VANISHING THEOREMS FOR THE LOGARITHMIC DE RHAM COMPLEX OF UNITARY LOCAL SYSTEM

A Dissertation

Submitted to the Faculty

of

Purdue University

by

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In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2019

Purdue University

West Lafayette, Indiana

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ACKNOWLEDGMENTS

This work is not possible without the mentorship of my academic advisor Donu Arapura. Thanks to his cool character and mathematical insight, collaborations with him have always been pleasant, and meetings with him have always been relaxing and fruitful.

I would also like to thank Prof. Kenji Matsuki for hosting algebraic geometry learning seminars for students every semester for the past few years. Thanks to his hard work and attitude towards taking care of every detail, the learning seminars were an invaluable experience for many graduate students including myself.

Also, special thanks to Prof. Jaroslaw Wlodarczyk. He joined our learning seminar last year and has offered many insightful comments since then.

I would also like to thank Prof. Deepam Patel for offering a well-organized class in Geometric Invariant Theory in the fall semester of 2015. Besides this, his habit for always being precise when making a mathematical statement has always been a positive influence to me.

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ABSTRACT

Li, Hongshan Ph.D., Purdue University, May 2019. Vanishing Theorems for the Logarithmic de Rham Complex of Unitary Local System. Major Professor: Donu Arapura.

Let X be a non-singular complex projective variety with $\dim_{\mathbb{C}} X = n$. We will prove two cohomology vanishing theorems for unitary vector bundle E on X with flat (integrable) connection ∇, which has at worst logarithmic singularities along some boundary divisor D. We will assume D is a simple normal crossing divisor.

Such an vector bundle has a de Rham complex $\mathrm{DR}_X(D, E)$

$$E \xrightarrow{\nabla} E \otimes \Omega_X(\log D) \xrightarrow{\nabla} E \otimes \Omega^2_X(\log D) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} E \otimes \Omega^n_X(\log D)$$

One approach for the vanishing theorems is to construct a mixed Hodge theory on the $\mathrm{DR}_X(D, E)$. Then, we will be able to apply the results from Deligne's study on abstract Hodge theory. The vanishing theorems are then the consequence spectral sequence degeneration as stated in [1] and [2].

Another approach is to interpret E as a semistable Higgs bundle with trivial Higgs field θ. Then the first vanishing theorem is a consequence of the main result of [3].

We will present both approaches in this work.

1. INTRODUCTION

Cohomology vanishing for coherent sheaves has always been one of central subjects in algebraic geometry and complex analytic geometry. The first substantial result in the field is the Kodaira Vanishing Theorem

Theorem 1.0.1 *(Kodaira Vanishing Theorem) Let X be a smooth complex projective variety of dimension n, and let D be an ample divisor on X. Then*

$$H^i(X, O_X(K_X + D)) = 0 \text{ for all } i > 0$$

Equivalently (by Serre Duality)

$$H^i(X, O_X(-D)) = 0 \text{ for all } i < n$$

One motivation behind Kodaira Vanishing Theorem is to determine whether a complex manifold has meromorphic functions with prescribed zeros and poles. For example, let X be a compact Riemann surface and let D be a divisor on X. By Riemann-Roch Theorem, we have

$$h^0(O_X(D)) - h^1(O_X(D)) = \deg D + 1 - g$$

If D is more "positive" than the canonical divisor K_X, *i.e.*

$$O_X(D) = K_X \otimes L$$

for some ample line bundle L, then by Kodaira Vanishing Theorem

$$h^1(O_X(D)) = 0$$

Therefore, if $\deg D > g$, then X admits a meromorphic function, whose zeros and poles are prescribed by D.

The original proof of Kodaira is differential-geometric in nature. He identified cohomology classes in $H^{n-q}(X, \Omega_X^{n-p} \otimes O_X(D))$ with $H^{p,q}(O_X(D))$, the harmonic forms of type (p, q) with value in $O_X(D)$ He wrote down explicitly the differential equations satisfied by a harmonic form φ of type (p, q) in terms of local coordinates on X, and he proved via explicit computation that if one can give a connection ∇ on $O_X(D)$ so that the associated curvature form of $O_X(D)$ is positive (ample in the language of algebraic geometry), then $\varphi = 0$ for $p + q \leq n$.

With modern technologies like Hodge theory, which links the topology of the space X to its complex analytic structure, some people would prefer to prove statements like Kodaira Vanishing Theorem without differential-geometric method. The trendy proof (due to C.P. Ramanujam) nowadays involves Lefschetz Hyperplane theorem (topological aspects) and classical Hodge decomposition on compact Kähler manifold (a link between singular cohomology and coherent cohomology):

Theorem 1.0.2 *(Lefschetz Hyperplane Theorem) Let X be a smooth complex projective variety of dimension n, and let D be an effective ample divisor on X. Then the restriction map*

$$r_i : H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

is an isomorphism for $i \leq n - 2$ and injective for $i = n - 1$

Theorem 1.0.3 *(Hodge Decomposition) Let X be compact Kähler manifold, then there is a decomposition*

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}$ denotes the harmonic forms of type (p, q). Moreover,

1. $H^{p,q}(X) = \overline{H^{q,p}(X)}$

2. $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$

Using the above results, we can prove a stronger version of Kodaira Vanishing Theorem

Theorem 1.0.4 *(Akizuki-Nakano Vanishing Theorem) Let X be a smooth complex projective variety of dimension n, and let D be an ample divisor on X. Then,*

$$H^q(X, \Omega_X^p(-D)) = 0 \textit{ for all } p + q < n$$

Equivalently

$$H^q(X, \Omega_X^p(D)) = 0 \textit{ for all } p + q > n$$

Proof (C.P. Ramanujam) Now, assuming D in Kodaira Vanishing Theorem is very ample, one can conclude that the restriction map

$$r_{p,q} : H^q(X, \Omega_X^p) \to H^q(D, \Omega_D^p)$$

induced from the map $r : H^{p+q}(X, \mathbb{C}) \to H^{p+q}(D, \mathbb{C})$ is an isomorphism for $p + q \leq n - 2$, and injective for $p + q = n - 1$.
Then, taking cohomology sequence of

$$0 \longrightarrow \Omega_X^p(-D) \longrightarrow \Omega_X^p \longrightarrow \Omega_D^p \longrightarrow 0$$

one gets

$$\cdots \longrightarrow H^q(X, \Omega_X^p(-D)) \longrightarrow H^q(X, \Omega_X^p) \longrightarrow H^q(D, \Omega_D^p) \longrightarrow \cdots$$

Therefore,

$$H^q(X, \Omega_X^p(-D)) = 0 \text{ for } p + q < n$$

which is equivalent to

$$H^q(X, \Omega_X^p(D)) = 0 \text{ for } p + q > n$$

■

The Hodge Decomposition stated above is equivalent to the degeneration of holomorphic de Rham complex with respect to the "naive" filtration (Hodge filtration) at E_1:

Let X be a complex manifold, the holomorphic de Rham complex $\Omega^{\cdot}_X$

$$0 \to O_X \xrightarrow{\partial} \Omega_X \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n_X \to 0$$

is a resolution of $\mathbb{C}$, the constant sheaf on X. This means

$$H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega^{\cdot}_X)$$

The holomoprhic de Rham complex $\Omega^{\cdot}_X$ is equipped with a "naive" filtration

$$F^p\Omega^{\cdot}_X := \Omega^{\geq p}_X$$

When X is a compact Kähler manifold, the Hodge filtration on $H^k(X, \mathbb{C})$ coincides with the "naive" filtration on $\Omega^{\cdot}_X$, *i.e.*

$$F^pH^k(X, \mathbb{C}) = \mathrm{Im}(\mathbb{H}^k(X, F^p\Omega^{\cdot}_X) \to \mathbb{H}^k(X, \Omega^{\cdot}_X))$$

Ω^p_X has a fine resolution by C^∞ differential forms $(\mathscr{A}^{p,\cdot}, \bar{\partial})$ (Debeault resolution). Therefore, one can use the cohomology of global sections of the double complex $\mathscr{A}^{\cdot,\cdot}$ to compute $\mathbb{H}^{\cdot}(X, \Omega^{\cdot}_X)$. The Hodge decomposition implies that the spectral sequence to $(R\Gamma(\Omega^{\cdot}_X), F)$

$$E^{p,q}_1 = H^q(X, \Omega^p_X) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^{\cdot}_X)$$

degenerates at E_1.

This seemingly more complicated way of stating Hodge decomposition generalizes to the study of Mixed Hodge Structure on the cohomology groups of open manifold admitting a compactification by Kähler manifold:

Consider a non-compact complex manifold U. Suppose U can be compactified to a Kähler manifold by normal crossing divisor, *i.e.*$U = X - D$ where X is a compact Kähler manifold and D is a normal crossing divisor on X. The holomorphic de Rham complex can be "enlarged" to the logarithmic de Rham complex

$$0 \to O_X \xrightarrow{\partial} \Omega_X(\log D) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n_X(\log D) \to 0$$

It is not exact, but it is a complex.

Let $j : U \to X$ be the inclusion map. The inclusion map

$$\Omega^{\cdot}_X(\log D) \to j_*\mathscr{A}^{\cdot}$$

is a quasi-isomorphism [4] [1]. This means

$$H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega^{\cdot}_X(\log D))$$

The advantage for using logarithmic de Rham complex $\Omega^{\cdot}_X(\log D)$ instead of holomorphic de Rham complex $\Omega^{\cdot}_U$ is that $\mathbb{H}^k(X, \Omega^{\cdot}_X(\log D))$ has a mixed Hodge structure defined by the "naive" filtration F and a rationally defined weight filtration W on $\Omega^{\cdot}_X(\log D)$.
As a general statement of mixed Hodge structure, one has

Theorem 1.0.5 *(P. Deligne) The spectral sequence to* $(R\Gamma(X, \Omega^{\cdot}_X(\log D)), F)$

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^{\cdot}_X(\log D))$$

degenerates at E_1*.*

Mixed Hodge Theory can be used to prove logarithmic version of various vanishing theorems. For example,

Theorem 1.0.6 *(Logarithmic Akizuki-Nakano Vanishing Theorem) Let X be a compact Kähler manifold, and let $D \subset X$ be a simple normal crossing divisor on X, then for any ample line bundle L on X*

$$H^q(X, \Omega^p_X(\log D) \otimes L) = 0$$

The goal of this thesis is to leverage the E_1-degeneration of mixed Hodge structure and topological vanishing theorem to prove a more general version of the logarithmic Akizuki-Nakano vanishing theorem:
Use the same notation as above

Theorem 1.0.7 *(Main Vanishing Theorem) Let V be a unitary local system on U and let E be its canonical extension on X. For any ample line bundle L on X*

$$H^q(X, \Omega_X^p(\log D) \otimes E \otimes L) = 0$$

for $p + q > n$.

We will review the notion of local system on complex manifold and its canonical extension in Chapter 2. Like the logarithmic de Rham complex for $\mathbb{C}$ on U, there is a logarithmic de Rham complex for V

$$0 \to E \xrightarrow{\nabla} E \otimes \Omega_X(\log D) \to \cdots \to E \otimes \Omega_X^n(\log D) \to 0$$

where ∇ is a flat connection on E with at most logarithmic singularities along D. We will denote this de Rham complex in the rest of this work by $\mathrm{DR}_X(D, E)$

The first step towards the proof of Theorem 1.0.7 is to construct a mixed Hodge structure on the hypercohomology groups of $\mathrm{DR}_X(D, E)$. This is done by defining a decreasing (Hodge) filtration F and an increasing (weight) filtration W on $\mathrm{DR}_X(D, E)$.

The Hodge filtration is quite straight-forward, it is the naive filtration on $\mathrm{DR}_X(D, E)$. The weight filtration on the de Rham complex $\mathrm{DR}_X(E, D)$ is defined as the kernel of residue maps (much like the case for the logarithmic de Rham complex with small caveat). The residue maps in this context will be defined in Chapter 2.3.

In Chapter 2.5, we will construct a mixed Hodge theory on the complex $\mathrm{DR}_X(E, D)$. The construction of Hodge theory on $W_0\mathrm{DR}_X(E, D)$, which is quasi-isomorphic to j_*V has been worked out by K.Timmerscheidt in Appendix D of [5]. The method he employed is to give the base space X a Kähler metric that is asymptotic to the Poincaré metric near the of L_2-integrable forms and a de Rham complex $L_2(U, V)$ of L_2-integrable forms with value in V. Then, the method for constructing classical Hodge theory apply verbatim to the cohomology groups of the complex $L_2(U, V)$. Lastly, K.Timmerscheidt proved that j_*V is quasi-isomorphic to $L_2(U, V)$. This gives a Hodge theory on $H^*(X, j_*V)$.

In fact, S.Zucker has given similar constructions for arbitrary variations of polarized Hodge structures over curves in [6], and the method used by K.Timmerscheidt is based on the work of S.Zucker.
To finish constructing a mixed Hodge structure on $\mathrm{DR}_X(E,D)$, we will need to construct a Hodge structure of weight m on the associated graded complex $\mathrm{Gr}_m^W\mathrm{DR}_X(E,D)$. But in the main result of Chapter 2.5, we will see that the residue maps defined in Chapter 2.3 will induce an isomorphism from $\mathrm{Gr}_m^W\mathrm{DR}_X(D,E)$ to $W_0\mathrm{DR}_{\tilde{D}_m}(E_m,C_m)[-m]$. Then, applying the method of K.Timmerschedit and S.Zucker, one can construct a Hodge theory on $\mathrm{Gr}_m^W\mathrm{DR}_X(E,D)$.
Once we have constructed a mixed Hodge theory on the de Rham complex $\mathrm{DR}_X(E,D)$, we can conclude that

1. The spectral sequence for $(R\Gamma(\mathrm{DR}_X(E,D)),W)$

$$E_1^{-m,k+m}=\mathbb{H}^k(X,\mathrm{Gr}_m^W\mathrm{DR}_X(E,D))\Rightarrow\mathbb{H}^k(X,\mathrm{DR}_X(E,D))$$

degenerates at E_2.

2. The spectral sequence for $(R\Gamma(\mathrm{DR}_X(E,D)),F)$

$$E_1^{p,q}=H^q(X,\Omega_X^p(\log D)\otimes E)\Rightarrow\mathbb{H}^{p+q}(X,\mathrm{DR}_X(E,D))$$

degenerates at E_1.

Then, the cohomology vanishing theorems of this work will be a formal consequence of cohomology exact sequences and cyclic cover trick.
Another approach for proving the vanishing theorem is to interpret E as a parabolic Higgs bundle. In Chapter 3.2, we will prove that E equiped with the trivial Higgs field is a parabolic Higgs bundle with trivial parabolic Cherns classes, and it is parabolic semistable. Vanishing theorem of such Higgs bundle has been studied by D. Arapura, F. Hao and H. Li in [3]. Theorem 1.0.7 is then a corollary of the main result from [3]
The weight filtration W defined on the complex $\mathrm{DR}_X(D,E)$ naturally defines a a weight filtration on the bundle $E\otimes L\otimes\Omega_X^*(\log D)$. In Chapter 3, we will prove the graded version of the main vanishting theorem

Theorem 1.0.8 *(Graded Vanishing Theorem)* $H^q(X, Gr_m^W \Omega_X^p(\log D) \otimes E \otimes L) = 0$ *for all* m *and for all* $p+q > \dim X$

This is a stronger statement than Theorem 1.0.7.

The proof to the graded vanishing theorem will follow the same pattern as the proof of the main vanishing theorem. We will first prove the case when L in Theorem 1.0.8 is very ample. In that case, let B be a smooth hyperplane divisor intersecting D transversally, *i.e.*$B + D$ form a normal crossing divisor. Then, we will construct a mixed Hodge theory on $\mathrm{DR}_X(D + B, E)$ and make use of the statements about the spectral sequence degeneration. We could follow the procedure in Chapter 2 and define a weight filtration W via residue map. This will give us a mixed Hodge structure. However, this is not the one that we want. For the technical reason of the proof, we in fact want to define a weight filtration W_B on $\mathrm{DR}_X(D + B, E)$ so that it is the kernel of the map

$$r_m \otimes \mathrm{id} : W_m(\mathrm{DR}_X(D, E)) \otimes O_X(B) \to W_m(\mathrm{DR}_B(D \cap B, E_B) \otimes O_X(B)$$

where r_m is the restriction map. In Chapter 4.2, we will prove this weight filtration W_B together with the usual Hodge filtration F define a mixed Hodge structure on $\mathrm{DR}_X(D + B, E)$.

This mixed Hodge structure will enable us to prove Theorem 1.0.8 in case that $L = O_X(B)$ is very ample. Then, applying the same cyclic cover trick as in the proof of Theorem 1.0.7 we can give a full proof of Theorem 1.0.8.

2. PRELIMINARIES

In this chapter, we will first review the theory of local system and its canonical extension. Then, we will give a more comprehensive study of unitary local sytem on the complement of a normal crossing divisor which is the subject of interest of this thesis work. This includes

- Residue map on the de Rham complex of a unitary local system;
- Weight filtration on the de Rham complex;
- Abstract Hodge theory on the de Rham complex;

This part of the preliminary work will show that the hypercohomology group of the de Rham complex carries a mixed Hodge structure. This allows us to obtain statement like degeneration of Hodge spectral sequence at E_1 stage which is an important component for the proof of all vanishing theorems in this thesis.

2.1 Local System and canonical connection

Let Y be a complex manifold. A local system $\mathscr{L}$ on Y with value in $\mathbb{C}^r$ is a sheaf on Y such that

- Y has an open cover by U_i, such that restriction of $\mathscr{L}$ on U_i is isomorphic to the constant sheaf $F_i = \mathbb{C}^r$;
- On the double overlap $U_{ij} = U_i \cap U_j$, there is an isomorphism
$$g_{ij} : F_i|_{U_{ij}} \to F_j|_{U_{ij}}$$
- g_{ij} satisfy cocycle condition on triple intersection;

Example 1 *Let Y be the punctured complex unit disk with coordinate z. The solution to the differential equation*

$$\frac{df}{dz} = \frac{1}{2}\frac{f}{z}$$

is generated by multi-valued function $f = z^{\frac{1}{2}}$ over $\mathbb{C}$. For each point $y \in Y$, there is an open set U_y on which one can choose a branch for $\log z$ and make f a well-defined holomorphic function. The solution to the above differental equation on Y form a local system with value in $\mathbb{C}$ on Y. It is clear that this local system does not have a global section.

Assume Y is connected.

Call g_{ij} the transition functions of $\mathscr{L}$.

Lemma 1 *Two local systems $\mathscr{L}$ and $\mathscr{L}'$ on Y are isomorphic if there is an linear map $A \in GL(\mathbb{C}, r)$ such that*

$$g_{ij} = Ag'_{ij}A^{-1}$$

Proof Let $\phi : \mathscr{L} \to \mathscr{L}'$ be an isomorphism. ■

Lemma 2 *If Y is a simply connected topological space, then Y admits no nontrivial local system.*

Then, we have

Theorem 2.1.1 *Fix a point $y \in Y$. Then, there is a natural bijection between isomorphism classes of local system with value in $\mathbb{C}^r$ and the set of representations*

$$\pi_1(Y, y) \to GL(\mathbb{C}, r)$$

modulo the action of $GL(\mathbb{C}, r)$ by conjugation.

Proof We give a sketch here. For more detailed discussion, see [7] Chapter 3. Fix a local system $\mathscr{L}$, we construct its corresponding representation $\rho(\mathscr{L})$: Let γ be a loop at y. For each point $z \in \gamma$, there is an open set U_z on which L is isomorphic to the constant sheaf. One can find finitely many points $y_1 = y, y_2, \cdots, y_n$ such that U_{y_i} cover γ. Take g_{in} to be the transition function of $\mathscr{L}$ on $U_1 \cap U_n$. g_{in} represents the image of γ in $\mathrm{GL}(\mathbb{C}, r)$.

From the lemma above, we see that if $\mathscr{L}'$ is isomorphic to $\mathscr{L}$, then its representation $\rho(\mathscr{L}')$ is conjugate to $\rho(\mathscr{L})$. ■

Remark 1 *For a local system $\mathscr{L}$ on Y, we don't get a representation $\pi_1(Y, y) \to GL(\mathbb{C}, r)$ out of box,* i.e.*there is no "canonical representation" associated to $\mathscr{L}$*

Work on the correspondence between local system and representation of fundamental groups

For each local system $\mathscr{L}$, we call any representation $\pi_1(Y, y) \to \mathrm{GL}(\mathbb{C}, r)$ corresponding to the isormophism class of $\mathscr{L}$ a monodromy representation of $\mathscr{L}$. Fix a monodromy representation

$$\rho : \pi_1(Y, y) \to \mathrm{GL}(\mathbb{C}, r)$$

For a loop $\tau \in \pi_1(Y, y)$, its image $\rho(\tau)$ is called the monodromy of $\mathscr{L}$ along τ. In Example 1, the monodromy of the solution to the differential equation

$$\frac{df}{dz} = \frac{1}{2}\frac{f}{z}$$

along the unit circle is -1, which is precisely the change undergoes $z^{\frac{1}{2}}$ when switching from one branch to the next.

2.1.1 Logarithmic extension of a local system

Let X be a complex manifold, and let $D \subset X$ be a normal crossing divisor. Let $\mathscr{L}$ be a local system of rank r defined on $Y := X - D$. In the following subsections, we will recall the construction of logarithmic extension of $\mathscr{L}$ and its properties. Most statements given here are due to Deligne. Readers can find more details about logarithmic extension in [8]

A logarithmic extension of $\mathscr{L}$ is a vector bundle E defined on X together with a logarithmic connection

$$\nabla : X \to X \otimes \Omega_X(\log D)$$

such that the flat sections of ∇ (kernel of ∇) on U coincide with $\mathscr{L}$. For future reference, we write the flat sections of ∇ as E^{∇}.

Logarithmic extension of $\mathscr{L}$ is uniquely determined once logarithms for the generalized eigenvalues of the monodromy of $\mathscr{L}$ are specified. We will illustrate this via explicit construction of (E, ∇).

We will construct (E, ∇) locally, then show these local objects are uniquely determined by the local system $\mathscr{L}$. This implies that the bundles constructed locally glue to a global object.

Fix a point $x \in X$, and let $U \subset X$ be a polydisk neighborhood of x. Let $(z_1, \cdots, z_n)$ be analytic coordinate on U such that $D \cap U = D_1 + \cdots + D_s$ is defined by $z_1 \cdots z_s = 0$. Let $Y_U = Y \cap U$. Then, Y_U is homotopic to

$$\overbrace{S^1 \times \cdots \times S^1}^{s}$$

Therefore, $\pi_1(Y_U)$ is a free abelian group of rank r, each generator is represented by δ_i, a small circle around D_i.

Fix a representation $\rho : \pi_1(Y) \to \mathrm{GL}(\mathbb{C}, r)$. We don't need to choose a base point, because we can assume Y is path-connected. Then, we have the local monodromy representation of $\mathscr{L}$

$$\pi_1(Y_U) \hookrightarrow \pi_1(Y) \xrightarrow{\rho} \mathrm{GL}(\mathbb{C}, r)$$

Let γ_j be the image of δ_i. Just like in Example 1, we will construct a system of differential equations, whose solution can be identified with the sections of $\mathscr{L}$. Consider the system

$$\frac{\partial f_i}{\partial z_j} = \sum_{k=1}^{r} a_{ik}^{j} f_k \frac{1}{z_j}$$

$k = 1, \cdots, r$, $j = 1, \cdots, s$. Let f denote the vector $(f_1, \cdots, f_r)^T$, and let A_j denote the matrix (a^j_{ik}). Then, the above system can be compactly written as

$$df - \sum_{j=1}^{r} A_j f \frac{dz_j}{z_j}$$

Just like Example 1, the solution to the above system can be represented by the multi-valued section

$$f = z_1^{A_1} \cdots z_r^{A_r}$$

The monodromy of f with respect to z_j is $e^{2\pi i A_j}$ (Here $i = \sqrt{-1}$).
This means,

$$e^{2\pi i A_j} = \gamma_j$$

and therefore, to construct the system

$$df - \sum_{j=1}^{r} A_j f \frac{dz_j}{z_j}$$

it remains to make sense of $\log A_j$.

2.1.2 Logarithm of complex-valued matrices

Given a matrix B, a matrix A is said to be a matrix logarithm of B if $e^A = B$ where exponential is defined in terms of power series expansion. Write $A = \log B$, if A is a matrix logarithm of B.

Lemma 3 *Let B be a complex-valued matrix. Write $B = VJV^{-1}$, where J is the Jordan canonical form of B. Then, if $\log J$ exists, then $V \log JV^{-1}$ is a logarithm of B*

Proof It is enough to show that

$$e^{V \log JV^{-1}} = VJV^{-1}$$

This follows directly from the power series expansion of of $e^{V \log JV^{-1}}$. ■

The above lemma shows that to define $\log B$, one can assume B is a Jordan block. Suppose B is a Jordan block of dimension n with generalized eigenvalue λ. Then,

$$B = \lambda(I + K)$$

where K is a $n \times n$ nilpotent matrix.
Use the formal power series expansion

$$\log(1 + x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

one gets

$$\log B = \log(\lambda(I + K)) = \log(\lambda I) \log(I + K) = (\log \lambda)I + K - \frac{K^2}{2} + \frac{K^3}{3} + \cdots$$

The infinite sum is actually finite, because K is a nilpotent matrix. Therefore, by specifying a branch for $\log \lambda$, one can define $\log B$.

2.1.3 Canonical extension and de Rham complex of a local system

Let B be a complex-valued matrix. Suppose all its generalized eigenvalues are non-zero. Then, as we see from above, one can define logarithm of B through its Jordan canonical form.
Back to the construction of logarithmic extension of $\mathscr{L}$. We have seen that locally on a polydisk open set U of X, ∇ can be defined as

$$d + \sum_{j=1}^{s} A_j \frac{dz_j}{z_j}$$

where $A_j = -\frac{1}{2\pi i} \log \gamma_j$. Choose the branch for $\log \gamma_j$ so that the real part of eigenvalues of A_j lie in $[0, 1)$
Set $v = z_1^{A_1} \cdots z_s^{A_s}$. v is a multi-valued function $U \cap Y \to \mathbb{C}^r$. The canonical extension of $\mathscr{L}$ over U consists of a coherent sheaf E_U and a connection $\nabla : E_U \to E_U \otimes \Omega_U(\log D)$. For any $p \in U$, $E_{U,p}$ consists of sections of the form

$$\sum_i f \cdot v \otimes e_i, i = 1, \cdots, r$$

where $f \in O_{U,p}$ and e_i are the standard basis of $\mathbb{C}^r$. And ∇ is of the form

$$d + \sum_{j=1}^{s} A_j \frac{dz_j}{z_j}$$

The flat sections of E_U can be identified with sections of $\mathscr{L}_U$.

Proposition 2.1.1 *The canonical extension constructed locally above glues to an global object.*

Proof Cover X by polydisks U_i over which $\mathscr{L}$ is isomorphic to the constant sheaf $\mathbb{C}^r$. Then, glue $O_{U_i}^r$ using transition matrices of $\mathscr{L}$, and denote the resulting global vector bundle by E.

Write ∇_i for the local definition of canonical connection on U_i and write ϕ_{ji} for the transition matrices of E on $U_i \cap U_j$, mapping the i-coordinate system to the j-coordinate system. Under these transition matrices, the flat sections over U_i are mapped to flat sections over U_j.

Now we will show these locally defined connections are compatible with the transition matrices. Take a section s_i defined over U_i, and write

$$s_i = \sum a_{ik} e_k$$

where e_k are flat sections. Then

$$\nabla_j(\phi_{ji} s_i) = \sum_k d(\phi_{ji} a_{ik}) e_k = \sum_k \phi_{ji} d(a_{ik} e_k = \phi_{ji} \nabla_i(s_i)$$

This is precisely what we want. ∎

2.1.4 Extension of monodromy action to the boundary

Let $\mathrm{res}_j : \Omega_X(\log D) \otimes E \to O_{D_j} \otimes E$ be the Poincaré residue. And let $\Gamma_j = \mathrm{res}_j \circ \nabla$. Locally on Δ, $\Gamma_j = A_j$.

Proposition 2.1.2 Γ_j *defines an endomorphism*

$$\Gamma_j : O_{D_j} \otimes E \to O_{D_j} \otimes E$$

Proof It is clear by from definition of Poincaré residue that $\mathrm{res}_j \circ \nabla = 0$ on $O_X(-D_j) \otimes E$. ■

This means A_i's constructed above are local representation of an endomorphisms of a vector bundle. Therefore, eigenvalues of A_i are global objects. As they are constant on Δ, they are constant globally.

2.2 Unitary local system on the complement of a normal crossing divisor

Let X be a compact Kähler manifold, $D \subset X$ a simple normal crossing divisor, and

$$j : U := X - D \hookrightarrow X$$

the inclusion map.

We have seen that a local system of rank r on U is equivalent to a representation

$$\rho : \pi_1(U) \to \mathrm{GL}(\mathbb{C}, r)$$

A local system of rank r is call *unitary* if its associated representation is unitary. One special feature about unitary local system is that one can define a Hodge structure on $H^k(X, j_*V)$ and it is the weight 0 part of the mixed Hodge structure on $H^k(U, V)$ which we will construct in Section 2.5. In this section, we will review the construction of the Hodge structure on $H^k(X, j_*V)$. Most results here can be found in [5](Appendix D) and [6]. The idea is to compute $H^k(X, j_*V)$ as the cohomology of square-integrable forms with value in V with respect to a carefully chosen Kähler metric on U.

2.2.1 Local decomposition of unitary local system

Let $\Delta = \Delta_1 \times \cdots \times \Delta_n$ be polydisk on X. Give each Δ_i the analytic coordinate z_i. Suppose Δ_i and z_i are chosen so that $D \cap \Delta$ is defined by $z_1 \cdots z_s$. Then $U \cap \Delta = \Delta_1^* \times \cdots \times \Delta_s^* \times \Delta_{s+1} \times \cdots \times \Delta_n$ which is topologically equivalent to

$$\overbrace{S^1 \times \cdots \times S^1}^{s \text{ times}}$$

As $\pi_1(\overbrace{S^1 \times \cdots \times S^1}^{s \text{ times}})$ is an Abelian group, the monodromy representation of V on $U \cap \Delta$ are commutative matrices. This means the monodromies of V on $U \cap \Delta$ can be simultaneously diagonalized, *i.e.*one can assume V decompose into direct sum of unitary local system of rank 1. This is a useful observation, because many statements latter can be proved via local computations on V and by the above argument, we can assume V is of rank 1.

2.2.2 Hermitian metric on the canonical connection of V

Let $E \to X$ be the canonical connection of V. On the polydisk Δ of X, let $\lambda_1, \cdots, \lambda_s$ be the local monodromy representation of V. As observed above, we can simultaneously diagonalize all λ_j, so we can assume they are all diagonal matrices. This makes taking logarithms of λ_j much easiser, and we can therefore get a more explicit representation of the canonical extension E:

The multi-valued C^∞-function f on $\Delta \cap U$ with the prescribed monodromies $\lambda_1, \cdots, \lambda_s$ satisfies the differential equations:

$$df + \sum_{j=1}^{s} A_j f \frac{dz_j}{z_j}$$

where $A_j = -\frac{1}{2\pi i} \log \lambda_j$. As λ_j are diagonal matrices, so are the A_j. Let a_i^j denote the eigenvalues of A_j. As λ_i are unitary, $a_i^j \in \mathbb{R}$. So $a_i^j \in [0, 1)$

On $\Delta \cap U$, holomorphic sections of E consists of

$$f \cdot \prod z_j^{-a_j^i} \otimes e_i$$

Set $v_i = \prod z_j^{-a_j^i} \otimes e_i$. On Δ, give E the Euclidean metric h

$$h(f \cdot v_i, f \cdot v_i) = |f|^2 \prod |z_j|^{-2a_j^i}$$

To show this metric extends globally, it is enough to show this metric, when restricted to the flat sections of ∇, is well defined globally.

Let Δ_1 and Δ_2 be two distinct polydisks with coordinate systems $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$. Use this coordinate systems and the monodromy representations of V on $U \cap \Delta_1$ and $U \cap \Delta_2$, one can write down the flat sections of E (sections of V), as v_z and v_w, respectively. On $\Delta_1 \cap \Delta_2$, the representations of v_z and v_w, under change of coordinates, differ by a unitary transformation. Therefore, $h(v_z, v_z) = h(v_w, v_w)$. This proves that h is well-defined globally.

2.2.3 A good metric on U

To give a Hodge theory on $H^k(X, j_*V)$, the idea is to represent every class of $H^k(X, j_*V)$ by a differential k-form, which is square-integrable with respect to a carefully chosen metric on U [6] [5].

Theorem 2.2.1 *[6](Prop 3.2) U has a Käher metric η, which on $\Delta \cap U$ is equivalent to*

$$\frac{i}{2}\left(\sum_{i=1}^{s} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z|^2} + \sum_{i=s+1}^{n} dz_i \wedge d\bar{z}_i\right)$$

η has some very nice properties

Theorem 2.2.2 *[6](Prop 3.3) U equipped with η is a complete manifold with finite volume*

Theorem 2.2.3 *[5](Appendix D) Sections of j_*V are square integrable with respect to η.*

Proof It is enough to show

$$\int_{U \cap \Delta} h(v, v) \frac{i}{2}\left(\sum_{i=1}^{s} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2 \log^2 |z|^2} + \sum_{i=s+1}^{n} dz_i \wedge d\bar{z}_i\right)$$

is finite. ∎

2.2.4 L_2-cohomology on U with value in V

Let h denote the Hermitian metric on E defined above. It extends to the C^∞ sections of E. And ∇ defined on the holomorphic sections of E extends to $\nabla + \bar{\partial}$ on C^∞ sections of E. Let $\mathscr{L}_2^k(V)$ be the sub-sheaf of $j_* \mathscr{A}_U^k \otimes V$ such that $\omega \in \mathscr{L}_2^k(V)$ if and only if both ω and $\nabla + \bar{\partial}$ are square-integrable with respect to η near D.
Let $L_2^k(V) = \Gamma(X, \mathscr{L}_2^k(V))$ and define the k-th L_2-cohomology group on U with value in V as

$$H^k(U,V)_2 := H^k(L_2^k(V), \nabla + \bar{\partial})$$

Theorem 2.2.4 *[6] [5](Appendix D) The L_2-complex*

$$\mathscr{L}_2^0(V) \xrightarrow{\nabla+\bar{\partial}} \mathscr{L}_2^1(V) \xrightarrow{\nabla+\bar{\partial}} \cdots \xrightarrow{\nabla+\bar{\partial}} \mathscr{L}_2^{2n}(V)$$

is exact in degree 1 and above, and its kernel is $j_* V$

$\mathscr{L}_2^k(V)$ is a fine sheaf by partition of unity. So

$$H^k(X, j_* V) = H^k(U,V)_2$$

Like in the classical Hodge theory, we can define the notion of Harmonic forms in $L_2^k(V)$. With the same reason as classical Hodge theory, one finds a decomposition

$$H^k(U,V)_2 = \bigoplus_{p+q=k} H^{p,q}(U,V)_2$$

where $H^{p,q}(U,V)$ is the space of Harmomic form in $L_2^k(V)$ of type (p,q).
This defines a Hodge theory on $H^k(X, j_* V)$.
Let $\mathscr{L}_2^{p,\cdot}(V)$ denote the Dolbeault complex

$$\mathscr{L}_2^{p,0}(V) \xrightarrow{\bar{\partial}} \mathscr{L}_2^{p,1}(V) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}_2^{p,n-p}(V)$$

Theorem 2.2.5 *[5](Appendix D) $\mathscr{L}_2^{p,\cdot}(V)$ is exact above (and include) degree 1.*

Let $W_0(\Omega_X^p(\log D)\otimes E)$ denote the kernel of $\mathscr{L}_2^{p,\cdot}(V)$ (the notation will be clear in Section 2.5). Then

$$H^{p,q}(U,V)_2 = H^q(X, W_0(\Omega_X^p(\log D)\otimes E))$$

$W_0(\Omega_X^p(\log D)\otimes E)$ is a subsheaf of $\Omega_X^p(\log D)$. Assume V is of rank 1, we can give more explicit description of $W_0(\Omega_X^p(\log D)\otimes E)$ (See [5](Appendix D))
Let δ be a local generator of E over Δ. Then $W_0(\Omega_X(\log D)\otimes E)$ is the sub-sheaf of $\Omega_X(\log D)\otimes E$ generated by sections of the form $\omega_i\otimes\delta$ for $i=1,\cdots n$, where

$$\omega_i = \frac{dz_i}{z_i} \text{ if monodromy of } V \text{ around } z_i = 0 \text{ is non-trivial}$$
$$\omega_i = dz_i \text{ if monodromy of } V \text{ around } z_i = 0 \text{ is trivial}$$

$W_0(\Omega_X^p(\log D)\otimes E) = \bigwedge^p W_0(\Omega_X(\log D)\otimes E)$
point out that unitary local system degenerates into rank 1 unitary local system.
point out one can give U a metric (poincare metric) on U so that $H^k(U,V)_2$ has a Hodge theory, and

$$H^k(X, j_*V) = H^k(U,V)_2$$

2.3 Residue Map

In this section we will define a residue map $\mathrm{Res}(E)$ on the complex $\mathrm{DR}_X(D,E)$. Similar to the usual residue map on the holomorphic de Rham complex, $\Omega_X^{\cdot}$, $\mathrm{Res}(E)$ will define a weight filtration on $\mathrm{DR}_X(D,E)$. $\mathrm{Res}(E)$ has been defined and studied in [9].
For $m=1,\cdots n$, let D_m be the union of m-fold intersection of components of D; Let $\tilde{D}_m$ be the disjoint union of components of D_m; Let $v_m:\tilde{D}_m\to X$ be the composition of the projection map onto D_m and the inclusion map. $\tilde{C}_m := v_m^* D_{m+1}$ is either empty or a normal crossing divisor in $\tilde{D}_m$

Theorem 2.3.1 *[9](Proposition 1.3)*

1. *$V_m := j_* V|_{D_m - D_{m+1}}$ is a unitary local system on $D_m - D_{m+1}$.*

2. *There exist a unique subvectorbundle E_m of $v_m^* E$ and a unique holomorphic integrable connection ∇_m on E_m with logarithmic poles along C_m such that*

$$\ker \nabla_m|_{\tilde{D}_m - \tilde{C}_m} = v_m^{-1} V_m$$

3. *There exists a unique subvectorbundle E_m^* of $v_m^* E$ with*

$$E_m \oplus E_m^* = v_m^* E$$

Proof All of the statements above are local. Therefore, we can assume X is a polydisk. Write $X = \Delta_1 \times \cdots \times \Delta_n$, and let z_i be the coordinate on Δ_i. Suppose D is defined by

$$z_1 \times \cdots \times z_s = 0$$

1. The local system V on U is equivalent to an unitary representation

$$T : \pi_1(U) \to \mathrm{GL}(r, \mathbb{C})$$

As $\pi_1(U)$ is abelian and T is unitary, we can simultaneously diagonalize all $T(\gamma_i)$, where γ_i's form a generating set of $\pi_1(U)$. Therefore, we can assume V is a direct sum of rank 1 unitary local systems. Write

$$V = V^1 \oplus \cdots \oplus V^r$$

For each V^i, let $\lambda_{i,j}$ be its monodromy around D^j. So V^i extends to D^j if and only if $\lambda_{i,j} = 1$.

Now let $D^{j1} \cap \cdots \cap D^{jm}$ be one component of D_m, and let $x \in D^{j1} \cap \cdots \cap D^{jm}$. Then, near x V_m is

$$\bigoplus_{\lambda_{i,j1} = \cdots = \lambda_{i,jm} = 1} V^i$$

This shows that V_m is a unitary local system.

2. The uniqueness of the subvectorbundle E_m follows from the uniqueness of canonical connection. Therefore, we only need to show the existence part. Use the notation from part 1, and assume V decomposes as direct sum of rank 1 unitary local system V^i. Let E^i be the canonical connection of V^i. Then, it is clear that

$$E_m = \bigoplus_{\lambda_{i,j1}=\cdots=\lambda_{i,jm}=1} v_m^* E^i$$

3. E inheits a flat Hermitian form from V. Define E_m^* as the complement of E_m with respect to this metric. On Δ, E_m^* is the direct sum of $v_m^* E^i$ not appearing in the definition of E_m. ■

Remark 2 *E_m could have different ranks on different component of $\tilde{D}_m$.*

For each $m \leq p \leq \dim D_m$, there exists a residue map crossing divisor in $\tilde{D}_m$.

$$\mathrm{Res}_m : \Omega_X^p(\log D) \to v_{m*}(\Omega_{\tilde{D}_m}^{p-m})$$

This map is defined as follow: Let D_{m1} be one of components of D_m, and suppose D_{m1} is the intersection of $D_{i1}, \cdots, D_{im}$. Then, the map Res_m sends dz_i/z_i to 1 if i appears in $i_1, \cdots, i_m$, and Res_m sends all other 1-form to 0. This map is well-defined independent of the chosen coordinate.
Res_m commutes with exterior derivative d, making it a homomorphism of complexes

$$\mathrm{Res}_m : \Omega_X^{\cdot}(\log D) \to v_{m*}\Omega_{\tilde{D}_m}^{\cdot}(\log \tilde{C}_m)[-m]$$

Consider the following variation of the residue map Res_m

$$\begin{aligned}\mathrm{Res}_m(E) : \Omega_X^p(\log D) \otimes E &\xrightarrow{\mathrm{Res}_m \otimes \mathrm{id}} v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m)) \otimes E \\ &= v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes v_m^* E) \\ &\xrightarrow{\mathrm{id} \otimes p_m} v_{m*}(\Omega_{\tilde{D}_m}^{p-m}(\log \tilde{C}_m) \otimes E_m)\end{aligned}$$

where $p_m : v_m^* E \to \tilde{E}_m$ is the projection onto the E_m component.

Lemma 4 *[9]* $Res_m(E) \circ \nabla = \nabla_m \circ Res_m(E)$, i.e. *$Res_m(E)$ is homomorphism of complexs*

$$DR_X(D,E) \to v_{m*}DR_{\tilde{D}_m}(\tilde{C}_m, E_m)[-m]$$

2.4 Weight Filtration on the de Rham Complex

The residue map

$$\mathrm{Res}_m(E) : \mathrm{DR}_X(D,E) \to v_{m*}\mathrm{DR}(\tilde{C}_m, E_m)[-m]$$

can be used to define a weight filtration $W_.$ on $\mathrm{DR}_X(D,E)$ [9]

$$W_m(\mathrm{DR}_X(D,E)) = \ker \mathrm{Res}_{m+1}(E) \qquad \text{if } m \geq 0$$

$$W_m(\mathrm{DR}_X(D,E)) = 0 \qquad \text{if } m < 0$$

Local descriptions of $W_m(\mathrm{DR}_X(D,E))$ have been given in [9]. We will review them here:

Let $\Delta = \Delta_1 \times \cdots \times \Delta_n$ be a polydisk of X with coordinate $z_1, \cdots, z_n$. Suppose D is defined as

$$z_1 \times \cdots \times z_s = 0$$

As in part 1 of Theorem 2.3.1, we assume V is the direct sum of rank 1 unitary local systems on Δ, and write

$$V = V^1 \oplus \cdots \oplus V^r$$

Definition 2.4.1 *We say $\frac{dz_j}{z_j}$ acts on V^i by identity if $\lambda_{i,j} = 1$, i.e. the monodromy of V^i by a small circle around D_j is the identity.*

Let E^i be be canonical extension of V^i on Δ;

Let μ_i be a generator of E^i, then

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p} \otimes \mu_i$$

is in $W_m(\mathrm{DR}_X(D,E))$ if and only if there are at most m log forms acting on V^i by identity.

Proposition 2.4.1 *[9]*

1. $W_{\cdot}(DR(D, E, \nabla))$ *is an increasing filtration.*

2. $Res_m(E)$ *induces an isomorphism*

$$Gr_m^W(DR(D, \nabla, E)) \to v_{m*}(W_0(DR_{\tilde{D}_m}(\tilde{C}_m, E_m))[-m])$$

Proof The statements are local. We can assume X is a polydisk and V is a unitary local system of rank 1.

1. From the local description of $W_m(\mathrm{DR}_X(D, E))$, it is clear that $W_{\cdot}$ is an increasing filtration.

2. Let s be a section $W_m(\mathrm{DR}_X(D, E))$. Use the local description above, s is of the form

$$\omega \otimes \mu$$

where

$$\omega = \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \cdots \wedge dz_{j_p}$$

and ω has at most m log 1-forms acting on V by identity. μ is a generating section of E.

First, we show $\mathrm{Res}_m(E)(s) \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m$.

$$\mathrm{Res}_m(E)(s) = \mathrm{Res}_m(\omega) \otimes \mu_m$$

By the construction of ω, $\mathrm{Res}_m(\omega)$ does not have log form $\frac{dz_j}{z_j}$ acting on V by identity. This shows that

$$\mathrm{Res}_m(E)(s) \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m)$$

If $\omega_0 \otimes \mu_m \in W_0(\Omega^{p-m}_{\tilde{D}_m}(\log \tilde{C}_m) \otimes E_m)$, to get a preimage in $W_m(\Omega^p_X(\log D) \otimes E)$, simply take

$$\omega_m \wedge \omega_0 \otimes \mu$$

where ω_m is any m-form. And $\omega_m \wedge \omega_0 \otimes \mu \in W_m(\Omega_X^p(\log D) \otimes E)$ by the construction of ω_0. This shows that

$$\mathrm{Res}_m(E) : W_m(\mathrm{DR}(D, E, \nabla)) \to W_0(\mathrm{DR}(\tilde{C}_m, \tilde{E}_m, \tilde{\nabla}_m))$$

is surjective.

If $\mathrm{Res}_m(E)(s) = 0$, that means in ω, there are at most $m-1$ log forms acting on V by identity. This is precisely the local description of $W_{m-1}(\Omega_X^p(\log D) \otimes E)$. ∎

2.5 Mixed Hodge Theory on the de Rham Complex

Let A be a Noetherian subring of $\mathbb{Q}$ such that $A \otimes \mathbb{Q}$ is a field. Throughout this section, assume the unitary local system V has an A-lattice, *i.e.*there is a unitary local system V_A with value in A such that

$$V = V_A \otimes_A \mathbb{C}$$

Recall that $j : U \to X$ is the inclusion map. Write $\Omega_U^{\cdot} \otimes E^o$ for the restriction of $\mathrm{DR}(D, E)$ on U. Then, $\Omega_U^{\cdot} \otimes E^o$ is a resolution of V.

Proposition 2.5.1 $\mathbb{R}j_*V$ *is quasi-isomorphic to* $j_*(\Omega_U^{\cdot} \otimes E^o)$

Proof Let $\mathscr{A}^{p,q}$ denote the sheaf of differential forms of type (p, q) with real analytic coefficient. The double complex $(\mathscr{A}^{\cdot\cdot} \otimes_{O_U} E^o$ gives a resolution of $\Omega_U^{\cdot} \otimes E^o$. Let $\mathscr{A}^{\cdot} \otimes E^o$ denote the associated single complex. It is a fine resolution of V, so we have

$$\mathbb{R}j_*V = j_*(\mathscr{A}^{\cdot} \otimes E^o)$$

But since

$$\Omega_U^{\cdot} \otimes E^o \hookrightarrow \mathscr{A}^{\cdot} \otimes E^o$$

is a quasi-isomorphism, it follows that

$$j_*\Omega_U^{\cdot} \otimes E^o \hookrightarrow j_*\mathscr{A}^{\cdot} \otimes E^o$$

∎

There is a natural inclusion map

$$i : \mathrm{DR}(D, E) \to j_*(\Omega^{\cdot}_U \otimes E^o)$$

Theorem 2.5.1 *[8](Corollary 3.14) The map i is a quasi-isomorphism*

This means

$$\mathbb{H}^k(X, \mathbb{R}j_*V) = \mathbb{H}^k(X, \mathrm{DR}(D, E))$$

Moreover, as V is assumed to have a real lattice V_A, $\mathbb{H}^k(X, \mathbb{R}j_*V_A)$ is a real lattice for $\mathbb{H}^k(X, \mathbb{R}j_*V) = \mathbb{H}^k(X, \mathrm{DR}(D, E))$

The objective of this section is to construct a mixed Hodge structure of weight k on $\mathbb{H}^k(X, \mathrm{DR}(D, E))$

The key result of this section is

Theorem 2.5.2 *Let $K_{\mathbb{C}}$ be $R\Gamma(DR(D, E))$ with the induced Hodge filtration F and weight filtration W. Then*

1. *The spectral sequence to $(K_{\mathbb{C}}, W)$ degenerates at E_2*

2. *The spectral sequence to $(K_{\mathbb{C}}, F)$ degenerates at E_1*

This result is the consequence of Lemma of Two Filtrations, which has been worked out by Deligne in [1] and [2]. We will also give a proof in this section.

The Hodge filtration F induces three descending filtrations on the spectral sequence to $(\mathrm{K}_{\mathbb{C}}, W)$: *Direct filtration* F_d is defined as:

$$F^p_d E_r = \mathrm{Im}(E(F^p\mathrm{K}_{\mathbb{C}}, W) \to E(\mathrm{K}_{\mathbb{C}}, W))$$

Dual of direct filtration F_{d^*} is defined as

$$F^p_{d^*} E_r = \ker(E_r(K, W) \to E_r(\mathrm{K}_{\mathbb{C}}/F\mathrm{K}_{\mathbb{C}}, W))$$

Inductive filtration F_{ind} is defined by induction on r. On E_0, take F_{ind} to be filtration defined by $F_d = F^*_d$, then

$$F_{\mathrm{ind}} E_{r+1}(\mathrm{K}_{\mathbb{C}}, W) = \mathrm{Im}(F_{\mathrm{ind}}\mathrm{Ker} d_r \to E_{r+1}(\mathrm{K}_{\mathbb{C}}, W))$$

Theorem 2.5.3 *[10](Theorem 3.12) Let $K^\cdot$ be a complex with two filtrations W and F, where W is biregular. Suppose for $r = 0, \cdots, r_0$ the differentials d_r of $E_r(K^\cdot, W)$ are strictly compatible with the inductive filtration F_{ind}.*

1. *Then for $r \leq r_0 + 1$ the sequence of complexes*

$$0 \to E_r(F^p K^\cdot, W) \to E_r(K^\cdot, W) \to E_r(K^\cdot / F^p K^\cdot, W) \to 0$$

   is exact. In particular $F_d = F_{ind} = F_d^*$ on $E_0, \cdots, E_{r_0+1}$

2. *Suppose for every $r \geq 0$, d_r is strictly compatible with the inductive filtration F_{ind}, then $F_d = F_{ind} = F_d^*$ and the spectral sequence to $(K_{\mathbb{C}}, F)$ degenerates at E_1*

Proof 1. For $r < r_0$, we will prove by induction on r the statements P_r and P_r^*:
P_r: $E_r(FK^\cdot, W)$ injects into $E_r(K^\cdot, W)$ and its image is $F_{\text{ind}} E_r(K, W)$.

P_r^*: $E_r(K, W)$ surjects onto $E_r(K^\cdot / FK^\cdot, W)$ and its kernel is $F_{\text{ind}} E_r(K, W)$
P_0 is true. Suppose P_r is proved. We want to show P_{r+1}

$$\begin{aligned} F_{\text{ind}} E_{r+1}(K^\cdot, W) := & \operatorname{Im}(\operatorname{Ker}(F_{\text{ind}} E_r(K^\cdot, W) \xrightarrow{d_r} E_r(K^\cdot, W)) \to E_{r+1}(K^\cdot, W)) \\ = & \operatorname{Im}(\operatorname{Ker}(E_r(FK^\cdot, W) \xrightarrow{d_r} E_r(K^\cdot, W)) \to E_{r+1}(K^\cdot, W)) \\ = & \operatorname{Im}(E_{r+1}(FK^\cdot, W) \to E_{r+1}(K^\cdot, W)) \end{aligned}$$

This shows that image of $E_{r+1}(FK^\cdot, W)$ in $E_{r+1}(K^\cdot, W)$ coincide with $F_{\text{ind}} E_{r+1}(K^\cdot, W)$.
Likewise, to show kernel of $E_{r+1}(K^\cdot, W)$ to $E_{r+1}(K^\cdot / FK^\cdot, W)$ is
To prove $E_{r+1}(FK^\cdot, W)$ injects into $E_{r+1}(K^\cdot, W)$ we use the strictness of d_r:

$$d_r E_r(K^\cdot, W) \cap E_r(FK^\cdot, W) = d_r E_r(FK^\cdot, W)$$

Suppose P_r^* is proved. We want to show kernel of $E_{r+1}(K, W) \to E_{r+1}(K/FK, W)$ can be identified as $F_{\text{ind}} E_{r+1}(K, W)$; and the map $E_{r+1}(K, W) \to E_{r+1}(K/FK, W$ is surjective.

Let $a_{r+1} \in F_d^* E_{r+1}(K, W)$, we will show $a_{r+1} F_{\text{ind}} E_{r+1}(K, W)$. From P_r and P_r^*, we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_r(FK^\cdot, W)[-1] & \longrightarrow & E_r(K^\cdot, W)[-1] & \longrightarrow & E_r(K^\cdot/FK^\cdot, W)[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_r(FK^\cdot, W) & \longrightarrow & E_r(K^\cdot, W) & \longrightarrow & E_r(K^\cdot/FK^\cdot, W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_r(FK^\cdot, W) & \longrightarrow & E_r(K^\cdot, W) & \longrightarrow & E_r(K^\cdot/FK^\cdot, W) & &
\end{array}$$

$E_r(K^\cdot, W)[-1]$ is the term such that $d_r : E_1(K^\cdot, W)[-1] \to E_1(K^\cdot, W)$ makes sense. Let $a_r \in E_r(K, W)$ such that $a_r \mapsto a_{r+1}$. Let $\bar{a_r}$ be the image of a_r in $E_r(K^\cdot/FK^\cdot, W)$. As $p(\bar{a_r}) = 0$, we know that

$$p(\bar{a_r}) = d_r(\bar{b})$$

for some $b \in E_r(K^\cdot, W)a[-1]$. Then, set $a_r' = a_r - d_r(b)$. It clear that $a_r' \in F_{\text{ind}} E_r(K^\cdot, W)$ and $p(a_r') = a_{r+1}$, $i.e. a_{r+1} \in F_{\text{ind}} E_{r+1}(K^\cdot, W)$.

To show $E_{r+1}(K^\cdot, W) \twoheadrightarrow E_{r+1}(K^\cdot/FK^\cdot, W)$, consider the following diagram

$$\begin{array}{ccccc}
E_r(FK^\cdot, W)[-1] & \longrightarrow & E_r(FK^\cdot, W) & \longrightarrow & E_r(FK^\cdot, W)[+1] \\
\downarrow & & \downarrow & & \downarrow \\
E_r(K^\cdot, W)[-1] & \longrightarrow & E_r(K^\cdot, W) & \longrightarrow & E_r(K^\cdot, W)[+1] \\
\downarrow & & \downarrow & & \downarrow \\
E_r(K^\cdot/FK^\cdot, W)[-1] & \longrightarrow & E_r(K^\cdot/FK^\cdot, W) & \longrightarrow & E_r(K^\cdot/FK^\cdot, W)[+1]
\end{array}$$

Take $\bar{b} \in E_r(K^\cdot/FK^\cdot, W)$ such that $d_r(b) = 0$. We want to construct an element $b \in E_r(K^\cdot, W)$ such that $p(b) = \bar{b}$ and $d_r(b) = 0$. Let $a \in E_r(K^\cdot, W)$ be any element such that $p(a) = \bar{b}$. $p(d_r(a)) = 0$. So there is some element $c \in E_r(FK^\cdot, W)$ such that $i(c) = d_r(a)$. Use the strictness of d_r:

$$d_r E_r(K^\cdot, W) \cap E_r(FK^\cdot, W) = d_r(E_r(FK^\cdot, W))$$

we conclude that there is an element $e \in E_r(FK^\cdot, W)$ such that

$$d_r(e) = c$$

Set $b = a - e$. We conclude that $d_r(b) = 0$ and $p(b) = \bar{b}$.

2. As W is assumed to be a biregular filtration on $K^\cdot$, $E_\infty = E_n$ for sufficiently large n.

At $r = \infty$, the exact sequence in 1 is

$$0 \to \mathrm{Gr}^W_\cdot H(FK^\cdot) \to \mathrm{Gr}^W_\cdot H(K^\cdot) \to \mathrm{Gr}^W_\cdot H(K^\cdot/FK^\cdot) \to 0$$

Use bootstrap, we can see that

$$0 \to H(FK^\cdot) \to H(K^\cdot) \to H(K^\cdot/FK^\cdot) \to 0$$

Apply the argument of 1 to the complex $(F^pK^\cdot, F, W)$ we conclude that

$$0 \to H(F^pK^\cdot) \to H(F^{p-1}K^\cdot) \to H(\mathrm{Gr}_F^{p-1}K^\cdot) \to 0$$

To show $E(K^\cdot, F)$ degenerates at E_1, we realize the differentials of $E_1(K^\cdot, F)$ as the connecting map of the cohomology of the sequence

$$0 \to \frac{F^{p+1}K^\cdot}{F^{p+2}K^\cdot} \to \frac{F^pK^\cdot}{F^{p+2}K^\cdot} \to \frac{F^pK^\cdot}{F^{p+1}K^\cdot} \to 0$$

Write $A = F^{p+2}K^\cdot, B = F^{p+1}K^\cdot, C = F^pK^\cdot$.

It is easy to see that the connecting homomorphism for the cohomology sequence of

$$0 \to \frac{B}{A} \to \frac{C}{A} \to \frac{C}{B} \to 0$$

is the composition of the connecting homomorphism of

$$0 \to B \to C \to \frac{C}{B} \to 0$$

and the natural map

$$H^\cdot(B) \to H^\cdot(\frac{B}{A})$$

$\blacksquare$

Let $\mathscr{F}^\cdot$ be a bounded complex of sheaf with an increasing filtration W, for each $n \in \mathbb{Z}$ Define

$$W_nR\Gamma(\mathscr{F}^\cdot) := \mathrm{Im}(R\Gamma(W_n\mathscr{F}^\cdot) \to R\Gamma(\mathscr{F}^\cdot))$$

Since $R\Gamma$ is a right-derived functor, one has an injection of complexes

$$R\Gamma(W_n\mathscr{F}^\cdot) \hookrightarrow R\Gamma(\mathscr{F}^\cdot)$$

Therefore, $W_n R\Gamma(\mathscr{F}^\cdot)$ can be identified with $R\Gamma(W_n\mathscr{F}^\cdot)$ and $\mathrm{Gr}_n^W R\Gamma(\mathscr{F}^\cdot)$ can be identified with $R\Gamma(\mathrm{Gr}_n^W\mathscr{F}^\cdot)$.

Now W and F on $\mathrm{DR}(D,E)$ induces an increasing and descreasing filtration on $(R\Gamma(\mathrm{DR}(D,E)),W)$, denote them by W and F as well.

F induces 3 filtrations $F_d \subset F_{\mathrm{ind}} \subset F_d^*$ on $E_r(R\Gamma(\mathrm{DR}(D,E)),W)$:

$$E_1^{-m,k+m} = \mathbb{H}^{k+m}(X, \mathrm{Gr}_m^W \mathrm{DR}(D,E))$$

We want to show d_r is strictly compatible with F_{ind} on all r. Then apply the Lemma of Two Filtrations, we conclude that the spectral sequence $(R\Gamma(\mathrm{DR}(D,E),F)$ (Hodge spectral sequence) degenerates at E_1 which is one of the most important steps to show the vanishing theorems in the following sections.

Proposition 2.5.2 *The vector space $E_1^{-m,k+m} = \mathbb{H}^{k+m}(Gr_m^{W.} DR(D,E))$ with the induced filtration F is a Hodge structure of weight k, and the morphism d_1 is a morphism of Hodge structure.*

Denote $W_0\mathrm{DR}(D,E)$ by $\tilde{\mathrm{DR}}(D,E)$ and write $\Omega_X^p(\log D)$ for its p-th component.

In Appendix D of [5] Timmerscheidt proved that $H^(X, j_*V)$ has a Hodge structure of weight k. We summarize his results here

Theorem 2.5.4 *[5](D.2)*

1. *j_*V is quasi-isomorphic to $\tilde{DR}(D,E)$*

2. *The spectral sequence to $(R\Gamma(\tilde{DR}(D,E)),F)$*

$$E_1^{p,q} = H^q(X, \tilde{\Omega}_X^p(\log D)\otimes E) \Rightarrow H^{p+q}(X, j_*V)$$

   degenerates at E_1, i.e.

$$H^k(X, j_*V) \cong \bigoplus_{p+q=k} H^q(X, \tilde{\Omega}_X^p(\log D)\otimes E)$$

3. There is a conjugate linear isomorphism

$$H^q(X, \tilde{\Omega}_X(\log D) \otimes E) \cong H^p(X, \tilde{\Omega}_X(\log D) \otimes E^\vee)$$

where $E^\vee$ is the canonical connection of $V^\vee$

Definition 2.5.1 *(Canonical filtration τ of a complex) Let $K^\cdot$ be a complex in an abelian category. Write K^p for its p-th component*
the canonical filtration τ on $K^\cdot$ is defined as

$$\tau_m K^p = \begin{cases} K^p & \textit{if } p < m \\ Kerd_p \subset K^p & \textit{if } p = m \\ 0 & \textit{if } p > m \end{cases}$$

Proof (Proof of Proposition 2.5.2) We need to construct a real lattice of $E_1^{-m,k+m}$ and show F and $\bar{F}$ are k-opposed.
To construct a real lattice for $\mathbb{H}^{p+q}(\mathrm{Gr}_p^W \mathrm{DR}(D, E))$, consider the complex $(\mathrm{K}_A := R\Gamma(\mathrm{K}_A), \tau)$ (τ is the canonical filtration on $R\Gamma(\mathrm{K}_A)$). We will show that:
Claim 1: The map

$$\alpha : (\mathbb{R}j_* V_A, \tau) \otimes \mathbb{C} \to (\mathrm{DR}(E, D), W)$$

is a quasi-isomorphism of filtered complex of sheaves
Proof of Claim 1 It is enough to show that the inclusion map

$$i : (\mathrm{DR}(E, D), \tau) \to (\mathrm{DR}(E, D), W)$$

is a quasi-isomorphism of filtered complexes. For a proof of this statement, see Proposition 2.1 of [9]
End of Proof of Claim 2
Claim 1 indicates that $\mathbb{H}^{p+q}(X, \mathrm{Gr}_p^\tau \mathbb{R}j_* V_A)$ is a real lattice for $\mathbb{H}^{p+q}(X, \mathrm{Gr}_p^W \mathrm{DR}(E, D))$.
Claim 2 The residue map $\mathrm{Res}_m(E)$ induces an isomorphism

$$\mathrm{Gr}_m^W(\mathrm{DR}(D, E)) \cong v_{m*}(\tilde{\mathrm{DR}}(C_m, E_m))[-m]$$

Proof of Claim 2 This is Proposition 2.4.1

End of Proof of Claim 2

Let F_m be the "naive" filtration on $\mathrm{DR}(C_m, E_m)$. We have seen that the spectral sequence to $(R\Gamma(\mathrm{DR}(C_m, E_m)), F_m)$ degenerates at page 1, and F_m defines a Hodge structure of weight i on $\mathbb{H}^i(D_m, \mathrm{DR}(C_m, E_m))$

A little calculation can show that $\mathrm{Res}_m(E)$ maps $F^p\mathbb{H}^{k+m}(\mathrm{Gr}^W_m \mathrm{DR}(D, E))$ isomorphically onto $F^{p-m}_m \mathbb{H}^k(D_m, \mathrm{DR}(C_m, E_m))$. Since F_m and $\bar{F_m}$ define a k-opposed filtration on $\mathbb{H}^k(D_m, \mathrm{DR}(C_m, E_m))$ so does F and $\bar{F}$ on $\mathbb{H}^{k+m}(X, \mathrm{Gr}^W_m \mathrm{DR}(D, E))$

■

Consider the spectral sequence for $(\mathrm{K}_{\mathbb{C}} := R\Gamma(\mathrm{DR}(D, E)), W)$:

$$E_1^{-m,k+m} = \mathbb{H}^k(X, \mathrm{Gr}^W_m \mathrm{DR}(D, E))$$

We have seen that the differentials on E_1 are morphism of Hodge structure of same weight.

Apply the "Lemma of two filtrations", we see that F_d, F_{ind} and F^*_d induce the same filtration F on E_2 and it is strictly compatible with d_2. $E_2^{p,q}$ with the induced filtration F is a Hodge structure of weight q. But

$$d_2 : E_2^{p,q} \to E_2^{p+r,q-r+1}$$

is then a morphism of HS of weight q and $q-1$ which is strictly compatible with F. So d_2 must be 0.

This means $d_r = 0$ for all $r \geq 2$. Then, it is trivial that d_r is strictly compatible with the F_{ind} on $E_r(\mathrm{K}_{\mathbb{C}}, W)$

So apply the Lemma of Two Filtrations, we conclude

Theorem 2.5.5 *The spectral sequence to* $(K_{\mathbb{C}} = R\Gamma(DR(D, E)), F)$ *degenerates at* E_1

This means we have an isomorphism

$$E_\infty^{p,q}(\mathrm{K}_{\mathbb{C}}, F) := \mathrm{Gr}^F_p \mathbb{H}^{p+q}(X, \mathrm{DR}(D, E)) \cong E_1^{p,q}(\mathrm{K}_{\mathbb{C}}, F) := \mathbb{H}^{p+q}(X, \mathrm{Gr}^F_p \mathrm{DR}(D, E))$$

It is easy to see that

$$\mathbb{H}^{p+q}(X, \mathrm{Gr}_p^F \mathrm{DR}(D, E)) = H^q(X, \Omega_X^p(\log D) \otimes E)$$

So as an abstract vector space $\mathbb{H}^k(X, \mathrm{DR}(D, E))$ can be written as

$$\bigoplus_{p+q=k} H^q(X, \Omega_X^p(\log D) \otimes E)$$

Put everything together, we have

Theorem 2.5.6 *Let* $\alpha : (\mathbb{R}j_* V_A, \tau, F) \to (DR(D, E), W, F)$ *be the natural map of filtered complex.*

1. $\mathbb{H}^k(X, \mathbb{R}j_* V_A)$ *is a real lattice for* $\mathbb{H}^k(X, DR(D, E))$
2. $R\Gamma(\alpha) \otimes \mathbb{C} : (K_A, \tau) \otimes \mathbb{C} \to (K_{\mathbb{C}}, W)$ *is a quasi-isomorphism of graded complexes*
3. *The spectral sequence to* $(R\Gamma(DR(D, E)), W)$ *degenerates at* E_2
4. *The spectral sequence to* $(R\Gamma(DR(D, E)), F)$ *degenerates at* E_1
5. *The system*

$$[(\mathbb{H}^k(X, \mathbb{R}j_* V_A), \tau[k], F), (\mathbb{H}^k(X, DR(D, E)), W[k], F)]$$

defines a mixed Hodge structure of weight k *on* $\mathbb{H}^k(X, \mathbb{R}j_* V_A)$.

Proof We prove (5) here: $W[k]$ on $\mathbb{H}^k(X, \mathrm{DR}_X(D, E))$ is defined as

$$W[k]^m \mathbb{H}^k(X, \mathrm{DR}_X(D, E)) = \mathrm{Im}(\mathbb{H}^k(X, W^{m-k} \mathrm{DR}_X(D, E)) \to \mathbb{H}^k(X, \mathrm{DR}_X(D, E)))$$

And $\tau[k]$ on $\mathbb{H}^k(X, \mathbb{R}j_* V_A)$ is defined likewise.
Let $(E_r^{-m,k+m}, d_r)$ be the spectral sequence to $(R\Gamma(\mathrm{DR}(D, E), W)$. We have seen $E_r^{-m,k+m}$ is a Hodge structure of weight $k + m$. Therefore

$$E_\infty^{-m,k+m} = \mathrm{Gr}_{k+m}^W \mathbb{H}^k(X, \mathrm{DR}(D, E))$$

is a Hodge structure of weight $k + m$.

By the definition of $W[k]$, we have

$$\mathrm{Gr}^W_{k+m}\mathbb{H}^k(X, \mathrm{DR}(D, E)) = \mathrm{Gr}^{W[k]}_m\mathbb{H}^k(X, \mathrm{DR}(D, E))$$

This is precisely a Hodge structure of weight k

■

For future reference, we will make some formal definitions here. The definitions can be found in [2] [11]

Let A be a Noetherian subring of $\mathbb{R}$

Definition 2.5.2 *(HC) A Hodge A-complex of weight n consists of*

1. *a complex of $K^{\cdot}_A$ of A-modules, such that $H^k(K_A)$ is an A-module of finite type for all k*
2. *a filtered complex $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces*
3. *an isomorphism $\alpha : K_A \otimes \mathbb{C} \to K_{\mathbb{C}}$ in $D^+(\mathbb{C})$*

The following two conditions must also be satisfied

1. *The differential d of $K_{\mathbb{C}}$ is strictly compatible with the filtration F*
2. *For all k, the filtration F on $H^k(K_{\mathbb{C}})$ defines an A-Hodge structure of weight $n + k$*

Definition 2.5.3 *(Cohomological HC) An A-cohomological Hodge complex of weight n consists of*

1. *a complex of sheaves $K^{\cdot}_A$ of A-modules, such that $H^k(K_A)$ is an A-module of finite type for all k*
2. *a filtered complex of sheaves $(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces*
3. *an isomorphism $\alpha : K_A \otimes \mathbb{C} \to K_{\mathbb{C}}$ in $D^+(\mathbb{C})$*

Moreover, the system $(R\Gamma(X, K_A), R\Gamma(X, K_{\mathbb{C}}, F), R\Gamma(\alpha))$ *is an A-Hodge complex of weight* n

The system

$$(\mathrm{Gr}^{\tau}_m \mathbb{R}j_* V_A, (\mathrm{Gr}^W_m \mathrm{DR}(D, E), F))$$

is an example of A-cohomological Hodge complex of weight m;

Definition 2.5.4 *(MHC) An A-mixed Hodge complex of weight* k *consists of:*

1. *a complex of* $K^{\cdot}_A$ *of A-modules, such that* $H^k(K_A)$ *is an A-module of finite type for all* k
2. *a filtered complex* $(K_{A\otimes\mathbb{Q}}, W)$ *of* $A\otimes\mathbb{Q}$*-vector spaces with an increasing filtration* W
3. *an isomorphism* $K_A \otimes \mathbb{Q} \to K_{A\otimes\mathbb{Q}}$ *in* $D^+(A \otimes \mathbb{Q})$
4. *a bi-filtered complex* $(K_{\mathbb{C}}, W, F)$ *of* $\mathbb{C}$*-vector spaces with an increasing (resp. decreasing) filtration* W *(resp.* F*) and an isomorphism*

$$\alpha : (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \to (K_{\mathbb{C}}, W)$$

in $D^+F(\mathbb{C})$

Moreover, for all n*, the system consisting of*

1. *the complex* $Gr^W_n(K_{A\otimes\mathbb{Q}}$ *of* $A \otimes \mathbb{Q}$*-vector spaces*
2. *the complex* $Gr^W_n(K_{\mathbb{C}}, F)$ *of* $\mathbb{C}$*-vector spaces*
3. *the isomorphism*

$$Gr^W_n(\alpha) : Gr^W_n(K_{A\otimes\mathbb{Q}}) \otimes \mathbb{C} \to Gr^W_n(K_{\mathbb{C}})$$

is an $A \otimes \mathbb{Q}$*-Hodge complex of weight* $n + k$

Definition 2.5.5 *(Cohomological MHC) An A-cohomological mixed Hodge complex of weight k consists of:*

1. *a complex of sheaves $K^{\cdot}_A$ of A-modules, such that $H^k(K_A)$ is an A-module of finite type for all k*

2. *a filtered complex of sheaves $(K_{A\otimes\mathbb{Q}}, W)$ of $A\otimes\mathbb{Q}$-vector spaces with an increasing filtration W*

3. *an isomorphism $K_A\otimes\mathbb{Q}\to K_{A\otimes\mathbb{Q}}$ in $D^+(A\otimes\mathbb{Q})$*

4. *a bi-filtered complex of sheaves $(K_{\mathbb{C}}, W, F)$ of $\mathbb{C}$-vector spaces with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism*

$$\alpha : (K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C}\to(K_{\mathbb{C}}, W)$$

in $D^+F(\mathbb{C})$

Moreover, for all n, the system consisting of

1. *the complex of sheaves $Gr^W_n(K_{A\otimes\mathbb{Q}}$ of $A\otimes\mathbb{Q}$-vector spaces*

2. *the complex of sheaves $Gr^W_n(K_{\mathbb{C}}, F)$ of $\mathbb{C}$-vector spaces*

3. *the isomorphism*

$$Gr^W_n(\alpha) : Gr^W_n(K_{A\otimes\mathbb{Q}})\otimes\mathbb{C}\to Gr^W_n(K_{\mathbb{C}})$$

is an $A\otimes\mathbb{Q}$-cohomological Hodge complex of weight $n+k$

The system

$$(\mathbb{R}j_*V_A, W), (\mathrm{DR}(D, E), W, F))$$

is one example of mixed Hodge complex of weight 0

3. MAIN VANISHING THEOREM

3.1 Vanishing Theorem on the de Rham Complex

we have seen in the previous section that if V has a real lattice V_A for some Noeatherian subring $A \subset \mathbb{R}$ then

$$(\mathbb{R}j_*V_A, (\mathbb{R}j_*V_{A\otimes\mathbb{Q}}, \tau), (\mathrm{DR}_X(D,E), F, W))$$

is an A-cohomological mixed Hodge complex. As the result 2.5.6 of the general theory developed in the previous section, we know the spectral sequence to $(R\Gamma(\mathrm{DR}(D,E)), F)$ degenerates at E_1, *i.e.*

$$\mathbb{H}^k(X, \mathbb{R}j_*V) = \mathbb{H}^k(X, \mathrm{DR}(D,E)) \cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p(\log D) \otimes E)$$

Theorem 3.1.1 *Assume there is a real-valued unitary locayl system $V_{\mathbb{R}}$ defined on U such that*

$$V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

Let V and $DR_X(D,E)$ be as above. The spectral sequence associated to the Hodge filtration on $DR_X(D,E)$.

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes E) => \mathbb{H}^{p+q}(X, DR_X(D,E))$$

degenerates at E_1

If V does not have an A-lattice with $A \subset \mathbb{R}$, then we cannot expect $\mathrm{DR}_X(D,E)$ to carry a mixed Hodge structure. However, the E_1-degeneration of Hodge spectral sequence still holds true. We will give a proof here.

Let $\bar{V}$ denote the conjugate of V, i.e. the monodromy representation of $\bar{V}$ is the complex conjugate of the monodromy representation of V

Lemma 5 *There exists a real unitary local system $W_{\mathbb{R}}$ of rank $2r$ such that*

$$V \oplus \bar{V} \cong W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

Proof We will construct $W_{\mathbb{R}}$ locally, and show it is canonically determined by V. Over a polydisk, we can assume V is diagonal, and we write

$$V = \bigoplus_{j=1}^{r} V^j$$

where V^i is a unitary local system of rank 1 with monodromy

$$\lambda_j = \cos\theta_j + i \sin\theta_j$$

We will construct $W^j_{\mathbb{R}}$ for each j. The monodromy of $\bar{V}^j$ is $\bar{\lambda}_j$ and the monodromy of $V^j \oplus \bar{V}^j$ is

$$\begin{bmatrix} \cos\theta_j + i\sin\theta_j & 0 \\ 0 & \cos\theta_j - i\sin\theta_j \end{bmatrix}$$

Since

$$\begin{bmatrix} \cos\theta_j + i\sin\theta_j & 0 \\ 0 & \cos\theta_j - i\sin\theta_j \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$

have the same characteristic polynomial over $\mathbb{C}$, they must be conjugate over $\mathbb{C}$. Therefore, we can take W^j to be

$$\begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$

Then,

$$W_{\mathbb{R}} = \bigoplus_{j=1}^{r} W^j$$

■

For any unitary local system V on $X - D$ defined over $\mathbb{C}$, we have constructed a real lattice $W_{\mathbb{R}}$ for $V \oplus \bar{V}$, this means we can constructed a $\mathbb{R}$-cohomological mixed Hodge complex out of

$$(W_{\mathbb{R}}, \mathrm{DR}(D, E \otimes \bar{E}))$$

Therefore, the main theorem 2.5.6 of Section 2.5 applies to $\mathrm{DR}(D, E \otimes \bar{E})$

Corollary 1 *Let V be any unitary local system on $X - D$, and let $DR_X(D, E)$ be its de Rham complex. The Hodge spectral sequence*

$$E_1^{p,q} := H^q(X, \Omega_X^p(\log D) \otimes E) => \mathbb{H}^{p+q}(X, DR(D, E)) = H^{p+q}(X, \mathbb{R}j_*V)$$

degenerates at E_1.

V can be viewed as a constructible sheaf on $X - D$.

Definition 3.1.1 *(Constructible sheaf) A constructible sheaf is a sheaf of abelian groups over some topological space X, such that X is the union of finite number of locally closed subsets on each of which the sheaf is a locally constant sheaf.*

Example 2

1. *Local systems on a topological space X are constructible sheaves on X*
2. *Let $f : X \to Y$ be a continuous map of topological spaces*

Theorem 3.1.2 *[12](Corollary 3.5) Suppose U is an affine variety of complex dimension n. Then, for any constructible sheaf $\mathcal{L}$ on U*

$$H^k(U, \mathcal{L}) = 0$$

for $k > n$

Corollary 2 *Let V and $DR_X(D, E)$ be as above. Suppose U is affine, then*

$$H^q(X, \Omega_X^p(\log D) \otimes E) = 0$$

for $p + q > \dim X$

Proof For $k > \dim$,

$$0 = H^k(U, V) = \mathbb{H}^k(X, \mathbb{R}j_*V) = \mathbb{H}^k(X, \mathrm{DR}(D, E)) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p(\log D) \otimes E)$$

■

Lemma 6 *Suppose B is a smooth divisor transversal to D. Then, there is short exact sequence*

$$0 \to \Omega_X^p(\log D + B) \otimes O_X(-B) \xrightarrow{i} \Omega_X^p(\log D) \xrightarrow{r} \Omega_B^p(\log D \cap B) \to 0$$

where i is the inclusion map, and r is the restriction map.

Proof For simplicity, we prove the case for $p = 1$. We may also assume X is affine. Let $X = \mathrm{Spec} A$, and let $f_1, \cdots, f_s$ be the regular sequence corresponding to D, and let b be the defining equation of B.

The basis of $\Omega_X^1(\log D + B) \otimes O_X(-B)$ as an A-module is

$$\frac{df_1}{f_1} \otimes b, \cdots, \frac{df_s}{f_s} \otimes b, \frac{db}{b} \otimes b$$

The basis of $\Omega_X(\log D)$ as an A-module is

$$\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}$$

The basis of $\Omega_B(\log D \cap B)$ as an $\frac{A}{b}$-module is

$$\frac{df_1}{f_1}, \cdots, \frac{df_s}{f_s}$$

where by abuse of notation f_i are regarded as their image in $\frac{A}{b}$.

Then, it is clear how to define i and r show that the above sequence is exact ■

By viewing $\Omega_X^p(\log D + B)$ as a sub-sheaf of $\Omega_X^p(\log D) \otimes O_X(B)$, the connection

$$\nabla : E \otimes \Omega_X^p(\log D) \to E \otimes \Omega_X^{p+1}(\log D)$$

clearly extends to a mp

$$E \otimes \Omega_X^p(\log D + B) \to E \otimes \Omega_X^{p+1}(\log D + B)$$

which we denote by ∇. Moreover, $\nabla \circ \nabla = 0$, making

$$E \to \Omega_X(\log D + B) \otimes E \to \cdots \to E \otimes \Omega_X^n(\log D + B)$$

a complex. We will denote this complex by

$$\mathrm{DR}(D+B,E)$$

It is clear that $\mathrm{DR}(D+B,E)$ is the canonical extension of V^o, the restriction of V on $Y-B$.

Lemma 7 *Suppose B is a smooth divisor transversal to D. Then, $E_B := E \otimes O_B$ is the canonical extension of $V_B := V|_{B-B\cap D}$.*

Proof The statement is local, therefore we may assume X is a polydisk

$$\Delta_1 \times \cdots \times \Delta_n$$

such that the analytic coordinate of Δ_i, for $i = 1, \cdots, s$, are defining equation of D_i, and the analytic coordinate of Δ_n is the defining equation of B.

First, we study V_B by computing its monodromy representation:

Let $T : \pi_1(X-D,x) \to \mathrm{GL}(r,\mathbb{C})$ be the monodromy representation of V. For each generator γ_i of $\pi_1(X-D,x)$, let $\Gamma_i = T(\gamma_i)$. As Γ_i are commuting and unitary, we can use one matrix to diagolize all of them. Therefore, we can assume all Γ_i are diagonal matrices. Moreover, as V is undefined only on D, so for each i, $\Gamma_i^{jj} = 1$, for $j = s+1, \cdots, n$.

Now, $B = \Delta_1 \times \cdots \times \Delta_{n-1}$, and the monodromy reprentation of $V|_{B-B\cap D}$ is given by

$$\pi_1(B - B\cap D) \xrightarrow{i} \pi_1(X-D) \xrightarrow{T} \mathrm{GL}(r,\mathbb{C})$$

where i is the natural inclusion map. It is clear that one can choose the basis of $\pi_1(B - B\cap D)$ and $\pi_1(X-D)$ such that i can be realized as the identity map. Therefore, the monodromy representations of $V_{B-B\cap D}$ are also Γ_i, for $i = 1, \cdots, s$.

To show $E|_B$ is the canonical extension of $V_{B-B\cap D}$, we compute the connection matrix of $E|_B$ and relate it to the monodromy representations of $V|_{B-B\cap D}$.

One can assume E is trivial over X. Choose a local frame of V on X, and use it as a trivialization of E. With respect to this trivialization, the connection ∇ can be realized as

$$d + N_1\frac{dz_1}{z_1} + \cdots + N_s\frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalues in the stripe

$$\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.

Now, restrict E to B, we see that the connection $\nabla|_B$ can still be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

As monodromy representations of $V_{B - B \cap D}$ are Γ_i, it follows that $E|_B$ is the canonical extension of $V_{B - B \cap D}$. ∎

Proposition 3.1.1 *Suppose L is very ample on X. Then*

$$H^q(X, E \otimes \Omega^p_X(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof Let B be a smooth divisor transversal to D such that $L \cong O_X(B)$. By Lemma 6 we have the following exact sequence

$$0 \to \Omega^p_X(\log D + B) \xrightarrow{i} \Omega^p_X(\log D) \otimes O_X(B) \xrightarrow{r} \Omega^p_B(\log D \cap B) \otimes O_X(B) \to 0$$

Tensor it by E and take the cohomology sequence, we get:

$$\begin{aligned} \cdots H^q(X, \Omega^p_X(\log D + B) \otimes E) &\to H^q(X, \Omega^p_X(\log D) \otimes O_X(B) \otimes E) \\ \to & H^q(X, \Omega^p_B(\log B \cap D) \otimes O_X(B) \otimes E) \cdots \end{aligned}$$

Therefore, to prove the proposition, it is enough to show

Claim 1: $H^q(X, \Omega^p_X(\log D + B) \otimes E) = 0$

Claim 2: $H^q(X, \Omega^p_B(\log B \cap D) \otimes O_X(B) \otimes E) = 0$ for $p + q > \dim X$.

Proof of Claim 1: Consider the maps

$$X - (B + D) \xrightarrow{f} X - B \xrightarrow{h} X$$

Let V^o be the restriction of V on $X-(B+D)$. The complex $\mathrm{DR}(D+B,E,\nabla)$ is quasi-isomorphic to $\mathbb{R}(h\circ f)_*V^o$. Therefore,

$$H^k(X-(B+D),V^o)=\mathbb{H}^k(X,\mathrm{DR}(D+B,E))$$

The claim then follows from Corollary 2.

End of Proof of Claim 1

Claim 2 follows from induction on the dimension of the variety.

Now to finish the proof, it remains to show the base case of Claim 2. One may assume now that X is a smooth projective curve over $\mathbb{C}$,

We need to show that

$$H^1(X,\Omega_X(\log D)\otimes E\otimes L)=0$$

But for the curve case, $\Omega_X(\log D)\otimes O_X(B)=\Omega_X(\log D+B)$. So the result follows again from Theorem 2 ■

Now suppose L is any ample line bundle. Let m be an integer such that $L^{\otimes m}$ is very ample. Take a smooth divisor B transversal to D such that $L^{\otimes m}\cong O_X(B)$. Let φ be the local equation of B on some affine open set, and let $\pi : X'\to X$ be the normalization of X in $\mathbb{C}(X)(\varphi^{\frac{1}{m}})$.

Proposition 3.1.2 *Let $\pi : X'\to X$, B and L be as above*

1. *X' is smooth.*
2. *$\pi^*B=m\tilde{B}$, where $\tilde{B}=(\pi^*B)_{red}$.*
3. *$D':=\pi^*D$ is a normal crossing divisor on X'.*
4. *$\tilde{B}$ is transversal to π^*D.*
5. *$\pi^*\Omega^p_X(\log D)=\Omega^p_{X'}(\log D')$.*
6. *π^*E is the canonical extension of $\pi^{-1}V$.*

Such covering space $\pi : X' \to X$ was constructed by Kawamata in [13]. Readers can find more details there.

Proof 1. We will construct X' by constructing its affine cover and specefiying the gluing morphisms. Let $U_i = \mathrm{Spec} A_i$ be an affine cover of X, and let f_i be the defining equation of D in A_i.

For each A_i, $\frac{A_i[Y]}{(Y^m - f_i)}$ is integrally closed in $\mathbb{C}(X)(f_i^{1/m})$. Therefore,

$$U_i' := \mathrm{Spec}\frac{A_i[Y]}{(Y^m - f_i)}$$

is the normalization of U_i in $\mathbb{C}(X)(f_i^{1/m})$

The same morphisms used to glue U_i into X can be used to glue U_i' into X'. Therefore, to show X' is smooth, it is enough to show $\frac{A_i[Y]}{(Y^m - f_i)}$ is a regular ring.

2. The local defining equation of $\tilde{B}$ is Y, and $\pi^*(f_i) = Y^m$

3. To see this, we describe $\pi^* D$ in $\pi^* U$ for any polydisk $U = \Delta_1 \times \cdots \times \Delta_n$. If $B \cap U \neq \varnothing$, then construct Δ_i such that defining equation of D_i, for $i = 1, \cdots, s$, are coordinates of D_i, for $i = 1, \cdots, s$; and the defining equation of B is the coordinate of D_n. Then,

$$\pi^* U = \Delta_1 \times \Delta_1 \cdots \Delta_{n-1} \times \Sigma^m$$

where Σ^m is the m-sheeted cover over a complex disk branched over the origin. In this case, $\pi^* D$ is still defined by $z_1 \times z_2 \times \cdots z_s$.

If $B \cap U = \varnothing$, then $\pi^* U$ is etale over U. Therefore, $\pi^* D$ is etale over D. So $\pi^* D$ is again a simple normal crossing divisor.

4.This is clear from the case 1 of part 3.

5. Straighforward computation. 6. We compute the monodromy representation of

$\pi^{-1} V$ first:

let $T : \pi_1(U - D, x) \to \mathrm{GL}(r, \mathbb{C})$ be the representation corresponding to the local system V.

Case 1: Suppose $x \notin B$, then $\pi^{-1}(U)$ is etale over U. Let U' be an component of $\pi^{-1}(U)$, and let $x' \in U'$ be a preimage of x. Then,

$$T' : \pi_1(U' - D', x') \xrightarrow{\pi_*} \pi_1(U - D, x) \xrightarrow{T} \mathrm{GL}(r, \mathbb{C})$$

is the representation corresponding to $\pi^{-1}V$.

Case 2: Suppose $x \in B$, then use the description from part 3, we know that

$$\pi^{-1}U = \Delta_1 \times \Delta_2 \times \cdots \times \Sigma^m$$

In both cases, $\pi^{-1}U - D'$ is homotopic to $S_1 \times S_2 \times \cdots \times S_s$ So we can define generators of $\pi_1(U' - D', x')$ and $\pi_1(U - D, x)$ such that π_* is the identity map.

To show π^*E is the canonical extension of $\pi^{-1}V$, we only need to compute the connection matrix of π^*E and relate it to the monodromies of $\pi^{-1}V$:

Let γ_i be a small circle around D_i, and let Γ_i be the monodromy $T(\gamma_i)$. As

$$\pi_* : \pi_1(U' - D', x') \to \pi_1(U - D, x)$$

is the identity map, Γ_i are also the monodromy representations of $\pi^{-1}V$. Next, we compute the connection matrix of E. Let U be small enough so that E is trivial over it. Choose a local frame of V, and use it as a trivialization of E. With respect to this trivialization, the connection ∇ can be realized as

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

where $N_1, \cdots, N_s$ are commuting matrices with eigenvalue in the stripe

$$\{z \in \mathbb{C} | 0 \leq \mathrm{Re} z < 1\}$$

such that $e^{-2\pi i N_i} = \Gamma_i$.

As $\pi^* z_i = z_i$, for $i = 1, \cdots, s$, we see that the $\pi^*\nabla$ over $\pi^{-1}U$ can be realized as:

$$d + N_1 \frac{dz_1}{z_1} + \cdots + N_s \frac{dz_s}{z_s}$$

This shows that π^*E is the canonical extension of $\pi^{-1}V$.

■

Theorem 3.1.3 *For any ample line bundle L on X,*

$$H^q(X, E \otimes \Omega_X^p(\log D) \otimes L) = 0$$

for $p + q > \dim X$

Proof Let m, B and $\pi : X' \to X$ be as above. By Theorem 3.1.1

$$H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) = 0$$

for $p + q > \dim X' = \dim X$.

$\pi : X' \to X$ is a finite morphism, so for $i > 0$, $R^i\pi_*\mathscr{F} = 0$ for any coherent sheaf $\mathscr{F}$ on X'. This implies

$$\begin{aligned} H^q(X', \pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) &= H^q(X, \pi_*(\pi^*(E \otimes \Omega_X^p(\log D) \otimes L)) \\ &= H^q(X, \pi_*(O_Y) \otimes E \otimes \Omega_X^p(\log D) \otimes L) \\ &= 0 \end{aligned}$$

for $p + q > \dim X$. The second equality follows from the projection formula.

As $\pi_*(O_Y) \cong \bigoplus_{i=0}^{m-1} O_X(-L^{\otimes i})$, the result follows. ■

3.2 Vanishing Theorem From the Perspective of Higgs Bundle

A Higgs bundle on the pair (X, D) is a vector bundle H together with an O_X-linear map $\theta : H \to H \otimes \Omega_X(\log D)$ such that $\theta \wedge \theta = 0$. We use $\mathrm{DR}(H, \theta)$ to denote the following complex

$$H \xrightarrow{\theta} H \otimes \Omega_X(\log D) \xrightarrow{\theta} H \otimes \Omega_X^2(\log D) \xrightarrow{\theta} \cdots \xrightarrow{\theta} H \otimes \Omega_X^n(\log D)$$

To study the behavior of H near the boundary divisor D, one can impose some additional structure on H

Definition 3.2.1 *(Parabolic Structure) A parablic structure on H is a decreasing $\mathbb{R}$-indexed filtration by coherent subsheaves, such that*

1. $H^0 = H$

2. $H^{\alpha+1} = H^{\alpha}(-D)$

3. $H^{\alpha-c} = H^{\alpha}$ *for any* $0 < c << 1$

4. The subset of α *such that* $Gr^{\alpha}H \neq 0$ *is discrete in* $\mathbb{R}$. *Here* $Gr^{\alpha}H\frac{H^{\alpha}}{H^{\alpha+\epsilon}}$

In this section, we will view the canonical extension E together with the zero Higgs field $(E, \theta = 0)$ as a parabolic Higgs bundle, and we will prove Theorem 1.0.7 from this perspective.

First off, we will give a natural filtration on E that constitutes a parabolic structure: Use the notation from Chapter 2.2, E is locally isomorphic to

$$\bigoplus_{i=1}^{r} O_{\Delta} \cdot v_i$$

where $v_i = \prod_{j=1}^{s} z_j^{-a_j^i}$.

For any holomorphic section $s \in E$, one knows precisely how $h(s, s)$ grows to infinity near the boundary divisor.

Define the filtration E^* such that: E^a consists of holomorphic sections s near defined at D such that $h(s, s) \to \infty$ faster than $\prod_{j=1}^{s} |1/z_j|^{2a+\epsilon}$ for $\epsilon > 0$

It can be easily checked that the above filtration defines a parabolic structure on E at least locally over Δ.

The jumps of E along D_j happens at $a_j^1, \cdots, a_j^r$ (eigenvalues of $\mathrm{res}_i \circ \nabla$) (see Proposition 2.1.2) This means the above parabolic structure is globally defined.

In the category of parabolic Higgs bundles, the "correct" notion of Cherns classes and semistability conditions are *parabolic Chern classes* and *parabolic semistable*(see [3] for the definitions)

We will show

Theorem 3.2.1 *The Higgs bundle* $(E, \theta = 0)$ *is a parabolic semistable with trivial parabolic Chern classes.*

Then the vanishing theorem Theorem 1.0.7 is a consequence of the main result in [3]

Theorem 3.2.2 *(D. Arapura, F. Hao, H. Li) Suppose* (E, θ) *is a nilpotent semistable Higgs bundle on* X *with* $c_i(E)$ *for all* i*. Let* L *be an ample line bunle, then*

$$\mathbb{H}^i(X, DR(E, \theta) \otimes L) = 0$$

for $i > \dim X$

Given the result above, we re-prove the vanishing theorem:

Proof (Proof of Theorem 1.0.7) For the Higgs bundle ($H = E, \theta = 0$), all the maps in the "de Rham" complex $\mathrm{DR}(H, \theta)$ are zero maps. Therefore, the "de Rham" complex can be written as

$$\mathrm{DR}(H, \theta) = \bigoplus_{k=1}^{n} E \otimes \Omega_X^k(\log D)[-k]$$

Apply Theorem 3.2.2, we have

$$\mathbb{H}^i(X, \bigoplus_{k=1}^{n} E \otimes \Omega_X^k(\log D)[-k]) = \bigoplus_{k=1}^{n} \mathbb{H}^{i-k}(X, \Omega_X^k(\log D) \otimes E \otimes L) = 0$$

■

Proposition 3.2.1 (E, θ) *has trivial parabolic Chern classes*

Proof There are formulas for computing Chern classes for canonical extension (e.g. see B.3 of [5]). One has

$$c_1(E) = -\sum_{j=1}^{s} \mathrm{Tr}(\Gamma_j) D_j$$

Suppose V is of rank 1. Then

$$\mathrm{par} - c_1(E) = c_1(E) + \sum a_j D_j = 0$$

Now, suppose rank of V ¿ 1. Then use the split principal, we see that all the parabolic Chern roots of E are 0. Therefore, all the higher Cherns classes of E are 0. ■

Proposition 3.2.2 $(E, \theta = 0)$ *is parabolic semistable.*

Proof By the work of Simpson [14], to show E is semistable, it is enough to show $E|_C$ is semistable for every smooth projective curve C on X. Therefore, it is enough to assume X is a curve.

In the curve case, the parabolic slope $\text{par} - \mu$ of a subsheaf F of E is as(see [3] section 4)

$$\text{par} - \mu(F) = \frac{\text{par} - c_1(F)}{\text{rank} F}$$

As $\text{par} - c_1(F) = c_1(F) + \sum_{j=1}^{s} \text{Tr}(\Gamma_j) D_j$, it only remains to show

$$c_1(F) \leq c_1(E)$$

Suppose F is a bundle, then by the well-known principal (see P79 of [15]) the curvature decreases in holomorphic subbundles. So $c_1(F) \leq c_1(E)$ follows.

Now, suppose F is any subsheaf of E. Then, its saturation F^s is a bundle, and there is an exact sequence

$$0 \to F \to F^s \to F^s/F \to 0$$

where F^s/F is a skyscraper sheaf. Then by the additive property of Chern classes, one has

$$c_1(F) \leq c_1(F^s)$$

Consequently, $c_1(F) \leq c_1(E)$ ■

4. GRADED VANISHING THEOREM

4.1 An Augmented Weight Filtration on $\mathbf{DR}(D+B,E)$

In the previous section, we proved the vanishing theorem for the complex

$$\mathrm{DR}_X(D,E)\otimes O_X(B)$$

where B is a smooth very ample divisor transversal to D. The intermediate step for the proof is a vanishing theorem for the complex

$$\mathrm{DR}_X(D+B,E)$$

In this section, we define another weight filtration W^B on the complex

$$\mathrm{DR}_X(D+B,E)$$

Definition 4.1.1 *(Augmented weight filtration) For each $m\in\mathbb{Z}$, consider the restriction map*

$$r_m : W_m DR_X(D,E)\to W_m DR_B(D\cap B,E_B)$$

define $W^B_m DR(D+B,E)$ to be the kernel of $r_m\otimes O_X(B)$ the restriction map

We will refer to it as *augmented weight filtration*, because for every $m\in\mathbb{Z}$,

$$W_m\mathrm{DR}(D+B,E)\subset W^B_m\mathrm{DR}_X(D+B,E)$$

We will illustrate this by giving local description of $W^B_m\mathrm{DR}_X(D+B,E)$

Proposition 4.1.1

$$\Omega^1_X(\log D)\wedge W_m DR_X(D,E)[-1]\cong W^B_m DR_X(D+B,E)$$

Proof Suppose for simplicity that E is of rank 1. On a small polydisk open set $U \subset X$, write μ for the local generator of E and z_n for the local defining equation of B.

Then, on U, $\Omega^1_X(\log B) \wedge W_m\Omega^{p-1}_X(\log D) \otimes E$ is generated by

$$\frac{dz_n}{z_n} \wedge \omega \otimes \mu$$

where ω contains at most m log forms acting trivially on V.

Let

$$\phi : \Omega^1_X(\log B) \wedge W_m\Omega^{p-1}_X(\log D) \otimes E) \to W_m\mathrm{DR}_X(D, E) \otimes O_X(B)$$

be the map such that

$$\phi(\frac{dz_n}{z_n} \wedge \omega \otimes \mu) = dz_n \wedge \omega \otimes \mu \otimes \frac{1}{z_n}$$

Then, it is clear that

$$\phi(\frac{dz_n}{z_n} \wedge \omega \otimes \mu) \in \mathrm{Ker}(r_m \otimes O_X(B))$$

On the other hand, $\mathrm{Ker}(r_m)$ is generated by sections of the form

$$dz_n \wedge \tau \otimes \mu$$

where $\tau \otimes \mu \in W_m\Omega^{p-1}_X(\log D) \otimes E$. So $\mathrm{Ker} r_m \otimes O_X(B)$ is generated by

$$dz_n \wedge \tau \otimes \mu \otimes \frac{1}{z_n}$$

which is precisely image of ϕ. ∎

4.2 Mixed Hodge Theory on $W^B_0\mathbf{DR}(D + B, E)$

Throughout this section, we assume the unitary local system V has a real lattice $V_\mathbb{R}$ such that

$$V = V_\mathbb{R} \otimes \mathbb{C}$$

We will study the mixed Hodge structure on the complex

$$W^B_0\mathrm{DR}_X(D + B, E)$$

Consider the maps

$$X - (D+B) \xrightarrow{f} X - B \xrightarrow{h} X$$

Write V^o (resp. $V^o_{\mathbb{R}}$) for the restriction of V (resp. $V_{\mathbb{R}}$) on $X - (D+B)$.

Let τ be the canonical filtration on $\mathbb{R}h_* f_* V^o_{\mathbb{R}}$; let W be the increasing filtration on $W_0^B \mathrm{DR}_X(D+B,E)$ defined as

$$W_m W_0^B \mathrm{DR}_X(D+B,E) = \begin{cases} 0 & \text{if } m < 0 \\ W_0 \mathrm{DR}_X(D+B,E) & \text{if } m = 0 \\ W_0^B \mathrm{DR}_X(D+B,E) & \text{if } m > 0 \end{cases}$$

The main result of this section is

Theorem 4.2.1

$$(\mathbb{R}h_* f_* V^o_{\mathbb{R}}, (\mathbb{R}h_* f_* V^o_{\mathbb{R}}, \tau), (W_0^B DR_X(D+B,E), F^\cdot, W_\cdot))$$

is a $\mathbb{R}$*-cohomological mixed Hodge complex.*

Proposition 4.2.1 $\mathbb{R}h_*(f_* V^o)$ *is quasi-isomoprhic to*

$$W_0^B DR_X(D+B,E)$$

Proof The statement is local, so we can assume X is a polydisk. For the basic case, one can assume V is of rank 1, D has two components D^1 and D^2 such that the monodromy of V around D^1 is trivial, and the monodromy of V around D^2 is nontrivial. General case can be proved similarly.

Let $Y = X - B$, and let $h : Y \to X$ be the natural inclusion map. Then, $\Omega^\cdot_Y(\log D^2) \otimes h^* E$ is a resolution of $f_* V^o$(see [9]).

Let $g : Y - D^2 \to Y$ be the inclusion map. By a theorem of Griffiths [4] and Deligne [1], the inclusion map

$$i : \mathrm{DR}_Y(\log D^2, h^* E) \to g_* \mathscr{A}^\cdot_{Y-D^2}$$

is a quasi-isomorphism (See [16]) Proposition 8.18). Therefore, $f_* V^o$ is quasi-isomorphic to

$$g_* \mathscr{A}^{\cdot}_{Y-D^2}$$

As $g_* \mathscr{A}^{\cdot}_{Y-D^2}$ is a complex of flasque sheaves, $\mathbb{R}h_* f_* V^o$ is quasi-isomorphic to

$$h_* g_* \mathscr{A}^{\cdot}_{Y-D^2}$$

Now,

$$\begin{aligned} W_0^B \Omega^{\cdot}_X(\log D + B) \otimes E &= \Omega_X(\log B) \wedge W_0 \mathrm{DR}_X(D, E)[-1] \\ &= \Omega_X(\log B) \wedge \mathrm{DR}_X(D^2, E)[-1] \\ &= \Omega^{\cdot}_X(\log D^2 + B) \otimes E \end{aligned}$$

But according the theorem of Griffiths and Deligne mentioned above, the complex $\Omega^{\cdot}_X(\log D^2 + B)$ is quasi-isomorphic to

$$(h \circ g)_* \mathscr{A}^{\cdot}_{Y-D^2}$$

So the result for the basic case follows.

Now, let V be of rank r. For each $i = 1, 2$, let Γ_i be the monodromy of V around D^i. As V is unitary, we can simultaneously diagonalize all Γ_1 and Γ_2. Therefore, we can assume V is the direct sum of two rank 1 unitary local systems. As $\mathbb{R}h_*$ and f_* commutes with direct sum. The result follows.

Now, let V be of rank 1 and let $D^1, \cdots, D^s$ be components of D. Now let D_1 be the subdivisor of D over which V has identity monodromy; and let D_2 be the subdivisor of D over which V has nontrivial monodromy. Then, the result follows after the same steps in the basic case. ■

Proposition 4.2.2 *The inclusion map*

$$i : (W_0^B DR_X(D + B, E), \tau_.) \to (W_0^B DR_X(D + B, E), W)$$

is a quasi-isomorphism of filtered complexes.

Proof This is again a local statement, so we can assume X is a polydisk and V is of rank 1. We need to show that the induced maps of i

$$H^k(i) : H^k(\mathrm{Gr}^{\tau}_m W_0^B \mathrm{DR}_X(D+B,E)) \to H^k(\mathrm{Gr}^{W}_m W_0^B \mathrm{DR}_X(D+B,E))$$

are isomorphisms.

$$H^i(\mathrm{Gr}^{\tau}_m W_0^B \mathrm{DR}_X(D+B,E)) = \begin{cases} H^m(W_0^B \mathrm{DR}_X(D+B,E)) & \text{if } i=m \\ 0 & \text{otherwise} \end{cases}$$

Claim 1 If $m > 1$, then $H^m(W_0^B \mathrm{DR}_X(D+B,E)) = 0$.

Proof of Claim 1 We have a short exact sequence of complexes

$$0 \to W_0 \mathrm{DR}_X(D+B,E) \to W_0^B \mathrm{DR}_X(D+B,E) \xrightarrow{res} W_0 \mathrm{DR}_B(B\cap D, E_B)[-1] \to 0$$

where the $\mathrm{DR}(D\cap B, E_B, \nabla_B)$ is the complex

$$\cdots \to \Omega^m_B(\log B\cap D)\otimes E_B \xrightarrow{\nabla_B} \Omega^{m+1}_B(\log B\cap D)\otimes E_B \to \cdots$$

and the map res is the residue map.

Taking cohomology, we get

$$\begin{aligned} \cdots \to H^k(W_0\mathrm{DR}_X(D+B,E)) &\to H^k(W_0^B \mathrm{DR}_X(D+B,E)) \\ \to H^{k-1}(W_0\mathrm{DR}_B(B\cap D, E_B)) &\to \cdots \end{aligned}$$

We have seen in Theorem 2.5.4 that $W_0\mathrm{DR}_X(D+B,E)$ is a resolution of $(h\circ f)_* V$. Therefore, $W_0^B\mathrm{DR}_X(D+B,E)$ is exact. Likewise,

$$W_0\mathrm{DR}_B(B\cap D, E_B)$$

is also exact.

So the conclusion follows.

Proof of claim 1 finished

The above proof also shows that

$$H^k(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = \begin{cases} H^1(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}$$

$$H^k(\mathrm{Gr}_0^W W_0^B \mathrm{DR}_X(D+B,E)) = \begin{cases} H^0(W_0 \mathrm{DR}(D+B,E)) & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$$

Therefore, to prove

$$i : (W_0^B \mathrm{DR}_X(D+B,E), \tau) \to (W_0^B \mathrm{DR}_X(D+B,E), W)$$

is a quasi-isomorphism of filtered complexes, it remains to prove that both

$$H^0(i) : H^0(\mathrm{Gr}_0^\tau W_0^B \mathrm{DR}_X(D+B,E)) \to H^0(\mathrm{Gr}_0^W W_0^B \mathrm{DR}_X(D+B,E))$$

and

$$H^1(i) : H^1(\mathrm{Gr}_1^\tau W_0^B \mathrm{DR}_X(D+B,E)) \to H^1(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E))$$

are isomorphisms.

Now,

$$H^0(\mathrm{Gr}_0^\tau W_0^B \mathrm{DR}_X(D+B,E)) = \ker(E \xrightarrow{\nabla} W_0^B(\Omega_X^1(\log D+B) \otimes E)$$

and

$$H^0(\mathrm{Gr}_0^W W_0^B \mathrm{DR}_X(D+B,E)) = \ker(E \xrightarrow{\nabla} W_0(\Omega_X^1(\log D+B) \otimes E)$$

It is clear then the map $H^0(i)$ is an isomorphism.

To simplify notations, write $K^\cdot$ for $W_0^B \mathrm{DR}_X(D+B,E)$, from the proof of Claim 1, we have a commutative diagram

$$\begin{tikzcd} H^1(\mathrm{Gr}_1^\tau K^\cdot) \arrow[r, "H^1(i)"] \arrow[d] & H^1(\mathrm{Gr}_1^W K^\cdot) \arrow[r, "res"] & H^1(W_0 \mathrm{DR}_B(B \cap D, E_B)[-1]) \arrow[d] \\ H^1(K^\cdot) \arrow[rr, "res"] & & H^1(W_0 \mathrm{DR}_B(B \cap D, E_B)[-1]) \end{tikzcd}$$

and the residue map on the second row is an isomorphism. As the residue map on the first row is an isomorphism (even on the complex level), we see that the map $H^1(i)$ is an isomorphism.

■

Now we prove the main theorem of this section:

Theorem 4.2.2

$$(\mathbb{R}h_* f_* V^o_{\mathbb{R}}, (\mathbb{R}h_* f_* V^o_{\mathbb{R}}, \tau), (W^B_0 DR_X(D+B,E), F^\cdot, W_\cdot))$$

is a cohomological mixed $\mathbb{R}$*-Hodge complex of weight 0.*

Proof The quasi-isomorphism

$$(\mathbb{R}h_* f_* V^o_{\mathbb{R}}, \tau) \otimes \mathbb{C} \to (W^B_0 \mathrm{DR}_X(D+B,E), W_\cdot)$$

was proved in the previous proposition.
It remains to show

$$(\mathrm{Gr}^\tau_m \mathbb{R}h_* f_* V^o_{\mathbb{R}}, (\mathrm{Gr}^W_m W^B_0 \mathrm{DR}_X(D+B,E), F))$$

is a cohomological $\mathbb{R}$-complex of weight m, i.e. the Hodge spectral sequence of $(\mathrm{Gr}^W_m W^B_0 \mathrm{DR}_X(D+B,E), F)$ degenerates at E_1, and the induced filtration on

$$\mathbb{H}^k(X, \mathrm{Gr}^W_m W^B_0 \mathrm{DR}_X(D+B,E)) = \mathbb{H}^k(X, \mathrm{Gr}^\tau_m \mathbb{R}h_* f_* V^o_{\mathbb{R}}) \otimes \mathbb{C}$$

defines a pure $\mathbb{R}$-Hodge structure of weight $k+m$ on

$$\mathbb{H}^k(X, \mathrm{Gr}^\tau_m \mathbb{R}h_* f_* V^o_{\mathbb{R}})$$

i.e. the induced filtration F on $\mathbb{H}^k(X, \mathrm{Gr}^W_m W^B_0 \mathrm{DR}_X(D+B,E))$ is $m+k$ opposed to its conjugate.
For $m > 1$, all $\mathrm{Gr}^W_m W^B_0 \mathrm{DR}_X(D+B,E)$ are 0, so we only need to show the case for $m = 0, 1$.

For $m = 0$,

$$(\mathrm{Gr}_m^W W_0^B \mathrm{DR}_X(D+B,E), F) = (W_0 \mathrm{DR}_X(D+B,E), F)$$

Timmerscheidt showed that it is a cohomological $\mathbb{R}$-complex of weight 0 in [5](Appendix D).

For $m = 1$, we have seen that

$$\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E) \cong W_0 \mathrm{DR}(B \cap D, E_B, \nabla_B)[-1]$$

Let F be the induced Hodge filtration on $\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)$, and let F_B be the usual Hodge filtration on $W_0 \mathrm{DR}(B \cap D, E_B, \nabla_B)$. let $\bar{F}$ and $\bar{F}_B$ be their conjugates. To show F and $\bar{F}$ are $k+1$ opposed on $\mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E))$, we show that

$$\mathrm{Gr}_q^{\bar{F}} \mathrm{Gr}_p^F \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = 0 \text{ if } p+q \neq k+1$$

As $\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E) \cong W_0 \mathrm{DR}_B(B \cap D, E_B)[-1]$,

$$\mathrm{Gr}_p^F \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = \mathrm{Gr}_{p-1}^{F_B} \mathbb{H}^{k-1}(B, W_0 \mathrm{DR}_B(B \cap D, E_B))$$
$$\mathrm{Gr}_q^{\bar{F}} \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = \mathrm{Gr}_{q-1}^{\bar{F}} \mathbb{H}^{k-1}(B, W_0 \mathrm{DR}_B(B \cap D, E_B))$$

Therefore, $\mathrm{Gr}_q^{\bar{F}} \mathrm{Gr}_p^F \mathbb{H}^k(X, \mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E)) = 0$ if $p-1+q-1 \neq k-1$. The E_1-degeneration of $(\mathrm{Gr}_1^W W_0^B \mathrm{DR}_X(D+B,E), F)$ follows from the E_1-degneration of $(W_0 \mathrm{DR}(B \cap D, E_B, \nabla_B), F_B)$.

■

4.3 Graded Vanishing Theorem on the de Rham Complex

We have proved in the previous section that

$$(\mathbb{R}h_*f_*V^o_{\mathbb{R}}, (\mathbb{R}h_*f_*V^o_{\mathbb{R}}, \tau), (W_0^B \mathrm{DR}_X(D+B, E), F^{\cdot}, W_{.}))$$

is a cohomological mixed Hodge complex of weight 0. By Theorem 2.5.6, this means

Proposition 4.3.1 *1. The spectral sequence to $(R\Gamma(W_0^B DR_X(D+B,E), W/)$ degenerates at E_2*

2. The spectral sequence to $(R\Gamma(W_0^B DR_X(D+B,E), F)$ degenerates at E_1

Remark 3 *Use the argument we have seen in Section 3.1, the above proposition is true even when V does not have a real lattice*

Now we are ready to prove the graded vanishing theorem 1.0.8

Proposition 4.3.2 *Let B be a smooth very ample divisor transveral to D, then for $m = 0, \cdots, n-1$*

$$H^q(X, Gr^W_m E \otimes \Omega^p_X(\log D) \otimes O_X(B)) = 0$$

for $p+q > n$.

Proof We show first that

$$H^q(X, W_0\Omega^p_X(\log D) \otimes E \otimes O_X(B)) = 0$$

for $p+q > n$.

For notational convenience, write

$$A = W_0^B \mathrm{DR}_X(D+B, E)$$

$$B = W_0 \mathrm{DR}_X(D, E) \otimes O_X(B)$$

$$C = W_0 \mathrm{DR}_B(B \cap D, E_B) \otimes O_X(B)$$

We have the exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Take cohomology sequence, we get

$$\cdots \to H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) \to H^q(X, W_0(\Omega_X^p(\log D) \otimes E) \otimes O_X(B)) \to$$

$$\to H^q(B, W_0(\Omega_B^p(\log B \cap D) \otimes E) \otimes O_X(B)) \to \cdots$$

Therefore, it is enough to show

Claim 1: $H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) = 0$ for $p + q > n$.

Proof of Claim 1: Consider the maps

$$X - (B + D) \xrightarrow{f} X - B \xrightarrow{h} X$$

Write V^o for the restriction of V on $X - B$. We have seen that the sequence

$$\begin{aligned} E_1^{p,q} = H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) => & \mathbb{H}^{p+q}(X, W_0^B \mathrm{DR}_X(D + B, E)) \\ = & H^{p+q}(X, \mathbb{R}h_* f_* V^o) \\ = & H^{p+q}(X - B, f_* V^o) \end{aligned}$$

degenerates at E_1, *i.e.*

$$\mathbb{H}^k(X, W_0^B \mathrm{DR}_X(D + B, E)) \cong \bigoplus_{p+q=k} H^q(X, W_0^B \Omega_X^p(\log D) \otimes E)$$

Now, consider the descending sequence

$$Y = Y_0 \supset Y_1 \supset \cdots \supset Y_n$$

where Y_m is the union of m-fold intersection of components of D on Y. We have seen in Theorem 2.3.1 that $f_* V^o$ on $Y_m - Y_{m-1}$ is a unitary local system. This means $f_* V^o$ is a constructible sheaf.

As $X - B$ is affine, it follows from Theorem 3.1.2 that

$$H^q(X, W_0^B(\Omega_X^p(\log D + B) \otimes E)) = 0$$

for $p+q>n$.

Proof of claim 1 finished

Then, to finish the proof, we can induct on the dimension of X. Therefore, we can assume X has dimension 1.

It remains to show that if X is a smooth projective curve, then

$$H^1(X, W_0(\Omega_X(\log D)\otimes E)\otimes O_X(B)) = 0$$

But for the curve case,

$$W_0(\Omega_X(\log D)\otimes E)\otimes O_X(B) = W_0^B(\Omega_X(\log D+B)\otimes E)$$

Therefore the result follows again from Theorem 3.1.2

To finish the rest of the proof, we use the identification from proposition 2.4.1

$$(W_m/W_{m-1})\mathrm{DR}_X(D,E) \cong W_0\mathrm{DR}_{\tilde{D}_m}(\tilde{C}_m, E_m)[-m]$$

and then apply the above argument to $\tilde{D}_m$. ■

Corollary 3 *For any $m \in \mathbb{Z}$,*

$$H^q(X, W_m(\Omega_X^p(\log D)\otimes E)\otimes O_X(B)) = 0$$

for $p+q > \dim X$

Proof Use the exact sequence

$$0 \to W_{m-1}\mathrm{DR}_X(D,E) \to W_m\mathrm{DR}_X(D,E) \to W_0\mathrm{DR}_{\tilde{D}_m}(\tilde{C}_m, E_m)[-m] \to 0$$

■

Corollary 4 *Let L be an ample line bundle on X, then*

$$H^q(X, Gr_m^W(\Omega_X^p(\log D)\otimes E)\otimes L) = 0$$

for any m and $p+q>n$.

Proof Like in Proposition 4.3.2, it is enough to show

$$H^q(X, W_0\mathrm{DR}^p(D, E, \nabla) \otimes L) = 0$$

for $p + q > n$.

Let m be a large enough integer such that $L^{\otimes m}$ is very ample. Let B be a smooth hyperplane divisor transversal to D so that

$$L \cong O_X(B)$$

Use the same argument in Theorem 3.1.3, we construct a cyclic cover of degree m branched over B

$$\pi : X' \to X$$

To finish the proof, it remains to show

$$\pi^* W_0\mathrm{DR}_X(D, E) = W_0\mathrm{DR}(\tilde{D}, \tilde{E})$$

But this is clear from the local description of W_0. ■

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