RATE ESTIMATORS FOR NON-STATIONARY POINT PROCESSES

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ABBREVIATIONS

NHPP	Nonhomogeneous Poisson Process
MPP	Marked Point Process
MLE	Maximum Likelihood Estimator
CLT	Central Limit Theorem
FCLT	Functional Central Limit Theorem
SLLN	Strong Law of Large Numbers
LIL	Law of Iterated Logarithms
iid	Independent and Identically Distributed
cdf	Cumulative Distribution Function
MSE	Mean squared error

ABSTRACT

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Non-stationary point processes are often used to model systems whose rates vary over time. Estimating underlying rate functions is important for input to a discrete-event simulation along with various statistical analyses. We study nonparametric estimators to the marked point process, the $M_t/G/\infty$ queueing model, and the $\Delta_{(i)}/G/1$ transitory queueing model. We conduct statistical inference for these estimators by establishing a number of asymptotic results.

For the marked point process, we consider estimating the offered load to the system over time. With direct observations of the offered load sampled at fixed intervals of width $\delta_n > 0$, we establish asymptotic consistency, rates of convergence, and asymptotic covariance through a Functional Strong Law of Large Numbers, a Functional Central Limit Theorem, and a Law of Iterated Logarithm. We also show that there exists an asymptotically optimal δ_n as the sample size approaches infinity.

The $M_t/G/\infty$ queueing model is central in many stochastic models. Specifically, the mean number of busy servers can be used as an estimator for the total load faced to a multi-server system with time-varying arrivals and in many other applications. Through an omniscient estimator based on observing both the arrival times and service requirements for n samples of an $M_t/G/\infty$ queue, we show asymptotic consistency and rate of convergence. Then, we establish the asymptotics for a non-parametric estimator based on observations of the busy servers at fixed intervals of width $\delta_n > 0$.

The $\Delta_{(i)}/G/1$ model is crucial when studying a transitory system, which arises when the time horizon or population is finite. We assume we observe arrival counts

at fixed intervals. We first consider a natural estimator which applies an underlying nonhomogeneous Poisson process. Although the estimator is asymptotically unbiased, we see that a correction term is required to retrieve an accurate asymptotic covariance. Next, we consider a nonparametric estimator that exploits the maximum likelihood estimator of a multinomial distribution to see that this estimator converges appropriately to a Brownian Bridge.

1. INTRODUCTION

Non-stationary point processes are often suggested to model systems whose rate varies over time. The nonhomogeneous Poisson process (NHPP) (Definition 1.6.1) is a common non-stationary point process that is used to model a variety of applications. It is common to use NHPPs to model call centers [1], software reliability [2], hospital systems [3], and many other systems outside of the classic queueing context.

Estimating the underlying intensity or cumulative intensity function of an NHPP has been of particular interest. If an NHPP is used as an input to a Monte Carlo simulation, inversion of the cumulative intensity function or thinning of the intensity function can be used to generate arrival times to an NHPP [4]. Simulation of an NHPP by thinning requires probabilistically rejecting points from an NHPP with intensity function that dominates over the entire domain [5]. The inverse transformation is the most fundamental way in which to generate random variates. To generate variates to an NHPP, the inverse transformation method inverts the cumulative intensity function and uses the exponentially distributed inter-arrival times [4]. Other work has been conducted to simulate NHPPs with cyclic or periodic behavior, where a "piecewise thinning" approach is used in conjunction with a polynomial exponential intensity function [6], [7]. Simulation techniques have also been developed for log-linear rate functions [8], for spatial point processes [9], and for piecewise linear cumulative intensity functions from count data [10]. In order to conduct these simulations, estimates of the intensity functions are required as input. Therefore, data-driven estimation techniques are needed to obtain consistent and efficient estimators of the underlying intensities.

Many techniques have been applied to estimate the intensity functions. Fitting a power intensity to the cumulative intensity function is a common parametric technique [11], [12], [13]. Optimization methods have also been used to fit a nonnegative cubic spline to the intensity function based on available arrival data [14]. Maximum likelihood estimation (MLE) is another common parametric technique to estimate the intensity functions [15], however, it is shown that the MLE need not be asymptotically consistent or normal for software reliability models [16]. MLE is also applied to estimation of a piecewise linear intensity function, where an optimization problem is formulated based on fully-observed NHPPs [17], [18].

Nonparametric methods have also been used to estimate the intensity functions. For fully-observed samples of an NHPP, a nonparametric estimator averages counts at observed times and linearly interpolates between times to retrieve a piecewise-linear cumulative intensity function estimator [19]. In practice, it may be more common to observe count data at fixed intervals. With the assumption of a piecewise-constant intensity function, the asymptotic consistency and normality of a nonparametric estimator which averages count data over intervals is shown [20]. To estimate a piecewise-constant intensity, a heuristic is also proposed in which points at which the intensity changes are user specified and count data is therefore averaged in the given intervals [21].

Although the intensity functions of the nonhomogeneous Poisson process are crucial to estimate, it is also of interest to estimate underlying parameters of other non-stationary processes.

1.1 Marked Point Processes

The marked point process (MPP) is composed of a point process and associated marks [22]. MPPs have been used to model image analysis [23], forest statistics [24], crowd counting [25], stock price variations [26], and various other applications [27], [28], [29]. Associated with the MPP is the cumulative marks or cumulative load to the system over time. Insurance risk, service systems, earthquake effects, and healthcare systems all benefit from estimating the cumulative load to the system over time. Methods for estimating the average load of a MPP have been developed based on dependence of service times and patience of customers to a service system [30]. Specific to multimedia cable network systems, a nonparametric estimator estimates the offered load at fixed times based on a fully-observed system [31].

1.2 $M_t/G/\infty$ Queueing Model

The $M_t/G/\infty$ queueing model has nonhomogeneous Poisson process arrivals, general service times, and infinite servers. Although seemingly unrealistic, infinite server models are central to many stochastic modeling applications. For multi-server systems with time-varying arrivals, the infinite server model characterizes the total load faced to the system, suggesting that infinite server models can also be used as a prototype [32]. Infinite server models are used to model repair mechanisms for damaged cells [33], software reliability analysis [34], [35], and call centers [36]. Parametric techniques have been used to estimate the queue length process based on count data using Little's Formula or based on busy/idle times of the system using regenerative models [37]. Nonparametric methods based on count data use Reynold's formula for estimating the queue length process [38].

1.3 Transitory Queueing Model

The transitory queueing model is important to study when drawing from a finite population or when the system operates in a finite window of time. This arises naturally in many applications such as the arrival of attendees to an event, the arrivals of customers to a store, or the arrivals of individuals in the peak hours of a 24-hour service system. The $\Delta_{(i)}/G/1$ model assumes that m customers independently sample arrival times from common distribution $F(t)$ [39]. A natural estimator for $F(t)$ is the empirical distribution function which assumes a fully-observed arrival process. Donsker's Theorem provides the well-known functional limit theorem for the empirical distribution function [40]. The empirical distribution has been well-studied to understand when parameters are estimated [41], when available data is grouped or truncated [42], or when incomplete data is used for maximum likelihood estimation [43].

1.4 Organization

We begin in section 1.5 and 1.6 with necessary notation, definitions, and theorems that will be used throughout the remaining sections. In section 2, we describe a nonparametric estimator for the offered load of a marked point process based on discrete observations of the process. We establish a number of asymptotic results. In section 3, we first consider an omniscient estimator of the mean number of busy servers in an $M_t/G/\infty$ queue. We also consider the natural estimator when the number of busy servers is observed at fixed intervals. We show asymptotic results for both nonparametric estimators. Section 4 considers the asymptotics of the natural estimator for the arrival time distribution in a transitory queue, showing asymptotics for nonparametric estimators based on count data.

1.5 Notations

Denote almost sure convergence by $\rightarrow$ and weak convergence by $\Rightarrow$. $\mathbb{1}\{\}$ denotes the indicator function. $x \wedge y$ and $x \vee y$ denote the minimum and maximum of x and y, respectively. Let $X := Y$ represent X is defined as Y. We let $(X(t) : t \geq 0)$ represent a stochastic process defined on $t \geq 0$.

1.6 Necessary Definitions and Theorems

Definition 1.6.1 (NHPP) *Suppose* $N = (N(t) : t \geq 0)$ *be an integer-valued, non-decreasing process with* $N(0) = 0$. *We say that* N *is a nonhomogeneous Poisson process (NHPP) with rate (or intensity) function* $\lambda(t) = (\lambda(t) : t \geq 0)$ *if:*

1. *the process* N *has independent increments,*
2. *for all* $t \geq 0$, $P(N(t+h) - N(t) \geq 2) = o(h)$, *and*

3. for all $t \geq 0$, $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$.

We will also use the following central limit theorem result for stochastic processes, Theorem 2 from [44].

Theorem 1.6.1 (Hahn's CLT [44]) *Let* X *be a stochastic process with sample paths in* D *such that* $E[X(t)] = 0$, $E[X(t)^2] < \infty \quad \forall \quad t \in [0,1]$. *For* $\alpha > 1/2$, $\beta > 1$, *and* $0 \leq s \leq t \leq u \leq 1$, *if*
1. $E\left[(X(u) - X(t))^2\right] \leq (G(u) - G(t))^\alpha$ *for* G *is a nondecreasing continuous function,*
2. $E\left[(X(u) - X(t))^2 (X(t) - X(s))^2\right] \leq (F(u) - F(s))^\beta$ *for* F *is a nondecreasing continuous function, then* X *satisfies a CLT.*

Theorem 8.1 and 8.2 from [40] will be used to show weak convergence.

Theorem 1.6.2 (Prokhorov's theorem) *Let* P_n, P *be probability measures on* $(C, \mathcal{C})$. *If the finite dimensional distributions of* P_n *converge weakly to those of* P, *and if* $\{P_n\}$ *is tight, then* $P_n \Rightarrow P$.

Theorem 1.6.3 (Tightness) *The sequence* $\{P_n\}$ *is tight if and only if these two conditions hold:*

(i) For each positive η, *there exists an* a *such that* $P_n\{x : |x(0)| > a\} \leq \eta$, $n \geq 1$.

(ii) For each positive ϵ *and* η, *there exists a* δ, *with* $0 < \delta < 1$, *and an integer* n_0 *such that* $P_n\{x : w_x(\delta) \geq \epsilon\} \leq \eta$, $n \geq n_0$, *where* $w_x(\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|$, $0 < \delta < 1$.

Theorem 1.6.4 (Continuous Mapping Theorem) *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be random variables. Suppose that* $X_n \Rightarrow X$ *and* $Y_n \to c$, *then* $(X_n, Y_n) \Rightarrow (X, c)$.

Theorem 1.6.5 (Slusky's Theorem) *If* $X_n \Rightarrow X$ *and* $Y_n \Rightarrow c$, *then*
1. $X_n + Y_n \Rightarrow X + c$
2. $X_n Y_n \Rightarrow Xc$
3. $X_n / Y_n \Rightarrow X/c$ *as long as* $c \neq 0$

2. MARKED POINT PROCESS ESTIMATOR

Recall the offered load to a stochastic system at time t: $(V(t) : t \geq 0)$ where $V(t) := \nu_1 + \cdots + \nu_{N(t)}$, where $\{\nu_j\}_{j=1}^{\infty}$ are a set of independent and identically distributed (i.i.d.) random variables where $E[\nu_1] < +\infty$, the moment generating function satisfies $M_\nu(u) := E\left[e^{u\nu}\right] < +\infty$, and $N_i(s)$ is the ith observation of a nonhomogenous Poisson process (NHPP) $(N(t) : t \geq 0)$ with rate function $(\lambda(t) : t \geq 0)$, at time s.

Given direct observations of the offered load at fixed intervals, our objective is to estimate the mean offered load $E[V(t)]$ at each time $t \geq 0$. The estimated mean offered load can be used as an input for simulation of a discrete event model as well as for statistical analysis of a service system.

Let $V_i(a, b) := V_i(b) - V_i(a)$ represent the increase in the offered load in the interval $[a, b)$ in the ith independent observed realization. We assume that we can observe increments of the offered load at fixed intervals of width $\delta_n > 0$. For $t \geq 0$,

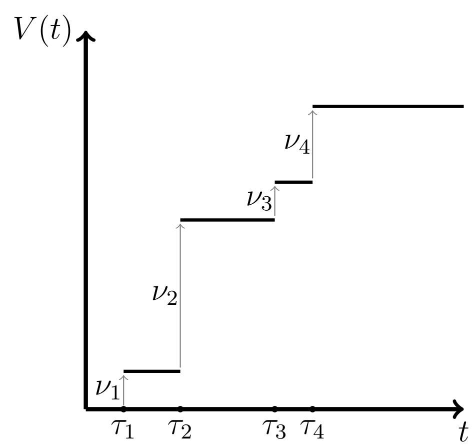

Fig. 2.1. Sample path realization of offered load $(V(t) : t \geq 0)$ where τ_i is ith arrival time and ν_i is ith service requirement.

let $l_n(t) = \left\lfloor \frac{t}{\delta_n} \right\rfloor \delta_n$ be the lower bound of the interval in which t falls such that $l_n(t) \leq t < l_n(t) + \delta_n$. Then, the natural estimator of the mean offered load at $t \geq 0$ is the random variable

$$V_n(t) := \frac{1}{n}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n).$$

We conduct statistical inference for the estimator by establishing a number of asymptotic results. We first demonstrate asymptotic consistency for $V_n(t)$ by proving a functional strong law of large numbers (FSSLN) result in Theorem 2.1.1. We then give the rate of convergence through a functional central limit theorem (FCLT) in Theorem 2.2.4 and a law of iterated logarithm (LIL) in Theorem 2.2.6. We also provide the order at which δ_n should shrink in order to minimize a bound on the mean-squared error in Theorem 2.3.1.

2.1 Asymptotic Consistency

We prove a functional strong law of large numbers (FSSLN) to show the asymptotic consistency of the estimator $V_n(t)$ to the true mean offered load $E[V(t)]$.

Theorem 2.1.1 (FSLLN) *If $\delta_n \to 0$ as $n \to \infty$, then $\sup_{t \epsilon [0,T]} |V_n(t) - E[V(t)]| \to 0$ a.s. as $n \to \infty$.*

Proof Note that

$$\begin{aligned} V_n(t) &= \frac{1}{n}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \\ &= \frac{1}{n}\sum_{i=1}^{n} V_i(0, t) - \frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n). \end{aligned}$$

Call the latter two terms R_n. Because $l_n \leq t < l_n + \delta_n$, it follows that

$$\begin{aligned} |R_n| &= \left| -\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \right| \\ &\leq \left| -\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), t) \right| + \left| \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \\
&\le \frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \\
&\le \frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) + \frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \\
&= \frac{2}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n).
\end{aligned}
$$

Now, for n sufficiently large, we have that for all $\epsilon > 0$, $t-\epsilon \le l_n(t) < l_n(t)+\delta_n \le t+\epsilon$. Taking the limsup, we have,

$$
\begin{aligned}
\limsup_{n\to\infty} |R_n| &\le \limsup_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} V_i(t-\epsilon, t+\epsilon) \\
&= 2E[V_1(t-\epsilon, t+\epsilon)], \qquad (2.1.1)
\end{aligned}
$$

which follows as a consequence of the strong law of large numbers. Next, from the independence of the service times and arrival process,

$$
\begin{aligned}
(2.1.1) &= 2E[\nu_1]E[N(t-\epsilon, t+\epsilon)] \\
&= 2E[\nu_1]\int_{t-\epsilon}^{t+\epsilon} \lambda(s)ds \\
&\to 0 \quad a.s. \quad as \quad \epsilon \downarrow 0.
\end{aligned}
$$

So, we have,

$$
\begin{aligned}
\lim_{n\to\infty} \sup |V_n(t)| &= \lim_{n\to\infty} \sup \left| \frac{1}{n}\sum_{i=1}^{n} V_i(0, t) + R_n \right| \\
&= E[V(t)],
\end{aligned}
$$

by the strong law of large numbers. We have shown pointwise almost sure convergence. To prove uniform convergence, note that $V(t)$ is a continuous, nondecreasing function. Therefore, for all $\epsilon > 0$, there exists $m(\epsilon) < \infty$ and points $0 = u_0 < u_1 < \cdots < u_{m(\epsilon)} = T$ such that $V(u_i) - V(u_{i-1}) \le \epsilon \quad \forall \quad i = 1, ..., m(\epsilon)$. For $t \in [0, T]$, let $a(t) = a_\epsilon(t)$ denote the index i such that $t \in [u_i, u_{i+1})$. Then,

$$
|V_n(t) - V(t)| = \max\left[V_n(t) - V(t), V(t) - V_n(t)\right]
$$

$$
\begin{aligned}
&\leq \max\left[V_n(u_{a(t)+1}) - V(u_{a(t)}), V(u_{a(t)+1}) - V_n(u_{a(t)})\right] \\
&\leq \max\left[V_n(u_{a(t)+1}) - V(u_{a(t)+1}) + V(u_{a(t)+1})\right. \\
&\qquad \left. -V(u_{a(t)}), V(u_{a(t)+1}) - V(u_{a(t)}) + V(u_{a(t)}) - V_n(u_{a(t)})\right] \\
&\leq \max\left[\left|V_n(u_{a(t)+1}) - V(u_{a(t)+1})\right| + V(u_{a(t)+1}) - V(u_{a(t)}),\right. \\
&\qquad \left. V(u_{a(t)+1}) - V(u_{a(t)}) + \left|V(u_{a(t)}) - V_n(u_{a(t)})\right|\right] \\
&\leq \epsilon + \max_{i=1,\ldots,m(\epsilon)} \left|V_n(u_i) - V(u_i)\right|.
\end{aligned}
$$

By pointwise convergence, we have $|V_n(u_i) - V(u_i)| \leq \epsilon$ for large enough n. Therefore, it follows that $\max |V_n(u_i) - V(u_i)| \leq \epsilon$. So, $\sup_{t\in[0,T]} |V_n(t) - V(t)| \leq \epsilon+\epsilon = 2\epsilon$. Therefore, for large enough n, $\limsup_{n\to\infty} \sup_{t\in[0,T]} |V_n(t) - V(t)| = 0$, since ϵ is arbitrary. This completes the proof. ■

Through a FSLLN, we have shown that $V_n(t)$ is an asymptotically consistent, and therefore an asymptotically unbiased, estimator of $E[V(t)]$.

2.2 Rates of Convergence

Understanding the rate of convergence of $V_n(t)$, along with its asymptotic covariance, is crucial for computing confidence intervals and other statistical measures. We establish the rates of convergence by proving a functional central limit theorem (FCLT) and a law of iterated logarithm (LIL).

2.2.1 Functional Central Limit Theorem

We start by proving a pointwise central limit theorem (CLT), under a specific scaling assumption on δ_n.

Lemma 2.2.1 (CLT) *Suppose $\delta_n = o(n^{-1/4})$ and the rate function $(\lambda(t) : t \geq 0)$ is Lipschitz continuous in a neighborhood of t with Lipschitz constant K. Then,*

$$\hat{V}_n(t) := \sqrt{n}\,(V_n(t) - E[V(t)]) \Rightarrow \mathcal{N}(0, \mathit{Var}(V(t))) \quad \mathit{as} \quad n \to \infty.$$

Proof First, recall that

$$
\begin{aligned}
\hat{V}_n(t) &:= \sqrt{n}\,(V_n(t) - E[V(t)]) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) - E[V(t)]\right) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) - \sqrt{n}E[V(t)].
\end{aligned}
$$

Then the log moment generating function of the lefthand side is

$$
\begin{aligned}
&\Psi_n(u) = \log E\left[\exp\{u\sqrt{n}(V_n(t) - E[V(t)])\}\right] \\
&= \log E\Bigg[\exp\Bigg\{\frac{u}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \\
&\qquad -u\sqrt{n}E[V(t)]\Bigg\}\Bigg] \\
&= \log \Bigg(E\left[\exp\left\{\frac{u}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t))\right\}\right] \\
&\qquad E\left[\exp\left\{\frac{t - l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n)\right\}\right] E\left[\exp\left\{-u\sqrt{n}E[V(t)]\right\}\right]\Bigg) \\
&= \log E\left[\exp\left\{\frac{u}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t))\right\}\right] \\
&\qquad + \log E\left[\exp\left\{\frac{t - l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n)\right\}\right] - u\sqrt{n}E[V(t)] \\
&= \log E\left[\exp\left\{\frac{u}{\sqrt{n}} V_1(0, l_n(t))\right\}\right]^n \\
&\qquad + \log E\left[\exp\left\{\frac{t - l_n(t)}{\delta_n}\frac{u}{\sqrt{n}} V_1(l_n(t), l_n(t) + \delta_n)\right\}\right]^n - u\sqrt{n}E[V(t)].
\end{aligned}
\tag{2.2.1.1}
$$

Consider the first term in (2.2.1.1). Observe that it is the log moment generating function of a random sum of random variables, $V_1(0, l_n(t)) = \nu_1 + \cdots + \nu_{N_1(0, l_n(t))}$. Next, conditioning the moment generating function (MGF) on the Poisson random variable $N_1(0, l_n(t))$, we have,

$$
E\left[\exp\left\{\frac{u}{\sqrt{n}} V_1(0, l_n(t))\right\} | N_1(0, l_n(t)) = m\right]
$$

$$= E\left[\exp\left\{\frac{u}{\sqrt{n}}\sum_{j=1}^{N_1(0,l_n(t))}\nu_j\right\}|N_1(0,l_n(t))=m\right]$$
$$= E\left[\exp\left\{\frac{u}{\sqrt{n}}\sum_{j=1}^{m}\nu_j\right\}\right]$$
$$= E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]^m$$
$$= \exp\left\{m\log E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}.$$

So, it follows that,

$$E\left[\exp\left\{\frac{u}{\sqrt{n}}V_1(0,l_n(t))\right\}\right]$$
$$= \sum_{m=1}^{\infty} E\left[\exp\left\{\frac{u}{\sqrt{n}}V_1(0,l_n(t))\right\}|N_1(0,l_n(t))=m\right]P(N_1(0,l_n(t))=m)$$
$$= \sum_{m=1}^{\infty}\exp\left\{m\log E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}P(N_1(0,l_n(t))=m)$$
$$= E\left[\exp\left\{N_1(0,l_n(t))\log E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}\right]. \tag{2.2.1.2}$$

Similarly, for the second term in (2.2.1.1),

$$E\left[\exp\left\{\frac{t-l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}V_1(l_n(t),l_n(t)+\delta_n)\right\}\right]$$
$$= E\left[\exp\left\{N_1(l_n(t),l_n(t)+\delta_n)\log E\left[\exp\left\{\frac{t-l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}\right]. \tag{2.2.1.3}$$

Returning to the log moment generating function $\Psi_n(u)$, and using (2.2.1.2) and (2.2.1.3), we have,

$$(1) = n\log E\left[\exp\left\{N_1(0,l_n(t))\log E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}\right]$$
$$+ n\log E\left[\exp\left\{N_1(l_n(t),l_n(t)+\delta_n)\log E\left[\exp\left\{\frac{t-l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}\right]$$
$$- u\sqrt{n}E[V(t)]. \tag{2.2.1.4}$$

Recall that if X is a Poisson random variable with mean μ, then $E[\exp\{uX\}] = \exp\{\mu(e^u-1)\}$. Also, recall that $E[N_1(a,b)] = \Lambda(b)-\Lambda(a)$ where $\Lambda(t) := \int_0^t \lambda(s)ds$. Therefore,

$$(2.2.1.4) = n\log\left(\exp\left\{\Lambda(l_n(t))\left(\exp\left\{\log E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}-1\right)\right\}\right)$$

$$
\begin{aligned}
&+ n\log\left(\exp\left\{\left(\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t))\right)\right.\right.\\
&\left.\left.\times\left(\exp\left\{\log E\left[\exp\left\{\frac{t-l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}-1\right)\right\}\right)\\
&- u\sqrt{n}E[V(t)]\\
= {} & n\left(\Lambda(l_n(t))\left(\exp\left\{\log E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}-1\right)\right)\\
&+ n\left(\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t))\right)\left(\exp\left\{\log E\left[\exp\left\{\frac{t-l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\nu_1\right\}\right]\right\}-1\right)\\
&- u\sqrt{n}E[V(t)]\\
= {} & n\Lambda(l_n(t))\left(E\left[\exp\left\{\frac{u}{\sqrt{n}}\nu_1\right\}\right]-1\right)\\
&+ n\left(\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t))\right)\left(E\left[\exp\left\{\frac{t-l_n(t)}{\delta_n}\frac{u}{\sqrt{n}}\nu_1\right\}\right]-1\right)\\
&- u\sqrt{n}E[V(t)]. \qquad (2.2.1.5)
\end{aligned}
$$

Next, using the Taylor series expansion for e^{tX}, it follows that

$$
\begin{aligned}
(2.2.1.5) = {} & n\Lambda(l_n(t))\left(\frac{u}{\sqrt{n}}E[\nu_1]+\frac{u^2}{2n}E[\nu_1^2]+O\left(n^{-3/2}\right)\right)+n\left(\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t))\right)\\
&\times\left(\frac{u}{\sqrt{n}}\frac{t-l_n(t)}{\delta_n}E[\nu_1]+\frac{u^2}{2n}\left(\frac{t-l_n(t)}{\delta_n}\right)^2E[\nu_1^2]+O\left(\left(\frac{t-l_n(t)}{\delta_n}\right)^3n^{-3/2}\right)\right)\\
&- u\sqrt{n}E[\nu_1]\Lambda(t)\\
= {} & u\sqrt{n}E[\nu_1]\left(\Lambda(l_n(t))-\Lambda(t)\right)\\
&+\frac{u^2}{2}E[\nu_1^2]\Lambda(l_n(t))+u\sqrt{n}\left(\frac{t-l_n(t)}{\delta_n}\right)\left(\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t))\right)E[\nu_1]\\
&+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2\left(\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t))\right)E[\nu_1^2]\\
&+O\left(n^{-1/2}\right)+O\left(\left(\frac{t-l_n(t)}{\delta_n}\right)^3n^{-1/2}\right). \qquad (2.2.1.6)
\end{aligned}
$$

By the Mean Value Theorem, for some $\zeta_n \in [l_n(t), l_n(t)+\delta_n]$ and $\theta_n \in [l_n(t), t]$, we know that $\Lambda(l_n(t)+\delta_n)-\Lambda(l_n(t)) = \lambda(\zeta_n)\delta_n$ and $\Lambda(t)-\Lambda(l_n(t)) = \lambda(\theta_n)(t-l_n(t))$. Then,

$$
(2.2.1.6) = u\sqrt{n}E[\nu_1](-\lambda(\theta_n))(t-l_n(t))+u\sqrt{n}\frac{t-l_n(t)}{\delta_n}\lambda(\zeta_n)\delta_nE[\nu_1]
$$

$$
\begin{aligned}
&+ \frac{u^2}{2}E[\nu_1^2]\Lambda(l_n(t)) + O\left(\frac{(t-l_n(t))^2}{\delta_n} + n^{-1/2} + \left(\frac{t-l_n(t)}{\delta_n}\right)^3 n^{-1/2}\right) \\
&= u\sqrt{n}E[\nu_1](t-l_n(t))(\lambda(\zeta_n)-\lambda(\theta_n)) + \frac{u^2}{2}E[\nu_1^2]\Lambda(l_n(t)) \\
&\quad + O\left(\frac{(t-l_n(t))^2}{\delta_n} + n^{-1/2} + \left(\frac{t-l_n(t)}{\delta_n}\right)^3 n^{-1/2}\right) \\
&\leq u\sqrt{n}E[\nu_1]\delta_n(\lambda(\zeta_n)-\lambda(\theta_n)) + \frac{u^2}{2}E[\nu_1^2]\Lambda(l_n(t)) \\
&\quad + O\left(\frac{(t-l_n(t))^2}{\delta_n} + n^{-1/2} + \left(\frac{t-l_n(t)}{\delta_n}\right)^3 n^{-1/2}\right).
\end{aligned} \tag{2.2.1.7}
$$

Recall K is the Lipschitz constant for $(\lambda(t) : t \geq 0)$. Also note that $\delta_n = o(n^{-1/4})$, so that $\delta_n^2 = o(n^{-1/2}) \to 0$ as $n \to \infty$. Then, we can bound as follows,

$$
\begin{aligned}
|(2.2.1.7)| &\leq u\sqrt{n}E[\nu_1]\delta_n K\,|\zeta_n - \theta_n| + \frac{u^2}{2}E[\nu_1^2]\Lambda(l_n(t)) \\
&\quad + O\left(\frac{(t-l_n(t))^2}{\delta_n} + n^{-1/2} + \left(\frac{t-l_n(t)}{\delta_n}\right)^3 n^{-1/2}\right) \\
&\leq u\sqrt{n}E[\nu_1]\delta_n^2 K + \frac{u^2}{2}E[\nu_1^2]\Lambda(l_n(t)) \\
&\quad + O\left(\frac{(t-l_n(t))^2}{\delta_n} + n^{-1/2} + \left(\frac{t-l_n(t)}{\delta_n}\right)^3 n^{-1/2}\right).
\end{aligned}
$$

Also note that (2.2.1.6) is bounded below by $\frac{u^2}{2}E\left[\nu_i^2\right]\Lambda(l_n(t))$ since $(t - l_n(t)) \geq 0$. It follows that $\Psi_n(u) \to \frac{u^2}{2}E[\nu_1^2]\Lambda(t)$ as $n \to \infty$, which is the moment generating function for a Gaussian random variable with mean 0 and variance $E[\nu_1^2]\Lambda(t) = \mathrm{Var}(V(t))$. ■

Therefore, for fixed $t \in [0, T]$, we know the rate of convergence and asymptotic variance for $V_n(t)$. However, we are still interested in how the entire process $(V_n(t) : t \geq 0)$ converges. The following two lemmas help to prove the FCLT. We start by showing the finite dimensional distributions (FDD's) of $(V_n(t) : t \geq 0)$ converge weakly to a multivariate Gaussian.

Lemma 2.2.2 (FDD's) *If* $\delta_n = o(n^{-1/4})$ *and the rate function* $(\lambda(t) : t \geq 0)$ *is Lipschitz continuous in a neighborhood of* t *with Lipschitz constant* K*, then for* $0 = t_0 < t_1 < \cdots < t_k \leq T$*,*

$$\left((\sqrt{n}(V_n(t_1) - E[V(t_1)]), ..., (\sqrt{n}(V_n(t_k) - E[V(t_k)])\right) \Rightarrow (Z(t_1), ..., Z(t_k))$$

where $(Z(t_1), ..., Z(t_k))$ *is a Gaussian vector with mean 0 and covariance matrix* $\Sigma = [\sigma_{ij}]$*, with* $\sigma_{ij} = Var(V(t_i \wedge t_j)) = E[\nu^2]\Lambda(t_i \wedge t_j)\ \forall\ 1 \leq i, j \leq k$*.*

Proof Consider the moment generating function for $k = 2$.

$$\begin{aligned}
&E\left[\exp\left\{\left\langle(u_1, u_2), \left(\sqrt{n}\left(V_n(t_1) - E[V(t_1)]\right), \sqrt{n}\left(V_n(t_2) - E[V(t_2)]\right)\right)\right\rangle\right\}\right] \\
&= E\left[\exp\left\{\frac{u_1}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t_1)) + \frac{u_1}{\sqrt{n}}\frac{t_1 - l_n(t_1)}{\delta_n}\sum_{i=1}^{n} V_i(l_n(t_1), l_n(t_1) + \delta_n)\right.\right. \\
&\quad + \frac{u_2}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t_2)) + \frac{u_2}{\sqrt{n}}\frac{t_2 - l_n(t_2)}{\delta_n}\sum_{i=1}^{n} V_i(l_n(t_2), l_n(t_2) + \delta_n) \\
&\quad \left.\left. - u_1\sqrt{n}E\left[V(t_1)\right] - u_2\sqrt{n}E\left[V(t_2)\right]\right\}\right] \\
&= E\left[\exp\left\{\frac{u_1 + u_2}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t_1))\right.\right. \\
&\quad + \left(\frac{u_1}{\sqrt{n}}\frac{t_1 - l_n(t_1)}{\delta_n} + \frac{u_2}{\sqrt{n}}\right)\sum_{i=1}^{n} V_i(l_n(t_1), l_n(t_1) + \delta_n) \\
&\quad + \frac{u_2}{\sqrt{n}}\sum_{i=1}^{n} V_i(l_n(t_1) + \delta_n, l_n(t_2)) + \frac{u_2}{\sqrt{n}}\frac{t_2 - l_n(t_2)}{\delta_n}\sum_{i=1}^{n} V_i(l_n(t_2), l_n(t_2) + \delta_n) \\
&\quad \left.\left. - u_1\sqrt{n}E\left[V(t_1)\right] - u_2\sqrt{n}E\left[V(t_2)\right]\right\}\right]. \qquad (2.2.2.1)
\end{aligned}$$

From the independence of nonoverlapping intervals and the i.i.d. observations of $(V(t) : t \geq 0)$,

$$\begin{aligned}
(2.2.2.1) = &E\left[\exp\left\{\frac{u_1 + u_2}{\sqrt{n}}\sum_{i=1}^{n} V_i(0, l_n(t_1))\right\}\right] \\
&\times E\left[\exp\left\{\left(\frac{u_1}{\sqrt{n}}\frac{t_1 - l_n(t_1)}{\delta_n} + \frac{u_2}{\sqrt{n}}\right)\sum_{i=1}^{n} V_i(l_n(t_1), l_n(t_1) + \delta_n)\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&\times E\left[\exp\left\{\frac{u_2}{\sqrt{n}}\sum_{i=1}^{n} V_i(l_n(t_1)+\delta_n, l_n(t_2))\right\}\right]\\
&\times E\left[\exp\left\{\frac{u_2}{\sqrt{n}}\frac{t_2-l_n(t_2)}{\delta_n}\sum_{i=1}^{n} V_i(l_n(t_2), l_n(t_2)+\delta_n)\right\}\right]\\
&\times E\left[\exp\left\{-u_1\sqrt{n}E\left[V(t_1)\right]-u_2\sqrt{n}\left[V(t_2)\right]\right\}\right]\\
= E&\left[\exp\left\{\frac{u_1+u_2}{\sqrt{n}}V_i(0, l_n(t_1))\right\}\right]^n\\
&\times E\left[\exp\left\{\left(\frac{u_1}{\sqrt{n}}\frac{t_1-l_n(t_1)}{\delta_n}+\frac{u_2}{\sqrt{n}}\right)V_i(l_n(t_1), l_n(t_1)+\delta_n)\right\}\right]^n\\
&\times E\left[\exp\left\{\frac{u_2}{\sqrt{n}}V_i(l_n(t_1)+\delta_n, l_n(t_2))\right\}\right]^n\\
&\times E\left[\exp\left\{\frac{u_2}{\sqrt{n}}\frac{t_2-l_n(t_2)}{\delta_n}V_i(l_n(t_2), l_n(t_2)+\delta_n)\right\}\right]^n\\
&\times E\left[\exp\left\{-u_1\sqrt{n}E\left[V(t_1)\right]-u_2\sqrt{n}\left[V(t_2)\right]\right\}\right]. \qquad (2.2.2.2)
\end{aligned}
$$

Taking the logarithm of (2.2.2.2) and applying the simplification for the moment generating function for a random sum of random variables as in Lemma 2.2.1, we obtain

$$
\begin{aligned}
\log\left((2.2.2.2)\right) = {}& n\Lambda(l_n(t_1))\left(E\left[\exp\left\{\frac{u_1+u_2}{\sqrt{n}}\nu_1\right\}\right]-1\right)\\
&+n\left(\Lambda(l_n(t_1)+\delta_n)-\Lambda(l_n(t_1))\right)\left(E\left[\exp\left\{\left(\frac{t_1-l_n(t_1)}{\delta_n}\frac{u_1}{\sqrt{n}}+\frac{u_2}{\sqrt{n}}\right)\nu_1\right\}\right]-1\right)\\
&+n\left(\Lambda(l_n(t_2))-\Lambda(l_n(t_1)+\delta_n)\right)\left(E\left[\exp\left\{\frac{u_2}{\sqrt{n}}\nu_1\right\}\right]-1\right)\\
&+n\left(\Lambda(l_n(t_2)+\delta_n)-\Lambda(l_n(t_2))\right)\left(E\left[\exp\left\{\frac{t_2-l_n(t_2)}{\delta_n}\frac{u_2}{\sqrt{n}}\nu_1\right\}\right]-1\right)\\
&-u_1\sqrt{n}E[V(t_1)]-u_2\sqrt{n}E[V(t_2)]. \qquad (2.2.2.3)
\end{aligned}
$$

By the Taylor series expansion for e^{tX}, it follows that

$$
\begin{aligned}
(2.2.2.3) = {}& n\Lambda(l_n(t_1))\left(\frac{u_1+u_2}{\sqrt{n}}E\left[\nu_1\right]+\frac{1}{2}\left(\frac{u_1+u_2}{\sqrt{n}}\right)^2 E\left[\nu_1^2\right]+O(n^{-3/2})\right)\\
&+n\left(\Lambda(l_n(t_1)+\delta_n)-\Lambda(l_n(t_1))\right)\left(\left(\frac{t_1-l_n(t_1)}{\delta_n}\frac{u_1}{\sqrt{n}}+\frac{u_2}{\sqrt{n}}\right)E\left[\nu_1\right]\right.\\
&\left.+\frac{1}{2}\left(\frac{t_1-l_n(t_1)}{\delta_n}\frac{u_1}{\sqrt{n}}+\frac{u_2}{\sqrt{n}}\right)^2 E\left[\nu_1^2\right]+O(n^{-3/2})\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ n\left(\Lambda(l_n(t_2)) - \Lambda(l_n(t_1)+\delta_n)\right)\left(\frac{u_2}{\sqrt{n}}E\left[\nu_1\right] + \frac{1}{2}\left(\frac{u_2}{\sqrt{n}}\right)^2 E\left[\nu_1^2\right] + O(n^{-3/2})\right) \\
&+ n\left(\Lambda(l_n(t_2)+\delta_n) - \Lambda(l_n(t_2))\right)\left(\frac{t_2 - l_n(t_2)}{\delta_n}\frac{u_2}{\sqrt{n}}E\left[\nu_1\right]\right. \\
&\left.+\frac{1}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\frac{u_2}{\sqrt{n}}\right)^2 E\left[\nu_1^2\right] + O(n^{-3/2})\right) \\
&- u_1\sqrt{n}E[V(t_1)] - u_2\sqrt{n}E[V(t_2)]. \qquad (2.2.2.4)
\end{aligned}
$$

Combining terms, (2.2.2.4) simplifies to

$$
\begin{aligned}
(2.2.2.4) = {} & u_1\sqrt{n}E\left[\nu_1\right]\Lambda(l_n(t_1)) + u_2\sqrt{n}E\left[\nu_1\right]\Lambda(l_n(t_2)) \\
&+ \frac{u_1^2}{2}E\left[\nu_1^2\right]\Lambda(l_n(t_1)) + \frac{u_2^2}{2}E\left[\nu_1^2\right]\Lambda(l_n(t_2)) + u_1u_2E\left[\nu_1^2\right]\Lambda(l_n(t_1)) \\
&+ u_1\sqrt{n}\frac{t_1 - l_n(t_1)}{\delta_n}E\left[\nu_1\right]\left(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))\right) \\
&+ u_2\sqrt{n}\frac{t_2 - l_n(t_2)}{\delta_n}E\left[\nu_1\right]\left(\Lambda(l_n(t_2)+\delta_n) - \Lambda(l_n(t_2))\right) \\
&+ \frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)^2 E\left[\nu_1^2\right]\left(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))\right) \\
&+ \frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)^2 E\left[\nu_1^2\right]\left(\Lambda(l_n(t_2)+\delta_n) - \Lambda(l_n(t_2))\right) \\
&+ u_1u_2\frac{t_1 - l_n(t_1)}{\delta_n}E\left[\nu_1^2\right]\left(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))\right) + O(n^{-1/2}) \\
&- u_1\sqrt{n}E[V(t_1)] - u_2\sqrt{n}E[V(t_2)]. \qquad (2.2.2.5)
\end{aligned}
$$

By the Mean Value Theorem, for $j = \{1,2\}$, $\theta_{nj} \in [l_n(t_j), t]$, and $\zeta_{nj} \in [l_n(t_j), l_n(t_j) + \delta_n]$,

$$
\begin{aligned}
&u_j\sqrt{n}E\left[\nu_1\right]\Lambda(l_n(t_j)) + u_j\sqrt{n}\frac{t_j - l_n(t_j)}{\delta_n}E\left[\nu_1\right]\left(\Lambda(l_n(t_j)+\delta_n) - \Lambda(l_n(t_j))\right) \\
&\qquad - u_j\sqrt{n}E[V(t_j)] \\
&= u_j\sqrt{n}E\left[\nu_1\right]\left(\Lambda(l_n(t_j)) - \Lambda(t_j)\right) + u_j\sqrt{n}\frac{t_j - l_n(t_j)}{\delta_n}E\left[\nu_1\right]\left(\Lambda(l_n(t_j)+\delta_n) - \Lambda(l_n(t_j))\right) \\
&= -u_j\sqrt{n}E\left[\nu_1\right]\lambda(\theta_{nj})(t_j - l_n(t_j)) + u_j\sqrt{n}\frac{t_j - l_n(t_j)}{\delta_n}E\left[\nu_1\right]\lambda(\zeta_{nj})\delta_n \\
&= u_j\sqrt{n}(t_j - l_n(t_j))E\left[\nu_1\right]\left(\lambda(\zeta_{nj}) - \lambda(\theta_{nj})\right),
\end{aligned}
$$

which is bounded in absolute value by $u_j\sqrt{n}\delta_n E\left[\nu_1\right] K\delta_n$, where K is the Lipschitz constant for $(\lambda(t) : t \geq 0)$. Since $\delta_n = o(n^{-1/4})$, the term converges to 0 as $n \to \infty$. Now consider the following terms from (2.2.2.5):

$$
\begin{aligned}
&\frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)^2 E\left[\nu_1^2\right]\left(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))\right)\\
&\qquad + \frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)^2 E\left[\nu_1^2\right]\left(\Lambda(l_n(t_2)+\delta_n) - \Lambda(l_n(t_2))\right)\\
&\qquad + u_1u_2\frac{t_1 - l_n(t_1)}{\delta_n}E\left[\nu_1^2\right]\left(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))\right)\\
&= \frac{u_1^2}{2}\frac{(t_1 - l_n(t_1))^2}{\delta_n}E\left[\nu_1^2\right]\lambda(\zeta_{n1}) + \frac{u_2^2}{2}\frac{(t_2 - l_n(t_2))^2}{\delta_n}E\left[\nu_1^2\right]\lambda(\zeta_{n2})\\
&\qquad + u_1u_2(t_1 - l_n(t_1))E\left[\nu_1^2\right]\lambda(\zeta_{n1})\\
&\leq \frac{u_1^2}{2}\delta_n E\left[\nu_1^2\right]\lambda(\zeta_{n1}) + \frac{u_2^2}{2}\delta_n E\left[\nu_1^2\right]\lambda(\zeta_{n2}) + u_1u_2\delta_n E\left[\nu_1^2\right]\lambda(\zeta_{n1})\\
&\to 0,
\end{aligned}
$$

where the limit holds since $\delta_n \to 0$ as $n \to \infty$. Therefore, since $l_n(t) \to t$ as $n \to \infty$, it follows that

$$
\begin{aligned}
(2.2.2.5) \to &\frac{u_1^2}{2}E\left[\nu_1^2\right]\Lambda(t_1) + \frac{u_2^2}{2}E\left[\nu_1^2\right]\Lambda(t_2) + u_1u_2E\left[\nu_1^2\right]\Lambda(t_1)\\
&= \frac{1}{2}\mathbf{u}^T\Sigma\mathbf{u},
\end{aligned}
$$

which is the moment generating function for a Gaussian vector with mean zero and covariance matrix $\Sigma = [\sigma_{ij}]$, with $\sigma_{ij} = \mathrm{Var}(V(t_i \wedge t_j))$. Now, since $V_n(t)$ can be written in nonoverlapping intervals of $0 = t_0 < t_1 < \cdots < t_k \leq T$, then the same simplifications will hold to prove that the moment generating function of an arbitrary finite k-dimensional vector converges to this Gaussian vector. ■

Next, we show that the sufficient conditions of Theorem 1.6.1 are satisfied straightforwardly by $(V_n(t) : t \geq 0)$.

Lemma 2.2.3 *Let*

$$
\bar{V}_i(s) = V_i(0, l_n(s)) + \frac{s - l_n(s)}{\delta_n}V_i(l_n(s), l_n(s) + \delta_n).
$$

If $\delta_n = o(n^{-1/4})$ and $E\left[\nu^2\right] < \infty$, then for $0 \leq s \leq t \leq u \leq T$, as $n \to \infty$,

$$(1) \quad E\left[\left(\left(\bar{V}_i(u) - E\left[V(u)\right]\right) - \left(\bar{V}_i(t) - E\left[V(t)\right]\right)\right)^2\right] \leq E\left[\nu^2\right]\left(\Lambda(u) - \Lambda(t)\right)$$

and

$$\begin{aligned}(2) \quad E\Big[&\left(\left(\bar{V}_i(u) - E\left[V(u)\right]\right) - \left(\bar{V}_i(t) - E\left[V(t)\right]\right)\right)^2 \\ &\left(\left(\bar{V}_i(t) - E\left[V(t)\right]\right) - \left(\bar{V}_i(s) - E\left[V(s)\right]\right)\right)^2\Big] \\ &\leq E\left[\nu^2\right]^2\left(\Lambda(u) - \Lambda(s)\right)^2.\end{aligned}$$

Proof We first prove (1). As in Theorem 2.1.1, let $\bar{V}_i(s) = V_i(0,s) + R_s$, where $R_s = -V_i(l_n(s), s) + \frac{s - l_n(s)}{\delta_n} V_i(l_n(s), l_n(s) + \delta_n)$. Note that,

$$\begin{aligned}&E\left[\left(\left(\bar{V}_i(u) - E\left[V(u)\right]\right) - \left(\bar{V}_i(t) - E\left[V(t)\right]\right)\right)^2\right] \\ &= E\left[\left(\left(V_i(0,u) + R_u - E\left[V(u)\right]\right) - \left(V_i(0,t) + R_t - E\left[V(t)\right]\right)\right)^2\right] \\ &= E\left[\left(\left(V_i(0,u) - E\left[V(u)\right]\right) - \left(V_i(0,t) - E\left[V(t)\right]\right) + \left(R_u - R_t\right)\right)^2\right] \\ &= E\left[\left(\left(V_i(0,u) - E\left[V(u)\right]\right) - \left(V_i(0,t) - E\left[V(t)\right]\right)\right)^2\right] + E\left[\left(R_u - R_t\right)^2\right] \\ &\qquad + 2E\left[\left(R_u - R_t\right)\left(\left(V_i(0,u) - E\left[V(u)\right]\right) - \left(V_i(0,t) - E\left[V(t)\right]\right)\right)\right]. \qquad (\star)\end{aligned}$$

We will first show an upper bound on the first expectation in $(\star)$.

$$\begin{aligned}&E\left[\left(\left(V_i(0,u) - E\left[V(u)\right]\right) - \left(V_i(0,t) - E\left[V(t)\right]\right)\right)^2\right] \\ &= E\left[\left(\left(V_i(0,u) - V_i(0,t)\right) - \left(E\left[V(u)\right] - E\left[V(t)\right]\right)\right)^2\right] \\ &= E\left[\left(V_i(0,u) - V_i(0,t)\right)^2\right] + E\left[\nu\right]^2\left(\Lambda(u) - \Lambda(t)\right)^2 \\ &\qquad - 2E\left[\nu\right]\left(\Lambda(u) - \Lambda(t)\right)E\left[V_i(0,u) - V_i(0,t)\right] \\ &= \mathrm{Var}\left(V_i(t,u)\right) + E\left[V_i(t,u)\right]^2 + E\left[\nu\right]^2\left(\Lambda(u) - \Lambda(t)\right)^2 \\ &\qquad - 2E\left[\nu\right]\left(\Lambda(u) - \Lambda(t)\right)E\left[V_i(0,u) - V_i(0,t)\right] \\ &= \mathrm{Var}\left(V_i(t,u)\right) \\ &= \mathrm{Var}\left(\sum_{j=1}^{N_i(t,u)} \nu_j\right) \\ &= \left(E[N_1(t,u)]\mathrm{Var}(\nu) + E[\nu]^2\mathrm{Var}(N_1(t,u))\right)\end{aligned}$$

$$= E\left[\nu^2\right](\Lambda(u) - \Lambda(t))$$
$$\leq E\left[\nu^2\right](\Lambda(u) - \Lambda(t)).$$

Now consider the second expectation in ($\star$). We will show that as $n \to \infty$, this expectation is bounded above by 0. Observe that,

$$\begin{aligned}
&E\left[(R_u - R_t)^2\right] \\
&= E\left[R_u^2\right] + E\left[R_t^2\right] - 2E\left[R_u R_t\right] \\
&\leq E\left[\left(-V_i(l_n(u), u) + \frac{u - l_n(u)}{\delta_n} V_i(l_n(u), l_n(u) + \delta_n)\right)^2\right] \\
&\qquad + E\left[\left(-V_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n} V_i(l_n(t), l_n(t) + \delta_n)\right)^2\right].
\end{aligned}$$

Since $l_n(s) \to s$ and $\delta_n \to 0$ as $n \to \infty$, this bound converges to 0. The third expectation in ($\star$) converges to 0 in a similar way.

$$\begin{aligned}
&2E\left[(R_u - R_t)\left((V_i(0,u) - E\left[V(u)\right]) - (V_i(0,t) - E\left[V(t)\right])\right)\right] \\
&= 2E\left[R_u\left(V_i(0,u) - E\left[V(u)\right]\right)\right] - 2E\left[R_u\left(V_i(0,t) - E\left[V(t)\right]\right)\right] \\
&\qquad - 2E\left[R_t\left(V_i(0,u) - E\left[V(u)\right]\right)\right] + 2E\left[R_t\left(V_i(0,t) - E\left[V(t)\right]\right)\right] \\
&= 2E\left[\left(-V_i(l_n(u), u) + \frac{u - l_n(u)}{\delta_n} V_i(l_n(u), l_n(u) + \delta_n)\right)\left(V_i(0,u) - E\left[V(u)\right]\right)\right] \\
&\qquad - 2E\left[\left(-V_i(l_n(u), u) + \frac{u - l_n(u)}{\delta_n} V_i(l_n(u), l_n(u) + \delta_n)\right)\left(V_i(0,t) - E\left[V(t)\right]\right)\right] \\
&\qquad - 2E\left[\left(-V_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n} V_i(l_n(t), l_n(t) + \delta_n)\right)\left(V_i(0,u) - E\left[V(u)\right]\right)\right] \\
&\qquad + 2E\left[\left(-V_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n} V_i(l_n(t), l_n(t) + \delta_n)\right)\left(V_i(0,t) - E\left[V(t)\right]\right)\right].
\end{aligned}$$

Since $l_n(s) \to s$ and $\delta_n \to 0$ as $n \to \infty$, these terms go to 0 in the limit. Now, to prove (2), we note that the expectations are over nonoverlapping intervals. Therefore, by independent increments property and the proof of (1), we have

$$\begin{aligned}
&E\left[\left(\left(\bar{V}_i(u) - E\left[V(u)\right]\right) - \left(\bar{V}_i(t) - E\left[V(t)\right]\right)\right)^2\right. \\
&\qquad\qquad \left.\left(\left(\bar{V}_i(t) - E\left[V(t)\right]\right) - \left(\bar{V}_i(s) - E\left[V(s)\right]\right)\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\left[\left(\left(\bar{V}_i(u) - E\left[V(u)\right]\right) - \left(\bar{V}_i(t) - E\left[V(t)\right]\right)\right)^2\right] \\
&\qquad\qquad\qquad E\left[\left(\left(\bar{V}_i(t) - E\left[V(t)\right]\right) - \left(\bar{V}_i(s) - E\left[V(s)\right]\right)\right)^2\right] \\
&\leq E\left[\nu^2\right]\left(\Lambda(u) - \Lambda(t)\right) E\left[\nu^2\right]\left(\Lambda(t) - \Lambda(s)\right) \\
&\leq E\left[\nu^2\right]^2 \left(\Lambda(u) - \Lambda(s)\right)^2,
\end{aligned}
$$

where the inequality holds in the limit as $n \to \infty$. ■

The two conditions from Theorem 1.6.1 [44] complete the necessary lemmas to prove a functional central limit theorem.

Theorem 2.2.4 (FCLT) *If $\delta_n = o(n^{-1/4})$, $(\lambda(t) : t \geq 0)$ is Lipschitz continuous in the neighborhood of t with Lipschitz constant K, and $E\left[\nu^2\right] < \infty$, then $\sqrt{n}\left(V_n(t) - E[V(t)]\right) \Rightarrow (Z(t) : t \geq 0)$, where $(Z(t) : t \geq 0)$ is a Gaussian process with mean 0 and covariance function $(\rho(s,t) : s, t \geq 0) := Var(V(s \wedge t))$.*

Proof Lemmas 2.2.2 and 2.2.3 show all the necessary conditions for Theorem 1.6.1, proving the claim. ■

Notice that Theorem 2.2.4 requires a specific order of convergence on δ_n and Lipschitz continuity on $(\lambda(t) : t \geq 0)$. Lipschitz continuity can be removed in this result but would in turn require a stronger condition on δ_n, specifically $\delta_n = o(n^{-1/2})$. However, we also notice that the condition on δ_n results from an upper bound. This condition can be relaxed to $\delta_n \to 0$ as $n \to \infty$ as seen in the following result.

Theorem 2.2.5 (CLT Relaxed Conditions) *For $t \in [0, T]$, if $\delta_n \to 0$ as $n \to \infty$, then $\sqrt{n}\left(V_n(t) - E\left[V(t)\right]\right) \Rightarrow \mathcal{N}\left(0, Var(V(t))\right)$.*

Proof Recall the estimator

$$
V_n(t) = \frac{1}{n}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n)
$$

for the offered load of a marked point process, observed at fixed intervals of width $\delta_n > 0$. Recall that

$$V_n(t) = \frac{1}{n}\sum_{i=1}^{n} V_i(0,t) + R_n$$

where

$$R_n = -\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t),t) + \frac{t-l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t)+\delta_n).$$

From the standard CLT, we see that

$$\tilde{V}_n(t) := \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} V_i(0,t) - E\left[V(t)\right]\right) \Rightarrow \mathcal{N}\left(0, \mathrm{Var}(V(t))\right).$$

Recall that if $\tilde{V}_n(t) \Rightarrow \mathcal{N}\left(0, \mathrm{Var}(V(t))\right)$ and $E\left[R_n^2\right] \to 0$ as $n \to \infty$, then $\tilde{V}_n(t) + R_n \Rightarrow \mathcal{N}\left(0, \mathrm{Var}(V(t))\right)$. We will show that $E\left[R_n^2\right] \to 0$ as $n \to \infty$ if $\delta_n \to 0$.

$$E\left[R_n^2\right] = \mathrm{Var}\left(R_n\right) + E\left[R_n\right]^2$$

Considering first the squared expectation, we have,

$$\begin{aligned}
&E\left[R_n\right]^2 \\
&= E\left[-\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t),t) + \frac{t-l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t)+\delta_n)\right]^2 \\
&= \left(E\left[-\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t),t)\right] + E\left[\frac{t-l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} V_i(l_n(t), l_n(t)+\delta_n)\right]\right)^2 \\
&= \left(-E\left[V_1(l_n(t),t)\right] + \frac{t-l_n(t)}{\delta_n}E\left[V_1(l_n(t), l_n(t)+\delta_n)\right]\right)^2 \\
&= \left(-E\left[\nu\right]\left(\Lambda(t) - \Lambda(l_n(t))\right) + \frac{t-l_n(t)}{\delta_n}E\left[\nu\right]\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))\right)\right)^2 \\
&= E\left[\nu\right]^2\Bigg(\left(\Lambda(t) - \Lambda(l_n(t))\right)^2 - 2\left(\Lambda(t) - \Lambda(l_n(t))\right)\frac{t-l_n(t)}{\delta_n}\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))\right) \\
&\qquad + \left(\frac{t-l_n(t)}{\delta_n}\right)^2\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))\right)^2\Bigg). \qquad (2.2.5.1)
\end{aligned}$$

Now, by the Mean Value Theorem, there exists $\theta_n \in [l_n(t), t]$ and $\zeta_n \in [l_n(t), l_n(t)+\delta_n]$ such that,

$$(2.2.5.1) = E\left[\nu\right]^2\Bigg(\left(\lambda(\theta_n)(t-l_n(t))\right)^2 - 2\lambda(\theta_n)(t-l_n(t))\frac{t-l_n(t)}{\delta_n}\lambda(\zeta_n)\delta_n$$

$$
\begin{aligned}
&\qquad\qquad + \left(\frac{t-l_n(t)}{\delta_n}\right)^2 (\lambda(\zeta_n)\delta_n)^2 \Bigg) \\
&= E\left[\nu\right]^2 \left(\lambda(\theta_n)^2(t-l_n(t))^2 - 2\lambda(\theta_n)(t-l_n(t))^2\lambda(\zeta_n)\right. \\
&\qquad\qquad \left. +(t-l_n(t))^2\lambda(\zeta_n)^2\right) \\
&= E\left[\nu\right]^2 (t-l_n(t))^2 \left(\lambda(\theta_n)^2 - 2\lambda(\theta_n)\lambda(\zeta_n) + \lambda(\zeta_n)^2\right) \\
&= E\left[\nu\right]^2 (t-l_n(t))^2 \left(\lambda(\theta_n) - \lambda(\zeta_n)\right)^2 . \qquad (2.2.5.2)
\end{aligned}
$$

Since $|\theta_n - \zeta_n| < \delta_n$ and $t - l_n(t) < \delta_n$, and $\delta_n \to 0$ as $n \to \infty$, (2.2.5.2)$\to 0$ as $n \to \infty$. Now, we consider the variance of R_n as follows.

$$
\begin{aligned}
&\mathrm{Var}\left(R_n\right) \\
&\quad = \mathrm{Var}\left(-\frac{1}{n}\sum_{i=1}^n V_i(l_n(t),t) + \frac{t-l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^n V_i(l_n(t), l_n(t)+\delta_n)\right) \\
&\quad = \mathrm{Var}\left(\left(-\frac{1}{n} + \frac{t-l_n(t)}{\delta_n}\frac{1}{n}\right)\sum_{i=1}^n V_i(l_n(t),t) + \frac{t-l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^n V_i(t, l_n(t)+\delta_n)\right) \\
&\quad = \frac{1}{n}\left(1 - \frac{t-l_n(t)}{\delta_n}\right)^2 E\left[\nu^2\right]\left(\Lambda(t) - \Lambda(l_n(t))\right) \\
&\qquad\qquad + \frac{1}{n}\left(\frac{t-l_n(t)}{\delta_n}\right)^2 E\left[\nu^2\right]\left(\Lambda(l_n(t)+\delta_n) - \Lambda(t)\right) \\
&\quad \le \frac{1}{n}E\left[\nu^2\right]\left(\Lambda(t) - \Lambda(l_n(t))\right) + \frac{1}{n}E\left[\nu^2\right]\left(\Lambda(l_n(t)+\delta_n) - \Lambda(t)\right), \qquad (2.2.5.3)
\end{aligned}
$$

where the inequality holds because $l_n(t) \le t < l_n(t) + \delta_n$. Since $\delta_n \to 0$ as $n \to \infty$, it is clear that (2.2.5.3)$\to 0$ as $n \to \infty$. Therefore, $E\left[R_n^2\right] \to 0$ as $n \to \infty$, proving the claim. ∎

This result removes the Lipschitz continuity and the specific rate of convergence of δ_n. Because $V_n(t)$ is well-structured, in that it is monotone increasing and continuous on a compact interval, the uniform convergence results will follow straightforwardly.

2.2.2 Law of Iterated Logarithm

The FCLT provides a rate of convergence for $V_n(t)$ which can be used to compute confidence intervals. We also identified the asymptotic covariance of the stochastic process. We now show a law of iterated logarithm (LIL) to describe the magnitude of the fluctuations of $V_n(t)$ away from the mean and provide a sense for how large n should be before the estimator is "accurate."

Theorem 2.2.6 (LIL) *Fix t and let $\tilde{V}_n(t) = nV_n(t)$. Then,*

$$\limsup_{n\to\infty} \frac{V_n^*(t)}{\sqrt{2\log\log n}} = 1$$

where $V_n^*(t) = \frac{\tilde{V}_n(t)-E[\tilde{V}_n(t)]}{\sqrt{\mathrm{Var}(\tilde{V}_n(t))}}$.

Proof Let $D := \left\{\tilde{V}_n(t) > E\left[\tilde{V}_n(t)\right] + \phi\sqrt{2\mathrm{Var}(\tilde{V}_n(t))\log\log n}\right\}$. The relation $\limsup_{n\to\infty} \frac{V_n^*(t)}{\sqrt{2\log\log n}} = 1$ is equivalent to showing that for $\phi > 1$, with probability one, the event D occurs only finitely many times, and for $\phi < 1$, with probability one, the event D occurs infinitely many times. Observe that,

(i) There exists a constant $c > 0$ that does not depend on n, such that

$$P\left(\tilde{V}_n(t) > E\left[\tilde{V}_n(t)\right]\right) > c.$$

Since a CLT holds for $V_n(t)$ (Lemma 2.2.1), inequality (i) clearly holds.

(ii) Let x be fixed, and let A be the event that for at least one k (with $k \le n$), $\tilde{V}_k(t) - E[\tilde{V}_k(t)] > x$. Then, $P(A) \le \frac{1}{c}P(\tilde{V}_n(t) - E[\tilde{V}_n(t)] > x)$.

To show (ii) is true, let A_v be the event that $\tilde{V}_k(t) - E[\tilde{V}_k(t)] > x$ holds for $k = v$, but not for $k = 1, ..., v-1$. So, $P(A) = P(A_1) + P(A_2) + \cdots + P(A_n)$. Let U_v be the event that $\tilde{V}_n(t) - \tilde{V}_v(t) > E[\tilde{V}_n(t) - \tilde{V}_v(t)]$. If both A_v and U_v occur, then $\tilde{V}_n(t) = \tilde{V}_v(t) + \tilde{V}_{n-v}(t) > E[\tilde{V}_v(t)] + x + E[\tilde{V}_n(t)] - E[\tilde{V}_v(t)] = E[\tilde{V}_n(t)] + x$. It follows that,

$$\begin{aligned} P(V_n(t) - \mu_{V_n} > x) &\ge P(A_1U_1) + P(A_2U_2) + \cdots + P(A_n) \\ &= P(A_1)P(U_1) + P(A_2)P(U_2) + \cdots + P(A_n). \qquad (\star) \end{aligned}$$

By inequality (i),

$$(\star) = c\sum_{v=1}^{n-1} P(A_v) + P(A_n)$$

$$\geq c \sum_{v=1}^{n} P(A_v)$$
$$= cP(A),$$

thereby, proving (ii). Now, let $\phi > 1$. Let $1 \leq \gamma < \phi$. Let n_r be the integer nearest to γ^r. Let B_r be the event that $\tilde{V}_n(t) - E[\tilde{V}_n(t)] > \phi\sqrt{2\mathrm{Var}(\tilde{V}_{n_r}(t)) \log\log n_r}$ for at least one n in $n_r \leq n < n_{r+1}$. Using inequality (ii), we have,

$$P(B_r) \leq \frac{1}{c} P\left(\tilde{V}_{n_{r+1}}(t) - E[\tilde{V}_{n_{r+1}}(t)] > \phi\sqrt{2\mathrm{Var}(\tilde{V}_{n_r}(t)) \log\log n_r}\right).$$

The event D can only occur infinitely many times if infinitely many B_r occur. Note that $\tilde{V}_n(t)$ is a sum of n random variables with finite variance, and $V_n^*(t) \Rightarrow N(0,1)$ for each $t \in [0,T]$ by Lemma 2.2.1. Then,

$$\begin{aligned} P(B_r) &\leq \frac{1}{c} P\left(\tilde{V}_{n_{r+1}}(t) - E[\tilde{V}_{n_{r+1}}(t)] > \phi\sqrt{2\mathrm{Var}(\tilde{V}_{n_r}(t)) \log\log n_r}\right) \\ &= \frac{1}{c} P\left(V^*_{n_{r+1}}(t) > \phi\sqrt{2\frac{\mathrm{Var}(\tilde{V}_{n_r}(t))}{\mathrm{Var}(\tilde{V}_{n_{r+1}}(t))} \log\log n_r}\right). \end{aligned} \tag{2.2.6.1}$$

Now, $\frac{\mathrm{Var}(\tilde{V}_{n_r}(t))}{\mathrm{Var}(\tilde{V}_{n_{r+1}}(t))} \approx \frac{n_r}{n_{r+1}}$. Also, note that since n_r is the integer nearest to γ^r, $n_r \approx \gamma^r$ and $n_{r+1} \approx \gamma^{r+1}$. So, $\frac{n_{r+1}}{n_r} \approx \gamma < \phi$. Continuing, we have,

$$\begin{aligned} (2.2.6.1) &\approx \frac{1}{c} P\left(V^*_{n_{r+1}}(t) > \phi\sqrt{2\frac{1}{\gamma} \log\log n_r}\right) \\ &\leq \frac{1}{c} P\left(V^*_{n_{r+1}}(t) > \phi\sqrt{2\frac{1}{\phi} \log\log n_r}\right) \\ &= \frac{1}{c} P\left(V^*_{n_{r+1}}(t) > \sqrt{2\phi \log\log n_r}\right). \end{aligned} \tag{2.2.6.2}$$

Using Lemma 2.2.1 , we have

$$\begin{aligned} (2.2.6.2) &\approx \frac{1}{c} \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{2\phi \log\log n_r}} \exp\left\{-1/2(\sqrt{2\phi \log\log n_r})^2\right\} \\ &\leq \frac{1}{c} \frac{1}{(\log n_r)^\phi} \\ &\approx \frac{1}{c} \frac{1}{(\log \gamma^r)^\phi} \\ &= \frac{1}{c} \frac{1}{(r \log \gamma)^\phi} \end{aligned}$$

$$\leq \frac{1}{c}\frac{1}{r^{\phi}}.$$

Since $\phi > 1$, $\sum_r B_r$ converges. By the first Borel-Cantelli Lemma [45], it follows that $P(\lim_{r\to\infty} B_r) = 0$. Now we will show that for $\phi < 1$, with probability one, the event D occurs infinitely many times. By the second Borel-Cantelli Lemma [45], we need to define mutually independent events such that if their sum diverges, then with probability one, infinitely many events occur.

Let $\frac{\gamma-1}{\gamma} > \eta > \phi$. Let $n_r = \gamma^r$. Let $\{A_r : r \geq 0\}$ be a sequence of independent events, where

$$\left(\tilde{V}_{n_r}(t) - \tilde{V}_{n-_{r-1}}(t)\right) - E[\tilde{V}_{n_r}(t) - \tilde{V}_{n-_{r-1}}(t)] > \eta\sqrt{2\mathrm{Var}(V_{n_r}(t))\log\log n_r}.$$

We need to show that $\sum_r P(A_r)$ diverges. First, note that $\frac{\mathrm{Var}(\tilde{V}_{n_r}(t))}{\mathrm{Var}(\tilde{V}_{n_r}(t)-\tilde{V}_{n_{r-1}}(t))} \approx \frac{n_r}{n_r - n_{r-1}}$.

$$\begin{aligned}
&P(A_r)\\
&= P\left(\left(\tilde{V}_{n_r}(t) - \tilde{V}_{n_{r-1}}(t)\right) - E[\tilde{V}_{n_r}(t) - \tilde{V}_{n_{r-1}}(t)] > \eta\sqrt{2\mathrm{Var}(V_{n_r}(t))\log\log n_r}\right)\\
&= P\left(\left(\tilde{V}_{n_r}(t) - \tilde{V}_{n-_{r-1}}(t)\right)^* > \eta\sqrt{2\frac{\mathrm{Var}(V_{n_r}(t))}{\mathrm{Var}(\tilde{V}_{n_r}(t) - \tilde{V}_{n_{r-1}}(t))}\log\log n_r}\right)\\
&\approx P\left(\left(\tilde{V}_{n_r}(t) - \tilde{V}_{n_{r-1}}(t)\right)^* > \eta\sqrt{2\frac{n_r}{n_r - n_{r-1}}\log\log n_r}\right)\\
&\geq P\left(\left(\tilde{V}_{n_r}(t) - \tilde{V}_{n_{r-1}}(t)\right)^* > \eta\sqrt{2\frac{1}{\eta}\log\log n_r}\right)\\
&= P\left(\left(\tilde{V}_{n_r}(t) - \tilde{V}_{n_{r-1}}(t)\right)^* > \sqrt{2\eta\log\log n_r}\right). \qquad (2.2.6.3)
\end{aligned}$$

And by Lemma 2.2.1,

$$\begin{aligned}
(2.2.6.3) &\approx \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\eta\log\log n_r}}\exp\left\{-1/2(\sqrt{2\eta\log\log n_r})^2\right\}\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\eta\log\log n_r}}\frac{1}{(\log n_r)^{\eta}}\\
&> \frac{1}{\sqrt{2\pi}}\frac{1}{2\eta\log\log n_r}\frac{1}{(\log n_r)^{\eta}}\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2\eta\log\log n_r}\frac{1}{(r\log\gamma)^{\eta}}
\end{aligned}$$

$$> \frac{1}{\sqrt{2\pi}} \frac{1}{r} \frac{1}{2}.$$

So, $\sum_r P(A) > \frac{1}{2\sqrt{2\pi}} \sum_r \frac{1}{r}$ diverges. ■

Showing an LIL allows control on the fluctuations from the mean. The LIL can be leveraged to obtain limits on the entire sequence of $V_n(t)$ as well as probability inequalities in confidence intervals and statistical tests [46].

2.3 δ_n Performance Analysis

We may be interested in understanding how large the intervals should be based on our sample size. We choose to analyze the mean-squared error (MSE) and show that there exists an asymptotically optimal δ_n as $n \to \infty$.

Theorem 2.3.1 (Optimal δ_n to minimize MSE) *If λ is continuously differentiable in a neighborhood of t and λ is Lipschitz continuous with Lipschitz constant K in the neighborhood of t, then for n sufficiently large, a bound on the mean-squared error of $V_n(t)$ is minimized by taking $\delta_n = \delta_n^*$ where*

$$\delta_n^* = \left(\frac{E[\nu_1^2]\lambda(\zeta_n)(t - l_n(t))^2}{4nE[\nu_1]^2 K^2} \right)^{1/5}.$$

Proof Recall the mean-squared error,

$$E\left[(V_n(t) - E[V(t)])^2\right] = \mathrm{Var}(V_n(t)) + (E[V_n(t)] - E[V(t)])^2.$$

We first consider the variance term. Note that $V_i(t)$ has independent increments, so the variance of the sums is the sum of the variances.

$$\begin{aligned}
\mathrm{Var}(V_n(t)) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} V_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n} \frac{1}{n} \sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \right) \\
&= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} V_i(0, l_n(t)) \right) + \mathrm{Var}\left(\frac{t - l_n(t)}{\delta_n} \frac{1}{n} \sum_{i=1}^{n} V_i(l_n(t), l_n(t) + \delta_n) \right) \\
&= \frac{1}{n}\mathrm{Var}\left(V_i(0, l_n(t))\right) + \left(\frac{t - l_n(t)}{\delta_n} \right)^2 \frac{1}{n} \mathrm{Var}\left(V_i(l_n(t), l_n(t) + \delta_n)\right).
\end{aligned}$$

Note that

$$\begin{aligned}\text{Var}(V_i(0,l_n(t))) &= E[N(l_n(t))]\text{Var}(\nu_1) + E[\nu]^2\text{Var}(N(l_n(t)) \\ &= \Lambda(l_n(t))\text{Var}(\nu_1) + E[\nu_1]^2\Lambda(l_n(t)) \\ &= \Lambda(l_n(t))E[\nu_1^2].\end{aligned}$$

So,

$$\text{Var}(V_n(t)) = \frac{1}{n}\Lambda(l_n(t))E[\nu^2] + \left(\frac{t-l_n(t)}{\delta_n}\right)^2 \frac{1}{n}\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t)\right) E[\nu_1^2].$$

The bias term is bounded in the following way, with $\zeta_n \in [l_n(t), l_n(t)+\delta_n]$ and $\theta_n \in [l_n(t), t]$.

$$\begin{aligned}&(E[V_n(t)] - E[V(t)])^2 \\ &= \left(\Lambda(l_n(t))E[\nu_1] - \Lambda(t)E[\nu_1] + \left(\frac{t-l_n(t)}{\delta_n}\right) E[\nu_1]\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))\right)\right)^2 \\ &= \left(\left(\frac{t-l_n(t)}{\delta_n}\right) E[\nu_1]\lambda(\zeta_n)\delta_n - E[\nu_1]\lambda(\theta_n)(t-l_n(t))\right)^2 \\ &= \left((t-l_n(t)E\left[\nu_1\right]\lambda(\zeta_n) - E\left[\nu_1\right]\lambda(\theta_n)(t-l_n(t))\right)^2 \\ &= (t-l_n(t))^2E[\nu_1]^2\left(\lambda(\zeta_n)-\lambda(\theta_n)\right)^2. \qquad (\star)\end{aligned}$$

Recall that K is the Lipschitz constant for $(\lambda(t) : t \geq 0)$, so that $(\lambda(\zeta_n) - \lambda(\theta_n))^2 \leq K^2|\zeta_n - \theta_n|^2 \leq K^2\delta_n^2$. Also, since $l_n(t) \leq t \leq l_n(t)+\delta_n$, $(t-l_n(t))^2 \leq \delta_n^2$, such that

$$(\star) \leq E[\nu_1]^2K^2\delta_n^4.$$

The mean-squared error is therefore bounded above by

$$\begin{aligned}&\frac{1}{n}\Lambda(l_n(t))E[\nu_1^2] + \left(\frac{t-l_n(t)}{\delta_n}\right)^2 \frac{1}{n}\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t)\right) E[\nu_1^2] + E[\nu_1]^2K^2\delta_n^3 \\ &\leq \frac{1}{n}\Lambda(l_n(t))E[\nu_1^2] + \frac{1}{n}E[\nu_1^2]\lambda(\zeta_n)\delta_n + E[\nu_1]^2K^2\delta_n^4,\end{aligned}$$

which is minimized at

$$0 = \frac{E[\nu_1^2]\lambda(\zeta_n)}{n} + 4E[\nu_1]^2K^2\delta_n^3$$

$$\implies \delta_n^* = \left(\frac{E[\nu_1^2]\lambda(\zeta_n)}{4nE[\nu_1]^2 K^2} \right)^{1/3}.$$

Choosing δ_n to be of the order $n^{-1/3}$ asymptotically minimizes the bound on the mean-squared error. ■

Through proofs of a FSSLN, FCLT, and LIL, we demonstrated the consistency, rate of convergence, and asymptotic covariance of $V_n(t)$ as a nonparametric estimator of $(V(t) : t \geq 0)$. We also considered the choice of δ_n to asymptotically minimize a bound on the mean squared error. Through these analyses, we can see that $V_n(t)$ is an appropriate nonparametric estimator for large n.

3. $M_t/G/\infty$ MEAN BUSY SERVER ESTIMATOR

Recall the number of busy servers in an $M_t/G/\infty$ queue at time t, $(Q(t) : t \geq 0)$, where $Q(s) := \sum_{k=1}^{N(0,s)} \mathbb{1}\{t \leq \tau_k + \nu_k\}$, where $N(0,s)$ is the NHPP $(N(t) : t \geq 0)$ with rate function $(\lambda(t) : t \geq 0)$ at time s, τ_k is the arrival time associated with individual k of the NHPP, and $\{\nu_k\}_{k=1}^{\infty}$ are i.i.d random variables from common distribution function $F(t)$ representing the service requirement associated with individual k of the queue.

Suppose we are interested in estimating the mean number of busy servers in an $M_t/G/\infty$ queue at time $t \geq 0$ based on either fully-observed samples of an $M_t/G/\infty$ queue or based on direct observations of the busy servers at fixed intervals. We consider two nonparametric estimators and establish asymptotic consistency and rates of convergence.

3.1 Notation and Key Insights

Before demonstrating the consistency and rates of convergence for the nonparametric estimators, we review key insights that will be used throughout the following proofs. We will use the fact that the arrival and service times of the $M_t/G/\infty$ queue generate a Poisson random measure in order to exploit the independent increments property of Poisson random measures [47].

First, recall that $E\left[Q(t)\right] = m(t) := \int_0^t \lambda(s)\bar{F}(t-s)ds$, where $\bar{F}(t) := 1-P(\nu \leq t)$ [47]. Note that $Q(t)$ is the number of individuals arriving before time t that are still in service at time t. Consider plotting points (τ_k, ν_k), then the number of points above the line $\nu = t - \tau$ and to the left of the line $\tau = t$ represents $Q(t)$ and is Poisson distributed with mean $m(t)$ (Figure 3.1) [47].

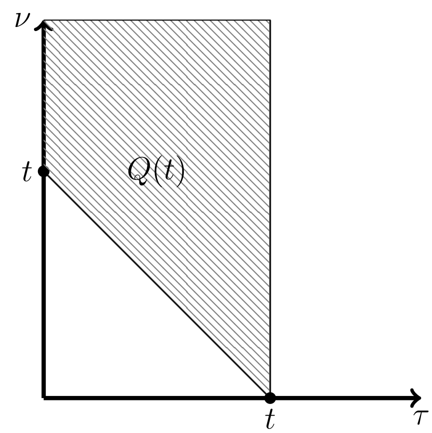

Fig. 3.1. Poisson Random Measure for $Q(t)$

Now, note that the number of points (τ_k, ν_k) in disjoint areas are independent Poisson random variables because independently splitting Poisson processes produces independent Poisson processes [47]. Therefore, we can write $Q(t)$ as the sum of independent Poisson random variables X, Y, Z as in Figure 3.2,

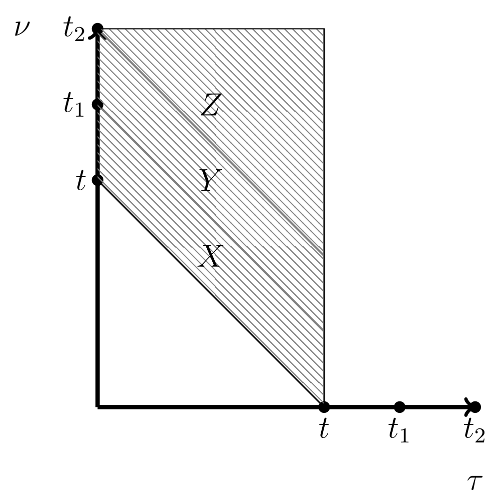

Fig. 3.2. $Q(t)$ Constructed of Disjoint Areas

where

$$E[X] = \int_0^t \lambda(u)\left(F(t_1-u) - F(t-u)\right)du.$$
$$E[Y] = \int_0^t \lambda(u)\left(F(t_2-u) - F(t_1-u)\right)du,$$
$$E[Z] = \int_0^t \lambda(u)\left(1 - F(t_2-u)\right)du.$$

Throughout the following analyses, there are many instances in which we analyze $(Q(t_1), Q(t_2), ..., Q(t_k))$ for any $0 \leq t_1 < t_2 < \cdots < t_k$. Consider $k = 3$, then Figure 3.3 demonstrates how to construct $Q(t_1), Q(t_2)$, and $Q(t_3)$ in terms of independent Poisson random variables, where

$$Q(t_1) = X^1_{1,2} + X^1_{2,3} + X^1_{3,4}$$
$$Q(t_2) = X^1_{2,3} + X^1_{3,4} + X^2_{2,3} + X^2_{3,4}$$
$$Q(t_3) = X^1_{3,4} + X^2_{3,4} + X^3_{3,4},$$

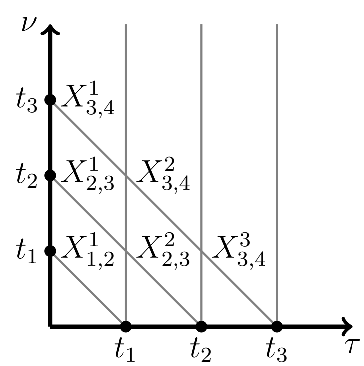

Fig. 3.3. Poisson Random Measure Example for $Q(t_1), Q(t_2), Q(t_3)$

with

$$E\left[X^l_{m,m+1}\right] = \int_{t_{l-1}}^{t_l} \lambda(u)\left(F(t_{m+1}-u) - F(t_m-u)\right)du.$$

Therefore, for any $j = \{1, ..., k\}$, $Q(t_j) = \sum_{l=1}^{j}\sum_{m=j}^{k} X^l_{m,m+1}$. We use this concept throughout the following analyses to exploit the independent increments property of Poisson random measures.

3.2 Nonparametric Omniscient Estimator

Suppose we observe both the arrival epochs and service requirements, (τ_{ik}, ν_{ik}), of individual k of observation $i = 1, ..., n$ of an $M_t/G/\infty$ queue. Then the natural estimator at $t \geq 0$ of the mean number of busy servers is the random variable

$$\hat{m}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{N_i(0,t)} \mathbb{1}\{t \leq \nu_{ik} + \tau_{ik}\}.$$

We conduct statistical inference by showing asymptotic consistency and rate of convergence of this estimator.

3.2.1 Asymptotic Consistency

We prove a functional strong law of large numbers (FSLLN) demonstrating the asymptotic consistency of $\hat{m}_n(t)$.

Theorem 3.2.1 (FSLLN) *For $\lambda(t) = \lambda \; \forall \; t \in [0, T]$, $\sup_{t \in [0,T]} |\hat{m}_n(t) - m(t)| \to 0$ a.s. as $n \to \infty$.*

Proof Note that $Q_i(t) = \sum_{k=1}^{N_i(0,t)} \mathbb{1}\{t \leq \nu_{ik} + \tau_{ik}\}$ is Poisson distributed with mean $m(t) = \int_0^t \lambda(s) \bar{F}(t-s) ds$. Applying the strong law of large numbers to $\hat{m}_n(t)$ gives a strong law for each t. We will use the Borel-Cantelli Lemma to show that the event

$$\left\{ \left| \sum_{i=1}^{n} Q_i(t) - nm(t) \right| \geq n\epsilon \right\}$$

occurs only finitely often. Recall Chebyshev's Inequality $P(|X| > a) \leq \frac{E[|X|^p]}{a^p}$. It follows that,

$$P\left(\left| \sum_{i=1}^{n} Q_i(t) - nm(t) \right| \geq n\epsilon \right) \leq \frac{1}{(n\epsilon)^4} E\left[\left(\sum_{i=1}^{n} Q_i(t) - nm(t) \right)^4 \right]. \qquad (3.2.1.1)$$

Using the fourth centralized moment of a Poisson random variable, it follows that

$$(3.2.1.1) = \frac{1}{(n\epsilon)^4} \left(nm(t)(1 + 3nm(t)) \right)$$

$$= \frac{m(t)}{n^3\epsilon^4} + \frac{3m(t)^2}{n^2\epsilon^4}.$$

Thus,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} Q_i(t) - nm(t)\right| \geq n\epsilon\right) \leq \sum_{n=1}^{\infty} \frac{m(t)}{n^3\epsilon^4} + \frac{3m(t)^2}{n^2\epsilon^4}$$
$$< \infty.$$

Therefore, by the first Borel-Cantelli lemma [45], $P\left(\left|\sum_{i=1}^{n} Q_i(t) - nm(t)\right| \geq n\epsilon \quad i.o.\right) = 0$. It is left to establish the uniform part of the result. Through pointwise convergence, we know that for each $t \in [0, T]$ and $\epsilon > 0$, there exists an $\bar{n}(\epsilon)$ such that $|\hat{m}_n(t) - m(t)| < \epsilon$ for $n \geq \bar{n}(\epsilon)$. Now, consider integers $k > l \geq \bar{n}(\epsilon)$, then,

$$\begin{aligned} |\hat{m}_k(t) - \hat{m}_l(t)| &= \left|\left(\frac{1}{k} - \frac{1}{l}\right)\sum_{i=1}^{l} Q_i(t) + \frac{1}{k}\sum_{i=l+1}^{k} Q_i(t)\right| \\ &\leq \frac{k-l}{kl}\sum_{i=1}^{l} Q_i(t) + \frac{1}{k}\sum_{i=l+1}^{k} Q_i(t), \end{aligned} \tag{3.2.1.2}$$

where the inequality holds by the triangle inequality. (3.2.1.2) is bounded below by 0. Observe that by pointwise convergence, $\frac{1}{l}\sum_{i=1}^{l} Q_i(t) \leq m(t) + \epsilon$. Considering the supremum over $t \in [0, T]$ for the first term in (2), we have,

$$\sup_{t\in[0,T]} \frac{k-l}{kl}\sum_{i=1}^{l} Q_i(t) \leq \left(1 - \frac{l}{k}\right) \sup_{t\in[0,T]} \left(m(t) + \epsilon\right). \tag{3.2.1.3}$$

For an $M/G/\infty$ queue, $m(t)$ is a nondecreasing, continuous function, converging to $\frac{\lambda}{E[\nu]}$ [48]. Now, choose a subsequence such that $l = \lfloor k(1-\epsilon)\rfloor$, then (3.2.1.3)$\leq$ $\frac{k\epsilon}{k}\frac{\lambda}{E[\nu]} \approx \epsilon\frac{\lambda}{E[\nu]}$. Next, recall that $Q_i(t) \leq N_i(t)$ where $(N(t) : t \geq 0)$ is the cumulative number of arrivals by time t, which is nondecreasing. Therefore, the second term in (3.2.1.2) is bounded above as follows:

$$\begin{aligned} \sup_{t\in[0,T]} \frac{1}{k}\sum_{i=l+1}^{k} Q_i(t) &\leq \frac{1}{k}\sum_{i=l+1}^{k} \sup_{t\in[0,T]} Q_i(t) \\ &\leq \frac{1}{k}\sum_{i=l+1}^{k} \sup_{t\in[0,T]} N_i(t) \end{aligned}$$

$$\leq \frac{1}{k}\sum_{i=l+1}^{k} N_i(T). \tag{3.2.1.4}$$

Since $N_i(T) < \infty \ \forall\ i$, $l = \lfloor k(1-\epsilon) \rfloor$, then as $k \to \infty$, (3.2.1.4)$\to 0$. Therefore,

$$\sup_{t\in[0,T]} |\hat{m}_k(t) - \hat{m}_l(t)| \leq \epsilon \frac{\lambda}{E[\nu]} + \epsilon. \tag{3.2.1.5}$$

Since, ϵ is arbitrary, the subsequence $|\hat{m}_k(t) - \hat{m}_l(t)|$ with $l = \lfloor k(1-\epsilon) \rfloor$ is uniformly Cauchy. Next, since the subsequence with $k > l \geq \bar{n}(\epsilon)$ has a further subsequence that is uniformly Cauchy,

$$\sup_{t\in[0,T]} |\hat{m}_k(t) - \hat{m}_l(t)| \to 0$$

for all $t \in [0, T]$ as $k, l \to \infty$. Since a uniformly Cauchy sequence of functions is uniformly convergent, the claim is proved [49]. ∎

Through a FSLLN, we have shown that $\hat{m}_n(t)$ is an asymptotically consistent, and therefore an asymptotically unbiased, estimator for $m(t)$.

3.2.2 Rate of Convergence

We show the rate of convergence and asymptotic covariance of $\hat{m}_n(t)$ by proving an FCLT. We first establish a pointwise CLT.

Lemma 3.2.2 (CLT) *For each $t \in [0, T]$, $\sqrt{n}\,(\hat{m}_n(t) - m(t)) \Rightarrow \mathcal{N}(0, m(t))$ as $n \to \infty$.*

Proof Recall that $Q_i(t)$ is a Poisson random variable with mean $m(t)$. Consider the log moment generating function of $\sqrt{n}\,(\hat{m}_n(t) - m(t))$,

$$\begin{aligned}\Psi_n(u) &= \log E\left[\exp\left\{u\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Q_i(t) - m(t)\right)\right\}\right] \\ &= \log E\left[\exp\left\{\frac{u}{\sqrt{n}}\sum_{i=1}^{n} Q_i(t)\right\}\right] - u\sqrt{n}m(t). \qquad (3.2.2.1)\end{aligned}$$

Recall that $\sum_{i=1}^{n} Q_i(t)$ is Poisson distributed with mean $nm(t)$. Using the MGF of a Poisson random variable, it follows that,

$$(3.2.2.1) = nm(t)(\exp\left\{u/\sqrt{n}\right\} - 1) - u\sqrt{n}m(t). \tag{3.2.2.2}$$

Next, using the Taylor expansion for e^t, we have,

$$\begin{aligned}(3.2.2.2) &= nm(t)\left(\frac{u}{\sqrt{n}} + \frac{u^2}{2n} + O(n^{-3/2})\right) - u\sqrt{n}m(t) \\ &= \frac{u^2}{2}m(t) + O\left(n^{-3/2}\right).\end{aligned}$$

It follows that $\Psi_n(u) \to \frac{u^2}{2}m(t)$ as $n \to \infty$, which is the log MGF of a Gaussian random variable with mean 0 and variance $m(t)$. ■

For fixed $t \in [0, T]$, Lemma 3.2.2 shows that the estimator is $O(\frac{1}{\sqrt{n}})$. Next, we show that the stochastic process $(\hat{m}_n(t) : t \geq 0)$ has the same order of convergence to a Gaussian stochastic process. First, we consider the FDD's and then tightness of the estimator.

Lemma 3.2.3 (FDD's) *If $(\lambda(t) : t \geq 0)$ is integrable, then for $0 = t_0 < t_1 < \cdots < t_k \leq T$,*

$$\left(\sqrt{n}\left(\hat{m}_n(t_1) - m(t_1)\right), ..., \sqrt{n}\left(\hat{m}_n(t_k) - m(t_k)\right)\right) \Rightarrow (Z(t_1), ..., Z(t_k))$$

where $(Z(t_1), ..., Z(t_k))$ is a Gaussian vector with mean 0 and covariance matrix $\Sigma = [\sigma_{ij}]$, where $\sigma_{ij} = \int_0^{t_i \wedge t_j} \lambda(u)\bar{F}(t_i \vee t_j - u)du \ \forall \ 1 \leq i, j \leq k$.

Proof Fix $k = 2$. For convenience, let $X = X_{1,2}^{t_1}$, $Y = X_{2,3}^{t_1}$, and $Z = X_{2,3}^{t_2}$. Note that $Q(t_1) = X + Y$ and $Q(t_2) = Y + Z$ (Figure 3.4). Recall the expectations

$$\begin{aligned}E[X] &= \int_0^{t_1} \lambda(u)\left(F(t_2 - u) - F(t_1 - u)\right)du \\ E[Y] &= \int_0^{t_1} \lambda(u)\bar{F}(t_2 - u)du \\ E[Z] &= \int_{t_1}^{t_2} \lambda(u)\bar{F}(t_2 - u)du.\end{aligned}$$

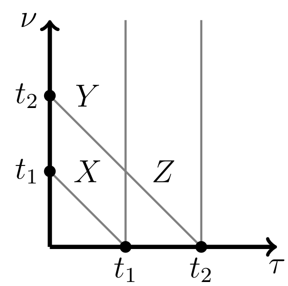

Fig. 3.4. Poisson random measure for $Q(t_1), Q(t_2)$

Next, recall the log MGF of the random vector

$$\left(\sqrt{n}\left(\hat{m}_n(t_1) - m(t_1)\right), \sqrt{n}\left(\hat{m}_n(t_2) - m(t_2)\right)\right),$$

$$\begin{aligned}
\Psi_n(u) &= \log E\left[\exp\left\{\left\langle (u_1, u_2), \left(\sqrt{n}(\hat{m}_n(t_1) - m(t_1)), \sqrt{n}(\hat{m}_n(t_2) - m(t_2))\right)\right\rangle\right\}\right] \\
&= \log E\left[\exp\left\{\sqrt{n}(u_1\hat{m}_n(t_1) + u_2\hat{m}_n(t_2)) - \sqrt{n}(u_1 m(t_1) + u_2 m(t_2))\right\}\right] \\
&= \log E\left[\exp\left\{\sqrt{n}(u_1 m_n(t_1) + u_2 m_n(t_2))\right\}\right] - \sqrt{n}(u_1 m(t_1) + u_2 m(t_2)) \\
&= \log E\left[\exp\left\{\frac{1}{\sqrt{n}}\left(u_1 \sum_{i=1}^{n} Q_i(t_1) + u_2 \sum_{i=1}^{n} Q_i(t_2)\right)\right\}\right] \\
&\qquad - \sqrt{n}(u_t m(t) + u_{t_1} m(t_1)). \qquad (3.2.3.1)
\end{aligned}$$

Recall that $Q_i(\cdot)$ is a Poisson random variable with mean $m(\cdot)$, but $Q_i(t_1)$ and $Q_i(t_2)$ are not independent. However,

$$\begin{aligned}
u_1 \sum_{i=1}^{n} Q_i(t_1) + u_2 \sum_{i=1}^{n} Q_i(t_2) &= u_1 \sum_{i=1}^{n}(X_i + Y_i) + u_2 \sum_{i=1}^{n}(Y_i + Z_i) \\
&= u_1 \sum_{i=1}^{n} X_i + (u_1 + u_2) \sum_{i=1}^{n} Y_i + u_2 \sum_{i=1}^{n} Z_i,
\end{aligned}$$

where X_i, Y_i, Z_i are independent Poisson random variables. Therefore, it follows that

(3.2.3.1)

$$= \log\left(E\left[\exp\left\{\frac{u_1}{\sqrt{n}} \sum_{i=1}^{n} X_i\right\}\right]\left[\exp\left\{\frac{(u_1+u_2)}{\sqrt{n}} \sum_{i=1}^{n} Y_i\right\}\right]\left[\exp\left\{\frac{u_2}{\sqrt{n}} \sum_{i=1}^{n} Z_i\right\}\right]\right)$$

$$
\begin{aligned}
&\quad - \sqrt{n}(u_1 m(t_1) + u_2 m(t_2)) \\
&= \log E\left[\exp\left\{\frac{u_1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right\}\right] + n\log E\left[\exp\left\{\frac{(u_1+u_2)}{\sqrt{n}}\sum_{i=1}^{n} Y_i\right\}\right] \\
&\quad + n\log E\left[\exp\left\{\frac{u_2}{\sqrt{n}}\sum_{i=1}^{n} Z_i\right\}\right] - \sqrt{n}(u_1 m(t_1) + u_2 m(t_2)).
\end{aligned}
\tag{3.2.3.2}
$$

Let the expectation of X_i, Y_i, Z_i be μ_X, μ_Y, μ_Z, respectively. By the MGF for a Poisson random variable, it follows that

$$
\begin{aligned}
(3.2.3.2) &= n\mu_X\left(\exp\left\{\frac{u_1}{\sqrt{n}}\right\} - 1\right) + n\mu_Y\left(\exp\left\{\frac{u_1+u_2}{\sqrt{n}}\right\} - 1\right) \\
&\quad + n\mu_Z\left(\exp\left\{\frac{u_2}{\sqrt{n}}\right\} - 1\right) - \sqrt{n}(u_1 m(t_1) + u_2 m(t_2)),
\end{aligned}
\tag{3.2.3.3}
$$

where we have used the fact that the mean of $\sum_{i=1}^{n} X_i = n\mu_X$ (and similarly for Y and Z). Next, using the Taylor series expansion for e^t, it follows that

$$
\begin{aligned}
(3.2.3.3) &= n\mu_X\left(\frac{u_1}{\sqrt{n}} + \frac{u_1^2}{2n} + O(n^{-3/2})\right) + n\mu_Y\left(\frac{(u_1+u_2)}{\sqrt{2n}} + \frac{(u_1+u_2)^2}{n} + O(n^{-3/2})\right) \\
&\quad + n\mu_Z\left(\frac{u_2}{\sqrt{n}} + \frac{u_2^2}{2n} + O(n^{-3/2})\right) - \sqrt{n}(u_1 m(t_1) + u_2 m(t_2)) \\
&= u_1\sqrt{n}\left(\mu_X + \mu_Y\right) + u_2\sqrt{n}\left(\mu_Y + \mu_Z\right) + \frac{u_1^2}{2}\left(\mu_X + \mu_Y\right) + \frac{u_2^2}{2}\left(\mu_Y + \mu_Z\right) \\
&\quad + 2u_1u_2\mu_Y - \sqrt{n}\left(u_1 m(t_1) + u_2 m(t_2)\right) + O(n^{-1/2}).
\end{aligned}
\tag{3.2.3.4}
$$

Recall that by construction, X, Y, Z satisfy $\mu_X + \mu_Y = m(t_1)$ and $\mu_Y + \mu_Z = m(t_2)$. Therefore,

$$
(3.2.3.4) = \frac{u_1^2}{2} m(t_1) + \frac{u_2^2}{2} m(t_2) + 2u_1u_2\mu_Y + O(n^{-1/2}). \tag{3.2.3.5}
$$

It follows that as $n \to \infty$,

$$
\begin{aligned}
(3.2.3.5) &\to \frac{u_1^2}{2} m(t_1) + \frac{u_2^2}{2} m(t_2) + 2u_1u_2\mu_Y \\
&= \frac{1}{2}\mathbf{u}^T\Sigma\mathbf{u},
\end{aligned}
$$

which is the log MGF for a Gaussian random vector with mean 0 and covariance matrix Σ, with $\sigma_{ij} = \int_0^{t_i \wedge t_j} \lambda(u)\bar{F}(t_i \vee t_j - u)du$. Now, since $Q_i(t_1), ..., Q_i(t_k)$ can be constructed from $\sum_{j=1}^{k} j$ independent Poisson random variables, the moment generating function can be simplified in the same way for arbitrary finite k, proving that the finite dimensional vector converges to a Gaussian vector. ∎

Next, we show that the sequence $\{\hat{m}_n(t)\}$ is tight following Theorem 1.6.3 [40].

Lemma 3.2.4 (Tightness) *(i) For each positive η, there exists an a such that $P(\hat{m}_n(t) : |\hat{m}_n(0)| > a) \leq \eta$, $n \geq 1$.*

(ii) For each positive ϵ and η, there exists a δ, with $0 < \delta < 1$, and an integer n_0 such that $P\left(\sup_{|s-t|<\delta} |\hat{m}_n(t) - \hat{m}_n(s)| \geq \epsilon\right) \leq \eta$, $n \geq n_0$, where $0 < \delta < 1$.

Proof First, since $\hat{m}_n(0) = 0$, $\{m_n(0)\}$ is tight, proving condition (i). To prove condition (ii), we show that for each $\epsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n} P\left(\sup_{s \leq t \leq s+\delta} |\hat{m}_n(t) - \hat{m}_n(s)| > \epsilon\right) = 0.$$

We have,

$$\begin{aligned} &P\left(\sup_{s \leq t \leq s+\delta} |\hat{m}_n(t) - \hat{m}_n(s)| > \epsilon\right) \\ &= P\left(\sup_{s \leq t \leq s+\delta} \left|\sum_{i=1}^{n} Q_i(t) - \sum_{i=1}^{n} Q_i(s)\right| > n\epsilon\right). \end{aligned} \tag{3.2.4.1}$$

As in Lemma 3.2.3, let $Q_i(s) = X_i + Y_i$ and $Q_i(t) = Y_i + Z_i$. It follows that,

$$\begin{aligned} (3.2.4.1) &= P\left(\sup_{s \leq t \leq s+\delta} \left|\sum_{i=1}^{n} (Y_i + Z_i) - \sum_{i=1}^{n} (X_i + Y_i)\right| > n\epsilon\right) \\ &= P\left(\sup_{s \leq t \leq s+\delta} \left|\sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} X_i\right| > n\epsilon\right) \\ &= P\left(\sup_{s \leq t \leq s+\delta} \max\left(\sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Z_i\right) > n\epsilon\right). \end{aligned} \tag{3.2.4.2}$$

Because $X_i \geq 0$ and $Z_i \geq 0$, we have,

$$(3.2.4.2) \leq P\left(\sup_{s \leq t \leq s+\delta} \max\left(\sum_{i=1}^{n} Z_i, \sum_{i=1}^{n} X_i\right) > n\epsilon\right). \tag{3.2.4.3}$$

Next, recall the construction of X_i and Z_i. Z_i is the number of arrivals to the NHPP $(N(t) : t \geq 0)$ in the interval $[s,t]$ with service time greater than $t - \tau_{ik}$. X_i is the number of arrivals to the NHPP $(N(t) : t \geq 0)$ in the interval $[0,s)$ with service times in the interval $s - \tau_{ik} \leq \nu_{ik} \leq t - \tau_{ik}$. Using these definitions, it follows that,

$$
\begin{aligned}
&(3.2.4.3)\\
&= P\left(\sup_{s\leq t\leq s+\delta} \max\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(s,t)} \mathbb{1}\{\nu_{ik} \geq t-\tau_{ik}\}\right),\right.\right.\\
&\qquad\qquad\left.\left.\sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \leq \nu_{ik} \leq t-\tau_{ik}\}\right)\right) > n\epsilon\right)\\
&\leq P\left(\sup_{s\leq t\leq s+\delta} \max\left(\sum_{i=1}^{n} N_i(s,t), \sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \leq \nu_{ik} \leq t-\tau_{ik}\}\right)\right) > n\epsilon\right)\\
&\leq P\left(\max \sup_{s\leq t\leq s+\delta}\left(\sum_{i=1}^{n} N_i(s,t), \sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \leq \nu_{ik} \leq t-\tau_{ik}\}\right)\right) > n\epsilon\right).
\end{aligned}
\tag{3.2.4.4}
$$

Because $N(t)$ is a nondecreasing function, $\sup_{s\leq t\leq s+\delta} N(s,t) = N(s, s+\delta)$. It follows that,

$$
\begin{aligned}
(3.2.4.4) &= P\left(\max\left(\sum_{i=1}^{n} N_i(s,s+\delta),\right.\right.\\
&\qquad\left.\left.\sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \leq \nu_{ik} \leq s+\delta-\tau_{ik}\}\right)\right) > n\epsilon\right)\\
&= 1 - P\left(\left(\sum_{i=1}^{n} N_i(s,s+\delta),\right.\right.\\
&\qquad\left.\left.\sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \leq \nu_{ik} \leq s+\delta-\tau_{ik}\}\right)\right) \leq n\epsilon\right)\\
&= 1 - P\left(\sum_{i=1}^{n} N_i(s,s+\delta) \leq n\epsilon\right)
\end{aligned}
$$

$$P\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \le \nu_{ik} \le s+\delta-\tau_{ik}\}\right) \le n\epsilon\right), \tag{3.2.4.5}$$

where the last equality holds because the two Poisson random variables are over nonoverlapping intervals. Next, consider the first probability in (3.2.4.5). Recall the expectation $E\left[N_i(s,t)\right] = \Lambda(t) - \Lambda(s)$. It follows that,

$$\begin{aligned} &P\left(\sum_{i=1}^{n} N_i(s,s+\delta) \le n\epsilon\right) \\ &= \left(\exp\left\{-n(\Lambda(s+\delta)-\Lambda(s))\right\}\sum_{j=0}^{\lfloor n\epsilon \rfloor}\frac{n(\Lambda(s+\delta)-\Lambda(s))^j}{j!}\right). \end{aligned} \tag{3.2.4.6}$$

Let $\theta_s \in [s, s+\delta]$. By the mean value theorem, it follows that

$$(3.2.4.6) = \exp\left\{-n\lambda(\theta_s)\delta\right\}\sum_{j=0}^{\lfloor n\epsilon \rfloor}\frac{(n\lambda(\theta_s)\delta)^j}{j!}. \tag{3.2.4.7}$$

For fixed $\epsilon > 0$, as $n \to \infty$, (7)$\to$ $\exp\left\{-n\lambda(\theta_s)\delta\right\}\exp\left\{n\lambda(\theta_s)\delta\right\} = 1$. Next, consider the second probability in (3.2.4.5).

$$\begin{aligned} &P\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{N_i(0,s)} \mathbb{1}\{s-\tau_{ik} \le \nu_{ik} \le s+\delta-\tau_{ik}\}\right) \le n\epsilon\right) \\ &= \left(\exp\left\{-n\int_0^s \lambda(u)(F(s+\delta-u)-F(s-u)du\right\}\right. \\ &\qquad \left.\sum_{j=0}^{\lfloor n\epsilon \rfloor}\frac{\left(n\int_0^s \lambda(u)(F(s+\delta-u)-F(s-u))du\right)^j}{j!}\right). \end{aligned} \tag{3.2.4.8}$$

Again, by the Taylor expansion for e^t, as $n \to \infty$,

$$\begin{aligned} (3.2.4.8) \to \exp&\left\{-n\int_0^s \lambda(u)(F(s+\delta-u)-F(s-u)du\right\} \\ &\exp\left\{n\int_0^s \lambda(u)(F(s+\delta-u)-F(s-u)du\right\} \\ = 1.& \end{aligned}$$

Therefore, by the convergence of (3.2.4.7) and (3.2.4.8), it follows that (3.2.4.5)$\to 0$ as $n \to \infty$, proving condition (ii). ■

Lemmas 3.2.3 and 3.2.4 complete the sufficient conditions to prove the FCLT for $\hat{m}_n(t)$.

Theorem 3.2.5 (FCLT) *$(\sqrt{n}\,(\hat{m}_n(t) - m(t)) : t \geq 0) \Rightarrow (Z(t) : t \geq 0)$ as $n \to \infty$, where $(Z(t) : t \geq 0)$ is a Gaussian process mean 0 with covariance function $\rho(s,t) = \int_0^{s \wedge t} \lambda(u)\bar{F}(s \vee t - u)du$.*

Proof First, Lemma 3.2.4 proves the conditions to [40, Theorem 1.6.3]. Then together, lemmas 3.2.3 and 3.2.4 prove the sufficient conditions to Theorem 1.6.2 [40], proving the claim. ■

Through functional asymptotic analysis, we see the asymptotic consistency and rate of convergence for the estimator $\hat{m}_n(t)$. However, as in Section 2, we note that in practice it is more likely to observe the state of the system at fixed intervals. Section 3.3 considers a nonparametric estimator for the mean number of busy servers with observations at fixed intervals.

3.3 Nonparametric Aggregated Estimator

Consider the case in which we do not observe all arrival epochs and associated service times. Instead, assume we directly observe the number of busy servers at fixed intervals of width $\delta_n > 0$. For $t \geq 0$, let $l_n(t) = \left\lfloor \frac{t}{\delta_n} \right\rfloor \delta_n$ be the lower bound of an interval in which t falls, as in the estimator $V_n(t)$ in Section 2. Then the natural estimator for the mean number of busy servers at time $t \geq 0$ is the random variable

$$m_n(t) = \frac{1}{n}\sum_{i=1}^{n} Q_i(l_n(t)) + \frac{t - l_n(t)}{\delta_n}\frac{1}{n}\sum_{i=1}^{n} Q_i(l_n(t) + \delta_n) - Q_i(l_n(t)).$$

Again, we are interested in proving asymptotic properties regarding the consistency and rate of convergence of this nonparametric estimator.

3.3.1 Asymptotic Consistency

Through FSLLN, we show that $m_n(t)$ is an asymptotically consistent and unbiased estimator of $m(t)$ at a specified rate of δ_n.

Theorem 3.3.1 (FSLLN) *If* $E\left[\left(\sum_{i=1}^{n} Q_i(t) - nm(t)\right)^4\right] < +\infty$ *and* $\delta_n \to 0$ *as* $n \to \infty$*, then* $\sup_t |m_n(t) - m(t)| \to 0$ *a.s. as* $n \to \infty$*.*

Proof We use the first Borel-Cantelli lemma [45] to show that for every $\epsilon > 0$,

$$P\left(|nm_n(t) - nm(t)| \geq n\epsilon \quad i.o.\right) = 0.$$

Consider the event above and note that

$$\begin{aligned} &P\left(|nm_n(t) - nm(t)| \geq \epsilon\right) \\ &= P\left(\left|\sum_{i=1}^{n} Q_i(l_n(t)) + \frac{t - l_n(t)}{\delta_n}\sum_{i=1}^{n} Q_i(l_n(t)+\delta_n) - Q_i(l_n(t)) - nm(t)\right| \geq n\epsilon\right). \end{aligned} \tag{3.3.1.1}$$

Now, we add and subtract $\sum_{i=1}^{n} Q_i(t)$, yielding,

$$\begin{aligned} (3.3.1.1) = P\Bigg(&\left|\sum_{i=1}^{n} Q_i(t) - nm(t) + \sum_{i=1}^{n} Q_i(l_n(t)) - \sum_{i=1}^{n} Q_i(t)\right. \\ &\left. + \frac{t - l_n(t)}{\delta_n}\sum_{i=1}^{n} Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right| \geq n\epsilon\Bigg) \\ \leq P\Bigg(&\left|\sum_{i=1}^{n} Q_i(t) - nm(t)\right| + \left|\sum_{i=1}^{n} Q_i(l_n(t)) - \sum_{i=1}^{n} Q_i(t)\right| \\ &+ \left|\frac{t - l_n(t)}{\delta_n}\sum_{i=1}^{n} Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right| \geq n\epsilon\Bigg) \\ \leq P\Bigg(&\left|\sum_{i=1}^{n} Q_i(t) - nm(t)\right| \geq n\epsilon\Bigg) + P\left(\left|\sum_{i=1}^{n} Q_i(l_n(t)) - \sum_{i=1}^{n} Q_i(t)\right| \geq n\epsilon\right) \\ &+ P\left(\left|\frac{t - l_n(t)}{\delta_n}\sum_{i=1}^{n} Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right| \geq n\epsilon\right), \end{aligned} \tag{3.3.1.2}$$

where the last inequality follows from the union bound. The first probability in (3.3.1.2) can be bounded using the Chebyshev inequality as in Theorem 3.2.1:

$$P\left(\left|\sum_{i=1}^{n} Q_i(t) - nm(t)\right| \geq n\epsilon\right) \leq \frac{E\left[\left|\sum_{i=1}^{n} Q_i(t) - nm(t)\right|^4\right]}{n\epsilon}$$
$$= \frac{m(t)}{n^3\epsilon^4} + \frac{3m(t)^2}{n^2\epsilon^4}. \tag{3.3.1.3}$$

Clearly, as $n \to \infty$, (3.3.1.3)$\to 0$. Next, consider the second term in (3.3.1.2). Construct $Q_i(l_n(t)) = X_i + Y_i$, $Q_i(t) = Y_i + Z_i$, where X_i, Y_i, Z_i are Poisson random variables with respective mean

$$E[X_i] = \int_0^{l_n(t)} \lambda(u)\left(F(t-u) - F(l_n(t)-u)\right) du$$
$$E[Y_i] = \int_0^{l_n(t)} \lambda(u)\bar{F}(t-u)du$$
$$E[Z_i] = \int_{l_n(t)}^{t} \lambda(u)\bar{F}(t-u)du.$$

Noting that $Q_i(\cdot) \geq 0$, it follows that

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n} Q_i(l_n(t)) - \sum_{i=1}^{n} Q_i(t)\right| \geq n\epsilon\right) \\
&= P\left(\max\left(\sum_{i=1}^{n} Q_i(l_n(t)) - \sum_{i=1}^{n} Q_i(t), \sum_{i=1}^{n} Q_i(t) - \sum_{i=1}^{n} Q_i(l_n(t))\right) \geq n\epsilon\right) \\
&\leq P\left(\max\left(\sum_{i=1}^{n} Q_i(l_n(t)), \sum_{i=1}^{n} Q_i(t)\right) \geq n\epsilon\right) \\
&= P\Bigg(\max\Bigg(\sum_{i=1}^{n} \sum_{k=1}^{N_i(l_n(t))} \mathbb{1}\{l_n(t) - \tau_{ik} \leq \nu_{ik} \leq t - \tau_{ik}\}, \\
&\qquad\qquad \sum_{i=1}^{n} \sum_{k=1}^{N_i(l_n(t),t)} \mathbb{1}\{\nu_{ik} \geq t - \tau_{ik}\}\Bigg) \geq n\epsilon\Bigg) \\
&\leq P\left(\max\left(\sum_{i=1}^{n} \sum_{k=1}^{N_i(l_n(t))} \mathbb{1}\{l_n(t) - \tau_{ik} \leq \nu_{ik} \leq t - \tau_{ik}\}, \sum_{i=1}^{n} N_i(l_n(t),t)\right) \geq n\epsilon\right).
\end{aligned} \tag{3.3.1.4}$$

Next, using the independent increments property of Poisson random variables, it follows that

$$(3.3.1.4) = 1 - P\left(\sum_{i=1}^{n}\sum_{k=1}^{N_i(l_n(t))} \mathbb{1}\{l_n(t) - \tau_{ik} \le \nu_{ik} \le t - \tau_{ik}\} \le n\epsilon\right)$$

$$P\left(\sum_{i=1}^{n} N_i(l_n(t), t) \ge n\epsilon\right)$$

$$= 1 - \left(\exp\{-nE[X]\}\sum_{j=0}^{\lfloor n\epsilon \rfloor}\frac{(nE[X])^j}{j!}\right)$$

$$\left(\exp\{-n(\Lambda(t) - \Lambda(l_n(t)))\}\sum_{j=0}^{\lfloor n\epsilon \rfloor}\frac{(n(\Lambda(t) - \Lambda(l_n(t))))^j}{j!}\right). \tag{3.3.1.5}$$

As in Lemma 3.2.4, we use the Taylor expansion to see that (3.3.1.5)$\to 0$ as $n \to \infty$. Consider the third term in (3.3.1.2). As in the previous simplification, we construct $Q_i(l_n(t)) = \bar{X}_i + \bar{Y}_i$ and $Q_i(l_n(t) + \delta_n) = \bar{Y}_i + \bar{Z}_i$ where

$$E[\bar{X}_i] = \int_0^{l_n(t)} \lambda(u)\left(F(l_n(t) + \delta_n - u) - F(l_n(t) - u)\right) du$$

$$E[\bar{Y}_i] = \int_0^{l_n(t)} \lambda(u)\bar{F}(l_n(t) + \delta_n - u) du$$

$$E[\bar{Z}_i] = \int_{l_n(t)}^{l_n(t)+\delta_n} \lambda(u)\bar{F}(l_n(t) + \delta_n - u) du.$$

Again, noting that $Q_i(\cdot) \ge 0$, it follows that

$$P\left(\left|\frac{t - l_n(t)}{\delta_n}\sum_{i=1}^{n} Q_i(l_n(t) + \delta_n) - Q_i(l_n(t))\right| \ge n\epsilon\right)$$

$$= P\left(\frac{t - l_n(t)}{\delta_n}\max\left(\sum_{i=1}^{n} Q_i(l_n(t) + \delta_n) - Q_i(l_n(t)),\right.\right.$$

$$\left.\left.\sum_{i=1}^{n} Q_i(l_n(t)) - Q_i(l_n(t) + \delta_n)\right) \ge n\epsilon\right)$$

$$\le P\left(\frac{t - l_n(t)}{\delta_n}\max\left(\sum_{i=1}^{n}\bar{Z}_i, \sum_{i=1}^{n}\bar{X}_i\right) \ge n\epsilon\right)$$

$$\le P\left(\max\left(\sum_{i=1}^{n} N_i(l_n(t), l_n(t) + \delta_n),\right.\right.$$

$$\sum_{i=1}^{n} \sum_{k=1}^{N_i(0,l_n(t))} \mathbb{1}\{l_n(t) - \tau_{ik} \leq \nu_{ik} \leq l_n(t) + \delta_n - \tau_{ik}\Bigg) \geq n\epsilon\Bigg). \tag{3.3.1.6}$$

Because of independent increments, it follows that

$$\begin{aligned}
(3.3.1.6) = 1 - P&\left(\sum_{i=1}^{n} N_i(l_n(t), l_n(t)+\delta_n) \leq n\epsilon\right) \\
&P\left(\sum_{i=1}^{n} \sum_{k=1}^{N_i(0,l_n(t))} \mathbb{1}\{l_n(t) - \tau_{ik} \leq \nu_{ik} \leq l_n(t) + \delta_n - \tau_{ik} \leq n\epsilon\right) \\
= 1 - &\left(\exp\left\{-n(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t)))\right\} \sum_{j=0}^{\lfloor n\epsilon \rfloor} \frac{(n(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))))^j}{j!}\right) \\
&\left(\exp\left\{-nE\left[\bar{X}\right]\right\} \sum_{j=0}^{\lfloor n\epsilon \rfloor} \frac{(nE\left[X\right])^j}{j!}\right).
\end{aligned} \tag{3.3.1.7}$$

Again, as $n \to \infty$, (3.3.1.7)$\to 0$. Therefore, it follows from (3.3.1.3), (3.3.1.5), and (3.3.1.7), that $\sum_{n=1}^{\infty}(2) < +\infty$. By the first Borel-Cantelli Lemma, $P\left(|nm_n(t) - nm(t)| \geq n\epsilon \quad i.o.\right) = 0$, proving the claim. ■

We have shown that $m_n(t)$ is an asymptotically consistent estimator for $m(t)$ at $t \geq 0$.

3.3.2 Rate of Convergence

We now consider the rate of convergence of $(m_n(t) : t \geq 0)$ to $(m(t) : t \geq 0)$. Before proving an FCLT, we first show a pointwise CLT for fixed t.

Lemma 3.3.2 (CLT) *If $\delta_n \to 0$ as $n \to \infty$, and $\delta_n = o(n^{-1/2})$, then for $t \in [0,T]$,*

$$\sqrt{n}\left(m_n(t) - m(t)\right) \Rightarrow \mathcal{N}\left(0, m(t)\right).$$

Proof Let X , Y, Z be independent Poisson random variables with respective mean

$$\mu_X = \int_0^{l_n(t)} \lambda(s)\left(F(l_n(t)+\delta_n - s) - F(l_n(t) - s)\right) ds$$

$$\mu_Y = \int_0^{l_n(t)} \lambda(s)\left(1 - F(l_n(t) - s)\right) ds$$

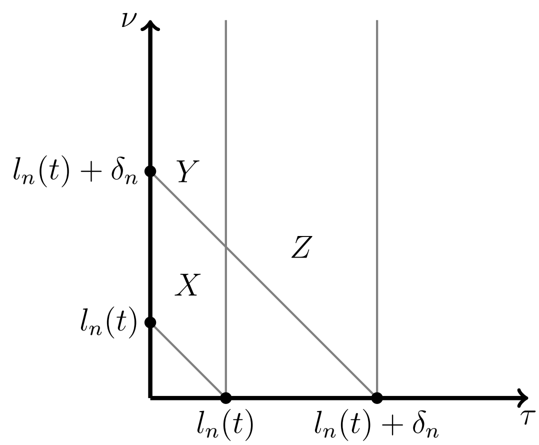

Fig. 3.5. Construction of $Q(l_n(t)), Q(l_n(t)+\delta_n)$

$$\mu_Z = \int_{l_n(t)}^{l_n(t)+\delta_n} \lambda(s)\left(1 - F(l_n(t)+\delta_n - s)\right) ds.$$

Construct $Q(l_n(t)) = X + Y$ and $Q(l_n(t)+\delta_n) = Y + Z$ as in Figure 3.5.

Consider the log moment generating function of $\sqrt{n}\,(m_n(t) - m(t))$,

$$\begin{aligned}
&\log E\left[\exp\left\{u\sqrt{n}\,(m_n(t)-m(t))\right\}\right] \\
&= \log E\left[\exp\left\{u\sqrt{n}m_n(t)\right\}\right] - u\sqrt{n}m(t) \\
&= \log E\left[\exp\left\{\frac{u}{\sqrt{n}}\left(\sum_{i=1}^{n} Q_i(l_n(t)) + \frac{t-l_n(t)}{\delta_n}\sum_{i=1}^{n} Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right)\right\}\right] \\
&\qquad - u\sqrt{n}m(t). \qquad (3.3.2.1)
\end{aligned}$$

Using the construction of $Q(l_n(t))$ and $Q(l_n(t)+\delta_n)$ (Figure 3.5),

$$\begin{aligned}
&(3.3.2.1) \\
&= \log E\left[\exp\left\{\frac{u}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i + Y_i + \frac{t-l_n(t)}{\delta_n}\sum_{i=1}^{n} Y_i + Z_i - Y_i - X_i\right)\right\}\right] \\
&\qquad - u\sqrt{n}m(t) \\
&= \log E\left[\exp\left\{\frac{u}{\sqrt{n}}\left(1 - \frac{t-l_n(t)}{\delta_n}\right)\sum_{i=1}^{n} X_i + \frac{u}{\sqrt{n}}\sum_{i=1}^{n} Y_i + \frac{u}{\sqrt{n}}\frac{t-l_n(t)}{\delta_n}\sum_{i=1}^{n} Z_i\right\}\right] \\
&\qquad - u\sqrt{n}m(t). \qquad (3.3.2.2)
\end{aligned}$$

Applying the MGF for a Poisson random variable followed by the Taylor series expansion for e^t, it follows that

(3.3.2.2)

$$\begin{aligned}
&= n\mu_X\left(e^{\frac{u}{\sqrt{n}}\left(1-\frac{t-l_n(t)}{\delta_n}\right)}-1\right)+n\mu_Y\left(e^{\frac{u}{\sqrt{n}}}-1\right)+n\mu_Z\left(e^{\frac{u}{\sqrt{n}}\frac{t-l_n(t)}{\delta_n}}-1\right)-u\sqrt{n}m(t)\\
&= n\mu_X\left(\frac{u}{\sqrt{n}}\left(1-\frac{t-l_n(t)}{\delta_n}\right)+\frac{1}{2}\frac{u^2}{n}\left(1-\frac{t-l_n(t)}{\delta_n}\right)^2+O(n^{-3/2})\right)\\
&\quad+n\mu_Y\left(\frac{u}{\sqrt{n}}+\frac{u^2}{2n}+O(n^{-3/2})\right)\\
&\quad+n\mu_Z\left(\frac{u}{\sqrt{n}}\frac{t-l_n(t)}{\delta_n}+\frac{u^2}{2n}\left(\frac{t-l_n(t)}{\delta_n}\right)^2+O(n^{-3/2})\right)\\
&\quad-u\sqrt{n}m(t)\\
&= u\sqrt{n}\mu_X-u\sqrt{n}\frac{t-l_n(t)}{\delta_n}\mu_X+\frac{u^2}{2}\mu_X-u^2\frac{t-l_n(t)}{\delta_n}\mu_X\\
&\quad+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2\mu_X+u\sqrt{n}\mu_Y+\frac{u^2}{2}\mu_Y+u\sqrt{n}\frac{t-l_n(t)}{\delta_n}\mu_Z\\
&\quad+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2\mu_Z+O(n^{-1/2})-u\sqrt{n}m(t)\\
&= u\sqrt{n}\left(\mu_X+\mu_Y\right)+\frac{u^2}{2}\left(\mu_X+\mu_Y\right)+u\sqrt{n}\frac{t-l_n(t)}{\delta_n}\left(\mu_Z-\mu_X\right)\\
&\quad-u^2\frac{t-l_n(t)}{\delta_n}\mu_X+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2\left(\mu_X+\mu_Z\right)-u\sqrt{n}m(t)+O(n^{-1/2}). \quad (3.3.2.3)
\end{aligned}$$

By the construction of $Q_i(l_n(t)), Q_i(l_n(t)+\delta_n)$, we have that $\mu_X+\mu_Y=m(l_n(t))$ and $\mu_Y+\mu_Z=m(l_n(t)+\delta_n)$, giving,

$$\begin{aligned}
(3.3.2.3) &= u\sqrt{n}m(l_n(t))+\frac{u^2}{2}m(l_n(t))+u\sqrt{n}\frac{t-l_n(t)}{\delta_n}(\mu_Z-\mu_X)-u^2\frac{t-l_n(t)}{\delta_n}\mu_X\\
&\quad+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2(\mu_X+\mu_Z)-u\sqrt{n}m(t)+O(n^{-1/2}).
\end{aligned} \tag{3.3.2.4}$$

Now, since $l_n(t)\le t<l_n(t)+\delta_n$, we have that

$$\begin{aligned}
&-u^2\frac{t-l_n(t)}{\delta_n}\mu_X+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2(\mu_X+\mu_Z)\\
&\qquad=-2\frac{u^2}{2}\frac{t-l_n(t)}{\delta_n}\mu_X+\frac{u^2}{2}\left(\frac{t-l_n(t)}{\delta_n}\right)^2(\mu_X+\mu_Z)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u^2}{2}\frac{t-l_n(t)}{\delta_n}\left(\frac{t-l_n(t)}{\delta_n}\mu_X - \mu_X + \frac{t-l_n(t)}{\delta_n}\mu_Z - \mu_X\right)\\
&\le \frac{u^2}{2}\frac{t-l_n(t)}{\delta_n}\left(\frac{t-l_n(t)}{\delta_n}\mu_Z - \mu_X\right)\\
&\le \frac{u^2}{2}\frac{t-l_n(t)}{\delta_n}\left(\mu_Z - \mu_X\right). \qquad (3.3.2.5)
\end{aligned}
$$

Also, consider the following inequality. We have that,

$$
\begin{aligned}
\mu_Z - \mu_X \le \mu_Z &\\
&= \int_{l_n(t)}^{l_n(t)+\delta_n} \lambda(s)\left(1 - F(l_n(t)+\delta_n - s)\right) ds\\
&\le \int_{l_n(t)}^{l_n(t)+\delta_n} \lambda(s) ds\\
&= \Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t)). \qquad (3.3.2.6)
\end{aligned}
$$

Therefore, by (3.3.2.5) and (3.3.2.6),

$$
\begin{aligned}
(3.3.2.4) \le u\sqrt{n}m(l_n(t)) + \frac{u^2}{2}m(l_n(t)) + u\sqrt{n}\frac{t-l_n(t)}{\delta_n}\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))\right)&\\
+ \frac{u^2}{2}\frac{t-l_n(t)}{\delta_n}\left(\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t))\right) - u\sqrt{n}m(t) + O(n^{-1/2}).&
\end{aligned}
\qquad (3.3.2.7)
$$

Next, let $\theta_n \in [l_n(t), l_n(t)+\delta_n]$. Then by the Mean Value Theorem, $\Lambda(l_n(t)+\delta_n) - \Lambda(l_n(t)) = \lambda(\theta_n)\delta_n$.

$$
\begin{aligned}
(3.3.2.7) &= u\sqrt{n}m(l_n(t)) + \frac{u^2}{2}m(l_n(t)) + u\sqrt{n}\frac{t-l_n(t)}{\delta_n}\lambda(\theta_n)\delta_n + \frac{u^2}{2}\frac{t-l_n(t)}{\delta_n}\lambda(\theta_n)\delta_n\\
&\qquad - u\sqrt{n}m(t) + O(n^{-1/2})\\
&\le u\sqrt{n}m(l_n(t)) + \frac{u^2}{2}m(l_n(t)) + u\sqrt{n}\lambda(\theta_n)\delta_n + \frac{u^2}{2}\lambda(\theta_n)\delta_n\\
&\qquad - u\sqrt{n}m(t) + O(n^{-1/2}). \qquad (3.3.2.8)
\end{aligned}
$$

Since $l_n(t) \to t$ as $n \to \infty$ and $\delta_n = o(n^{-1/2})$, (3.3.2.8)$\to \frac{u^2}{2}m(t)$, which is the log MGF of a Gaussian with mean 0 and variance $m(t)$. ■

We continue the rate of convergence analysis with the following two lemmas to prove an FCLT.

Lemma 3.3.3 (FDD's) *If $\delta_n = o(n^{-1/2})$ and $\lambda(\cdot)$ is Lipschitz continuous with Lipschitz constant K, then as $n \to \infty$, for $0 = t_0 < t_1 < \cdots < t_k \leq T$,*

$$\left(\sqrt{n}(m_n(t_1) - m(t_1)), ..., \sqrt{n}(m_n(t_k) - m(t_k))\right) \Rightarrow (Z(t_1), ..., Z(t_k)),$$

where $(Z(t_1), ..., Z(t_k))$ is a Gaussian vector with mean 0 and covariance matrix $\Sigma = [\sigma_{ij}]$ with $\sigma_{ij} = \int_0^{t_i \wedge t_j} \lambda(u)\bar{F}(t_i \vee t_j - u)du \ \forall\ 1 \leq i, j \leq k$.

Proof Consider k=2. As in the previous analyses of $m_n(t)$, we will construct $Q_i(\cdot)$ as a sum of Poisson random variables. Consider the Poisson random measure decomposition of the following terms (Figure 3.6).

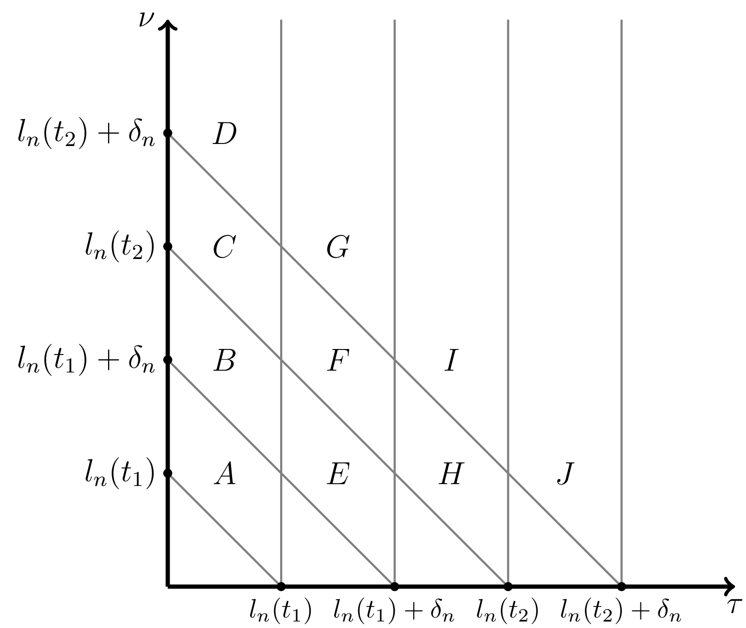

Fig. 3.6. FDD's Poisson Random Measure Decomposition

$$Q_i(l_n(t_1)) = A_i + B_i + C_i + D_i$$

$$Q_i(l_n(t_1) + \delta_n) = B_i + C_i + D_i + E_i + F_i + G_i$$

$$Q_i(l_n(t_2)) = C_i + D_i + F_i + G_i + H_i + I_i$$

$$Q_i(l_n(t_2) + \delta_n) = D_i + F_i + G_i + I_i + J_i$$

We will analyze the log moment generating function as in Lemma 3.3.2. It simplifies similarly to the following:

$$\begin{aligned}
&\log E\left[e^{\left\langle (u_1,u_2),\left(\sqrt{n}(m_n(t_1)-m(t_1)),\sqrt{n}(m_n(t_2)-m(t_2))\right)\right\rangle}\right] \\
&= u_1\sqrt{n}m(l_n(t_1)) + u_2\sqrt{n}m(l_n(t_2)) - u_1\sqrt{n}m(t_1) - u_2\sqrt{n}m(t_2) \\
&\qquad + \frac{u_1^2}{2}m(l_n(t_1)) + \frac{u_2^2}{2}m(l_n(t_2)) + R. \qquad (3.3.3.1)
\end{aligned}$$

Clearly, as $n \to \infty$, the above expression converges to $\frac{u_1^2}{2}m(t_1) + \frac{u_2^2}{2}m(t_2) + \lim_{n\to\infty} R$. We break down R separately. For simplicity, let $A, B, C, D, E, F, G, H, I, J$ be the expected value of their respective Poisson random variables. We have that

$$\begin{aligned}
R &= u_1\sqrt{n}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(E+F+G-A) + u_1^2\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(-A) \\
&+ \frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)^2(A+E+F+G) + u_2\sqrt{n}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(J-C-H-F) \\
&+ \frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)^2(C+F+H+J) + u_2^2\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(-C-F-H) \\
&+ u_1u_2(C+D) + u_1u_2\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(-C) + u_1u_2\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(F+G) \\
&+ u_1u_2\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(-F).
\end{aligned}$$

First, we analyze the following two terms of R. Let $\theta_1 \in [l_n(t_1), l_n(t_1)+\delta_n]$ and $\theta_2 \in [l_n(t_2), l_n(t_2)+\delta_n]$. By the Mean Value Theorem,

$$\begin{aligned}
&u_1\sqrt{n}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(E+F+G-A) + u_2\sqrt{n}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(J-C-H-F) \\
&\le u_1\sqrt{n}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(E+F+G) + u_2\sqrt{n}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(J) \\
&\le u_1\sqrt{n}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))) \\
&\qquad + u_2\sqrt{n}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(\Lambda(l_n(t_2)+\delta_n) - \Lambda(l_n(t_2))) \\
&\le u_1\sqrt{n}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)\lambda(\theta_1)\delta_n + u_2\sqrt{n}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)\lambda(\theta_2)\delta_n \\
&\le u_1\sqrt{n}\lambda(\theta_1)\delta_n + u_2\sqrt{n}\lambda(\theta_2)\delta_n. \qquad (3.3.3.2)
\end{aligned}$$

Since $\delta_n = o(n^{-1/2})$, (3.3.3.2)$\to 0$ as $n \to \infty$. Now, consider the following two terms of R. We have,

$$\begin{aligned}
&\frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)^2 (A+E+F+G) + u_1^2\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(-A) \\
&= \frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)\left(\frac{t_1 - l_n(t_1)}{\delta_n}(A+E+F+G) - 2A\right) \\
&\leq \frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(E+F+G) \\
&\leq \frac{u_1^2}{2}\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(\Lambda(l_n(t_1)+\delta_n) - \Lambda(l_n(t_1))) \\
&\leq \frac{u_1^2}{2}\lambda(\theta_1)\delta_n, \qquad (3.3.3.3)
\end{aligned}$$

where the last inequality holds again by the Mean Value Theorem. Clearly, since $\delta_n \to 0$ as $n \to \infty$, (3.3.3.3)$\to 0$ as $n \to \infty$. We analyze the following term of R in the same way. Let $\theta_2 \in [l_n(t_2), l_n(t_2)+\delta_n]$. It follows that,

$$\begin{aligned}
&\frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)^2 (C+F+H+J) + u_2^2\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(-C-F-H) \\
&= \frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)\left(\frac{t_2 - l_n(t_2)}{\delta_n}(C+F+H+J) - 2(C+F+H)\right) \\
&\leq \frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(J) \\
&\leq \frac{u_2^2}{2}\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(\Lambda(l_n(t_2)+\delta_n) - \Lambda(l_n(t_2))) \\
&\leq \frac{u_1^2}{2}\lambda(\theta_2)\delta_n. \qquad (3.3.3.4)
\end{aligned}$$

Again, (3.3.3.4)$\to 0$ as $n \to \infty$ since $\delta_n \to 0$ as $n \to \infty$. We are now left with four remaining terms of R,

$$\begin{aligned}
&u_1u_2(C+D) + u_1u_2\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(-C) + u_1u_2\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)(F+G) \\
&\qquad + u_1u_2\left(\frac{t_1 - l_n(t_1)}{\delta_n}\right)\left(\frac{t_2 - l_n(t_2)}{\delta_n}\right)(-F) \\
&= u_1u_2\left(D + \frac{t_1 - l_n(t_1)}{\delta_n}G\right) \\
&\qquad + u_1u_2\left(C + \frac{t_1 - l_n(t_1)}{\delta_n}F - \frac{t_2 - l_n(t_2)}{\delta_n}C - \frac{t_1 - l_n(t_1)}{\delta_n}\frac{t_2 - l_n(t_2)}{\delta_n}F\right).
\end{aligned}$$
$$(3.3.3.5)$$

Recall the expectations,

$$G = \int_{l_n(t_1)}^{l_n(t_1)+\delta_n} \lambda(u)\bar{F}(l_n(t_2) + \delta_n - u)du$$

$$C = \int_0^{l_n(t_1)} \lambda(u)\left(F(l_n(t_2) + \delta_n - u) - F(l_n(t_2) - u)\right)du$$

$$F = \int_{l_n(t_1)}^{l_n(t_1)+\delta_n} \lambda(u)\left(F(l_n(t_2) + \delta_n - u) - F(l_n(t_2) - u)\right)du.$$

Because $l_n(t_i) \to t_i$ as $n \to \infty$, (3.3.3.5)$\to u_1u_2 \lim_{n\to\infty} D$, where $D = \int_0^{l_n(t_1)} \lambda(u)\bar{F}\left(l_n(t_2) + \delta_n - u\right)du$. Therefore,

$$(2) \to \frac{1}{2}\left(u_1^2 m(t_1) + 2u_1u_2 \int_0^{t_1} \lambda(u)\bar{F}\left(t_2 - u\right)du + u_2^2 m(t_2)\right),$$

which is the log MGF of a Gaussian vector with mean 0 and covariance matrix Σ with $\sigma_{ij} = \int_0^{t_i \wedge t_j} \lambda(u)\bar{F}(t_i \vee t_j - u)du$. As in Lemma 3.2.5, $Q_i(l_n(t_1)), Q_i(l_n(t_1) + \delta_n) ..., Q_i(l_n(t_k)), Q_i(l_n(t_k) + \delta_n)$ can be constructed from independent Poisson random variables. Therefore, the log MGF can be simplified in the same way for arbitrary finite k, proving that the finite dimensional vector converges to a Gaussian vector as long as $\delta_n = o(n^{-1/2})$. ■

We now show the tightness of $(m_n(t) : t \geq 0)$, which is the second condition necessary to proving an FCLT for the stochastic process.

Lemma 3.3.4 *(i) For each positive η, there exists an a such that $P\left(m_n(t) : |m_n(0)| > a\right) \leq \eta$, $n \geq 1$.*

(ii) For each positive ϵ and η, there exists a δ, with $0 < \delta < 1$, and an integer n_0 such that $P\left(\sup_{|s-t|<\delta} |m_n(t) - m_n(s)| \geq \epsilon\right) \leq \eta$, $n \geq n_0$, where $0 < \delta < 1$.

Proof First, because $m_n(0) = 0$, $\{m_n(0)\}$ is tight, proving condition (i). To prove condition (ii), we show that for each $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_n P\left(\sup_{s\leq t\leq s+\delta} |m_n(t) - m_n(s)| > \epsilon\right) = 0.$$

It follows that,

$$P\left(\sup_{s\leq t\leq s+\delta} |m_n(t) - m_n(s)| > \epsilon\right)$$

$$
\begin{aligned}
&= P\left(\sup_{s\le t\le s+\delta}\left|\left(\sum_{i=1}^n Q_i(l_n(t)) + \frac{t-l_n(t)}{\delta_n}\sum_{i=1}^n Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right)\right.\right.\\
&\qquad \left.\left. - \left(\sum_{i=1}^n Q_i(l_n(s)) + \frac{s-l_n(s)}{\delta_n}\sum_{i=1}^n Q_i(l_n(s)+\delta_n) - Q_i(l_n(s))\right)\right| > n\epsilon\right)\\
&= P\left(\sup_{s\le t\le s+\delta}\max\left(\left(\left(\sum_{i=1}^n Q_i(l_n(t)) + \frac{t-l_n(t)}{\delta_n}\sum_{i=1}^n Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right)\right.\right.\right.\\
&\qquad \left. - \left(\sum_{i=1}^n Q_i(l_n(s)) + \frac{s-l_n(s)}{\delta_n}\sum_{i=1}^n Q_i(l_n(s)+\delta_n) - Q_i(l_n(s))\right)\right),\\
&\qquad \left(\left(\sum_{i=1}^n Q_i(l_n(s)) + \frac{s-l_n(s)}{\delta_n}\sum_{i=1}^n Q_i(l_n(s)+\delta_n) - Q_i(l_n(s))\right)\right.\\
&\qquad \left.\left.\left. - \left(\sum_{i=1}^n Q_i(l_n(t)) + \frac{t-l_n(t)}{\delta_n}\sum_{i=1}^n Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right)\right)\right) > n\epsilon\right)\\
&\le P\left(\sup_{s\le t\le s+\delta}\max\left(\left(\left(\sum_{i=1}^n Q_i(l_n(t)) + \frac{t-l_n(t)}{\delta_n}\sum_{i=1}^n Q_i(l_n(t)+\delta_n) - Q_i(l_n(t))\right),\right.\right.\right.\\
&\qquad \left.\left.\left.\left(\sum_{i=1}^n Q_i(l_n(s)) + \frac{s-l_n(s)}{\delta_n}\sum_{i=1}^n Q_i(l_n(s)+\delta_n) - Q_i(l_n(s))\right)\right)\right)\right). \qquad (3.3.4.1)
\end{aligned}
$$

Construct the terms as in Lemma 3.3.3. Let $Q_i(l_n(s)) = A_i + B_i + C_i + D_i$, $Q_i(l_n(s) + \delta_n) = B_i + C_i + D_i + E_i + F_i + G_i$, $Q_i(l_n(t)) = C_i + D_i + F_i + G_i + H_i + I_i$, and $Q_i(l_n(t) + \delta_n) = D_i + G_i + I_i + J_i$. Since $l_n(t) \le t \le l_n(t) + \delta_n$, it follows that,

$$
\begin{aligned}
(3.3.4.1) &\le P\left(\sup_{s\le t\le s+\delta}\max\left(\sum_{i=1}^n (F_i + G_i + H_i + I_i + J_i),\right.\right.\\
&\qquad\qquad \left.\left.\sum_{i=1}^n (2A_i + B_i + C_i + E_i + F_i)\right) > n\epsilon\right)\\
&\le P\left(\max\sup_{s\le t\le s+\delta}\left(\sum_{i=1}^n (F_i + G_i + H_i + I_i + J_i),\right.\right.\\
&\qquad\qquad \left.\left.\sum_{i=1}^n (2A_i + B_i + C_i + E_i + F_i)\right) > n\epsilon\right). \qquad (3.3.4.2)
\end{aligned}
$$

Let H_i^s, I_i^s, B_i^s, E_i^s be their respective Poisson random variables with $t = s + \delta$. Now, let $\sum_{i=1}^n (G_i + H_i^s + I_i^s + J_i) = X$ which is a Poisson random variable with mean $nE\left[G_i + H_i^s + I_i^s + J_i\right]$ and let $\sum_{i=1}^n (2A_i + B_i^s + C_i + E_i^s) = Y$ which is a

Poisson random variable with mean $nE\left[2A_i + B_i^s + C_i + E_i^s\right]$. Let $F = \sum_{i=1}^n F_i$ be the Poisson random variable with mean $nE\left[F_i\right]$. It follows that,

$$\begin{aligned}
(3.3.4.2) &= P\left(\max(X,Y) + F > n\epsilon\right) \\
&= \sum_f P(\max(X,Y) > n\epsilon - f | F = f)P(F = f) \\
&= \sum_f \left(1 - P(X \le n\epsilon - f | F = f)P(Y \le n\epsilon - f | F = f)\right) P(F = f) \\
&= \sum_f P(F = f) - \sum_f \left(P(X \le n\epsilon - f | F = f)P(Y \le n\epsilon - f | F = f)\right) P(F = f) \\
&= 1 - \sum_f \left(P(X \le n\epsilon - f | F = f)P(Y \le n\epsilon - f | F = f)\right) P(F = f) \\
&= 1 - e^{-(E[X]+E[Y]+E[F])} \sum_f \frac{E[F]^f}{f!} \left(\sum_{j=0}^{n\epsilon - f} \frac{E[X]^j}{j!} \sum_{j=0}^{n\epsilon - f} \frac{E[Y]^j}{j!} \right).
\end{aligned}$$

Note that for fixed f and fixed $\epsilon > 0$, $(n\epsilon - f) \to \infty$ as $n \to \infty$. Therefore,

$$\begin{aligned}
\limsup_n \sum_{j=0}^{n\epsilon - f} \frac{E[X]^j}{j!} &= e^{-E[X]} \\
\limsup_n \sum_{j=0}^{n\epsilon - f} \frac{E[Y]^j}{j!} &= e^{-E[Y]} \\
\limsup_n \sum_{j=0}^{n\epsilon - f} \frac{E[F]^j}{j!} &= e^{-E[F]}.
\end{aligned}$$

So, $\lim_{\delta \to 0} \limsup_n P\left(\sup_{s \le t \le s+\delta} |m_n(t) - m_n(s)| > \epsilon\right) = 0$, proving condition (ii). ■

Lemma 3.3.4 proves condition (ii) of Theorem 1.6.2, allowing us to conclude the following theorem.

Theorem 3.3.5 (FCLT) *If $\delta_n = o(n^{-1/2})$, then as $n \to \infty$, $\sqrt{n}\,(m_n - m) \Rightarrow (Z(t) : t \ge 0)$ where $(Z(t) : t \ge 0)$ is a Gaussian process with mean 0 and covariance function $\rho(s,t) = \int_0^{s \wedge t} \lambda(u)\bar{F}(s \vee t - u)ds$.*

Proof Lemmas 3.3.3 and 3.3.4 prove the sufficient conditions to Theorem 1.6.2 [40], proving the claim. ■

The FCLT proves the rate of convergence for $(m_n(t) : t \geq 0)$ to $(m(t) : t \geq 0)$ which can be used to compute confidence intervals and other statistical measures. We have shown two nonparametric estimators for the mean number of busy servers in an $M_t/G/\infty$ queue assuming first omniscience and then only observations of the busy servers at fixed intervals of width δ_n. For the omniscient estimator $\hat{m}(t)$, we established asymptotic consistency and an FCLT. For the aggregated estimator $m_n(t)$, we established asymptotic consistency and an FCLT as well, but with the restriction that $\delta_n = o(n^{-1/2})$.

4. ARRIVAL DISTRIBUTION $\Delta_{(i)}/G/1$ ESTIMATOR

When analyzing service systems, it is often the case that we are interested in the transitory period of the system over a finite window of time. We consider the transitory queueing model, $\Delta_{(i)}/G/1$, where a finite population of m customers choose arrival times, τ_i, as i.i.d. samples from common distribution $F(t)$ to a single-server queue [39]. Our objective is to estimate the arrival process distribution $F(t)$ as an input to a discrete-event simulation and for further statistical analysis.

Given all arrival times for a finite population m, the empirical sum, $(\bar{F}_m(t) : t \geq 0)$ with $\bar{F}_m(t) := \sum_{i=1}^{m} \mathbb{1}\{\tau_i \leq t\}$, is the natural nonparametric estimator for the number of arrivals. It is well-known that through an FSLLN and FCLT, $\hat{F}_m(t) := \frac{1}{m}\bar{F}_m(t)$ is an asymptotically consistent and efficient estimator for $F(t)$ [40]. However, we consider the case in which arrival count data is the only data available for n i.i.d. observations of the arrival process. We consider two nonparametric estimators and show their asymptotic consistency, rates of convergence, and asymptotic covariance.

4.1 Nonparametric Omniscient Estimator

Let $(\lambda(t) : t \geq 0)$ be a non-negative integrable function and as before, define $(\Lambda(t) : t \geq 0)$ as $\Lambda(t) = \int_0^t \lambda(s)ds$. Observe that $\frac{\Lambda(t)}{\Lambda(T)}$ is a distribution function. Recall the hazard function $h(t) := \frac{f(t)}{1-F(t)}$. Let $f(t) := \frac{\lambda(t)}{\Lambda(T)}$ such that $h(t) = \frac{\lambda(t)/\Lambda(T)}{1-\Lambda(t)/\Lambda(T)} = \frac{\lambda(t)}{\Lambda(T)-\Lambda(t)}$. Then, it is well known that $F(t) = 1 - \exp\left\{\int_0^t h(s)ds\right\} = 1 - \exp\left\{-\int_0^t \frac{\lambda(s)}{\Lambda(T)-\Lambda(s)}ds\right\} = \frac{\Lambda(t)}{\Lambda(T)}$. Now, let $\{\bar{N}_i\}$ be a sequence of i.i.d. unit-rate Poisson processes. Then, $N_i(t) := \left(\bar{N}_i \cdot \Lambda\right)(t)$ is a NHPP with rate $(\lambda(t))$. Then the natural estimator for $\frac{\Lambda(t)}{\Lambda(T)}$ is the random variable

$$\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)} := \frac{\frac{1}{n}\sum_{i=1}^{n} N_i(0,t)}{\frac{1}{n}\sum_{i=1}^{n} N_i(0,T)}.$$

We demonstrate the asymptotic consistency of the estimator and the rate of convergence by proving a FCLT.

Lemma 4.1.1 *For* $0 \leq s \leq t \leq u \leq T$,

$$(i) \quad E\left[((N_i(0,u) - \Lambda(u)) - (N_i(0,t) - \Lambda(t)))^2\right] \leq \Lambda(u) - \Lambda(t)$$

and

$$(ii) \quad E\left[((N_i(0,u) - \Lambda(u)) - (N_i(0,t) - \Lambda(t)))^2 \right.$$
$$\left. ((N_i(0,t) - \Lambda(t)) - (N_i(0,s) - \Lambda(s)))^2\right] \leq (\Lambda(u) - \Lambda(s))^2 .$$

Proof We first consider (i). It follows that,

$$\begin{aligned}
&E\left[((N_i(0,u) - \Lambda(u)) - (N_i(0,t) - \Lambda(t)))^2\right] \\
&\qquad = E\left[(N_i(0,u) - N_i(0,t))^2\right] - (\Lambda(u) - \Lambda(t))^2 \\
&\qquad = E\left[N_i(t,u)^2\right] - (\Lambda(u) - \Lambda(t))^2 \\
&\qquad = \mathrm{Var}(N_i(t,u)) + E\left[N_i(t,u)\right]^2 - (\Lambda(u) - \Lambda(t))^2 \\
&\qquad = \Lambda(u) - \Lambda(t).
\end{aligned}$$

Now, using independent, non-overlapping intervals and the result from (i), we show (ii) holds. Since $(\Lambda(t) : t \geq 0)$ is a nondecreasing function, we have,

$$\begin{aligned}
&E\left[((N_i(0,u) - \Lambda(u)) - (N_i(0,t) - \Lambda(t)))^2 ((N_i(0,t) - \Lambda(t)) - (N_i(0,s) - \Lambda(s)))^2\right] \\
&\qquad = E\left[((N_i(0,u) - \Lambda(u)) - (N_i(0,t) - \Lambda(t)))^2\right] \\
&\qquad\qquad E\left[((N_i(0,t) - \Lambda(t)) - (N_i(0,s) - \Lambda(s)))^2\right] \\
&\qquad \leq (\Lambda(u) - \Lambda(t))(\Lambda(t) - \Lambda(s)) \\
&\qquad \leq (\Lambda(u) - \Lambda(s))^2 .
\end{aligned}$$

We have shown that both inequalities (i) and (ii) hold. ■

Now, we consider the finite-dimensional distributions of $\hat{\Lambda}_n(t)$ and their rate of convergence.

Lemma 4.1.2 (FDD's) *Let* $0 = t_0 < t_1 < \cdots < t_k = T$, *and* $\hat{\Lambda}_n(t) = \frac{1}{n}\sum_{i=1}^{n} N_i(0,t)$ *be the estimator for* $\Lambda(t)$, *then as* $n \to \infty$,

$$\left(\sqrt{n}(\hat{\Lambda}_n(t_1) - \Lambda(t_1)), ..., \sqrt{n}(\hat{\Lambda}_n(t_k) - \Lambda(t_k))\right) \Rightarrow (Z(t_1), ..., Z(t_k))$$

where $Z(t)$ *is a Gaussian vector with mean 0 and covariance matrix* $\Sigma = [\sigma_{ij}]$, *with* $\sigma_{ij} = \Lambda(t_i \wedge t_j)\ \forall\ 1 \le i, j \le k$.

Proof To prove the weak convergence of the finite-dimensional distributions, it is enough to check the convergence of moment generating functions. It follows for arbitrary k,

$$\begin{aligned}
&E\left[\exp\left\{\left\langle (u_1, ..., u_k), (\sqrt{n}\left(\hat{\Lambda}_n(t_1) - \Lambda(t_1)\right), ..., \sqrt{n}\left(\hat{\Lambda}_n(t_k) - \Lambda(t_k)\right)\right\rangle\right\}\right] \\
&= E\left[\exp\left\{\sqrt{n}\left(u_1\hat{\Lambda}_n(t_1) + \cdots + u_k\hat{\Lambda}_n(t_k)\right) - \sqrt{n}\left(u_1\Lambda(t_1) + \cdots + u_k\Lambda(t_k)\right)\right\}\right] \\
&= E\left[\exp\left\{\sqrt{n}\left(u_1\hat{\Lambda}_n(t_1) + \cdots + u_k\hat{\Lambda}_n(t_k)\right)\right\}\right] \\
&\qquad\qquad E\left[\exp\left\{-\sqrt{n}\left(u_1\Lambda(t_1) + \cdots + u_k\Lambda(t_k)\right)\right\}\right]. \qquad (4.1.2.1)
\end{aligned}$$

Consider the first expectation in (4.1.2.1). We have,

$$\begin{aligned}
&E\left[\exp\left\{\sqrt{n}\left(u_1\hat{\Lambda}_n(t_1) + \cdots + u_k\hat{\Lambda}_n(t_k)\right)\right\}\right] \\
&= E\left[\exp\left\{\frac{1}{\sqrt{n}}\left(u_1\sum_{i=1}^{n} N_i(0,t_1) + u_2\left(\sum_{i=1}^{n} N_i(0,t_1) + \sum_{i=1}^{n} N_i(t_1,t_2)\right)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \cdots + u_k\left(\sum_{i=1}^{n} N_i(0,t_1) + \cdots + \sum_{i=1}^{n} N_i(t_{k-1},t_k)\right)\right)\right\}\right] \\
&= E\left[\exp\left\{\frac{u_1 + \cdots + u_k}{\sqrt{n}}\sum_{i=1}^{n} N_i(0,t_1) + \frac{u_2 + \cdots + u_k}{\sqrt{n}}\sum_{i=1}^{n} N_i(t_1,t_2)\right.\right. \\
&\qquad\qquad \left.\left. + \cdots + \frac{u_k}{\sqrt{n}}\sum_{i=1}^{n} N_i(t_{k-1},t_k)\right\}\right]. \qquad (4.1.2.2)
\end{aligned}$$

Because of the independent increments property for Poisson random variables, it follows that

$$(4.1.2.2) = E\left[\exp\left\{\frac{u_1 + \cdots + u_k}{\sqrt{n}}\sum_{i=1}^{n} N_i(0,t_1)\right\}\right] E\left[\exp\left\{\frac{u_2 + \cdots + u_k}{\sqrt{n}}\sum_{i=1}^{n} N_i(t_1,t_2)\right\}\right]$$

$$\times \cdots \times E\left[\exp\left\{\frac{u_k}{\sqrt{n}}\sum_{i=1}^{n} N_i(t_{k-1}, t_k)\right\}\right]. \tag{4.1.2.3}$$

Note that $\sum_{i=1}^{n} N_i(t_{j-1}, t_j)$ is a Poisson random variable with mean $n\left(\Lambda(t_j) - \Lambda(t_{j-1})\right)$, letting $\Lambda(t_0) := 0$. By the MGF of a Poisson random variable,

$$\begin{aligned}\log\left((4.1.2.3)\right) = n\Lambda(t_1)\left(e^{\frac{u_1+\cdots+u_k}{\sqrt{n}}} - 1\right) + n\left(\Lambda(t_2) - \Lambda(t_1)\right)\left(e^{\frac{u_2+\cdots+u_k}{\sqrt{n}}} - 1\right)&\\ + \cdots + n\left(\Lambda(t_k) - \Lambda(t_{k-1})\right)\left(e^{\frac{u_k}{\sqrt{n}}} - 1\right).&\end{aligned} \tag{4.1.2.4}$$

By the Taylor series expansion of e^X, it follows,

$$\begin{aligned}(4.1.2.4) = n\Lambda(t_1)&\left(\frac{u_1 + \cdots + u_k}{\sqrt{n}} + \frac{(u_1 + \cdots + u_k)^2}{2n} + O(n^{-3/2})\right)\\ &+ n\left(\Lambda(t_2) - \Lambda(t_1)\right)\left(\frac{u_2 + \cdots + u_k}{\sqrt{n}} + \frac{(u_2 + \cdots + u_k)^2}{2n} + O(n^{-3/2})\right)\\ &+ \cdots + n\left(\Lambda(t_k) - \Lambda(t_{k-1})\right)\left(\frac{u_k}{\sqrt{n}} + \frac{(u_k)^2}{2n} + O(n^{-3/2})\right).\end{aligned} \tag{5}$$

Exploiting the telescoping sum, it follows that all $u\sqrt{n}$ terms in (4.1.2.5) simplify to

$$\begin{aligned}&(u_1 + \cdots + u_k)\sqrt{n}\Lambda(t_1) + (u_2 + \cdots + u_k)\sqrt{n}\left(\Lambda(t_2) - \Lambda(t_1)\right)\\ &\qquad + \cdots + u_k\sqrt{n}\left(\Lambda(t_k) - \Lambda(t_{k-1})\right)\\ &= \sqrt{n}\sum_{j=1}^{k} u_j \sum_{m=1}^{j}\left(\Lambda(t_m) - \Lambda(t_{m-1})\right)\\ &= \sqrt{n}\sum_{j=1}^{k} u_j\Lambda(t_j).\end{aligned} \tag{4.1.2.6}$$

Analyzing the leftover terms in (4.1.2.5), we have,

$$\begin{aligned}&\Lambda(t_1)\left(\frac{(u_1 + \cdots + u_k)^2}{2}\right) + \left(\Lambda(t_2) - \Lambda(t_1)\right)\left(\frac{(u_2 + \cdots + u_k)^2}{2}\right)\\ &\qquad + \cdots + \left(\Lambda(t_k) - \Lambda(t_{k-1})\right)\left(\frac{(u_k)^2}{2}\right) + O(n^{-1/2})\\ &= \frac{1}{2}\left(\Lambda(t_1)(u_1^2 + 2u_1(u_2 + \cdots + u_k)) + \cdots + \Lambda(t_j)(u_j^2 + 2u_j(u_{j+1} + \cdots + u_k)\right.\\ &\qquad \left. + \cdots + \Lambda(t_k)u_k^2\right) + O(n^{-1/2}).\end{aligned} \tag{4.1.2.7}$$

Therefore, from (4.1.2.6) and (4.1.2.7),

$$\begin{aligned}\log((1)) &= \sqrt{n}\sum_{j=1}^{k} u_j\Lambda(t_j) - \sqrt{n}\sum_{j=1}^{k} u_j\Lambda(t_j) + \frac{1}{2}\langle(\Lambda(t_1),...,\Lambda(t_j),...,\Lambda(t_k)),\\ &\qquad ((u_1^2+2u_1(u_2+\cdots+u_k)),...,(u_j^2+2u_j(u_{j+1}+\cdots+u_k)),...,(u_k^2))\big\rangle\\ &\qquad + O(n^{-1/2})\\ &= \frac{1}{2}\langle(\Lambda(t_1),...,\Lambda(t_j),...,\Lambda(t_k)),\\ &\qquad ((u_1^2+2u_1(u_2+\cdots+u_k)),...,(u_j^2+2u_j(u_{j+1}+\cdots+u_k)),...,(u_k^2))\big\rangle\\ &\qquad + O(n^{-1/2}), \end{aligned} \tag{4.1.2.8}$$

where as $n \to \infty$, (4.1.2.8) converges to the log MGF for a Gaussian vector with mean 0 and covariance matrix $\Sigma = [\sigma_{ij}]$ with $\sigma_{ij} = \Lambda(t_i \wedge t_j)$ for $1 \leq i, j \leq k$. ■

We finish the analysis of $\hat{\Lambda}_n(t)$ through a FCLT which will be used to analyze the asymptotics of the estimator $\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)}$.

Lemma 4.1.3 (FCLT) $\sqrt{n}\left(\hat{\Lambda}_n(t) - \Lambda(t)\right) \Rightarrow (Z(t) : t \geq 0)$ *as* $n \to \infty$, *where* $(Z(t) : t \geq 0)$ *is a Gaussian process with mean 0 and covariance function* $\rho(s,t) = \Lambda(s \wedge t)$.

Proof Lemmas 4.1.1 and 4.1.2 prove the necessary conditions for Theorem 1.6.1 [44]. ■

Lemma 4.1.3 along with Theorems 1.6.5 and 1.6.4 allow us to prove a FCLT for the estimator $\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)}$.

Theorem 4.1.4 (FCLT) *As* $n \to \infty$,

$$\sqrt{n}\left(\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\Lambda(T)}\right) \Rightarrow \left(\frac{1}{\Lambda(T)}Z(t) - \frac{\Lambda(t)}{\Lambda(T)^2}Z(T) : t \geq 0\right) := \left(\hat{Z}(t) : t \geq 0\right),$$

where $(Z(t) : t \geq 0)$ *is a Gaussian process with mean 0 and covariance function* $\rho(s,t) = \Lambda(s \wedge t)$ *such that the covariance function of* $\left(\hat{Z}(t) : t \geq 0\right)$ *is* $\rho(s,t) = \frac{1}{\Lambda(T)}\left(\frac{\Lambda(s\wedge t)}{\Lambda(T)} - \frac{\Lambda(s)\Lambda(t)}{\Lambda(T)^2}\right)$.

Proof Observe that

$$\begin{aligned}
&\sqrt{n}\left(\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\Lambda(T)}\right) \\
&\quad = \sqrt{n}\left(\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\hat{\Lambda}_n(T)} + \frac{\Lambda(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\Lambda(T)}\right) \\
&\quad = \sqrt{n}\left(\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\hat{\Lambda}_n(T)}\right) + \sqrt{n}\left(\frac{\Lambda(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\Lambda(T)}\right) \\
&\quad = \frac{\sqrt{n}}{\hat{\Lambda}_n(T)}\left(\hat{\Lambda}_n(t) - \Lambda(t)\right) + \frac{\sqrt{n}}{\hat{\Lambda}_n(T)}\frac{\Lambda(t)}{\Lambda(T)}\left(\Lambda(T) - \hat{\Lambda}_n(T)\right) \qquad (4.1.4.1)
\end{aligned}$$

From Lemma 4.1.3, $\left(\sqrt{n}\left(\hat{\Lambda}_n(t) - \Lambda(t)\right) : t \geq 0\right) \Rightarrow (Z(t) : t \geq 0)$. We also know by the FSLLN that $\sup_{t\in[0,T)}\left|\hat{\Lambda}_n(t) - \Lambda(t)\right| \to 0$ [20]. Therefore, by Theorems 1.6.5 and 1.6.4,

$$\sqrt{n}\frac{1}{\hat{\Lambda}_n(T)}\left(\hat{\Lambda}_n(t) - \Lambda(t)\right) \Rightarrow \frac{1}{\Lambda(T)}(Z(t) : t \geq 0) \quad \text{as} \quad n \to \infty.$$

Now we consider the second term in (4.1.4.1). Once again, by the FSLLN [20], Lemma 4.1.3 and Theorem 1.6.5,

$$\frac{\sqrt{n}}{\hat{\Lambda}_n(T)}\frac{\Lambda(t)}{\Lambda(T)}\left(\Lambda(T) - \hat{\Lambda}_n(T)\right) \Rightarrow -\frac{\Lambda(t)}{\Lambda(T)^2}Z(T).$$

Because the sum operator is continuous on $D \times R$, by Theorem 1.6.4,

$$\begin{aligned}
&\left(\sqrt{n}\frac{1}{\hat{\Lambda}_n(T)}\left(\hat{\Lambda}_n(t) - \Lambda(t)\right) + \frac{\sqrt{n}}{\hat{\Lambda}_n(T)}\frac{\Lambda(t)}{\Lambda(T)}\left(\Lambda(T) - \hat{\Lambda}_n(T)\right) : t \geq 0\right) \\
&\qquad\qquad \Rightarrow \left(\frac{1}{\Lambda(T)}Z(t) - \frac{\Lambda(t)}{\Lambda(T)^2}Z(T) : t \geq 0\right),
\end{aligned}$$

proving the claim. ■

Through an FCLT, we established the asymptotic rate of convergence for a nonparametric estimator with observed count data in a finite interval $[0, T]$. Note that the asymptotic variance of $\sqrt{n}\left(\frac{\hat{\Lambda}_n(t)}{\hat{\Lambda}_n(T)} - \frac{\Lambda(t)}{\Lambda(T)}\right)$ is $\frac{1}{\Lambda(T)}F(t)(1 - F(t))$. By Donsker's theorem, we should anticipate that the appropriate covariance of the estimator should be $F(t)(1-F(t))$. Indeed, to see this, we consider a more "natural" estimator, based on the multinomial structure of the arrival process.

4.2 Nonparametric Multinomial Estimator

Now, suppose we observe arrival count data at fixed intervals of width $\delta_n > 0$ for n i.i.d. $\Delta_{(i)}$ arrival processes with arrival epoch distribution $F(t)$. Our objective is to estimate $F(t)$ from available data. Let $X_i(s,t)$ be the number of arrivals occurring in the interval $(s,t]$ for observation i. Assume a population size of m such that $\sum_{j=1}^{k} X_j(t_{j-1}, t_j) = m$ for $i = \{1, ..., n\}$, where $t_0 = 0$, and $k = \left\lfloor \frac{T}{\delta_n} \right\rfloor$ is the number of intervals. Also, for $t \geq 0$, let $l_n(t) = \left\lfloor \frac{t}{\delta_n} \right\rfloor \delta_n$ be the lower bound of the interval for which t falls. It is well-known that the count data has a multinomial distribution,

$$P\left(X_i(0,t_1) = x_1, X_i(t_1,t_2) = x_2, ..., X_i(t_{k-1},t_k) = x_k\right) = \frac{m!}{x_1! \cdots x_k!} p_1^{x_1} \times \cdots \times p_k^{x_k},$$

where $p_j = F(t_j) - F(t_{j-1})$, $\sum_{j=1}^{k} p_j = 1$, and $F(t_j) = \sum_{l=1}^{j} p_l$. The maximum likelihood estimator (MLE) for the multinomial distribution is well-known to be $\hat{p}_j = \frac{x_j}{m}$ for $j = \{1, .., k\}$ [50]. Therefore, the natural estimator for the arrival distribution at $t \geq 0$ is the random variable

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{m} X_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{m} X_i(l_n(t), l_n(t) + \delta_n).$$

Let τ_{il} be the arrival time of the lth individual for observation i. We note that at interval edges $t_j := j\delta_n$, for $j = 1, ..., k$,

$$\begin{aligned} F_n(t_j) &= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{m} X_i(0, t_j) \\ &= \frac{1}{nm} \sum_{i=1}^{n} \sum_{l=1}^{m} \mathbb{1}\{\tau_{il} \leq t_j\} \\ &\stackrel{D}{=} \frac{1}{nm} \sum_{z=1}^{nm} \mathbb{1}\{\tau_z \leq t_j\}, \end{aligned} \tag{$\star$}$$

where the τ_{il}s are iid random variables from common distribution $F(t)$. So $(\star)$ is simply the empirical distribution function. Therefore, we know the asymptotic properties of $F_n(t)$ at interval edges. What is left to show is the asymptotic properties for the linearly interpolated portions as the interval size $\delta_n \to 0$ as $n \to \infty$. We start with a FSLLN to show asymptotic consistency and then establish a FCLT.

Theorem 4.2.1 (FSLLN) *If $\delta_n \to 0$ as $n \to \infty$, then $\sup_t |F_n(t) - F(t)| \to 0$ a.s. as $n \to \infty$.*

Proof Observe that $F_n(t) = \frac{1}{nm} \sum_{i=1}^n X_i(0,t) + R_n$, where

$$R_n = -\frac{1}{nm} \sum_{i=1}^n X_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n} \frac{1}{nm} \sum_{i=1}^n X_i(l_n(t), l_n(t) + \delta_n).$$

Since $l_n(t) \leq t < l_n(t) + \delta_n$,

$$|R_n| \leq \frac{1}{nm} \sum_{i=1}^n X_i(l_n(t), l_n(t) + \delta_n).$$

Note that since $l_n(t) \to t$ and $\delta_n \to 0$ as $n \to \infty$, for each $\epsilon > 0$, $t - \epsilon \leq l_n(t) \leq l_n(t) + \delta_n \leq t + \epsilon$ for n sufficiently large. Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} |R_n| &\leq \limsup_{n\to\infty} \frac{1}{nm} \sum_{i=1}^n X_i(l_n(t), l_n(t) + \delta_n) \\
&\leq \limsup_{n\to\infty} \frac{1}{nm} \sum_{i=1}^n X_i(t - \epsilon, t + \epsilon) \\
&\stackrel{D}{=} \limsup_{n\to\infty} \frac{1}{nm} \sum_{i=1}^n \sum_{l=1}^m \mathbb{1}\{t - \epsilon \leq \tau_{il} \leq t + \epsilon\} \\
&= F(t + \epsilon) - F(t - \epsilon),
\end{aligned}$$

by the strong law of large numbers. Therefore, $\lim_{\epsilon\to 0} \limsup_{n\to\infty} |R_n| = 0$. By the strong law of large numbers, $\limsup_{n\to\infty} \frac{1}{nm} \sum_{i=1}^n X_i(0,t) \to F(t)$. Since $F(t)$ is a non-decreasing continuous function, the pointwise almost sure convergence implies the uniform convergence. ∎

Next, we establish an FCLT to identify the rate of convergence and the asymptotic covariance. First, we show that the finite dimensional distribution's (FDD's) converge.

Lemma 4.2.2 (FDD's) *For $f(t)$ is Lipschitz continuous in the neighborhood of t with Lipschitz constant K, $\delta_n = o(n^{-1/4})$, and $0 = t_0 < t_1 < \cdots < t_k = T$ such that $\sum_{j=1}^k X_i(l_n(t_{j-1}), l_n(t_{j-1} + \delta_n)) = m \quad \forall \quad i = 1, ..., n$,*

$$\left(\sqrt{n}\,(F_n(t_1) - F(t_1))\,, \cdots, \sqrt{n}\,(F_n(t_k) - F(t_k))\right) \Rightarrow (Z(t_1), ..., Z(t_k))$$

as $n \to \infty$, where $(Z(t_1), ..., Z(t_k))$ is a Gaussian vector with covariance matrix $\Sigma = [\sigma_{ij}]$, where $\sigma_j^2 = F(t_j)(1 - F(t_j))$, and $\sigma_{ij} = (F(t_i \wedge t_j) - F(t_i)F(t_j))$ for $i \neq j$.

Proof Let $k = 2$. Observe the characteristic function of the random vector $(\sqrt{n}\,(mF_n(t_1) - mF(t_1))\,, \sqrt{n}\,(mF_n(t_2) - mF(t_2)))$, for $s \in \mathcal{R}$ and $i^2 = -1$, is

$$\begin{aligned}
&E\left[\exp\left\{is\left\langle(u_1, u_2), \left(\sqrt{n}\,(mF_n(t_1) - mF(t_1))\,, \sqrt{n}\,(mF_n(t_2) - mF(t_2))\right)\right\rangle\right\}\right] \\
&= E\left[\exp\left\{is\left(u_1 m\sqrt{n}F_n(t_1) + u_2 m\sqrt{n}F_n(t_2) - u_1 m\sqrt{n}F(t_1) - u_2 m\sqrt{n}F(t_2)\right)\right\}\right] \\
&= E\left[\exp\left\{is\left(u_1 m\sqrt{n}F_n(t_1) + u_2 m\sqrt{n}F_n(t_2)\right)\right\}\right] \\
&\qquad\qquad \exp\left\{is\left(-u_1 m\sqrt{n}F(t_1) - u_2 m\sqrt{n}F(t_2)\right)\right\} \\
&= E\left[\exp\left\{is\left(\frac{u_1}{\sqrt{n}}\sum_{i=1}^{n} X_i(0, l_n(t_1)) + \frac{u_1}{\sqrt{n}}\frac{t_1 - l_n(t_1)}{\delta_n}\sum_{i=1}^{n} X_i(l_n(t_1), l_n(t_1) + \delta_n)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \frac{u_2}{\sqrt{n}}\sum_{i=1}^{n} X_i(0, l_n(t_2)) + \frac{u_2}{\sqrt{n}}\frac{t_2 - l_n(t_2)}{\delta_n}\sum_{i=1}^{n} X_i(l_n(t_2), l_n(t_2) + \delta_n)\right)\right\}\right] \\
&\qquad\qquad \exp\left\{is\left(-u_1 m\sqrt{n}F(t_1) - u_2 m\sqrt{n}F(t_2)\right)\right\} \\
&= E\left[\exp\left\{is\left(\frac{u_1 + u_2}{\sqrt{n}}\sum_{i=1}^{n} X_i(0, l_n(t_1))\right.\right.\right. \\
&\qquad\qquad + \left(\frac{u_2}{\sqrt{n}} + \frac{u_1}{\sqrt{n}}\frac{t_1 - l_n(t_1)}{\delta_n}\right)\sum_{i=1}^{n} X_i(l_n(t_1), l_n(t_1) + \delta_n) \\
&\qquad\qquad + \frac{u_2}{\sqrt{n}}\sum_{i=1}^{n} X_i(l_n(t_1) + \delta_n, l_n(t_2)) \\
&\qquad\qquad \left.\left.\left. + \frac{u_2}{\sqrt{n}}\frac{t_2 - l_n(t_2)}{\delta_n}\sum_{i=1}^{n} X_i(l_n(t_2), l_n(t_2) + \delta_n)\right)\right\}\right] \\
&\qquad\qquad \exp\left\{is\left(-u_1 m\sqrt{n}F(t_1) - u_2 m\sqrt{n}F(t_2)\right)\right\}. \qquad (4.2.2.1)
\end{aligned}$$

Next, let $Y \sim \text{Mult}(m, \mathbf{p})$, with

$$\begin{aligned}
Y = [&X_i(0, l_n(t_1)), X_i(l_n(t_1), l_n(t_1) + \delta_n), \\
&\qquad X_i(l_n(t_1) + \delta_n, l_n(t_2)), X_i(l_n(t_2), l_n(t_2) + \delta_n)]
\end{aligned}$$

and

$$\mathbf{p} = [F(l_n(t_1)), F(l_n(t_1) + \delta_n) - F(l_n(t_1)),$$

$$F(l_n(t_2)) - F(l_n(t_1)+\delta_n), F(l_n(t_2)+\delta_n) - F(l_n(t_2))] \,.$$

Also, let

$$\mathbf{v} = \left[\frac{u_1+u_2}{\sqrt{n}}, \left(\frac{u_2}{\sqrt{n}} + \frac{u_1}{\sqrt{n}}\frac{t_1 - l_n(t_1)}{\delta_n}\right), \frac{u_2}{\sqrt{n}}, \frac{u_2}{\sqrt{n}}\frac{t_2 - l_n(t_2)}{\delta_n}\right].$$

Observe that we can rewrite (4.2.2.1) as

$$\begin{aligned}
(4.2.2.1) &= E\left[\exp\left\{is\sum_{i=1}^{n}\left\langle \mathbf{v}, Y^i\right\rangle\right\}\right]\exp\left\{is\left(-u_1 m\sqrt{n}F(t_1) - u_2 m\sqrt{n}F(t_2)\right)\right\} \\
&= E\left[\exp\left\{is\sum_{i=1}^{n}\left\langle \mathbf{v}, Y^i\right\rangle - isu_1 m\sqrt{n}F(t_1) - isu_2 m\sqrt{n}F(t_2)\right\}\right], \\
&= E\left[\exp\left\{is\sum_{i=1}^{n}\left(\left\langle \mathbf{v}, Y^i\right\rangle - \frac{u_1 m}{\sqrt{n}}F(t_1) - \frac{u_2 m}{\sqrt{n}}F(t_2)\right)\right\}\right], \qquad (4.2.2.2)
\end{aligned}$$

where Y^i is the ith i.i.d. observation of the multinomial random variable Y. Hence,

$$\begin{aligned}
(4.2.2.2) &= E\left[\exp\left\{is\left(\langle \mathbf{v}, Y\rangle - \frac{u_1 m}{\sqrt{n}}F(t_1) - \frac{u_2 m}{\sqrt{n}}F(t_2)\right)\right\}\right]^n \\
&= \left(\phi_{\bar{Y}}(s)\right)^n. \qquad (4.2.2.3)
\end{aligned}$$

$\phi_{\bar{Y}}(s) = \int_{-\infty}^{+\infty} e^{is\bar{y}} f(\bar{y})d\bar{y}$, where $\bar{Y} = \langle \mathbf{v}, Y\rangle - \frac{u_1 m}{\sqrt{n}}F(t_1) - \frac{u_2 m}{\sqrt{n}}F(t_2)$. Recall the second order Taylor expansion around 0, $\phi_{\bar{Y}}(s) = \phi_{\bar{Y}}(0) + s\phi_{\bar{Y}}'(0) + \frac{s^2}{2}\phi_{\bar{Y}}''(0) + o(s^2)$, where $\phi_{\bar{Y}}'(s) = i\int_{-\infty}^{+\infty} \bar{y}e^{-is\bar{y}} f(\bar{y})d\bar{y}$ and $\phi_{\bar{Y}}''(s) = i^2\int_{-\infty}^{+\infty} \bar{y}^2 e^{is\bar{y}} f(\bar{y})d\bar{y}$. Therefore,

$$\begin{aligned}
\phi_{\bar{Y}}(s) &= 1 + iE\left[\bar{Y}\right] - \frac{s^2}{2}E\left[\bar{Y}^2\right] + o(s^2) \\
&= 1 + \frac{i}{n}E\left[n\bar{Y}\right] - \frac{s^2}{2n}E\left[n\bar{Y}^2\right] + \frac{\eta(s,n)}{n}, \qquad (4.2.2.4)
\end{aligned}$$

where $\eta(s,n) \to 0$ as $n \to \infty$. Using (4) and letting $x = \frac{i}{n}E\left[n\bar{Y}\right] - \frac{s^2}{2n}E\left[n\bar{Y}^2\right] + \frac{\eta(s,n)}{n}$, we have

$$\begin{aligned}
(4.2.2.3) &= (1+x)^{\frac{i}{x}E\left[n\bar{Y}\right] - \frac{s^2}{2x}E\left[n\bar{Y}^2\right] + \frac{\eta(s,n)}{x}} \\
&= (1+x)^{\frac{i}{x}E\left[n\bar{Y}\right]}(1+x)^{-\frac{s^2}{2x}E\left[n\bar{Y}^2\right]}(1+x)^{\frac{\eta(s,n)}{x}}. \qquad (4.2.2.5)
\end{aligned}$$

Next, consider $E\left[n\bar{Y}\right]$.

$$E\left[n\bar{Y}\right] = nE\left[\langle \mathbf{v}, Y\rangle - \frac{u_1 m}{\sqrt{n}}F(t_1) - \frac{u_2 m}{\sqrt{n}}F(t_2)\right]$$

$$
\begin{aligned}
&= nE\left[v_1Y_1 + v_2Y_2 + v_3Y_3 + v_4Y_4\right] - \sqrt{n}u_1mF(t_1) - \sqrt{n}u_2mF(t_2) \\
&= nv_1mp_1 + nv_2mp_2 + nv_3mp_3 + nv_4mp_4 - \sqrt{n}u_1mF(t_1) - \sqrt{n}u_2mF(t_2) \\
&= \sqrt{n}(u_1+u_2)mp_1 + \sqrt{n}\left(u_2 + u_1\frac{t_1 - l_n(t_1)}{\delta_n}\right)mp_2 \\
&\qquad + \sqrt{n}u_2mp_3 + u_2\frac{t_2 - l_n(t_2)}{\delta_n}mp_4 - \sqrt{n}u_1mF(t_1) - \sqrt{n}u_2mF(t_2) \\
&= u_1\sqrt{n}\left(mp_1 - mF(t_1)\right) + u_2\sqrt{n}\left(mp_1 + mp_2 + mp_3 - mF(t_2)\right) \\
&\qquad + u_1\sqrt{n}\frac{t_1 - l_n(t_1)}{\delta_n}mp_2 + u_2\sqrt{n}\frac{t_2 - l_n(t_2)}{\delta_n}mp_4 \\
&= u_1\sqrt{n}m\left(F(l_n(t_1)) - F(t_1)\right) + u_2\sqrt{n}m\left(F(l_n(t_2)) - F(t_2)\right) \\
&\qquad + u_1\sqrt{n}\frac{t_1 - l_n(t_1)}{\delta_n}mp_2 + u_2\sqrt{n}\frac{t_2 - l_n(t_2)}{\delta_n}mp_4. \qquad (4.2.2.5)
\end{aligned}
$$

Next, let $\theta_1 \in [l_n(t_1), t_1)$, $\theta_2 \in [l_n(t_2), t_2)$, $\theta_3 \in [l_n(t_1), l_n(t_1)+\delta_n)$ and $\theta_4 \in [l_n(t_2), l_n(t_2)+\delta_n)$. By the Mean Value Theorem, it follows that

$$
\begin{aligned}
(4.2.2.5) &= -u_1\sqrt{n}mf(\theta_1)\left(t_1 - l_n(t_1)\right) - u_2\sqrt{n}mf(\theta_2)\left(t_2 - l_n(t_2)\right) \\
&\qquad + u_1\sqrt{n}\frac{t_1 - l_n(t_1)}{\delta_n}mf(\theta_3)\delta_n + u_2\sqrt{n}\frac{t_2 - l_n(t_2)}{\delta_n}mf(\theta_4)\delta_n \\
&= -u_1\sqrt{n}mf(\theta_1)\left(t_1 - l_n(t_1)\right) - u_2\sqrt{n}mf(\theta_2)\left(t_2 - l_n(t_2)\right) \\
&\qquad + u_1\sqrt{n}(t_1 - l_n(t_1))mf(\theta_3) + u_2\sqrt{n}(t_2 - l_n(t_2))mf(\theta_4) \\
&= u_1\sqrt{n}m\left(t_1 - l_n(t_1)\right)\left(f(\theta_3) - f(\theta_1)\right) + u_2\sqrt{n}m\left(t_2 - l_n(t_2)\right)\left(f(\theta_4) - f(\theta_2)\right). \qquad (4.2.2.6)
\end{aligned}
$$

Since $f(\cdot)$ is Lipschitz continuous with constant K and $\theta_3 - \theta_1 \le \delta_n$, $\theta_4 - \theta_2 \le \delta_n$, (4.2.2.6)$\le$ $u_1\sqrt{n}mK\delta_n^2 + u_2\sqrt{n}mK\delta_n^2 \to 0$ as $n \to \infty$. Next, consider $E\left[n\bar{Y}^2\right]$. Observe that

$$
\begin{aligned}
E\left[n\bar{Y}^2\right] &= nE\left[\left(\langle \mathbf{v}, Y\rangle - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)^2\right] \\
&= nE\left[\left(v_1Y_1 + v_3Y_3 - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)^2\right] + nE\left[(v_2Y_2 + v_4Y_4)^2\right] \\
&\qquad + 2nE\left[(v_2Y_2 + v_4Y_4)\left(v_1Y_1 + v_3Y_3 - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)\right]. \qquad (4.2.2.7)
\end{aligned}
$$

Consider the second two terms in (4.2.2.7). Recall that $E[Y_2] = mp_2, E[Y_4] = mp_4$, $\text{Var}(Y_2) = mp_2(1-p_2)$, $\text{Var}(Y_4) = mp_4(1-p_4)$, and $\text{Cov}(Y_i, Y_j) = -mp_ip_j$. Since $p_2 = F(l_n(t_1)+\delta_n) - F(l_n(t_1)) \to 0$ and $p_4 = F(l_n(t_2)+\delta_n) - F(l_n(t_2)) \to 0$ as $n \to \infty$, we have,

$$nE\left[(v_2Y_2 + v_4Y_4)^2\right] \\ + 2nE\left[(v_2Y_2+v_4Y_4)\left(v_1Y_1 + v_3Y_3 - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)\right] \to 0$$

as $n \to \infty$. Next, consider the first expectation in (4.2.2.7). We have,

$$\begin{aligned}
&nE\left[\left(v_1Y_1 + v_3Y_3 - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)^2\right]\\
&= nE\left[\left(\frac{u_1}{\sqrt{n}}X(0,l_n(t_1)) + \frac{u_2}{\sqrt{n}}X(0,l_n(t_1)) + \frac{u_2}{\sqrt{n}}X(l_n(t_1)+\delta_n, l_n(t_2))\right.\right.\\
&\qquad\qquad \left.\left. - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)^2\right]\\
&= nE\left[\left(\frac{u_1}{\sqrt{n}}X(0,l_n(t_1)) + \frac{u_2}{\sqrt{n}}\left(X(0,l_n(t_1)) + X(l_n(t_1)+\delta_n, l_n(t_2))\right)\right.\right.\\
&\qquad\qquad \left.\left. - \frac{u_1m}{\sqrt{n}}F(t_1) - \frac{u_2m}{\sqrt{n}}F(t_2)\right)^2\right]\\
&= nE\left[\left\langle\left(\frac{u_1}{\sqrt{n}}, \frac{u_2}{\sqrt{n}}\right),\right.\right.\\
&\qquad\qquad \left.\left.(X(0,l_n(t_1)) - mF(t_1), X(0,l_n(t_1)) + X(l_n(t_1)+\delta_n, l_n(t_2)) - mF(t_2))\right\rangle^2\right].
\end{aligned} \tag{4.2.2.8}$$

Now, since $l_n(t) \to t$ and $\delta_n \to 0$ as $n \to \infty$, $X(0, l_n(t_1)) \to X(0,t_1)$ and $X(0,l_n(t_1)) + X(l_n(t_1)+\delta_n, l_n(t_2)) \to X(0,t_2)$ as $n \to \infty$. So as $n \to \infty$,

$$\begin{aligned}
(4.2.2.8) \to nE&\left[\left\langle\left(\frac{u_1}{\sqrt{n}}, \frac{u_2}{\sqrt{n}}\right), (X(0,t_1) - mF(t_1), X(0,t_2) - mF(t_2))\right\rangle^2\right]\\
&= n\text{Var}\left(\left\langle\left(\frac{u_1}{\sqrt{n}}, \frac{u_2}{\sqrt{n}}\right), (X(0,t_1), X(0,t_2))\right\rangle\right)\\
&= \mathbf{u}^T\Sigma\mathbf{u},
\end{aligned}$$

where $\mathbf{u}^T$=$[u_1, u_2]$ and Σ is the covariance matrix for $X(0,t_1), X(0,t_2)$, such that

$$\text{Cov}\left(X(0,t_1), X(0,t_2)\right) = \text{Cov}\left(X(0,t_1), X(0,t_1)\right) + \text{Cov}\left(X(0,t_1), X(t_1,t_2)\right)$$

$$= mF(t_1)(1 - F(t_1)) - mF(t_1)\left(F(t_2) - F(t_1)\right)$$
$$= mF(t_1)\left(1 - F(t_2)\right).$$

Therefore, taking limits as $x \to 0$, $(n \to \infty)$, and using $e = \lim_{x\to 0}(1+x)^{1/x}$

$$\lim_{x\to 0}(1+x)^{\frac{i}{x}E\left[n\bar{Y}\right]}(1+x)^{-\frac{s^2}{2x}E\left[n\bar{Y}^2\right]}(1+x)^{\frac{\eta(s,n)}{x}}$$
$$= e^{-\frac{s^2}{2}\mathbf{u}^T\Sigma\mathbf{u}}$$

which is the characteristic function of a zero mean Gaussian random vector with covariance matrix Σ.

■

Next, we show the two conditions of Theorem 1.6.1, which will allow us to conclude an FCLT for the estimator $F_n(t)$.

Lemma 4.2.3 *Let* $\bar{X}_i(t) := X_i(0, l_n(t)) + \frac{t - l_n(t)}{\delta_n} X_i(l_n(u), l_n(u) + \delta_n)$ *and* $0 \le s \le t \le u$ *such that* $X_i(0, u) = m$ *for all* $i = 1, ..., n$. *Then,*

$$(i)\quad E\left[\left(\left(\bar{X}_i(u) - mF(u)\right) - \left(\bar{X}_i(t) - mF(t)\right)\right)^2\right] \le 2m^2\left(F(u) - F(t)\right)$$

and

$$(ii)\quad E\Big[\left(\left(\bar{X}_i(u) - mF(u)\right) - \left(\bar{X}_i(t) - mF(t)\right)\right)^2$$
$$\left(\left(\bar{X}_i(t) - mF(t)\right) - \left(\bar{X}_i(s) - mF(s)\right)\right)^2\Big] \le 8m^2\left(F(u) - F(s)\right)^2$$

Proof We begin by proving (i). First, we note that $\bar{X}_i(u) = X_i(0, u) + R_u$, where $R_u = -X_i(l_n(u), u) + \frac{u - l_n(u)}{\delta_n} X_i(l_n(u), l_n(u) + \delta_n)$, and similarly for $\bar{X}_i(t)$. We have,

$$E\left[\left(\left(\bar{X}_i(u) - mF(u)\right) - \left(\bar{X}_i(t) - mF(t)\right)\right)^2\right]$$
$$= E\left[\left((X_i(0,u) + R_u - mF(u)) - (X_i(0,t) + R_t - mF(t))\right)^2\right]$$
$$= E\left[\left((X_i(u) - mF(u)) - (X_i(t) - mF(t)) + (R_u - R_t)\right)^2\right]$$
$$= E\left[\left((X_i(0,u) - mF(u)) - (X_i(0,t) - mF(t))\right)^2\right] + E\left[(R_u - R_t)^2\right]$$
$$+ 2E\left[\left((X_i(0,u) - mF(u)) - (X_i(0,t) - mF(t))\right)(R_u - R_t)\right]. \qquad (\star)$$

We will first show the upper bound on the first term in $(\star)$. To reduce notation, we will drop the 0 in the count interval, letting $X_i(t) := X_i(0,t)$. We have,

$$E\left[\left((X_i(u) - mF(u)) - (X_i(t) - mF(t))\right)^2\right]$$
$$= E\left[(X_i(u) - X_i(t))^2\right] + (mF(u) - mF(t))^2 - 2E\left[X_i(u) - X_i(t)\right](mF(u) - mF(t)). \tag{4.2.3.1}$$

Recall that for $t \geq 0$, $(X_i(u) - X_i(t)) = X_i(t,u)$ which has mean $m\,(F(u) - F(t))$ and variance $m\,(F(u) - F(t))\,(1 - (F(u) - F(t)))$. Hence,

$$\begin{aligned}(4.2.3.1) &= E\left[(X_i(t,u))^2\right] - (mF(u) - mF(t))^2 \\ &= m\,(F(u) - F(t))\,(1 - (F(u) - F(t))) + m^2\,(F(u) - F(t))^2 - (mF(u) - mF(t))^2.\end{aligned} \tag{4.2.3.2}$$

Because $F(\cdot)$ is a nondecreasing function taking values in $[0,1]$,

$$\begin{aligned}(4.2.3.2) &\leq m\,(F(u) - F(t))\,(1 - (F(u) - F(t))) + m^2\,(F(u) - F(t))^2 \\ &\leq m\,(F(u) - F(t)) + m^2\,(F(u) - F(t))^2 \\ &\leq m^2\left[(F(u) - F(t)) + (F(u) - F(t))^2\right] \\ &\leq 2m^2\,(F(u) - F(t)).\end{aligned}$$

Now consider the second expectation in $(\star)$. We will show that as $n \to \infty$, this expectation is bounded above by 0. Observe that,

$$\begin{aligned}&E\left[(R_u - R_t)^2\right] \\ &= E\left[R_u^2\right] + E\left[R_t^2\right] - 2E\left[R_u R_t\right] \\ &\leq E\left[\left(-X_i(l_n(u), u) + \frac{u - l_n(u)}{\delta_n} X_i(l_n(u), l_n(u) + \delta_n)\right)^2\right] \\ &\qquad + E\left[\left(-X_i(l_n(t), t) + \frac{t - l_n(t)}{\delta_n} X_i(l_n(t), l_n(t) + \delta_n)\right)^2\right].\end{aligned}$$

Since $l_n(s) \to s$ and $\delta_n \to 0$ as $n \to \infty$, this bound converges to 0. The third expectation in $(\star)$ converges to 0 in a similar way.

$$2E\left[(R_u - R_t)\left((X_i(0,u) - mF(u)) - (X_i(0,t) - mF(t))\right)\right]$$

$$= 2E\left[R_u\left(X_i(0,u) - mF(u)\right)\right] - 2E\left[R_u\left(X_i(0,t) - mF(t)\right)\right]$$
$$- 2E\left[R_t\left(X_i(0,u) - mF(u)\right)\right] + 2E\left[R_t\left(X_i(0,t) - mF(t)\right)\right].$$

Since $l_n(s) \to s$ and $\delta_n \to 0$ as $n \to \infty$, these terms go to 0 in the limit. Therefore, we have proved inequality (i) as $n \to \infty$.

Next, we consider inequality (ii).

$$E\left[\left(\left(\bar{X}_i(u) - mF(u)\right) - \left(\bar{X}_i(t) - mF(t)\right)\right)^2 \left(\left(\bar{X}_i(t) - mF(t)\right) - \left(\bar{X}_i(s) - mF(s)\right)\right)^2\right]$$
$$= E\big[\left(\left(X_i(u) + R_u - mF(u)\right) - \left(X_i(t) + R_t - mF(t)\right)\right)^2$$
$$\left(\left(X_i(t) + R_t - mF(t)\right) - \left(X_i(s) + R_s - mF(s)\right)\right)^2\big]$$
$$= E\big[\left(\left(X_i(u) - mF(u)\right) - \left(X_i(t) - mF(t)\right) + \left(R_u - R_t\right)\right)^2$$
$$\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right) + \left(R_t - R_s\right)\right)^2\big]$$
$$= E\big[\big(\left(\left(X_i(u) - mF(u)\right) - \left(X_i(t) - mF(t)\right)\right)^2 + \left(R_u - R_t\right)^2$$
$$+2\left(\left(X_i(u) - mF(u)\right) - \left(X_i(t) - mF(t)\right)\right)\left(R_u - R_t\right)\big)$$
$$\big(\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)^2 + \left(R_t - R_s\right)^2$$
$$+2\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)\left(R_u - R_t\right)\big)\big]$$
$$= E\big[\left(\left(X_i(u) - mF(u)\right) - \left(X_i(t) - mF(t)\right)\right)^2$$
$$\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)^2\big] + R, \tag{$\star\star$}$$

where

$$R = E\big[\left(\left(X_i(u) - mF(u)\right) - \left(X_i(t) - mF(t)\right)\right)^2 \big(\left(R_t - R_s\right)^2$$
$$+2\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)\left(R_u - R_t\right)\big)\big]$$
$$+ E\big[\left(R_u - R_t\right)^2 \big(\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)^2$$
$$+\left(R_t - R_s\right)^2 + 2\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)\left(R_u - R_t\right)\big)\big]$$
$$+ E\big[2\left(\left(X_i(u) - mF(u)\right) - \left(X_i(t) - mF(t)\right)\right)\left(R_u - R_t\right)$$
$$\big(\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)^2 + \left(R_t - R_s\right)^2$$
$$+2\left(\left(X_i(t) - mF(t)\right) - \left(X_i(s) - mF(s)\right)\right)\left(R_u - R_t\right)\big)\big],$$

which converges to 0 as $n \to \infty$ since $l_n(t) \to t$ and $\delta_n \to 0$.

Analyzing the first expectation of $(\star\star)$, it follows that,

$$\begin{aligned}
&E\left[((X_i(u)-mF(u))-(X_i(t)-mF(t)))^2\left((X_i(t)-mF(t))-(X_i(s)-mF(s))\right)^2\right]\\
&=E\left[(X_i(t,u)-(mF(u)-F(t)))^2\left(X_i(s,t)-(mF(t)-F(s))\right)^2\right]\\
&=E\Big[\big(X_i(t,u)^2+(mF(u)-mF(t))^2-2X_i(t,u)\left(mF(u)-mF(t)\right)\big)\\
&\qquad\qquad\big(X_i(s,t)^2+(mF(t)-mF(s))^2-2X_i(s,t)\left(mF(t)-mF(s)\right)\big)\Big]. \qquad (4.2.3.3)
\end{aligned}$$

Now, we introduce notation in order to simplify the writing. Let $p_1=F(u)-F(t)$ and $p_2=F(t)-F(s)$. Multiplying through and using the linearity of expectation, we have,

$$\begin{aligned}
(4.2.3.3)={}&E\left[X_i(t,u)^2X_i(s,t)^2\right]+E\left[X_i(t,u)^2\right]m^2p_2^2-2E\left[X_i(t,u)^2X_i(s,t)\right]mp_2\\
&+m^2p_1^2E\left[X_i(s,t)^2\right]+m^4p_1^2p_2^2-2E\left[X_i(s,t)\right]m^4p_1^2p_2^2\\
&-2mp_1E\left[X_i(t,u)X_i(s,t)^2\right]-2m^3p_1p_2^2E\left[X_i(t,u)\right]\\
&+4E\left[X_i(t,u)X_i(s,t)\right]m_1^2p_2. \qquad (4.2.3.4)
\end{aligned}$$

Recall $(X_i(0,s),X_i(s,t),X_i(t,u)$ is a multinomial random variable and that for $X_1,X_2,...,X_k$, a k-dimensional multinomial random variable with parameters m, $[p_1,...,p_k]$, for $i\neq j$,

$$\begin{aligned}
&E[X_i]=mp_i\\
&E[X_iX_j]=m(m-1)p_ip_j\\
&E[X_i^2]=m(m-1)p_i^2+mp_i\\
&E[X_i^2X_j]=m(m-1)(m-2)p_i^2p_j+m(m-1)p_ip_j\\
&E[X_i^2X_j^2]=m(m-1)(m-2)(m-3)p_i^2p_j^2\\
&\qquad\qquad+m(m-1)(m-2)(p_i^2p_j+p_ip_j^2)+m(m-1)p_ip_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(4.2.3.4)={}&m(m-1)(m-2)(m-3)p_1^2p_2^2+m(m-1)(m-2)(p_1^2p_2+p_1p_2^2)\\
&+m(m-1)p_1p_2+m^3(m-1)p_1^2p_2^2+m^3p_1p_2^2-2m^2(m-1)(m-2)p_1^2p_2^2
\end{aligned}$$

$$
\begin{aligned}
&- 2m^2(m-1)p_1p_2^2 + m^3(m-1)p_1^2p_2^2 + m^3p_1^2p_2 + m^4p_1^2p_2^2 - 2m^4p_1^2p_2^2 \\
&- 2m^2(m-1)(m-2)p_1^2p_2^2 - 2m^2(m-1)p_1^2p_2 - 2m^4p_1^2p_2^2 \\
&+ 4m^3(m-1)p_1^2p_2^2. \qquad (4.2.3.5)
\end{aligned}
$$

Simplifying the terms, it follows that

$$
(4.2.3.5) = p_1^2p_2^2(3m^2-6m) + p_1^2p_2(-m^2+2m) + p_1p_2^2(-m^2+2m) + p_1p_2(m^2-m). \tag{6}
$$

Next, since $p_1, p_2 \geq 0$, it follows that

$$
\begin{aligned}
(4.2.3.6) &\leq 3m^2p_1^2p_2^2 + 2mp_1^2p_2 + 2mp_1p_2^2 + m^2p_1p_2 \\
&= 3m^2\left(F(u)-F(t)\right)^2\left(F(t)-F(s)\right)^2 + 2m\left(F(u)-F(t)\right)^2\left(F(t)-F(s)\right) \\
&\quad + 2m\left(F(u)-F(t)\right)\left(F(t)-F(s)\right)^2 + m^2\left(F(u)-F(t)\right)\left(F(t)-F(s)\right).
\end{aligned} \tag{4.2.3.7}
$$

Because $F(\cdot)$ is nondecreasing, $\left(F(t)-F(s)\right), \left(F(u)-F(t)\right) \leq \left(F(u)-F(s)\right)$. Therefore,

$$
\begin{aligned}
(4.2.3.7) &\leq 3m^2\left(F(u)-F(s)\right)^2\left(F(u)-F(s)\right)^2 + 2m\left(F(u)-F(s)\right)^2\left(F(u)-F(s)\right) \\
&\quad + 2m\left(F(u)-F(s)\right)\left(F(u)-F(s)\right)^2 + m^2\left(F(u)-F(s)\right)\left(F(u)-F(s)\right) \\
&= 3m^2\left(F(u)-F(s)\right)^4 + 4m\left(F(u)-F(s)\right)^3 + m^2\left(F(u)-F(s)\right)^2.
\end{aligned} \tag{4.2.3.8}
$$

Lastly, since $F(\cdot) \in [0,1]$,

$$
\begin{aligned}
(4.2.3.8) &\leq \left(4m^2+4m\right)\left(F(u)-F(s)\right)^2 \\
&\leq 8m^2\left(F(u)-F(s)\right)^2,
\end{aligned}
$$

proving the claim. ■

Now, we have the necessary lemmas to prove an FCLT.

Theorem 4.2.4 (FCLT) *If $\delta_n \to 0$ as $n \to \infty$ such that $\delta_n = o(n^{-1/4})$, then $\sqrt{n}\left(F_n(t)-F(t)\right) \Rightarrow (Z(t) : t \geq 0)$, where $(Z(t) : t \geq 0)$ is a Gaussian process with mean 0 and covariance function $(\rho(s,t) : s,t \geq 0) := (F(s \wedge t) - F(s)F(t))$.*

By Lemmas 4.2.2 and 4.2.3, we have satisfied conditions for Theorem 1.6.1, proving the claim.

Through Theorem 4.2.4, we see that the asymptotic covariance function of the multinomial estimator matches the covariance structure for the empirical distribution function. However, we saw in Theorem 4.1.4 that the asymptotic covariance function included an extra scaling factor. We note that if the distribution $F(t)$ is defined on a finite interval, then the estimators are the same. Otherwise, by assuming an underlying NHPP driving the arrival process, the ratio estimator underestimates the covariance.

5. CONCLUSION

Non-stationary point processes are naturally modeled in real-world applications such as healthcare, natural disasters, or traffic patterns. The marked point process is of interest any time cumulative effects are crucial to understanding a process. We established asymptotic consistency and rates of convergence for a nonparametric estimator from observations of the offered load to a MPP at fixed intervals. We also show that an asymptotic rate to shrink the interval width exists to minimize a bound on the mean-squared error.

Another important non-stationary point process is the $M_t/G/\infty$ queue, which is central to many modeling applications. Classically, call centers are modeled with an infinite-server model, yet the $M_t/G/\infty$ queue can be leveraged for any multi-server process with time-varying arrivals. We analyzed the mean number of busy servers and established asymptotic consistency and rate of convergence for a nonparametric estimator based on discrete observations of the queue at fixed intervals. Through further research, the mean number of busy servers in an $M_t/G/\infty$ queue is to be used as an estimator for the arrival rate stochastic process in a doubly stochastic process. We plan to assume observed arrival count data at fixed intervals and apply an underlying $M_t/G/\infty$ queue to an arrival process. We are also interested in obtaining an asymptotically optimal δ_n to minimize a bound on the mean-squared error, as in the offered load estimator. We also note that the offered load estimator and the mean busy servers estimator can be constructed from the same Poisson random measure. We are interested in generalizing results for nonparametric estimators of Poisson random measures.

Lastly, we considered the transitory queueing model, which arises in any finite time or population situation. Although the empirical distribution function is a well-known nonparametric estimator that is consistent and efficient, we assume the only available data is arrival counts in fixed intervals. By assuming an underlying NHPP arrival process, we find that the natural estimator underestimates the covariance of the limit process. Instead, using the maximum likelihood estimator for the multinomial distribution, we show convergence of the arrival process distribution to the well-known Brownian motion. We are interested in understanding how the two estimators compare for various scenarios of the underlying arrival distribution $F(t)$. We plan to conduct a simulation analysis to support this analytic comparison. Further research will also understand how the population size and sample size scale together in the asymptotic results.

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