ. A possibly interesting question for the future would be to turn the proofs of Lemmas 3.5.5 and 3.5.6 into an algorithm for computing such a λ.

The following corollary follows immediately from Theorem 3.5.8:

Corollary 3.5.9 *Assume S_A is normal, let $F \preceq A$ be a face, and let $\beta \in \mathbb{C}^d$. Then at least one of $i_F^+\mathcal{M}_A(\beta)$ and $\pi_{F+}\mathcal{M}_A(\beta)$ is zero.*

3.6 Duality of normal GKZ systems

Throughout this section, S_A is assumed to be normal. In Theorem 3.6.3, we assume in addition that A is homogeneous (Recall that A is *homogeneous* if its columns all lie in a hyperplane).

Lemma 3.6.1 shows that for all parameters β, there is a parameter $\beta' \in -\beta + \mathbb{Z}^d$ such that $\hat{\mathcal{M}}_A(\beta')$ has the cofiber support one would expect for the holonomic dual of $\hat{\mathcal{M}}_A(\beta)$. Proposition 3.6.2 uses this to prove that this $\hat{\mathcal{M}}_A(\beta')$ is indeed the holonomic dual of $\hat{\mathcal{M}}_A(\beta)$. The Fourier–Laplace transform of this result, together with a monodromicity argument, gives Theorem 3.6.3.

Lemma 3.6.1 *Let $\beta \in \mathbb{C}^d$. Then there exists a $\beta' \in -\beta + \mathbb{Z}^d$ such that*

$$\operatorname{cofSupp} \hat{\mathcal{M}}_A(\beta') = \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta).$$

If β does not lie on the $\mathbb{C}$-span of any facet, then β' may be taken to be $-\beta$.

Proof By Lemma 2.9.1(c) and (d), it suffices to show that there exists a $\beta' \in -\beta + \mathbb{Z}^d$ for all facets $G \preceq A$,

$$h_G(\beta') \in \mathbb{N} \quad \text{if and only if} \quad h_G(\beta) \in \mathbb{Z}_{<0}.$$

Choose $\gamma \in \mathbb{Z}^d$ as in Lemma 3.5.5. Then $\hat{\mathcal{M}}_A(\beta)$ and $\hat{\mathcal{M}}_A(\beta+\gamma)$ have the same fiber support (by Lemma 2.9.1(c)) and are therefore isomorphic by Theorem 2.9.3. Moreover, $\beta+\gamma$ does not lie on the $\mathbb{C}$-span of any facet. Replacing β with $\beta+\gamma$, we may assume that β itself does not lie on the $\mathbb{C}$-span of any facet.

Let $\beta' = -\beta$. Then $h_G(\beta)$ is never zero, so $h_G(\beta') \in \mathbb{N}$ if and only if $h_G(\beta) \in \mathbb{Z}_{<0}$, as hoped. ■

Proposition 3.6.2 *Let $\beta \in \mathbb{C}^d$. Then there exists a $\beta' \in -\beta + \mathbb{Z}^d$ such that $\mathbb{D}\hat{\mathcal{M}}_A(\beta) \cong \hat{\mathcal{M}}_A(\beta')$. If β does not lie on the $\mathbb{C}$-span of any facet, then β' may be taken to be $-\beta$.*

Proof By Theorem 2.9.3, there exists an open $U \subseteq \mathbb{C}^n$ with $U \cap X_A = \operatorname{fSupp} \hat{\mathcal{M}}_A(\beta)$ such that $\hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}^*(U, \beta)$. Applying the holonomic duality functor gives $\mathbb{D}\hat{\mathcal{M}}_A(\beta) \cong \mathrm{MGM}(U, -\beta)$. Now use Theorem 2.9.3 again along with Lemma 3.6.1. ■

Theorem 3.6.3 *Assume that A is homogeneous. Let $\beta \in \mathbb{C}^d$. Then there exists a $\beta' \in -\beta + \mathbb{Z}^d$ such that $\mathbb{D}\mathcal{M}_A(\beta) \cong \mathcal{M}_A(\beta')$. If β does not lie on the $\mathbb{C}$-span of any facet, then β' may be taken to be $-\beta$.*

Proof By [9, Lem. 1.13], the homogeneity condition implies that every A-hypergeometric system is monodromic. By [31, Prop. 6.13] (or rather the restatement of it for D-modules which appears in [9, Th. 1.4]), if $\mathcal{M}$ is monodromic, then $\mathbb{D}\,\mathrm{FL}\,\mathcal{M} \cong \mathrm{FL}\,\mathbb{D}\mathcal{M}$. Now use Proposition 3.6.2. ■

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