QUASI-TOROIDAL VARIETIES AND RATIONAL LOG STRUCTURES IN CHARACTERISTIC 0

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Andres Figuerola

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2019

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Professor Jarosław Włodarczyk, Chair
Department of Mathematics
Professor Donu Arapura
Department of Mathematics
Professor Tong Liu
Department of Mathematics
Professor Sai-Kee Yeung
Department of Mathematics

Approved by:
Professor David Goldberg
Head of the School Graduate Program

ACKNOWLEDGMENTS

I would like to thank Professor Tong Liu, Professor Sai-Kee Yeung and Professor Donu Arapura for agreeing to be part of my thesis committee and for helpful comments in the presentation of this work.

To my fellow graduate students I met along the way Feng Hao, Hongshan Li, Artur Jackson, Heng Du, Youngsu Kim, Partha Solapurkar, Roberto Ulloa, Jake Desmond, Justin Katz, Joe Knight, Pavel Coupek and particularly to Nick Miller, I thank them for all the conversations we shared over the years.

I would also like to thank Professor Deepam Patel, whose classes I particularly benefited from. A very special thank you is indebted to Professor Kenji Matsuki for mentoring me since the beginning of the program and inculcating his view of mathematics in me.

To my parents Luz Maria and Marcelo for their support along this journey, and to my roommate and best friend Luzma for always encouraging me, I thank them deeply.

Finally I would like to thank my advisor Jarosław Włodarczyk, his constant support, patience, insight and guide were most essential to my education, finding a research path and succeeding in the program.

This work was partially supported by BSF grant 2014365.

TABLE OF CONTENTS

Page

ABSTRACT vi

1 INTRODUCTION 1

1.1 Outline 2

1.2 Conventions and pre-requisites 3

2 RATIONAL LOG VARIETIES 5

2.1 Notation and Background 5

2.2 Toric varieties 8

2.3 Toroidal embeddings and regular log schemes 8

2.4 A motivating example 9

2.5 Quasi toroidal varieties 10

2.6 Simplicial Kummer fs varieties 11

2.7 Conditions for being a chart 12

2.8 Reduced induced monoid 13

2.9 Rational log varieties 13

2.10 Charts of saturated rational log varieties 14

2.11 Zariski optimized charts 15

2.12 The rank stratification 16

2.13 First assumption on the log structure of rational log varieties 18

2.14 A finer stratification for rational log varieties 19

2.15 Second assumption on the log structure of rational log varieties 22

2.16 Condition for being coherent 23

2.17 Kummer modification towards fs 24

3 KUMMER FS VARIETIES 27

3.1 Normal rational log varieties 27

Page

3.2 Normally log smooth morphisms 31

3.3 Saturation of log structures and definition of Kummer fs variety 39

3.4 Quasi log smooth morphisms 42

3.5 Quasi log smooth varieties 43

3.6 Minimal Kummer covers 47

3.7 Quasi log smooth morphisms of quasi-toroidal varieties. 51

3.8 Quasi Kummer étale site for normal rational varieties 54

4 LOG RESOLUTION OF KUMMER FS VARIETIES 58

4.1 Embeddings into quasi-toroidal varieties 58

4.2 Functoriality of log resolutions 64

4.3 Étale independence upon embedding of the same dimension 64

4.4 Quotient log smooth morphisms 65

4.5 Central Charts 71

4.6 Functorial non embedded log resolution of Kummer fs varieties 72

REFERENCES 81

ABSTRACT

Figuerola, Andres Ph.D., Purdue University, August 2019. Quasi-Toroidal Varieties And Rational Log Structures In Characteristic 0 . Major Professor: Jarosław Włodarczyk.

We study log varieties, over a field of characteristic zero, which are generically logarithmically smooth and fs in the Kummer normally log étale topology. As an application, we prove an analog of Abramovich-Temkin-Wlodarczyk's log resolution of singularities of fs log schemes in the Kummer fs setting.

1. INTRODUCTION

Log structures, as conceived by Fontaine and Illusie, were developed by Kato in [1]. Most of the applications of logarithmic geometry focus on the case where log structures are coherent. A log structure is coherent if, étale locally at every point, we can find a strict morphism (called a chart) $X \to spec(\mathbb{Z}[P])$, where P is a finitely generated monoid. That is, the log structure on X coincides with the pullback of the canonical log structure on $spec(\mathbb{Z}[P])$. However, in the context of logarithmic resolution of singularities, one is naturally led to consider log schemes which are not necessarily coherent.

Already in [2, II (2.6.4)], certain examples of non-coherent log structure are presented. In [3], motivated by functorial logarithmic resolution of singularities, a class of log schemes where we can find a morphism $X \to spec(\mathbb{Z}[P])$ of Kummer type (but not necessarily strict) was introduced, they were called rational log schemes. Rational log schemes were defined to be normal, and they come with an associated notion of log smooth morphism which is called normally log smooth. Also in [3], a subclass of rational log schemes which should be interpreted as being log smooth over a field was identified. A "log smooth" rational log variety was called quasi-toroidal. Quasi-toroidal varieties can locally be viewed as a quotient of a toroidal scheme by a finite abelian group action. However, quasi-toroidal varieties are not, in general, normally log smooth over a field.

The main construction of this thesis is a category of rational varieties over a field of characteristic zero, whose log structures are fs (i.e. coherent with saturated charts) in the Kummer normally log étale topology. We call its objects, Kummer fs varieties. This category has a notion of log smooth morphism, containing the class of normally log smooth morphisms, which we call quasi log smooth.

Proposition 1.0.1 *Let (X, M) be a generically fs rational log variety over a field k of characteristic zero. Then (X, M) is quasi-toroidal at $x \in X$, if and only if X is Kummer fs and the structure morphism $X \to spec(k)$ is quasi log smooth.*

One of the main theorems in [3], Theorem 1.3.4, states that there is a functorial logarithmic desingularization of (generically logarithmically smooth) fs log varieties over a field of characteristic zero. Starting from an fs log variety X, it produces a quasi toroidal variety X' and a projective morphism $X' \to X$, which is an isomorphism on the toroidal locus. This desingularization is functorial with respect to logarithmically smooth morphisms. In fact, requesting the functoriality of the algorithm naturally forces one to depart from the category of fs varieties. One cannot expect X' to be toroidal and still have functoriality of the resolution with respect to log smooth morphisms (see 2.4). This motivates the need to study Kummer fs varieties.

In the same spirit, restricting to a subclass of quasi log smooth morphisms (which we call quotient log smooth), we show the following result.

Theorem 1.0.2 *Let (X, M) be a Kummer fs variety over a field k of characteristic zero. Suppose that M is generically fs. Then, there exists a quasi-toroidal variety $D(X)$ and a projective morphism $D(X) \to X$, which is an isomorphism over the quasi-toroidal locus of X. The assignment $X \mapsto D(X)$ is functorial with respect to quotient log smooth morphisms.*

A similar theorem, in a more general context, is proved in [4] using stacks. This thesis circumvents the language of stacks and develops a different framework using only basic algebraic geometry.

1.1 Outline

The study is divided into three chapters.

In the first chapter we develop the necessary background and introduce rational log varieties in the non necessarily normal setting. We recall the rank stratification associated to its log structure and show that any rational generically fs log variety X has a Kummer cover from an fs variety $p : X' \to X$.

In the second chapter we restrict to normal rational varieties. We recall the notion of normally log smooth morphism from [3], and list some of its properties. Then, we define Kummer fs varieties and quasi log smooth morphisms. The definition of quasi log smooth morphisms extends the notion of logarithmically smooth morphism of toroidal varieties. We show that they are stable under composition and dominant base change.

We later go on to study minimal fs covers of Kummer fs varieties and analyze the structure of quasi log smooth morphisms of quasi toroidal varieties. In particular, we deduce that a morphism of quasi toroidal varieties is quasi log smooth if and only if it has a lift on minimal Kummer covers which is classically log smooth (see 3.7.4). We end with a discussion on the Kummer quasi log étale site of Kummer fs varieties.

In the third chapter, we start by constructing local embeddings into quasi toroidal varieties. We define the notion of quotient log smooth morphisms and show that they are stable under composition. The class of quotient log smooth morphisms contains strict smooth morphisms of Kummer fs varieties and log smooth morphisms of fs varieties. Quotient log smooth morphisms can be thought of as quotients of classically log smooth morphisms of fs varieties. Then, using the main theorems in [3], we construct a functorial logarithmic desingularization of Kummer fs varieties.

1.2 Conventions and pre-requisites

After a brief review of the definitions of log structures and toroidal varieties, starting in 2.5, we work exclusively with varieties over a field k of characteristic 0.

Although we recall the definitions and standard theorems in basic logarithmic geometry, we assume familiarity with log structures as presented in [1] and [5].

For details of the algorithm of logarithmic resolution of singularities in characteristic 0, we refer to [6] and [3]. For a general (non-logarithmic) algorithm of functorial resolution of singularities in characteristic 0, we refer to [7].

2. RATIONAL LOG VARIETIES

2.1 Notation and Background

All monoids considered are commutative with unit. A monoid P is called integral if $ac = ab$ implies $c = b$. We say P is fine if it is finitely generated and integral.

For any monoid P, there is an associated group P^{gp} which is constructed from $P \times P$ modulo the equivalence relation: $(a, b) \sim (c, d)$ if and only if there exists $f \in P$ such that $acf = dbf$. The morphism $P \to P^{gp}$ is initial with respect to morphisms from P to a group.

A monoid P is integral if and only if the canonical map $P \to P^{gp}$, sending $p \mapsto (p, 1)$, is injective.

For any monoid P, there is a canonical integral monoid P^{int} and a morphism $P \to P^{int}$ which is universal with respect to maps from P to integral monoids. P^{int} can be constructed as the image of P in P^{gp}.

Given a monoid P, we let $P^* \subset P$ be the subgroup of units and denote $\overline{P} := P/P^*$.

An integral monoid P is said to be saturated if for any $p \in P^{gp}$, such that $p^n \in P$ (for some $n > 0$), then $p \in P$. A fine and saturated monoid is said to be fs.

Similarly as in the case of $()^{int}$, for an integral monoid P, there is a saturated monoid P^{sat} and a morphism $P \to P^{sat}$, which is universal with respect to morphisms from P to saturated monoids. P^{sat} can be constructed as $\{p \in P^{gp} \mid p^n \in P$ for some $n > 0\}$.

An ideal of an fine monoid P, is a subset $I \subset P$, such that $pI \subset I$ for all $p \in P$. An ideal $I \subset P$ is prime, if $P \setminus I$ is a submonoid of P. The height of a prime ideal I is defined as $ht(I) := rank(P^{gp}) - rank((P \setminus I)^{gp})$. See [5] for details.

A morphism of monoids $P \to Q$ is said to be Kummer, if it is injective and for any $q \in Q$, the exists $n > 0$, such that $q^n \in P$.

A morphism of fine monoids $P \to Q$ is called log smooth (étale) if the induced morphism on groups $P^{gp} \to Q^{gp}$ has finite kernel and torsion part of the cokernel (finite kernel and cokernel).

For any monoid P, we let $\mathbb{Z}[P]$ denote the monoid algebra generated by P. We have an associated scheme $spec(\mathbb{Z}[P])$.

Given a scheme X, a log structure is a sheaf of monoids M on the étale site of X, together with a monoid sheaf map $\alpha : M \to \mathcal{O}_X$, such that $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$.

A log scheme is a pair (X, M), where X is a scheme and M is a log structure on X.

Let $\pi : X \to Y$ be a morphism of schemes. Given a log structure M_Y on Y there is a canonical way to construct a pullback log structure $\pi^* M_Y$ on X, this is the log structure associated to $\pi^{-1}M_Y \to \pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Similarly, for a log structure M_X on X, there is a pushforward log structure $\pi_* M_X$ on Y, it is the log structure associated to $\pi_* M_X \times_{f_*\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{O}_Y$ (see [1, 1.4]).

Given $x \in X$, $\overline{x}$ will denote a geometric point above x and the étale stalks will be denoted as $M_{\overline{x}}$. Whenever we write M_x, for a point $x \in X$, we mean the Zariski stalk of M at x. Same goes for $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^*$.

The logarithmically trivial locus of (X, M) is the maximal Zariski open subset $U \subset X$ such that $M_{X,\overline{u}} = \mathcal{O}_{X,\overline{u}}^*$ for all $u \in U$. We sometimes denote it as X_{tr}.

We say M is integral (saturated) if for any $x \in X$, we have that $\overline{M_{\overline{x}}}$ is integral (saturated).

Given a scheme X and a monoid morphism $\alpha : P \to \mathcal{O}_X$, there is a associated log structure P^a which is constructed as the monoid pushout $P \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$. A log structure M is said to be coherent if, étale locally, we can find a finitely generated monoid P and a morphism $P \to M$, such that $P^a \cong M$. Such a morphism $P \to M$, is called a chart of M. If we can choose P which is fine, we say that M is fine, and similarly M is called fs if we can find P fs.

In particular, a fine log structure is integral and an fs log structure is saturated.

There is an adjoint functor to the inclusion functor {coherent log schemes }$\to$ {fine log schemes}, which sends a coherent log scheme X to a fine log scheme denoted as X^{int}. It can be described, étale locally, when we have a chart $X \to A_P$ as $(X)^{int} := (X \times_P P^{int})$. Similarly, the inclusion functor {fs log schemes} $\to${fine log schemes} has an adjoint which sends a fine log scheme X to a fs log scheme X^{sat}. It can also be described, étale locally, when we have a fine chart $X \to A_P$ as $(X)^{sat} := (X \times_P P^{sat})$ (see [8]).

The schemes $spec(\mathbb{Z}[P])$, for P finitely generated, come equipped with a canonical coherent log structure associated to $P \to \mathbb{Z}[P]$.

A morphism of log schemes $(X, M_X) \to (Y, M_Y)$ is a morphism of underlying schemes, together with a compatible morphism of log stuctures $M_Y \to M_X$.

A morphism of log schemes $f : (X, M_X) \to (Y, M_Y)$ is said to be strict if $f^*(M_Y) \cong M_X$. In particular, a chart of a fine log structure M can be identified with a strict morphism $U \to spec(\mathbb{Z}[P])$, on some étale neighborhood U of X.

A morphism of fine log schemes $f : X \to Y$ is said to be log smooth (étale) if, étale locally, we can find charts $P \to M_Y$, $Q \to M_X$, and an induced monoid morphism $P \to Q$ such that:

1. $P \to Q$ is log smooth (étale)

2. The order of the kernel, and the torsion part of the cokernel, of $P^{gp} \to Q^{gp}$ is invertible on X.

3. $X \to Y \times_{spec(\mathbb{Z}[P])} spec(\mathbb{Z}[Q])$ is étale.

We usually denote $Y \times_{spec(\mathbb{Z}[P])} spec(\mathbb{Z}[Q])$ as $Y \times_P Q$.

A morphism of log schemes $f : (X, M_X) \to (Y, M_Y)$ is Kummer if for any $y \in Y$, $x := f(y) \in X$, the induced morphism of monoids $(\overline{M_{Y,\overline{y}}})^{gp} \to (\overline{M_{X,\overline{x}}})^{gp}$ is Kummer and the order of its cokernel is invertible in X.

For a log scheme (X, M) and a point $x \in X$, we denote by $I(M, x)$ the ideal generated by the image on $M_x \setminus M_x^*$ in $\mathcal{O}_{X,x}$.

Let P be a fine monoid. Whenever we are working over a field k, we let A_P denote the scheme $spec(k[P])$.

2.2 Toric varieties

Definition 2.2.1 *A toric variety is a normal variety X, containing a torus $T = (k^*)^n$ as a dense open subset and such that the action of the torus extends to X.*

This gives rise to a logarithmic pair (X, T), with log structure $\mathcal{M} := \mathcal{O}_X \cap \mathcal{O}_T^*$.

Affine toric varieties can be described as $X_\sigma := spec(K[\sigma^\vee \cap M])$, where $M := Hom(T, k^*)$ is the lattice of characters of the torus, $N := Hom(M, \mathbb{Z})$ is the dual, σ is a strongly convex rational polyhedral cone in $N \otimes \mathbb{Q}$ and $\sigma^\vee := \{F \in M \otimes \mathbb{Q} \mid F|_\sigma \geq 0\}$.

Setting $P_\sigma := \mathcal{M}(X) = \sigma^\vee \cap M$, we see that giving an affine toric variety X_σ is the same as giving A_P, where P is an fs torsion free monoid which can be identified with P_σ. For the converse direction, we identify $M := P^{gp}$ and $T := A_{P^{gp}} = spec(k[P^{gp}])$.

2.3 Toroidal embeddings and regular log schemes

Let X be a scheme over a field k, and let $U \subset X$ be a non-empty open subset. Consider the log structure on X given by $M := \mathcal{O}_X \cap \mathcal{O}_U^*$. Then, we say that (X, U) is a toroidal embedding if M is fs, and for any point $x \in X$, étale locally we can find a chart $f_P : X \to spec(k[P])$, where P is an fs monoid, such that f is étale. This is equivalent to the definition given in [9].

The notion of logarithmic regularity was introduced in [5], and its definition independent of the base scheme.

Definition 2.3.1 *([5, 2.1] [10, II 4.5]) An fs log scheme (X, M) is regular at a point $x \in X$ if the following two conditions are satisfied.*

1. *$\mathcal{O}_{X,\overline{x}}/I(M, \overline{x})$ is a regular local ring.*

2. *$dim(\mathcal{O}_{X,\overline{x}}) = dim(\mathcal{O}_{X,\overline{x}}/I(M, \overline{x})) + rank(\overline{M_{X,\overline{x}}}^{gp})$*

A log scheme is regular, if it is regular at every point.

It is shown in [5, 8.3], that for an fs log scheme (X, M) over a perfect field, the notion of toroidal embedding, as above, is equivalent to (X, M) being logarithmically regular.

Whenever we have a chart $X \to A_P$ for X as above, the definition of regularity is independent of whether we take the étale or Zariski topology for M ([10, II lemma 4.6]).

2.4 A motivating example

The following example is taken from [3], it justifies the need for defining non-coherent log structures in the context of log resolution of singularities.

The usual Hironaka approach to embedded resolution of singularities in characteristic zero, is to solve the problem of principalization of ideals on smooth ambient schemes. In the logarithmic context, the analogue result is to consider principalization of ideals on toroidal schemes. Let $X := spec(\mathbb{C}[x, y])$, with log structure coming from $y = 0$ and consider the ideal $I := (x^2, y^2)$. This ideal is principalized after blowing up the origin (x, y). Let X' be another copy of X, and consider the log smooth morphism of $p : X \to X'$, induced from the ramified cover $y \mapsto y^2$. Note also, that $p^*(x^2, y) = I$ in X. Hence, if the algorithm of principalization of ideals were to be functorial with respect to logarithmically smooth morphisms, then the principalization of (x^2, y) should naturally restrict to the principalization of I on X. This forces to blow up an ideal of the form $(x, y^{1/2})$ on X'. We can interpret this modification as blowing up the ideal (x^2, y) on X'. Locally, on an affine chart of this blow up, the variety that we get is $spec(\mathbb{C}[u^2, uv, v^2])$, with log structure induced from $v = 0$. This variety is no longer toroidal. Moreover, its log structure is not even coherent.

2.5 Quasi toroidal varieties

From now on we will work over a field k of characteristic zero.

We recall the definition and some properties of quasi-toroidal varieties. For further details we refer to [3, 4.].

Definition 2.5.1 *A toric doubleton is a pair (X_σ, X_τ) where:*

- *X_σ is a toric variety corresponding to a cone σ with torus embedding given by a lattice N_σ. That is, $X_\sigma = spec(k[\sigma^\vee \cap M_\sigma])$ where M_σ is the dual lattice to N_σ, and $\sigma^\vee$ is the dual cone to σ in $M_\sigma \otimes \mathbb{Q}$.*

- *τ is a regular face of σ, for which there exists a face δ of σ, such that $\sigma = \tau + \delta$ and there is a $\mathbb{Q}$-isomorphism $\sigma \cong \tau \times \delta$ (preserving τ and δ but not necessarily the lattices). We let X_τ be the open toric subvariety of X_σ induced from the face inclusion.*

We give X_σ the log structure coming from regular functions which are invertible on X_τ.

Let $p : \sigma \to \delta$ be the face projection and let $N_{\delta^0} := p(N_\sigma)$. Denote by X_{δ^0} the toric variety corresponding to δ with lattice structure N_{δ^0}. We give X_{δ^0} the log structure induced from the torus. Then we get an induced Kummer morphism of log varieties $X_\sigma \to X_{\delta^0}$.

Denote by X_δ the toric variety associated to δ with lattice structure $N_\delta = N_\sigma \cap \delta$. We give X_δ the log structure induced from the torus embedding. There is a Kummer morphism of fs varieties $X_\delta \to X_{\delta^0}$ induced from the inclusion $N_\delta \to N_{\delta^0}$.

The normalized fiber product of the maps $X_\sigma \to X_{\delta^0} \leftarrow X_\delta$, corresponds to $X_{\tau\times\delta}$ with it's induced log structure coming from the projection $N_\tau \times N_\delta \to N_\delta$. That is, the morphism $X_{\tau\times\delta} \to X_\delta$ is strict. The log structure on $X_{\tau\times\delta}$ gives rise to a toroidal embedding since τ is a regular face of σ.

Let $G_\delta := N_{\delta^0}/N_\delta$, then G_δ is a finite group acting on X_δ with quotient X_{δ^0}. The action of G_δ, extends to $X_{\tau\times\delta}$ with quotient X_σ.

Definition 2.5.2 *A quasi-toroidal variety is a log variety (X, M) for which, étale locally at each point, there is a strict étale map to a toric doubleton $(X, M) \to (X_\sigma, X_\tau)$.*

Example 2.5.3 *The simplest example of a quasi toroidal variety which is not log regular is given by spec$(k[x^2, xy, y^2])$, with log structure induced from regular functions which are invertible at points where $y \neq 0$. Note that this log structure is not coherent at the origin.*

2.6 Simplicial Kummer fs varieties

As a generalization of quasi toroidal varieties, we can consider toric pairs (X_σ, X_τ), such that:

- τ is a simplicial face of σ with respect to the N_σ lattice structure.
- There exists a face δ, such that $\tau \cap \delta = 0$, $\tau + \delta = \sigma$ and there is an $\mathbb{Q}$-isomorphism $\sigma \cong \tau \times \delta$ (preserving τ and δ but not necessarily the lattices).

We call (X_σ, X_τ) a simplicial toric doubleton.

Remark. Toric doubletons, as in the above section, should perhaps be thought of as simplicial toric doubletons which are regular in the Kummer topology. That is, after Kummer extension they become toroidal, and hence regular on the logarithmically trivial locus. Analogously, after Kummer extension, simplicial toric doubletons become fs log varieties with at most simplicial toroidal singularities on the locus where the log structure is trivial.

As above, we get an induced morphisms $X_\sigma \to X_{\delta^0} \leftarrow X_\delta$ whose normalized product gives $X_{\delta\times\tau}$ with log structure coming from regular functions which are invertible on X_τ .

Example 2.6.1 *As an example of a simplicial toric doubleton which is not quasi toroidal, we can take spec$(k[x^2, y^2, z^2, xyz])$ with log structure given by regular func-*

tions which are invertible away from points where $z \neq 0$. In this case the face τ corresponds to $x = y = 0$, which is simplicial but not regular in the associated cone $\sigma = \mathbb{R}^3_{\geq 0}$, $N_\sigma =< [1,0,0],[1/2,1/2,0],[0,1,0],[0,1/2,1/2],[0,0,1],[1/2,0,1/2] >$.

2.7 Conditions for being a chart

Lemma 2.7.1 *Let (X, M) be a log scheme, and let P be a monoid together with a morphism $\alpha : P \to M$. Then $\alpha : P \to M$ is a chart (in particular M is coherent) if and only if the following two conditions are satisfied for all $x \in X$:*

1. *$P \to \overline{M}_{X,x}$ is surjective .*
2. *For any $p, p' \in P$ such that $\alpha(p') = \alpha(p)u$ in M_x, for some $u \in \mathcal{O}_X^*$, there exists $v, v' \in \alpha^{-1}(\mathcal{O}_X^*)$, such that $\alpha(v') = u\alpha(v)$ and $pv' = pv$ in P.*

Remark. As we shall see in the proof, the second condition is equivalent to the fact that the induced map $\alpha^a : P^a \to M$ is injective.

Proof If $P \to M$ is a chart, then the induced map $P^a \to M$ is an isomorphism, so we can replace α by $P \to P^a$. It is clear then that $P \to \overline{P^a} = (P \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*)/\mathcal{O}_X^*$ must be surjective. On the other hand if $\alpha(p') = \alpha(p)u$ in P^a, then the second condition is the definition of $(p', 1) \sim (p, u)$ in $P \oplus \mathcal{O}_X^*$ to obtain P^a.

Conversely, suppose $\alpha : P \to M$ satisfies 1) and 2). Denote $\beta := \alpha^a : P^a \to M$ the induced map. We will show that β is an isomorphism. By 1), given $m \in M_x$ there exists $p \in P$, $u \in \mathcal{O}_{X,x}^*$ such that $\alpha(p)u = m$ in M_x. So β is surjective since is surjective on stalks.

If we have $(p, u), (p', u') \in P^a$ such that $\beta(p, u) = \beta(p', u')$ then, denoting $u'' := u(u')^{-1}$, we have that $\alpha(p)u'' = \alpha(p)$ in M. So by 2) we can find $v, v' \in \alpha^{-1}(\mathcal{O}_X^*)$ such that $pv' = pv$ and $\alpha(v') = u''\alpha(v)$. Therefore $(p, u'') \sim (p', 1)$ in P^a and hence $(p, u) \sim (p', u')$, so β is injective. ■

Corollary 2.7.2 *Let $\alpha : P \to M$ be a chart and $p, p' \in P$ such that $\alpha(p) = \alpha(p')$. Then, there exists $v', v \in \alpha^{-1}(\mathcal{O}_X^*)$ such that $pv = p'v'$. In particular, if P is integral and α restricted to $\alpha^{-1}(\mathcal{O}_X^*)$ is injective, then α is injective.*

Proof Immediate from condition 2. in 2.7.1. ■

2.8 Reduced induced monoid

Given an integral monoid P, we consider on it the equivalence relation given by $p_1 \sim p_2$ if and only if there exists $n > 0$ such that $np_1 = np_2$. Then $P^{red} := P/\sim$ has an induced structure of an integral monoid and $P \to P/\sim$ is a monoid morphism.

Example. Let $P = [(1,0),(0,1)] \subset \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}$. Then $P^{red} = \mathbb{N}$ and the map $P \to P^{red}$, corresponds to the projection onto first component.

For any monoid P (not necessarily integral) we write P^{red} for $(P^{int})^{red}$.

2.9 Rational log varieties

By a variety we mean a finite disjoint union of reduced, irreducible schemes of finite type over a field k (in our case $char(k) = 0$).

Definition 2.9.1 *A rational log variety (X, M), consists of a variety X, together with a log structure $M \subset \mathcal{O}_X \setminus \{0\}$, such that at every point $x \in X$, there exists an étale neighborhood $U \to X$ of x, and a morphism $P \to M|_U$ from a finitely generated monoid P, satisfying the following conditions:*

1. *The induced morphism $P^a \to M|_U$ is Kummer.*
2. *$\overline{P} \cong \overline{M_{\overline{x}}}$.*
3. *The restriction $M|_U$ is Zariski (i.e. for any étale morphism $\pi : V \to U$ we have $M|_V = \pi^*(M|_{U^{Zar}})$, where U^{Zar} denotes U with its Zariski topology).*

We say $P \to M$ is a rational chart optimized at x. A rational log variety X will be called rationally fine (respectively fs) if we can choose P to be integral (saturated).

A monoid morphism $P \to M$ satisfying 1) in the definition above, will be called a rational chart.

2.9.2 *Remark.* If X is rationally fine (fs) then, from choosing a rational chart optimized at $x \in X$, we see that $\overline{M_{\overline{x}}} = \overline{M_{\overline{x}}}^{int}$ ($\overline{M_{\overline{x}}} = \overline{M_{\overline{x}}}^{sat}$).

Lemma 2.9.3 *(cf. [3, 4.5.8]) Suppose that $\overline{M_{\overline{x}}}$ is finitely generated and saturated. Let $P \to M$ be a rational chart in a neighborhood of $x \in X$, with P saturated. Then, étale locally, we can construct a rational chart $P \subset Q \to M$ which is optimized at x. In particular, condition 2 in the above definition is automatically satisfied.*

Proof After localizing P, we can assume $\overline{P} \to \overline{M_{\overline{x}}}$ is Kummer. As P is saturated, we have a morphism $\overline{P} \to P$ giving a section to the projection $P \to \overline{P}$. Let Q be the pushout $\overline{M_{\overline{x}}} \leftarrow \overline{P} \to P$. Then the induced morphism $P \to Q$ is Kummer. We have an induced morphism $i : Q \to \overline{M_{\overline{x}}}$ factoring the identity on $\overline{M_{\overline{x}}}$. Hence i is surjective. We must show there is an injective map $Q^a \to M$.

Let $q \in Q$ and choose a lifting $m \in M_x$ of $i(q)$. Since $P \to Q$ is Kummer, we must have that $q^n \in P$. Hence there is a unit $v \in \mathcal{O}_X^*$, such that $q^n = vm^n$. By the characteristic 0 assumption, we can find an étale neighborhood where q/m is defined, and we have a lifting $m' \in M_x$ of $i(q)$ such that $m'^n = q^n$. Repeating the argument for a set of generators of Q we get an inclusion morphism $Q \subset M(X')$ (where $X' \to X$ is an étale neighborhood) lifting $Q \to \overline{M}_x$.

Thus, $Q \to M$ is a rational chart with $\overline{Q} = \overline{M_x}$. ■

2.10 Charts of saturated rational log varieties

2.10.1 . Let (X, M) be a saturated rational log variety. That is, X is a variety and M a rational log structure on X, for which $\overline{M_{\overline{x}}}$ is fs for every $x \in X$. By the remark following definition 2.9.2, this is equivalent to M being rationally fs.

Lemma 2.10.2 *Let (X, M) be a saturated rational log variety. Étale locally, for any $x \in X$, we can find a neighborhood of x with a sharp and optimized rational chart. That is, a rational chart $P \to M$, such that $P = \overline{M_{\overline{x}}}$.*

Proof We can find a rational chart $Q \to M$ optimized at x. Since Q is saturated, then we have a splitting $Q = \overline{Q} \oplus Q^*$. Thus the inclusion $\overline{Q} \to Q \to M$ gives a sharp and optimized rational chart. Let $P := \overline{Q}$. ■

2.11 Zariski optimized charts

Lemma 2.11.1 *Let (X, M) be a rationally fine log variety and let $x \in X$. Then in an étale neighborhood $U \to X$ of x, there is an rational chart $U \to A_P$ optimized at a point u (above x), such that $(M|_{U^{Zar}})_u = P\mathcal{O}^*_{U^{Zar},u}$. Here, $(-)^{Zar}$ means in the restriction to the Zariski topology of U.*

Proof At any $x \in X$, we can find an étale neighborhood $V \to X$, and a chart $V \to A_P$, optimized at a point $v \in V$ above x, such that $M|_V$ is Zariski. If we take the stalk of $M|_{V^{Zar}}$ at v and take the sharpification $\overline{M|_{V^{Zar},v}} := M|_{U^{Zar},v}/\mathcal{O}^*_{U^{Zar},v}$, then we get an induced map $\overline{M|_{V^{Zar},v}} \to \overline{M_{V,\overline{v}}}$ (where $\overline{v}$ is a geometric point above x). Since M is rationally fine, $P \cong \overline{M_{V,\overline{v}}}$, $P^a \to M|_V$ is Kummer, then $\overline{M|_{V^{Zar},v}}^{gp} \to \overline{M_{V,\overline{v}}}^{gp}$ is surjective and it's kernel is torsion. Let $t \in \overline{M|_{V^{Zar},v}}^{gp}$ be a torsion element. Étale locally at v, we can lift t to an element in $\mathcal{O}_{(V^{Zar})^*}$. Repeating the same process for a set of generators of the torsion of $\overline{M|_{V^{Zar},v}}^{gp}$, and since $M|_V$ is Zariski, we get an étale neighborhood $U \to V$, and a point $u \in U$ above $v \in V$, such that $\overline{M|_{U^{Zar},u}}^{gp} \to \overline{M_{U,\overline{u}}}^{gp}$ is an isomorphism. Hence, $\overline{M_{U^{zar},u}} = P$. ■

Remark. By the above lemma and condition 3 in the definition of rational log variety, we may assume that (up to an étale cover) rationally fine log varieties are Zariski with optimized rational charts in Zariski topology. Hence, we will usually think of M as defined in Zariski topology, where we can find optimized charts up to pulling back the log structure to an étale neighborhood. Moreover, whenever we have an optimized rational chart $X \to A_P$ at x, we will always assume that $P\mathcal{O}^*_{X^{Zar},x} = M_{X^{Zar},x}$.

Example 2.11.2 *Consider the Whitney umbrella $X := spec(k[x^2, xy, y])$ with $y \neq 0$ log structure. Then at the point p with coordinates $x = 1$, $y = 0$, the log structure is fine but not saturated in Zariski topology. In the étale topology the log structure is fs. This log structure is Zariski.*

Example 2.11.3 *Consider $\mathbb{A}^2$ with log structure coming from the nodal curve $\{f = y^2 - x^2(x+1) = 0\}$. Then Zariski locally, the log structure is fs given by $\mathbb{N} \to k[x,y]$ $(1 \mapsto f)$. At the origin, considering the étale neighborhood*

$$k[x,y] \to \Big(\frac{k[x,y][T]}{(T^2 - (x+1))}\Big)_{x+1} =: R$$

the log structure is given by $\mathbb{N}^2 \to R$ ($e_1 \mapsto y - Tx$, $e_2 \mapsto y + Tx$). This log structure on $spec(R)$ is not the pullback of the log structure on $\mathbb{A}^2$.

2.12 The rank stratification

Lemma 2.12.1 *Let X be a variety and $x \in X$. Let $f : X \to A_P$ be a morphism, such that $(P \setminus P^*)$ maps into the maximal ideal m_x of $\mathcal{O}_{X,x}$. Then for any $z \in X$, such that $f(z) = f(x)$, we have $\overline{P} = \overline{(P^a)_z}$.*

Proof (cf. [2, Proposition 1.1.8 (1)]) For any z, such that $f(z) = f(x)$, the subset $(P \setminus P^*) \subset k[P]$ maps to maximal ideal m_z in $\mathcal{O}_{X,z}$.

Consider the induced map $\overline{P} \to \overline{P^a_z}$, we will show it is injective. Let $p_1, p_2 \in P$, be such that $\overline{p_1} = \overline{p_2}$ in $\overline{P^a_z}$. Then we can find $u \in \mathcal{O}^*_{X,z}$, such that $(p_1, u) \sim (p_2, 1)$ in $P^a_z = P \oplus_{P^*} \mathcal{O}^*_{X,z}$ (In this description we are using that $(P \setminus P^*)$ maps to m_z, since then the inverse image of $\mathcal{O}^*_{X,z}$ in P coincides with P^*). Thus, by definition of pushout, there exists $z, w \in P^*$, such that $zp_1 = wp_2$. But then $\overline{p_1} = \overline{p_2}$ in $\overline{P}$.

Since $P \to \overline{P^a_z}$ must also be surjective, then $\overline{P} = \overline{(P^a)_z}$. ∎

Lemma 2.12.2 *Let (X, M) be a rational log variety and $P \to M$ be a rational chart optimized at x. Then (locally) the set $\{y \in X \mid rank(\overline{M_y}^{gp}) = rank(\overline{M_x}^{gp})\}$, coincides with $\{z \in X \mid (P \setminus P^*)$ maps to m_z in $\mathcal{O}_{X,z}\}$.*

Proof Suppose $z \in \{z \in X \mid (P \setminus P^*)$ maps to m_z in $\mathcal{O}_{X,z}\}$. Since $P_z^a \to M_z$ is Kummer, then $rank(\overline{M_z}^{gp}) = rank(\overline{P_z^a}^{gp})$. By the above lemma $\overline{P_z^a} = \overline{P}$, so $rank(\overline{M_z}^{gp}) = rank(\overline{P}^{gp}) = rank(\overline{M_x}^{gp})$.

Conversely, let $z \in \{y \in X \mid rank(\overline{M_y}^{gp}) = rank(\overline{M_x}^{gp})\}$. Suppose there is an element $p \in (P \setminus P^*)$ which maps to $\mathcal{O}^*_{X,z}$. Then $\overline{p}$ cannot be a torsion element in $\overline{P}^{gp}$, because that would imply $p \in P^*$. The morphism $\overline{P} \to \overline{M_z}$ factors as $\overline{P} \to \overline{P_z} \to \overline{M_z}$. Since $\overline{P} \to \overline{P_z}$ is surjective, then so is $\overline{P}^{gp} \to \overline{P_z}^{gp}$. On the other hand, the non torsion element $\overline{p} \in \overline{P}^{gp}$ is in the kernel of $\overline{P}^{gp} \to \overline{P_z}^{gp}$, hence $rnk(\overline{P}^{gp}) > rnk(\overline{P_z}^{gp})$. But since $P_z \to M_z$ is Kummer, then $rank(\overline{M_z}^{gp}) = rank(\overline{P_z}^{gp})$. This contradicts the fact that z is in the same stratum as x. ■

Lemma 2.12.3 (Rank Stratification) *Let (X, M) be a fine rational log variety. Then the subsets $\{x \in X \mid rank(\overline{M_x}^{gp}) = n\}$ define a stratification on X.*

Proof Étale locally, we can find a rational chart P optimized at a point x' above x. Note that the rank of $\overline{M_{\overline{x}}}^{gp}$ and $\overline{M_{x'}}^{gp}$ are the same for any geometric point $\overline{x}$ above x' and is given by the rank of $\overline{P}^{gp}$. Hence, we may assume that the optimized chart exists Zariski locally.

First consider the case when $X = A_P$ where P is a fine monoid. Let $n := rank((P \setminus P^*)^{gp})$, and note that the rank n locus corresponds to $V(P \setminus P^*)$, the vanishing of the ideal generated by $P \setminus P^*$. Then $\{x \in X \mid rank(\overline{M}_x^{gp}) \geq n - 1\}$ coincides with the union $\cup_{ht(\mathfrak{p})=1} V(P \setminus (\mathfrak{p} \cup P^*))$, where each $\mathfrak{p} \in spec(P)$ is a prime ideal of P, and $V(P \setminus (\mathfrak{p} \cup P^*))$ is the vanishing locus of the ideal generated by $P \setminus (\mathfrak{p} \cup P^*)$. Inductively, by the analogue formula, we see that each set $\{x \in X \mid rank(\overline{M}_x^{gp}) \geq n - k\}$ is closed in A_P. Hence A_P has a rank stratification.

Now let X be a general fine rational log variety. Let $x \in X$, then (by definition) we can find an optimized chart $X \to A_P$. The inverse image of the rank strata from

A_P form a (local) stratification for X. Since any point has an optimized chart, we see from 2.12.2, this stratification glues on all of X. In particular, the local stratification is independent of the chart and the point x at which P is optimized at. ■

2.13 First assumption on the log structure of rational log varieties

2.13.1 . From now on, we make the following assumption on rational log varieties.

In the rank stratification of (X, M), every point is in a unique irreducible component of its stratum.

For example on $X := spec(k[x, y])$, the log structure $\mathbb{N} \to k[x, y]$ $(1 \mapsto xy)$ does not satisfy this requirement at the origin, but the log structure $\mathbb{N} \to k[x, y]$ $(1 \mapsto y)$ does.

Assuming the above condition and considering the irreducible components of each stratum, we get a finer stratification on (X, M). For a point $x \in X$, we denote by $s(x)$ the generic point of the stratum of x.

We give an example of a kind of log scheme for which this condition is always satisfied.

Lemma 2.13.2 *Let X be a variety and $U \subset X$ be the complement of a codimension 1 closed subset. Let M be the log structure $\mathcal{O}_X \cap \mathcal{O}_U^*$ induced on X, and suppose that M is rationally fs. Then, condition 2.13.1 is satisfied for (X, M).*

Proof Let s be a stratum and let $x \in s$. Suppose s has two irreducible components s_1, s_2 containing x. By abuse of notation, we denote the generic points of this components as s_1 and s_2. Let $P \to M$ be a rational chart optimized at x. Then, since X is separated, there is $p \in P$ which vanishes on s_1, but not on s_2. Thus the image of p in M_x, lies in the kernel of the induced cospecialization morphism $\overline{M}_x^{gp} \to \overline{M}_{s_2}^{gp}$. Since $P^a \to M$ is Kummer in a neighborhood of x, and $\overline{P} = \overline{M_x}$ is saturated, then the induced morphism $\overline{P} \cong \overline{M_x} \to \overline{M}_{s_2}$ is Kummer (see the argument in 2.14.3) . Therefore p must be a unit at x. Given that X is reduced, and p vanishes on s_1, then p must vanish at x, a contradiction. ■

A pair (X, U) as above is called a solid log scheme.

2.14 A finer stratification for rational log varieties

Lemma 2.14.1 *Let (X, M) be a rationally fine log variety and let $x \in X$. Let $s(x)$ denote the the generic point of the rank stratum of x. Suppose $\overline{M_x} \to \overline{M_{s(x)}}$ is injective, then $\overline{M_x} \to \overline{M_{s(x)}}$ is Kummer.*

Proof Locally, we can find a rational chart $P \to M$, such that $\overline{P} = \overline{M_x}$. Then we have the following commutative diagram :

$$\begin{array}{ccc} \overline{P^a_x} & \longrightarrow & \overline{M_x} \\ \downarrow & & \downarrow \\ \overline{P^a_{s(x)}} & \longrightarrow & \overline{M_{s(x)}} \end{array}$$

From the surjectivity of $P \to \overline{P^a_{s(x)}}$, and the factorization $P \to \overline{P^a_x} \to \overline{P^a_{s(x)}}$, we get that $\overline{P^a_x} \to \overline{P^a_{s(x)}}$ is surjective. Since by assumption $\overline{M_x} \to \overline{M_{s(x)}}$ is injective, then $\overline{P^a_x} = \overline{P^a_{s(x)}}$. On the other hand $\overline{P^a_{s(x)}}^{gp} \to \overline{M_{s(x)}}^{gp}$ is Kummer (up to torsion) since P is a rational chart. Thus $\overline{M_x} \to \overline{M_{s(x)}}$ is Kummer. ■

2.14.2 . *Remark*. As a corollary of the proof, note that, if $\overline{M_x} \to \overline{M_{s(x)}}$ is injective then $\overline{P^a_x} = \overline{P^a_{s(x)}}$.

Lemma 2.14.3 *If $\overline{M_x}$ is saturated, then $\overline{M_x}^{gp} \to \overline{M}^{gp}_{s(x)}$ is injective.*

Proof As above, let P be a rational chart optimized at x. Recall that $\overline{P^a_x} \to \overline{P^a_{s(x)}}$ is surjective (see the proof of 2.14.1), and that both $\overline{M_x}^{gp}$ and $\overline{M}^{gp}_{s(x)}$ have the same rank. Considering the following diagram

$$\begin{array}{ccc} \overline{P^a_x}^{gp} & \longrightarrow & \overline{M_x}^{gp} \\ \downarrow & & \downarrow \\ \overline{P^a_{s(x)}}^{gp} & \longrightarrow & \overline{M_{s(x)}}^{gp} \end{array}$$

we see that, since the horizontal arrows are Kummer, any element in the kernel of $\overline{M_x}^{gp} \to \overline{M}^{gp}_{s(x)}$, must be torsion in $\overline{M_x}^{gp}$. On the other hand, given that $\overline{M_x}$ is saturated, then $\overline{M_x}^{gp}$ is torsion free. ■

Remark. In general, if $\overline{M}_x$ is not necessarily saturated, since any element in the kernel of $\overline{M_x}^{gp} \to \overline{M}^{gp}_{s(x)}$ is torsion in $\overline{M_x}^{gp}$, then we have that $\overline{M_x}^{red} \to \overline{M}^{red}_{s(x)}$ is Kummer.

Definition 2.14.4 *For a monoid M, we denote by $M^{(n)}$, the image of M under the n-th power map $M \to M$ $(m \mapsto m^n)$. Let (X, M) be a rationally fine log variety with a rational chart $X \to A_P$. We define the multiplicity of P at a point $x \in X$ as $m_P(x) := min\{n > 0 \mid (\overline{M_{X,x}}^{gp})^{(n)} \subset \overline{P^a_x}^{gp}\}$. Since P is a rational chart, $m_p(x)$ exists and is finite.*

Lemma 2.14.5 *Let (X, M) be a rationally fine log variety and $P \to M$ a rational chart at $x \in X$. If $M^{(n)}_{X,s(x)} \subset P^a_{s(x)}$, and $\overline{M_x} \to \overline{M_{s(x)}}$ is injective, then $m_p(x) \leq n$. Moreover, the multiplicity m_P is lower semi-continuous on the stratum corresponding to $s(x)$.*

Proof Let $s := s(x)$, and s_α be the stratum corresponding to s. Since (by 2.14.2) $\overline{P^a_x} = \overline{P^a_s}$, we have the following diagram,

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{P^a_x}^{gp} & \longrightarrow & \overline{M_{X,x}}^{gp} \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{P^a_s}^{gp} & \longrightarrow & \overline{M_{X,s}}^{gp} \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

Hence, given $m \in \overline{M_{X,x}}^{gp} \subset \overline{M_{X,s}}^{gp}$, if $m^j \in \overline{P^a_s}^{gp}$, then $m^j \in \overline{P^a_x}^{gp}$. This proves the first claim.

To prove semi-continuity, let $m \in \overline{M_{X,x}}^{gp}$ be such that $m^j \notin \overline{P^a_x}^{gp}$ for some $j > 0$. There exist a neighborhood U of x, such that for any $y \in U \cap s_\alpha$, we have $m \in \overline{M_{X,y}}^{gp}$. So for every $y \in U \cap s_\alpha$ we must also have $m^j \notin \overline{P^a_y}^{gp}$, because $s \in U \cap s_\alpha$. Hence $\{y \in s_\alpha \mid m_p(y) > j\}$ is open in s_α. ■

Corollary 2.14.6 *Let $x \in X$. If $\overline{M}_x = \overline{M}_{s(x)}$, then in $s(x)$ (the stratum) there exists a neighborhood $U \subset s(x)$ of x , such that for any $y \in U$, we have $\overline{M}_y = \overline{M}_{s(x)}$.*

Proof This is a direct consequence of 2.14.5. ■

Lemma 2.14.7 *Let X be a rationally fine log variety. Let s_α, s_β be two generic points of a strata, such that s_α specializes to s_β. Let $P \to M$ be a rational chart at $s_\beta \in X$. Furthermore suppose the cospecialization map $\overline{M_{s_\beta}} \to \overline{M_{s_\alpha}}$ is surjective, then $m_P(s_\alpha)$ divides $m_P(s_\beta)$.*

Proof Denote $n_\alpha := m_P(s_\alpha)$ and $n_\beta := m_P(s_\beta)$. By definition of n_β, we have that $\overline{M}^{n_\beta}_{s_\beta} \subset \overline{P^a}_{s_\beta}$. Any two elements $m_1, m_2 \in M^{n_\beta}_{s_\beta}$, mapping to the same $\overline{m}$ in $\overline{M}_{s_\alpha}$, must differ by a unit in s_α. So there exists a morphism $\overline{M}^{n_\beta}_{s_\alpha} \to \overline{P^a}_{s_\alpha}$ filling in the diagram.

$$\begin{tikzcd} \overline{M}^{n_\beta}_{s_\beta} \arrow[r] \arrow[d] & \overline{M}^{n_\beta}_{s_\alpha} \arrow[d, dashed] \\ \overline{P^a}_{s_\beta} \arrow[r] & \overline{P^a}_{s_\alpha} \end{tikzcd}$$

Moreover, $\overline{M}^{n_\beta}_{s_\alpha} \to \overline{P^a}_{s_\alpha}$ must be injective. To see this, consider any two elements $\overline{m_1}, \overline{m_2} \in \overline{M^{n_\beta}_{s_\alpha}}$ mapping to the same element $\overline{m} \in \overline{P^a}_{s_\alpha}$. Let $m_1', m_2' \in M^{n_\beta}_{s_\beta}$ be liftings of $\overline{m_1}, \overline{m_2}$ respectively. As m_1', m_2' both map to $\overline{m} \in \overline{P^a_{s_\alpha}}$, then m_1', m_2' must differ by a unit in s_α and so $\overline{m_1} = \overline{m_2}$.

Hence $n_\alpha \leq n_\beta$. Let $n_\beta = kn_\alpha + r$ for $0 \leq r < n_\alpha$. For any $m \in \overline{M}_{s_\alpha}$, we have $m^r = m^{n_\beta}/m^{kn_\alpha} \in \overline{P^a}^{gp}_{s_\alpha}$, and since n_α is minimal with respect to this property, we must have $r = 0$. ■

2.15 Second assumption on the log structure of rational log varieties

2.15.1 . Let (X, M) be a rationally fine log scheme satisfying 2.13.1. In the rest of the paper, in order to avoid pathologies, we will generally require that (X, M) satisfies the following extra assumption:

(*) **For any two generic points of strata s, s', such that s' specializes to s, the cospecialization map $\overline{M_s} \to \overline{M_{s'}}$ is surjective.**

Lemma 2.15.2 *Let (X, M) be a rationally fine log variety satisfying 2.13.1. Then (X, M) satisfies 2.15.1 if and only if at every generic point of strata, we can find a neighborhood where the log structure is fine.*

Proof Sufficiency is clear since condition 2.15.1 holds for a fine log structure.

We prove necessity. Let s be a generic point of strata, consider a rational chart $P \to M$ optimized at s, i.e. $\overline{M_s} = \overline{P}$. Then by 2.14.5, there is a neighborhood U of s, where P is defined, and such that for every $y \in U \cap s$, we have that $m_P(y) = 1$. Restricting U, we may suppose that it contain only points of strata specializing to s. Let s' be a stratum specializing to s, then for any point $y \in U$ in the same stratum of s', we have the following factorization $P \to \overline{M_y} \to \overline{M_{s'}}$. By 2.14.7, we must have that $P \to \overline{M_{s'}}$ is surjective and hence, so is $\overline{M_y} \to \overline{M_{s'}}$. The cospecialization morphism must also be Kummer $\overline{M_y} \to \overline{M_{s'}}$, therefore an isomorphism, so $P \to \overline{M_y}$ is surjective. Thus by 2.7.1 we have $M|_U = P^a$. ■

The above lemma suggests the following definition.

Definition 2.15.3 *We say a rational log structure is generically fine (fs) if it is rationally fine (fs) and it satisfies 2.13.1 and 2.15.1 (*) above.*

Example 2.15.4 *Let* $X = spec(k[x,y])$*, with log structure along* $y = 0$ *defined by* (x^2, y)*, and at* $y \neq 0$*, to be induced from* $[x] \to k[x,y]$*. Then* $(x^2,y)^a \to M$ *is Kummer, but it does not satisfy* $(*)$ *for the rank strata* $s' := \{x = y = 0\}$ *and* $s := \{x = 0, y \neq 0\}$*.*

2.16 Condition for being coherent

2.16.1 . Let (X, M) be a rationally fs log variety. Given $x \in X$, let $s(x)$ denote the generic point of the rank stratum of x. We let $\underline{s(x)}$ denote the actual stratum (not just the generic point). Consider the cospecialization map $\alpha : \overline{M_x} \to \overline{M_{s(x)}}$. Since X is rationally fs, this morphism is a Kummer extension (2.14.1, 2.14.3).

Denote by X^{coh} the points in X where the log structure is coherent, by definition X^{coh} is open.

Lemma 2.16.2 *Let* (X, M) *be a generically fs rational log variety and let* $x \in X$*. Then,* $x \in X^{coh}$ *if and only if* $\alpha : \overline{M_x} \to \overline{M_{s(x)}}$ *is an isomorphism.*

Proof Necessity is clear since any chart P at x must also be a chart at $s(x)$, hence $P\mathcal{O}^*_{X,x}/\mathcal{O}^*_{X,x} = \overline{M_x} \to \overline{M_{s(x)}} = P\mathcal{O}^*_{X,s(x)}/\mathcal{O}^*_{X,s(x)}$ is surjective. Since by hypothesis α is also injective (it is Kummer), then it is an isomorphism.

For the sufficiency, note that for each stratum s the set $U_s := \{y \in \underline{s} \mid \overline{M}_y \to \overline{M}_s \text{ is an isomorphism }\}$ is open in $\underline{s}$ by 2.14.6. Choose an open neighborhood U of x containing only points in strata specializing to $s(x)$ and which has an optimized rational chart P at x. Given a stratum s specializing to $s(x)$, then we claim that $\underline{s} \setminus U_s$ (as defined above) does not contain any component specializing to x.

To see this, suppose that l is the generic point of a component of $\underline{s} \setminus U_s$, specializing to x. Then $\overline{M}_x \to \overline{M}_s$ factors as $\overline{M}_x \to \overline{M}_l \to \overline{M}_s$. Since by assumption $\overline{M_x} = \overline{M_{s(x)}}$, and $\overline{M_{s(x)}} \to \overline{M_s}$ is surjective, then $\overline{M_l} \to \overline{M_s}$ must be surjective. Given that $\overline{M_l} \to \overline{M_s}$ is Kummer, then $\overline{M_l} \to \overline{M_s}$ must be an isomorphism, and so $l \in U_s$, a contradiction.

Therefore, no component of $\underline{s} \setminus U_s$ specializes to x. So there is an open $V \subset \cup_s(U_s \cap U)$ containing x. For any $y \in V$ we have a surjective morphism $\overline{M}_x = \overline{M}_{s(x)} \to \overline{M}_{s(y)} = \overline{M}_y$. The optimized rational chart P must actually be a chart on V by 2.7.1, since $P^a \to M$ is injective (it is Kummer) and $P \to \overline{M}$ is surjective on V.

■

Remark. In fact, in the above proof, we do not need that the strata are irreducible. One could instead take the generic points of the components of a stratum, and require that the cospecialization map is surjective on the sharpened stalks of the log structure at those points.

2.17 Kummer modification towards fs

In this section, (X, M) will denote an irreducible generically fs rational log variety, such that $M \subset \mathcal{O}_X \setminus \{0\}$.

Proposition 2.17.1 *Let (X, M) be a saturated rational log variety over a field k of characteristic zero, which is generically fs and $M \subset \mathcal{O}_X \setminus \{0\}$. Then, étale locally at every $x \in X$, there exists a finite morphism of Kummer type $f : X' \to X$, where X' is fs above x. If X is normal, then f is étale over the logarithmically trivial locus of X.*

2.17.2 Outline of the strategy. In view of 2.16.2, we seek to make $\alpha : \overline{M_x} \to \overline{M_{s(x)}}$ an isomorphism. Our approach is given by constructing a normalized Kummer cover of X. That is, given a point $x \in X$ and an optimized rational chart $X \to A_P$, we want to find a Kummer extension $P \subset P'$, such that on some components of the normalized fiber product $X_1 := (X \times_P P')^n$, we have that M_{X_1} is fs.

We emphasize that $X \times_P P'$ might not be reduced or irreducible. By the normalization $(X \times_P P')^n$, we mean the disjoint union of the normalized components of $X \times_P P'$ with reduced induced structure.

Example 2.17.3 *Consider $A^1 = spec(k[x])$ with the log structure induced from $x \neq 0$. Let $A^1 \to A^1$ be the Kummer morphism induced from $x \to x^2$. Then, in the category of varieties $Z := A^1 \times_{A^1} A^1 = A^1 \sqcup A^1$ and M_Z is induced from $x \neq 0$ on each copy of A^1.*

Proof (Proof of Proposition 2.17.1)

Étale locally, we can find a rational chart $X \to A_P$, with $P \cong \overline{M_{X,x}}$. Denoting $P' := \overline{M_{s(x)}}$ and $X_0 := X \times_P P'$, we have the following diagram where the vertical arrows are Kummer.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & A_{P'} \arrow[d] \\ X \arrow[r] & A_P \end{tikzcd}$$

Let X_1 be the closure of $X_{tr} \times_P P'$ in the normalized fiber product $(X \times_P P')^n$, whose log structure is that associated to $(M_X \oplus_P P')^{sat}$. We will show that for any point $x_1 \in X_1$, above $x \in X$, we have that X_1 is fs above x.

Since X_1 is normal, then $(M_X \oplus_P P')^{sat}$ does in fact have a map to $\mathcal{O}_{X_1}$.

For any point x_1, above x, the map $\overline{M_{X,x}} \to \overline{M_{X_1,x_1}}$ is Kummer. Therefore, the groups $\overline{M_{X,x}}^{gp}$, $\overline{M_{X_1,x_1}}^{gp}$ have the same rank. Hence, $X_1 \to X_0$ preserves the rank strata.

Consider a generic point of a component of strata $s' \in X_1$, lying above the generic point of strata $s \in X$, which specializes to $s(x)$. Then as X_0 is generically fs at $s(x)$, étale locally, we have $M_s = P'\mathcal{O}^*_{X,s}$. Hence, the stalk of $M_{X_1,s'}$ is associated to $(P'\mathcal{O}^*_{X,s} \oplus_P P')^{sat}$. Since $\overline{M_s}$ is saturated, we have a map $\overline{(M_s \oplus_P P')^{sat}} \to \overline{M_s}$ (induced from a chart $P' \to M_s$) which is a section to the Kummer map $\overline{M_s} \to \overline{(M_s \oplus_P P')^{sat}}$. Therefore, $\overline{(M_s \oplus_P P')^{sat}}^{red} \to \overline{M_s}$ is an isomorphism. As $\overline{(M_s \oplus_P P')^{sat}}$ is saturated, then $\overline{(M_s \oplus_P P')^{sat}}^{red} = \overline{(M_s \oplus_P P')^{sat}}$. Thus $\overline{M_{X_1,s'}} = \overline{M_{X,s}}$.

Let s'_1, s'_2 be generic points of components of strata on X_1, such that s'_1 specializes to s'_2. Let s_1 and s_2 be their corresponding images in X. Then, by the above, the cospecialization morphism $\overline{M_{X_1,s'_2}} \to \overline{M_{X_1,s'_1}}$ coincides with $\overline{M_{X,s_2}} \to \overline{M_{X,s_1}}$, which is surjective.

Thus, by 2.16.2, and the remark immediately afterwards, it is enough to show that $\overline{M_{X_0,x_0}} = \overline{M_{X_0,s(x_0)}}$.

Since X is rationally fs then, by 2.9.3, étale locally we may construct a Kummer extension $P \subset Q$, such that Q is a saturated optimized chart at $s(x)$. As Q is saturated, moding out by the torsion, we may suppose that Q is reduced, so $Q = \overline{Q} = \overline{M_{X,s(x)}}$. By definition of X_1, we have that $(Q \oplus_P P')^{sat}$ is an optimized chart at $s(x_0)$. Given that $Q = \overline{M_{X,s(x)}} = P'$, then we may compute $\overline{M_{X_0,s(x_0)}}$ as $\overline{(P' \oplus_P P')^{sat}}$. Since P' is saturated, we have an induced morphism $(P' \oplus_P P')^{sat} \to P'$ which is a section to the Kummer map $P' \to (P' \oplus_P P')^{sat}$. Hence, as $(P' \oplus_P P')^{sat}$ is saturated, then $(P' \oplus_P P')^{sat} = P' \oplus \tau$, where τ is a torsion group. Therefore, $\overline{M_{X_0,s(x_0)}} = \overline{(P' \oplus_P P')^{sat}} = P' = \overline{M_{X_0,x_0}}$.

Suppose the X that we started with, is already normal. Given that $X_{tr} \times_P P' \to X_{tr}$ is étale (since all the elements in P are invertible in X_{tr}), then $X_1 \to X$ is étale over the logarithmically trivial locus. ■

Remark. The condition of taking the closure of $(X_{tr} \times_P P')$ in $(X \times_P P')^n$ is imposed so that we don't get components in X_1, where the structure map $M_{X_1} \to \mathcal{O}_{X_1}$ sends an element $m \in M_1$ to 0. It is not strictly necessary at this point, but it will become important in the following sections. Hence, we prefer to include it.

3. KUMMER FS VARIETIES

In this section, unless stated otherwise, all log varieties considered will be normal and rationally fs over a field k of characteristic zero. Moreover, we assume that for every rational log variety (X, M), the log structure M is locally (on every component) contained in $\mathcal{O}_X \setminus \{0\}$. Note that this last condition implies that the logarithmically trivial locus X_{tr} is dense in X.

3.1 Normal rational log varieties

Notation. For a variety X, non necessarily irreducible, we denote it's normalization (i.e. the disjoint union of normalized components) as X^n.

Example 3.1.1 *Saturating a rational log variety with respect to rational charts, as in the case fine schemes, is not well defined. For example, consider* $X = spec(k[x^2, xy, y])$ *with log structure* $y \neq 0$*. At the origin* $\mathbb{N} \to k[x^2, xy, y]$ $(1 \mapsto y)$ *is an optimized rational chart. Hence* $X \times_{\mathbb{N}} \mathbb{N}^{sat} = X$*. On the other hand at any point* $(x_0, 0)$ $(x_0 \neq 0)$*, we have that* $P := [x^{\pm 2}, xy, y]$ *is an optimized rational chart. So* $X \times_P P^{sat} = spec(k[x^{\pm 1}, y])$ *which is not a Zariski open neighborhood of* $X \times_{\mathbb{N}} \mathbb{N}^{sat} = X$*. Thus saturation does not glue in Zariski topology.*

Even if we consider a normal variety, saturation of rational log schemes is not well defined. For example, let $X = spec(k[x, y])$*, and consider the morphism* $\pi : spec(k[x, y]) \to spec(k[x^2, xy, y^2])$*. Give* $spec(k[x, y])$ *the pullback log structure* $\pi^{-1}(y \neq 0)$*. Even though, as in the lemma below, for an optimized rational chart* $P \to M_X$*,* X *is a component of* $(X \times_P P^{sat})^n$*, the induced sheaves do not glue for charts optimized at different points.*

Lemma 3.1.2 *Let (X, M) be a normal rational log variety (not necessarily saturated) and let $X \to A_P$ be an rational chart optimized at some point $x \in X$. Then (scheme theoretically) X is a component of $(X \times_P P^{sat})^n$, and the rational log structure induced by $(M \oplus_P P^{sat})$ (on X) is independent of P.*

Remark. Note that this log structure **does** depend on the point x.

Proof We show first that X is a component of $(X \times_P P^{sat})^n$. Note that $A_{P^{sat}}$ corresponds to some components of A_P^n. Given that $X \to A_P$ is a chart, then $X \times_P P^{gp}$ is a nonempty open subset of X. Since X is normal, we see by 3.2.1, that $X \to A_P$ lifts to a morphism $X \to A_{P^{sat}}$. Also by 3.2.1, we get an induced morphism $X \to (X \times_P P^{sat})^n$ which gives a right inverse to the projection $(X \times_P P^{sat})^n \to X$. Thus X is a component of $(X \times_P P^{sat})^n$.

To see that the log structure induced by $M \oplus_P P^{sat}$ is independent of P, note that given any other rational chart $R \to M$ (which we can assume injective), for any $r \in R$, there is $p \in P$ and $u \in M_x^*$ such that $r = pu$. Moreover for any $r' \in R^{sat}$, then as $\mathcal{O}_X$ is normal, then $r' \in \mathcal{O}_X$ and there exists an n such that $(r')^n := r \in M_x$. So $(r')^n = up$ for some $u \in \mathcal{O}_X^*$ and $p \in P$. On the other hand since P is optimized, there exists a unit $v \in \mathcal{O}_x^*$, such that $r' = vp_1/p_2$ for some $p_1, p_2 \in P$. Since $(p_1/p_2)^n = (r'/v)^n \in M_x$ and P is localized (i.e. $p = uq \Rightarrow u \in P$ for $p, q \in P$, $u \in \mathcal{O}_x^*$), then $p' = p_1/p_2 \in P^{sat}$. Hence $\mathcal{O}_{X,x}^* \oplus_R R^{sat} \subset \mathcal{O}_{X,x}^* \oplus_P P^{sat}$. ■

To address the situation in the above remark, we instead consider the sheaf M^{sat} (i.e. the sheaf associated to $U \mapsto M(U)^{sat}$).

Note first that for a normal variety, M^{sat} is a well defined log structure, since $M \subset \mathcal{O}_X \setminus \{0\}$ and $\mathcal{O}_X$ is normal, then $M^{sat} \subset \mathcal{O}_X \setminus \{0\}$.

We check that for a normal, fine rational log variety (X, M), the sharpened stalks $\overline{(M^{sat})_x}$ are finitely generated, and so M^{sat} defines a rational log structure on X. Let $U \subset X$ be an open subset. Since $M(U) \subset K(X)$ (the function field of X), $M_x^{gp} \subset K(X)$ and we have a factorization $Lim\ M(U)^{gp} \to M_x^{gp} \subset K(X)$, then $Lim\ M(U)^{gp} \subset M_x^{gp}$. On the other hand, we have that each $M(U)^{sat} \subset M(U)^{gp}$

so $Lim\ M(U)^{sat} \subset M_x^{sat} \subset M_x^{gp}$. Hence, we must have that $Lim\ M(U)^{sat} \subset M_x^{sat}$. Let P be an optimized rational chart at x. Since M is rational then, P is finitely generated. The morphism $P^{sat} \to \overline{M_x^{sat}}$ must be surjective. To see this, take m_1, m_2 in M_x such that $(m_1/m_2)^n \in M_x$. Because P is an optimized chart, we get that $m_1/m_2 = up_1/p_2$, where $p_1, p_2 \in P$ and $u \in \mathcal{O}_X^*$. Hence $(p_1/p_2)^n \in M_x$, and since $M_{X,x} = P\mathcal{O}_X^*$, then $p_1^n/p_2^n \in P$. Thus $p_1/p_2 \in P^{sat}$. Hence, as P is finitely generated, then P^{sat} is finitely generated and so $\overline{M^{sat}}$ also finitely generated.

Definition 3.1.3 *Let $f : X \to Y$, $g : Z \to X$ be a morphisms of rational log varieties. Let $U_Z \subset Z$ (resp. $U_X \subset X$) denote the locus where the log structure on Z (resp. X) is trivial. We define the fiber product, denoted by $(Z \times_Y X)^{n-log}$, as the closure of $(U_Z \times_Y U_X)^n$ in $(Z \times_Y X)^n$. Let $\pi_X : (X \times_Y Z)^{n-log} \to X$, $\pi_Z : (X \times_Y Z)^{n-log} \to Z$, $\pi_Y : (X \times_Y Z)^{n-log} \to Y$ be the induced projections. The log structure on $(X \times_Y Z)^{n-log}$ is given by $((\pi_Z^{-1}M_Z \oplus_{\pi_Y^{-1}M_Y} \pi_X^{-1}M_X)^a)^{sat}$ under the induced morphism $\phi : \pi_Z^{-1}M_Z \oplus_{\pi_Y^{-1}M_Y} \pi_X^{-1}M_X \to \mathcal{O}_{(Z\times_Y X)^{n-log}}$.*

Lemma 3.1.4 *(cf. [3, 4.6.16]) The log structure defined above on $(X \times_Y Z)^{n-log}$ is rational.*

Proof We follow the argument in [3, 4.6.16] and adapt it to our setting.

Let $W := (X \times_Y Z)^{n-log}$, $w \in W$. Denote $x \in X$, $z \in Z$, $y \in Y$ the corresponding images of w. Pick a rational chart $P \to M_Y$, together with compatible rational charts $R \to M_Z$, $P \to R$, $Q \to M_X$, $P \to Q$ at the points y, x, z respectively. Denote $T := (R \oplus_P Q)^{sat}$. Let $P \subset P_1$ be an optimization of P. Note that $(Q \oplus_P P_1)^{sat}$ is still a rational chart on some étale neighborhood of X (similarly for $(R \oplus_P P_1)^{sat}$ and Z). Let $(Q \oplus_P P_1)^{sat} \subset Q_1$ be an optimization of $(Q \oplus_P P_1)^{sat}$ at some point x_1 above x, and $(R \oplus_P P_1)^{sat} \subset R_1$ an optimization of $(R \oplus_P P_1)^{sat}$ at some point

z_1 above z. Let $T_1 := (R_1 \oplus_{P_1} Q_1)^{sat}$, then for some point w_1 above w, $(T_1)^a_{w_1} = ((M_{Z,z_1} \oplus_{M_{Y,y_1}} M_{x,x_1})^{sat})^a_{w_1}$. Here is a diagram illustrating the situation.

$$\begin{tikzcd}
T_1 & (T \oplus_Q Q_1)^{sat} \arrow[l] & & Q_1 \arrow[ll] \\
(T \oplus_R R_1)^{sat} \arrow[u] & T \arrow[l] \arrow[u] & Q \arrow[l] \arrow[r] & (Q \oplus_P P_1)^{sat} \arrow[u] \\
 & R \arrow[u] \arrow[d] & P \arrow[l] \arrow[u] \arrow[dr] & \\
R_1 \arrow[uu] & (R \oplus_P P_1)^{sat} \arrow[l] & & P_1 \arrow[ll] \arrow[uu]
\end{tikzcd}$$

The morphisms $T \to (T \oplus_Q Q_1)^{sat}$, $T \to (T \oplus_R R_1)^{sat}$ correspond to optimizations, hence $T^a \to ((T \oplus_Q Q_1)^{sat})^a$, $T^a \to ((T \oplus_R R_1)^{sat})^a$ are both Kummer. Since $T_1 = (T \oplus_Q Q_1)^{sat} \oplus_T (T \oplus_R R_1)^{sat}$, then $(T)^a_{w_1} \to (T_1)^a_{w_1}$ must be a Kummer. This works for any points $x \in X$, $y \in Y$, $z \in Z$, where Q, P, R are rational charts (even though the Q_1, P_1, R_1 might be different depending on the points chosen), hence $\alpha : (T)^a \to ((\pi_Z^{-1} M_Z \oplus_{\pi_Y^{-1} M_Y} \pi_X^{-1} M_X)^a)^{sat}$ is Kummer. ■

Remarks. We note that $(Z \times_X Y)^{n-log}$ might be empty even if the underlying scheme of $(Z \times_Y X)$ is non empty.

Let $U_Y \subset Y$ be the logarithmically trivial locus. Then $U_Z \times_Y U_X = U_Z \times_{U_Y} U_X$, because logarithmic morphisms must restricts to morphism on logarithmically trivial loci (these follows because morphisms of schemes induce local homomorphisms of local rings).

This definition does not give a categorical fiber product of rational log varieties. However, it will satisfy the universal property of fiber product when we restrict to Kummer étale maps (see 3.8.6).

Example 3.1.5 *Let $P \to Q$, $P \to R$ be morphisms of fs monoids and let $T := (R \oplus_P Q)$. Suppose R, P, Q are torsion free, then $(A_R \times_{A_P} A_Q)^{n-log} = A_{T^{sat}}$. To see this, note that $A_{T^{int}}$ is the closure of $A_{T^{gp}}$ in A_T ($A_{T^{gp}} \to A_T$ is always an open immersion). Then we have that $A_{T^{sat}}$ is the normalization of $A_{T^{int}}$ (this can be seen* *from the universal property of normalization and using the fact that $A_{T^{sat}}$ is normal). Finally note that $(R^{gp} \oplus_P Q^{gp})$ is a group, and hence $(R^{gp} \oplus_P Q^{gp}) = (R \oplus_P Q)^{gp} = (R^{gp} \oplus_{P^{gp}} Q^{gp})$ by the universal property of pushouts and groupification.*

3.2 Normally log smooth morphisms

We revisit normally log smooth morphisms introduced in [3]. Most of the results in this section can be found there. We rewrite the proofs for completeness.

Lemma 3.2.1 *Let X, Y be irreducible varieties. Denote by $Y_{nor} \subset Y$, the maximal open subset of Y which is normal. Let $f : X \to Y$ be a morphism such that $f^{-1}(Y_{nor}) \neq \emptyset$. Then there is a unique component C in $(X \times_Y Y^n)$ which dominates X, and moreover $C^n \cong X^n$.*

Proof Let $X_0 := f^{-1}(Y_{nor})$. Since $Y^n \to Y$ is birational, the morphism $X \times_Y Y^n \to X$ is an isomorphism above X_0. Given that X_0 is not empty, there is a unique component C on $X \times_Y Y^n$ which dominates X. Since the morphism $C \to X$ is dominant, then by the universal property of normalization we get an induced morphism $C^n \to X^n$. Moreover $C^n \to X^n$ must be birational and finite, thus $C^n \cong X^n$. ∎

Remark. The hypothesis for $f : X \to Y$ in the above lemma is satisfied, in particular, if f is dominant.

Lemma 3.2.2 *Let $P \to Q$ be an injective log smooth morphism of fs monoids. Then $A_{Q^{gp}} \to A_{P^{gp}}$ is surjective in the underlying schemes. Moreover, given a rational log variety Z, together with a log morphism $Z \to A_P$, then $Z \times_P P^{gp}$ is non empty and $Z \times_Q Q^{gp} \to Z \times_P P^{gp}$ is surjective.*

Proof Consider the exact sequence $0 \to P^{gp} \to Q^{gp} \to Q^{gp}/P^{gp} \to 0$. Choosing a splitting $Q^{gp}/P^{gp} = \mathbb{Z}^n \oplus T$ (where T is torsion), we get a factorization $P^{gp} \to \mathbb{Z}^n \oplus P^{gp} \to Q^{gp}$ of the inclusion $P^{gp} \subset Q^{gp}$. The first map is the canonical inclusion and the second map has finite cokernel. Therefore $A_{P^{gp}\oplus\mathbb{Z}^n} \to A_{P^{gp}}$ is surjective and $A_{Q^{gp}} \to A_{P^{gp}}$ is finite surjective. Hence $A_{Q^{gp}} \to A_{P^{gp}}$ is surjective.

For the second assertion, since $Z \to A_P$ is a log morphism, and no element of P goes to zero on $\mathcal{O}_Z$, the localization $\mathcal{O}_Z[P^{-1}]$ is non zero ($\mathcal{O}_Z$ has no nilpotent elements since Z is a variety) and thus $Z \times_P P^{gp}$ is non empty. Since surjectivity is preserved by base change, then $Z \times_Q Q^{gp} \to Z \times_P P^{gp}$ is surjective. ■

Lemma 3.2.3 *Let $f : X \to Y$ be a morphism of non-empty normal irreducible rational log varieties. Suppose there exists a chart $P \to Q$ of f, which is log smooth as a morphism of monoids. Then the induced morphism $j : X \to (Y \times_P Q)$, has a factorization $X \to (Y \times_P Q)^{n-log} \to (Y \times_P Q)$.*

Proof Let Y_0 be the logarithmically trivial locus, that is $Y_0 := Y \times_P P^{gp}$. Note that, since we assume $M_Y \subset \mathcal{O}_Y \setminus \{0\}$, then Y_0 is non-empty. Since $P \to Q$ is log smooth, then the restriction $A_{Q^{gp}} \to A_{P^{gp}}$ is classically smooth. Therefore $Y_0 \times_{P^{gp}} Q^{gp} \to Y_0$ is also classically smooth. Since $A_{Q^{gp}} \to A_Q$ is an open immersion, then $Y_0 \times_{P^{gp}} Q^{gp}$ is open in $Y \times_P Q$. Given that $Y_0 \times_{P^{gp}} Q^{gp} \to Y_0$ is smooth, then $Y_0 \times_{P^{gp}} Q^{gp}$ must lie within the locus where $Y \times_P Q$ is normal.

On the other hand, let $X_0 := X \times_Q Q^{gp}$ be the logarithmically trivial locus of X. Similarly as above X_0 is non-empty. Since $j(X_0) \subset Y_0 \times_{P^{gp}} Q^{gp}$, we get that $j^{-1}(Y_0 \times_{P^{gp}} Q^{gp})$ is non empty in X. Thus by 3.2.1, we have an induced map $X \to (Y \times_P Q)^n$ factoring j. Now, as X_0 is non empty and maps to Y_0, then (from the definition of $(Y \times_P Q)^{n-log}$) we see that the morphism $X \to (Y \times_P Q)^n$, factors as $X \to (Y \times_P Q)^{n-log} \to (Y \times_P Q)^n$. ■

Definition 3.2.4 *[3] Let $f : X \to Y$ be a morphism of normal rationally fs varieties. We say f is normally log smooth (étale) if we can find a rational chart $(P \to M_X, Q \to M_Y, P \to Q)$, which is log smooth (étale) and such that the induced map $X \to (Y \times_P Q)^{n-log}$ is étale.*

Such a chart $P \to Q$, will be called a smooth (étale) chart of f.

Lemma 3.2.5 *Let $f : X \to Y$ be a morphism of normal rationally fs varieties. Let $(P \to Q, P \to M_Y, Q \to M_X)$ be a rational chart of f. There exists an optimized rational chart of f containing $(P \to Q)$.*

Proof By 2.9.1, étale locally, we can find an optimized rational chart $P' \to M_Y$, containing $P \to M_Y$. Let Q' be the saturated pushout $P' \leftarrow P \to Q$. Then, since M_X is saturated, there is an induced map $\alpha : Q' \to M_X$, factoring $Q \to M_X$. For any $q_1, q_2 \in Q'$ such that $\alpha(q_1) = \alpha(q_2)$, we have that $q_1^n = q_2^n$ (because $Q \to Q'$ is Kummer and $Q^a \to M_X$ is injective). Thus after localizing $\alpha : Q' \to M_X$ (that is, replacing Q' by $Q'[(\alpha^{-1}(M_X^*))^{gp}]$) then $Q' \to M_X$ becomes a rational chart. Let Q'' be the optimization of Q'. Considering the induced map $P' \to Q''$, we get an optimized chart of f containing the original $P \to Q$. ■

Lemma 3.2.6 *(Kato's Lemma) [8](lemma 3.1.6), (c.f. [11] (lemma 2.8)). Let $\phi : P \to Q$ be an injective smooth morphism of fs monoids. Let $P' \to P$ be an inclusion of fs monoids such that $(P')/(P')^* = P/P^*$. Then, after an étale extention of P, there exists an fs monoid $Q' \subset Q$, such that:*

1. *$P' \subset \phi^{-1}(Q')$*
2. *the restriction $P' \to Q'$ is smooth.*
3. *$Q = P \oplus_{P'} Q'$.*

Proof Consider the induced group exact sequence $0 \to P'^{gp} \to P^{gp} \to T \to 0$. After applying $Hom(-, P^{gp}/Q^{gp})$, we get a short exact sequence

$$Ext^1((P')^{gp}, P^{gp}/Q^{gp}) \to Ext^1(P^{gp}, P^{gp}/Q^{gp}) \to Ext^1(T, P^{gp}/Q^{gp}).$$

We must show that the class in $Ext^1(P^{gp}, P^{gp}/Q^{gp})$, corresponding to

$$0 \to P^{gp} \to Q^{gp} \to P^{gp}/Q^{gp} \to 0,$$

maps to zero in $Ext^1(T, P^{gp}/Q^{gp})$.

Let R denote the pushout $T \leftarrow P^{gp} \to Q^{gp}$. We will show that, after an étale extension of P, the exact sequence

$$0 \to T \to R \to P^{gp}/Q^{gp} \to 0$$

splits.

Let $\overline{m} \in P^{gp}/Q^{gp}$ be a torsion element. Let $m \in Q^{gp}$ be a lift of $\overline{m}$. Then, for some $n > 0$, we have that $m^n \in P^{gp}$. Since $(P')/(P')^* = P/P^*$, multiplying by an appropriate $u \in P^*$, we can lift um^n to P'.

Consider the extension $P^* \to P^*[u^{1/n}]$, and let P'' be the pushout $P^*[u^{1/n}] \leftarrow P^* \to P$. By definition $P \subset P''$, is an étale extension.

Now replace Q with the (saturated) pushout $P'' \leftarrow P \to Q$. Replace T with the pushout $(P'')^{gp} \leftarrow P^{gp} \to T$. Then replace R with the pushout $T \leftarrow (P'')^{gp} \to Q^{gp}$. Note that, after pushout, P^{gp}/Q^{gp} remains unaltered.

In P'' we have $v := u^{1/n} \in (P'')^*$, thus $(vm)^n \in P$. So the image of $(vm)^n$ must be zero in T (and hence also in R). Thus the image of $vm \in Q^{gp}$ in R is a n-th torsion element above $\overline{m}$.

Repeating this process for a set of generators of the torsion part of P^{gp}/Q^{gp}, we can construct a splitting of $R \to P^{gp}/Q^{gp}$. ∎

Corollary 3.2.7 *Let $f : X \to Y$ be a normally log smooth (étale) morphism. Let $Y \to A_P$ be a rational chart. Then, étale locally on X, we can find a smooth (étale) rational chart $P \to Q, Q \to M_X$ of f, extending P.*

Proof We only address the smooth case, the étale case has the same proof after replacing "smooth" by "étale".

Let $R \to T, R \to M_Y, T \to M_X$ be a smooth chart of f. Recall that we always assume that charts are saturated, hence both R and T are saturated.

Let K be the kernel of $R^{gp} \to T^{gp}$. Since $R \to T$ is smooth, then K is finite. Since R is saturated, then $K \subset R^*$. Hence, considering a section $\overline{R} \to R$, and replacing T with $(T \oplus_R \overline{R})^{sat}$, we get an injective log smooth chart $\overline{R} \to T$ of f.

Hence, we can assume $R^{gp} \to T^{gp}$ is injective.

Find a chart $S \to M_Y$ containing both, R and P. For instance consider $\alpha : (R \oplus T)^{gp} \to M_x^{gp}$ and let S be the inverse image of M_x (S is saturated by construction).

Let $L := (S \oplus_R T)$ (it is already saturated since $S = RS^*$). Then $S \to L$ is a log smooth chart of f.

By Kato's lemma, applied to $P \to S$, $S \to L$, étale locally on X, we get a chart $Q \to M_X$ and a smooth morphism $P \to Q$ such that $L = (Q \oplus_P L)$. Thus $P \to Q$ is a smooth rational chart of f, extending P. ■

3.2.8 . By the above corollary, 3.2.5, and Kato's lemma (see the proofs in [8] [lemma 3.1.6], c.f. [11](lemma 2.8)), any normally log smooth morphism has an optimized smooth chart.

Corollary 3.2.9 *Let $f : X \to Y$ be normally log étale and of Kummer type. Then, étale locally we can find a log smooth chart $P \to Q$ of f which is Kummer. We call such morphisms Kummer normally log étale.*

Proof Étale locally, for any points $x \in X$, $y := f(x)$, we can find a log étale optimized rational chart $P \to Q$ of f. Moreover, we can assume that P is sharp. Hence, as $\overline{M_{Y,\overline{y}}} \to \overline{M_{X,\overline{x}}}$ is Kummer and the cokernel of $P \to Q$ is finite, then $P \to Q$ must actually be Kummer. ■

Lemma 3.2.10 *Let $X \to Y$, $Z \to Y$ be morphisms of normal rational log varieties. Assume that $X_{tr} \to Y_{tr}$ is smooth (e.g. if $X \to Y$ is normally log smooth). Suppose that $(X_{tr} \times_{Y_{tr}} Z_{tr}) \neq \emptyset$. Then for any normal rational log variety W with morphisms $W \to Z$, $W \to X$ which coincide after composing with $X \to Y$, $Z \to Y$ respectively, we get a unique factorization $W \to (X \times_Y Z)^{n-log}$. That is, if $X_{tr} \to Y_{tr}$ is classically smooth and $(X_{tr} \times_{Y_{tr}} Z_{tr}) \neq \emptyset$, then the $n-$log fiber product is a colimit in the category of normal rational log varieties.*

Proof Given that under out assumptions of rational log varieties are $M_X \subset \mathcal{O}_X \setminus \{0\}$ (same for Y and Z) then W_{tr} maps to X_{tr} and Y_{tr} and so we get a morphism $W_{tr} \to$

$X_{tr} \times_{Y_{tr}} Z_{tr}$. Since $X_{tr} \to Y_{tr}$ is smooth, then so is $X_{tr} \times_{Y_{tr}} Z_{tr} \to Z_{tr}$ and hence $X_{tr} \times_{Y_{tr}} Z_{tr}$ is already normal. Thus if $W \neq \emptyset$, then by 3.2.1, and the definition of $()^{n-log}$, we get a unique morphism $W \to (X \times_Y Z)^{n-log}$. ■

Corollary 3.2.11 *Let $X \to Y$, $Y \to Z$, $W \to Z$ be morphisms of normal rational log varieties, such that $X \to Y$ and $Y \to Z$ are smooth on their respective logarithmically trivial loci. Suppose that $(W_{tr} \times_{Z_{tr}} X_{tr}) \neq \emptyset$. Then $((W \times_Z Y)^{n-log} \times_Y X)^{n-log} = (W \times_Z X)^{n-log}$*

Proof This is immediate from the universal property above. ■

Proposition 3.2.12 *[3] Normally log smooth (étale, Kummer étale) morphisms are preserved under composition.*

Proof Let $f : X \to Y$, $g : Y \to Z$, be normally log smooth morphisms. By 3.2.7, we can construct smooth optimized charts $P \to M_Z$, $Q \to M_Y$, $R \to M_X$, $P \to Q$, $Q \to R$ (same monoid Q) of f and g respectively.

The following diagram, with normalized Cartesian squares (see 3.2.11), illustrates the situation.

$$\begin{array}{ccccccc}
X & \longrightarrow & (Y \times_Q R)^{n-log} & \longrightarrow & (Z \times_P R)^{n-log} & \longrightarrow & A_R \\
& & \downarrow & & \downarrow & & \downarrow \\
& & Y & \longrightarrow & (Z \times_P Q)^{n-log} & \longrightarrow & A_Q \\
& & & & \downarrow & & \downarrow \\
& & & & Z & \longrightarrow & A_P
\end{array}$$

Since $Y \to (Z \times_P Q)^{n-log}$ is étale, base changing with respect to R, we get that $(Y \times_Q R)^{n-log} \to (Z \times_P R)^{n-log}$ is also étale. Now, given that $Q \to R$ is a smooth chart of f, we get that the composition $X \to (Y \times_Q R)^{n-log} \to (Z \times_P R)^{n-log}$ is étale.

The cases of étale and Kummer étale follow from the analogous argument. ■

Lemma 3.2.13 *Let $f : Y \to X$ be a normally log smooth morphism of normal rational log varieties , and $g : Z \to X$ a morphism of normal rational varieties. If $(Z \times_Y X)^{n-log}$ is non empty, then the induced morphism $(Z \times_Y X)^{n-log} \to Z$ is normally log smooth.*

Example 3.2.14 *(cf. [12, Appendix B.1.]) Consider the morphism of monoids* $\mathbb{N}^2 \to \mathbb{N}^2$ ($(1,0) \mapsto (1,0)$, $(0,1) \mapsto (1,1)$), *inducing the morphism of monoschemes* $spec(k[x,y/x]) \to spec(k[x,y])$. *This morphism is classically log étale. Consider the open subset* $U \subset spec(k[x,y/x])$ *where* $y/x \neq 1$. *Let* $\mathbb{N}^2 \to \mathbb{N}$ *be the sum inducing the diagonal morphism* $spec(k[t]) \to spec(k[x,y])$. *Denote* $Y := spec(k[x,y])$, $Z := spec(k[t])$, *then* $(U \times_Y Z)^{n-log}$ *is empty.*

Proof (Proof of Lemma 3.2.13)

Let $z \in Z$, $y \in y$ such that $f(y) = x = g(z)$. Étale locally we can find a smooth injective rational chart $P \to Q$ of f, and a rational chart $P \to R$ of g (same P for both charts).

Since $P \to Q$ is log smooth, then $X \times_P Q^{gp} \to X \times_P P^{gp}$ is classically smooth. Hence, as $X \times_P P^{gp} \to X$ is an open immersion, $X \times_P Q^{gp}$ must be normal. By 3.2.2, $Z \times_P Q^{gp}$ is nonempty.

Given that $Z \times_P Q^{gp}$ is the inverse image of $X \times_P Q^{gp}$ under $Z \times_P Q \to X \times_P Q$, then (by 3.2.1) the folowing diagram is normalized Cartesian (upto components of $Z \times_X (X \times_P Q)^n$ which do not dominate Z).

$$\begin{tikzcd}
(Z \times_P Q)^n \arrow[r] \arrow[d] & (X \times_P Q)^n \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

Similarly, letting $T := (P \oplus_Q R)^{sat}$, we get that the following diagram is normalized Cartesian (upto non dominating components above Z)

$$\begin{tikzcd}
(Z \times_P Q)^n \arrow[r] \arrow[d] & A_T \arrow[d] \\
Z \arrow[r] & A_R
\end{tikzcd}$$

In particular, the induced morphism $(Z \times_P Q)^n \to (Z \times_R T)^n$ is an open immersion.

Let $(Z \times_X Y)^n$, denote only those components in the normalized fiber product which dominate Z. Since $Y \to (X \times_P Q)^n$ is étale ($P \to Q$ was chosen as a smooth chart), then the following diagram is Cartesian.

$$\begin{array}{ccc}
(Z \times_X Y)^n & \longrightarrow & Y \\
\downarrow & & \downarrow \\
(Z \times_P Q)^n & \longrightarrow & (X \times_P Q)^n
\end{array}$$

Hence $(Z \times_X Y)^n \to (Z \times_P Q)^n \to (Z \times_R T)^n$ is étale. We conclude that $R \to T$ is a log smooth chart making $(Z \times_X Y)^n \to Z$ normally log smooth.

Since log morphisms restrict to morphisms of logarithmically trivial loci, we can replace all normalizations $()^n$ by $()^{n-log}$ and get that $(Z \times_X Y)^{n-log} \to Z$ is normally log smooth (whenever $(Z \times_X Y)^{n-log} \neq \emptyset$). ■

Finally we end with a lemma which we shall use in the following sections.

Lemma 3.2.15 *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of normal rational log varieties. If $g \circ f$ is normally log smooth (étale, Kummer normally log étale) and g is Kummer normally log étale, then f is normally log smooth (étale, Kummer normally log étale).*

Proof By Kato's lemma, up to an étale cover we can assume that there are common smooth charts $(P \to P', P \to M_Z, P' \to M_Y)$, $(P \to T, P \to M_Z, T \to M_X)$ (same P in both charts, $P \to P'$ Kummer) optimized at points $x \in X$, $y \in Y$, $z \in Z$ above each other.

Let $R := (T \oplus_P P')^{sat}$.

Consider the normalized Cartesian diagram

$$\begin{array}{ccccc}
(X \times_T R)^{n-log} & \longrightarrow & (Z \times_P R)^{n-log} & \longrightarrow & A_R \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & (Z \times_P T)^{n-log} & \longrightarrow & A_T
\end{array}$$

Since $X \to (Z \times_P T)^{n-log}$ is étale, then $(X \times_P R)^{n-log} \to (Z \times_P R)^{n-log}$ is also étale.

Similarly from the diagram

$$\begin{array}{ccccc}
(Y \times_{P'} R)^{n-log} & \longrightarrow & (Z \times_P R)^{n-log} & \longrightarrow & A_R \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & (Z \times_P P')^{n-log} & \longrightarrow & A_{P'}
\end{array}$$

We conclude that $(Y \times'_P R)^{n-log} \to (Z \times_P R)^{n-log}$ is étale.

Then letting X' be the base change

$$\begin{array}{ccc} X' & \longrightarrow & (X \times_T R)^{n-log} \\ \downarrow & & \downarrow \\ (Y \times_{P'} R)^{n-log} & \longrightarrow & (Z \times_P R)^{n-log} \end{array}$$

We see that both $X' \to (X \times_T R)^{n-log}$ and $X' \to (Y \times'_P R)^{n-log}$ are étale .

Since X is saturated, we get a morphism $X \to A_R$. Let $X \to (X \times_T R)^{n-log}$ be the induced morphism. This is a right inverse to $(X \times_T R)^{n-log} \to X$. So X must be a component of $(X \times_T R)^{n-log}$, since $(X \times_T R)^{n-log} \to X$ is finite. Moreover, the image of X' in $(X \times_T R)^{n-log}$ contains the X component since the morphism $X \to (X \times_T R)^{n-log}$ factors through X'. Hence, restricting to a component of $(X \times_T R)^{n-log}$, we have found an étale neighborhood $X' \to X$, such that $X' \to Y$ is normally log smooth (étale, Kummer normally log étale).

Note also that the induced morphism $X \to R$ is in fact a rational chart. To see this, note first that the composition $T^a \to R^a \to M_X$ and the morphism $T \to R$ are Kummer. Let $\alpha : R \to M_X$ be the induced morphism. Since R is saturated, any two elements $r_1, r_2 \in R$ mapping to the same monomial in M_X must differ by a torsion element in R^*. Therefore, r_1 and r_2 are identified in R^a, so $R^a \to M_X$ is Kummer.

■

3.3 Saturation of log structures and definition of Kummer fs variety

Definition 3.3.1 *Let (X, M) be a normal rational log variety. We denote by X^{sat} the log variety whose underlying scheme is X, and whose log structure $M_{X^{sat}}$ is the sheaf associated to the presheaf by $U \mapsto \{f \in \mathcal{O}_X(U) \mid f^n \in M\}$. Since X is normal, then $M^{sat} \subset M_{X^{sat}} \subset \mathcal{O}_X$. We call $M_{X^{sat}}$ the saturation of M in $\mathcal{O}_X$. We say M is saturated in $\mathcal{O}_X$ if $M = M_{X^{sat}}$.*

Example 3.3.2 *Consider* $X := spec(k[x^2, xy, y^2])$ *with fs log structure* M *induced by the chart* $spec(k[x^2, xy, y^2]) \to spec(k[y^2])$. *Then,* (X, M) *is not saturated since away from the origin* $(xy)^2$ *is in the log structure, but* (xy) *is not.*

Remark. Let (X, M) is an rational normal log variety. Then $(X)^{sat} \to X$ need not be strict, even if X and $(X)^{sat}$ are fs. For example, $X = spec(k[x])$ with log structure associated to $\mathbb{N} \to k[x]$ $(1 \mapsto x^2)$. Then $X^{sat} \to X$ is not strict. If $M = M_{X^{sat}}$ we say M is saturated in $\mathcal{O}_X$.

3.3.3 . We recall the conditions satisfied by generically fs rational log varieties from the last section.

- 2.13.1 *In the rank stratification of* (X, M), *every point is in a unique irreducible component of its stratum.*
- 2.15.1 *If* s', s *are generic points of rank strata, such that* s' *specializes to* s, *then the cospecialization map* $\overline{M}_s \to \overline{M}_{s'}$ *is surjective.*

Definition 3.3.4 *Let* X *be a rational log variety,* $X \to A_P$ *a rational chart and* $P \subset P'$ *be a Kummer extension. We call the induced morphism* $(X \times_P P')^{n-log} \to X$ *a Kummer cover.*

Definition 3.3.5 *A Kummer fs variety is a normal rational log variety* (X, M) *with* M_X *saturated in* $\mathcal{O}_X$ *and such that, étale locally at every point* $x \in X$, *there is a Kummer cover* $X' \to X$ *which is fs.*

Remark. Let X be a generically fs rational log variety. By 2.17.1, étale locally at every $x \in X$, there is a Kummer cover $X' \to X$, such that X' is fs. Hence, a saturated generically fs rational log variety is Kummer fs.

Definition 3.3.6 *Let* X *be a Kummer fs variety and* $U \to X$ *an étale neighborhood. A Kummer cover* $U' \to U$, *where* U' *is fs, will be called an fs Kummer neighborhood.*

3.3.7 . Let (X, M) be a generically fs rational log variety over a field K. For $x \in X$, denote $n(x) := |\overline{M_{s(x)}}^{gp} : \overline{M_x}^{gp}|$. We say K has sufficient roots of unity with respect to (X, M) if K has $n(x)$ distinct $n(x)$-th roots of unity for every $x \in X$.

3.3.8 . Let X be a rational log variety. Given a point $x \in X$, whenever we speak of the generic point of it's stratum $s(x)$, we implicitly assume that X satisfies 2.13.1. This is satisfied for all log structures induced from a codimension one closed subset D in a variety X (see 2.13.2). In particular, quasi-toroidal varieties satisfy 2.13.1.

Lemma 3.3.9 *Let X be a Kummer fs variety. Given an fs Kummer neighborhood $f : X' \to X$, and an optimized smooth Kummer chart $P \to P'$ (at $x' \in X'$, $x = f(x') \in X$ respectively), we must have that P' is actually a chart of X' at x'.*

Proof The log structure on $(X \times_P P')^{n-log}$ is locally induced at any $y \in (X \times_P P')^{n-log}$ by taking $(M_X \oplus_P P')^{sat}$. Since P' is an optimized rational chart at x', it must also be an optimized chart along the rank stratum of x' because $(X \times_P P')^n$ is fs at x'. Therefore, in a neighborhood of x', $P' \to \overline{(M_X \oplus_P P')^{sat}}$ is surjective (since $(X \times_P P')^n$ satisfies 2.15.1). The result follows from 2.7.1 and because $X' \to (X \times_P P')^n$ is étale strict. ■

3.3.10 . If X is fs and $X' \to X$ is normally Kummer log étale, then X' is fs. This follows because, locally, we can find an optimized chart $P \to Q$, such that $X' \to (X \times_P Q)^{n-log}$ is étale strict, and $(X \times_P Q)^{n-log}$ is fs.

3.3.11 . As a consequence of 3.3.10, we have the following. Suppose X is Kummer fs and $Y \to X$ is Kummer normally log étale morphism of rational log varieties. Then Y is has an fs Kummer neighborhood. To see this, let $X' \to X$ be an fs Kummer neighborhood and let $Y' := (Y \times_X X')^{n-log}$. Then $Y' \to Y$ is Kummer normally log étale. Moreover, Y' must be fs since X' is fs and $Y' \to X'$ is Kummer normally log étale.

3.4 Quasi log smooth morphisms

Definition 3.4.1 *A morphism $f : X \to Y$ of rationally fs varieties will be called quasi log smooth is at every $x \in X$, we can find a Kummer normally log étale neighborhood $X' \to X$, such that the composition $X' \to X \to Y$ is normally log smooth.*

Proposition 3.4.2 *Let $X \to Y$ be a quasi log smooth morphism and let $Z \to Y$ be a morphism of rationally fs log varieties. Suppose that $(Z \times_Y X)^{n-log} \neq \emptyset$. Then $(Z \times_Y X)^{n-log} \to Z$ quasi log smooth.*

Proof Let $X' \to X$ be a Kummer étale neighborhood, such that $X' \to Y$ is normally log smooth. Restricting X, we have that $X' \to X$ is surjective and smooth above X_{tr}. Similarly as $X' \to Y$ is normally log smooth, then $X'_{tr} \to Y_{tr}$ is smooth. Therefore $X_{tr} \to Y_{tr}$ must also be smooth [13, Tag 02K5]. So, by 3.2.11, we have the following normalized pullback diagram.

$$\begin{tikzcd}
(Z \times_Y X')^{n-log} \arrow[r] \arrow[d] & X' \arrow[d] \\
(Z \times_Y X)^{n-log} \arrow[r] \arrow[d] & X \arrow[d] \\
Z \arrow[r] & Y
\end{tikzcd}$$

Given that normally log smooth (Kummer normally log étale) morphisms are preserved under base change (when the n-log product is nonempty), then $(Z \times_Y X')^{n-log} \to (Z \times_Y X)^{n-log}$ is Kummer normally log étale. Similarly, as $X' \to Y$ is normally log smooth, then so is $(Z \times_Y X')^{n-log} \to Z$. ■

Proposition 3.4.3 *Quasi log smooth morphisms are preserved under composition.*

Proof Let $X \to Y$, $Y \to Z$ be quasi log smooth morphisms. Consider a Kummer normally log étale neighborhood $Y' \to Y$ such that $Y' \to Z$ is normally log smooth. Since $(X_{tr} \times_{Y_{tr}} Y'_{tr}) \neq \emptyset$, then $(X \times_Y Y')^{n-log} \to Y'$ is quasi log smooth. Find a Kummer étale neighborhood $X'' \to (X \times_Y Y')^{n-log}$, for which the composition

$X'' \to Y'$ is normally log smooth. Hence the composition $X'' \to Y' \to Z$ is normally log smooth. On the other hand, by base change, $(X \times_Y Y')^{n-log} \to X$ is Kummer normally log étale. Hence, $X'' \to X$ is a Kummer étale neighborhood. ■

Remark. It follows by definition that normally log smooth morphism are quasi log smooth.

3.4.4 . Quasi log smooth morphisms are not necessarily normally log smooth. Consider the following example $X := spec(k[x^2, xy, y^2]) \to spec(k)$. Give X the log structure induced from $y \neq 0$ and $spec(k)$ the trivial log structure. Then we can find an fs Kummer neighborhood $X' := spec(k[x, y]) \to X$ such that $X' \to spec(k)$ is log smooth (hence normally log smooth since the log structure on $spec(k)$ is trivial). However, $f : X \to spec(k)$ is not normally log smooth since the normalized base change $(spec(k) \times_P Q)^n$ of $spec(k)$, with respect to any rational chart $P \to Q$ of f, is fs and X is not.

Example 3.4.5 *Let $X = spec(k[x^2, xy, y^2])$ with log structure coming from $X \to \mathbb{A}^1 = spec(k[y^2])$. Give $\mathbb{A}^1$ the standard log structure at the origin. Then the following square shows that $X \to \mathbb{A}^1$ is Kummer log smooth.*

$$\begin{tikzcd} spec(k[x,y]) \arrow[r] \arrow[d] & spec(k[y]) \arrow[d] \\ spec(k[x^2, xy, y^2]) \arrow[r] & spec(k[y^2]) \end{tikzcd}$$

However, $X = spec(k[x^2, xy, y^2]) \to spec(k[y^2]) = \mathbb{A}^1$ cannot be normally log smooth. To see this, choose the standard chart on $\mathbb{A}^1$ and notice that no rational chart Q of X at the origin will give a strict étale map $X \to A_Q$, since the log structure on X is not fs.

3.5 Quasi log smooth varieties

As a preliminary, we recall the definition of strongly equivariant morphism, and Luna's fundamental lemma for equivariant étale morphisms (for details see [14]).

Since we will be working only with finite (hence diagonalizable) group actions on affine varieties, we restrict to that situation in the definition and the statement of the lemma, even though in [15] and [14], they are stated in much greater generality. In particular, for an action of G (finite) on X, we may speak of the geometric quotient $X/G := spec(k[X]^G)$.

Definition 3.5.1 *([15], [14]) Let G be a diagonalizable algebraic group acting on normal affine varieties X, Y, and let $f : X \to Y$ be a G-equivariant morphism. We say that f is strongly G-equivariant if $X = (X//G \times_{Y//G} Y)$. Furthermore, if we also have that $f//G : X//G \to Y//G$ is étale, then we say that f is strongly étale.*

Proposition 3.5.2 *(Luna's Fundamental lemma for étale morphisms)([15], [14, 5.6.4])*

Let G be a diagonalizable algebraic group acting on normal affine varieties X, Y, and let $f : X \to Y$ be a G-equivariant morphism. Then f is strongly étale if and only if f is étale, it sends closed orbits to closed orbits and it preserves stabilizers.

Using Luna's fundamental lemma, we can characterize generically fs rational log varieties which are, Kummer locally, smooth over a field.

Proposition 3.5.3 *Let (X, M) be a Kummer fs variety over field k (of characteristic zero) with enough roots of unity. Suppose M is generically fs. Let x be a point in X. Let P be a sharp optimized chart at x, and let $P' \supset P$ be a Kummer extension, with P' sharp, such that the associated Kummer cover is $X' := (X \times_P P')^{n-log}$ is fs. If X' is classically log smooth over k, then (X, M) is quasi-toroidal.*

Proof *Step* 1. By [5](proof of 8.3) we have that the underlying map of schemes $X' \to A_{P'}$ is smooth.

Step 2. Let $x' \in X'$ be a point above $x \in X$. Then $G_P = (P'^{gp}/P^{gp})^\vee$ acts on X'. The induced map from the quotient $X'/G_P \to X$ must be finite and birational, because G_P acts transitively on the generic points of X' and these correspond to the inverse image of the generic point of X. Thus since X is normal, then $X'/G_P \to X$ is an isomorphism.

Step 3. Let $G_x \subset G_P$ be the stabilizer at x'. Since $X' \to A_{P'}$ is G_P-equivariant, we get a $\overline{G} := G_P/G_x$-equivariant map in the quotient $\overline{X} := X'/G_x \to A_{(P')^{G_x}}$. Let $\phi : X' \to \overline{X}$ be the quotient morphism. Now, $\overline{G}$ acts on $\overline{X}$ with no fixed points above x, hence (restricting X) the quotient morphism $\overline{X} \to X$ is étale at $x_0 := \phi(x')$.

Since P is an optimized sharp chart, P' is sharp, and $\overline{X} \to X$ is étale, then $P \cong \overline{M_{X,x}} \cong \overline{M_{\overline{X},x_0}} \cong (P')^{G_x}$. Since G_P acts effectively on P' this can only be the case if $G_P = G_x$. Thus x' is fixed by G_P.

Step 4. Given that the stratum $s(x')$ is fixed by the G_P action, restricting X, we can choose global semi-invariant functions generating the cotangent space to the stratum $s(x')$. In this way, we obtain a factorization of $X' \to A_{P'}$ through an G_P-equivariant morphism $g : X' \to \mathbb{A}^n \times A_{P'}$. Given that, by hypothesis, X' is log regular (toroidal), g is in fact étale, as $X \to A_{P'}$, $\mathbb{A}^n \times A_{P'} \to A_{P'}$ are both smooth and $X \to \mathbb{A}^n \times A_{P'}$ is unramified. Since x' is fixed by the G_P action, by further restricting X, we can assume that g respects stabilizers. The morphism g must send closed orbits to closed orbits since G_P is finite. Hence, by Luna's fundamental lemma [14](5.6.4), g is strongly étale and descends to an étale morphism $X \to A_Q := (\mathbb{A}^n \times A_{P'})/G_P$.

Step 5. Consider $A_{P'} \times A^n$ with the toric structure coming from P' and the semi-invariant coordinates in A^n. Let σ be an associated cone to the toric structure on $A_{P'} \times A^n$, with lattice structure N.

The action of G_P factors through the action of the torus. That is, letting $M := Hom(N, \mathbb{Z})$, then $G_P \subset Hom(M, k^*)$. Let $M_\sigma := \{m \in M \mid g(m) = 1 \text{ for all } g \in G_P\}$, and denote $N_\sigma := Hom(M_\sigma, \mathbb{Z})$.

Then, A_Q coincides with a toric variety X_σ associated to σ and lattice $N_\sigma \supset N$. The toric structure on X_σ is compatible with that on $A_{P'} \times \mathbb{A}^n$, making the quotient morphism $A_{P'} \times \mathbb{A}^n \to X_\sigma$ toric. We emphasize that the rational log structure on X_σ is coming from the quotient of the log structure induced by P' on $A_P \times \mathbb{A}^n$ (not the toric structure).

The toric structure on $A_P \times \mathbb{A}^n$, coincides with that induced by σ, but with different lattice structure than in the quotient X_σ. The triviality locus of the log structure induced by $A_{P'}$ on $A_{P'} \times \mathbb{A}^n$, comes from a face τ in σ. The chart $A_{P'} \times \mathbb{A}^n \to A_{P'}$, corresponds to a face δ of σ. It is readily seen that $\sigma = \delta + \tau$ and $\tau \cap \delta = 0$. The lattice structure on $A_{P'} \times \mathbb{A}^n$ is then given by $N = N_\tau \times N_\delta$.

Since the action of G_P on $A_{P'} \times \mathbb{A}^n$ is toric, the locus where the rational log structure on the quotient X_σ is trivial comes from the same face $\tau \leq \sigma$ as above. The rational chart A_P corresponds to the face $\delta \leq \sigma$.

Step 6. The morphism $\mathbb{A}^n \times A_{P'} \to X_\sigma$ is étale above X_τ (the locus where the rational log structure is trivial on X_σ). This follows because the elements of P are invertible on X_τ, and the fact that $A_{P'} \times A^n$ is obtained as a component of the normalized fiber product $(X_\sigma \times_P P')^n$.

Step 7. In particular, since $\mathbb{A}^n \times A_{P'}$ is toroidal, X_τ must be regular. Therefore, τ must be a regular face of σ and so $(X_\sigma, X_\tau) = (A_Q, (P'^a)^{G_P})$ is a toric doubleton.

Step 8. Let $M' := (P'^a)^{G_P}$. Since M is invariant under the G_P-action, then $M \subset M' \subset \mathcal{O}_X \setminus \{0\}$. Moreover, $M \subset M'$ must be Kummer as P is a rational chart and $P \subset P'$ is Kummer. Therefore, as M is saturated in $\mathcal{O}_X$, we get that $M = M'$. Hence, (X, M) is quasi-toroidal. ∎

Remark. The hypothesis on sufficient roots of unity was used to ensure that the group G_P acts on X' with quotient X. If k does not have enough roots of unity, this may not be the case. For example, consider $X := spec(\mathbb{R}[x^3, x^2y, xy^2, y^3])$ with log structure $y \neq 0$. It has a Kummer fs neighborhood at the origin given by $A^3 \to X$, with induced group $\mathbb{Z}/3\mathbb{Z}$. However, this group acts trivially on A^3, and hence X is not a quotient of A^3.

Lemma 3.5.4 *[3, lemma 4.6.23] Let $f : X \to Y$ be a normally log smooth morphism, with Y toroidal. Then X is toroidal and f is classically log smooth.*

Proof Since Y is toroidal, at any point $y \in Y$ we can find a chart $Y \to A_P$ which is étale. By Kato's lemma, there exists a smooth chart $P \to Q$ of f, containing P. Then

$Y \times_P Q \to Q$ is étale, and hence $Y \times_P Q$ (before taking normalization) is toroidal. Since, étale locally , the morphism $X \to (Y \times_P Q)^n = Y \times_P Q$ is étale strict, we get that X must be fs, with étale chart $X \to A_Q$. Moreover, $X \to Y$ is classically log smooth. ■

Lemma 3.5.5 *Let (X, M) be a Kummer fs variety over a field k of characteristic zero, with enough roots of unity, and M generically fs. Suppose there exists a Kummer cover $X' \to X$, such that $X' \to k$ is log smooth. Then, there exists a Kummer cover $X'' \to X$, induced from a sharp optimized chart Kummer extension $P \subset P'$ (with P' sharp), such that $X'' \to spec(k)$ is classically log smooth.*

Proof Given a sharp optimized rational chart P at $x \in X$, by Kato's lemma for normally log smooth morphisms, we can find a Kummer extension $P \subset P'$ (P' a saturated optimized chart of X') such that $(X \times_P P')^{n-log}$ is fs and log smooth over $spec(k)$. Since P' is saturated, P is sharp, and $P \subset P'$ is Kummer, then the monoid morphism $P' \to \overline{P'} := P'/P'^*$ is log smooth. Therefore $(X \times_P \overline{P'})^n$ must be normally log smooth over $spec(k)$. Since $spec(k)$ is toroidal then, by 3.5.4, $(X \times_P \overline{P'})^{n-log} \to spec(k)$ is log smooth. ■

Proof [of Proposition 1.0.1.] From the discussion in 2.5, we see that quasi-toroidal varieties are Kummer fs varieties and are quasi-log smooth over k. The strata are locally irreducible since they coincide étale locally with toric strata of a toric variety.

Conversely, since the property of being quasi-toroidal is étale local, we can take a finite field extension $k \subset k'$, where k' has sufficient roots of unity and work instead with $X \times_k k'$. Hence, we may assume that k sufficient roots of unity.

The result now follows immediately from 3.5.5 and 3.5.3. ■

3.6 Minimal Kummer covers

We define the notion of minimal Kummer cover at a point. This extends the notion of minimal toroidal cover of quasi-toroidal varieties in [3, 5.2.8., 5.2.10.]. In this section all Kummer fs varieties are assumed to be generically fs.

Lemma 3.6.1 *Let (X, M) be a Kummer fs variety and let $P \to M$ be a rational chart optimized at $x \in X$. Let $P \subset P'$ be a Kummer extension, such that $(P')^* = P^*$, giving a Kummer fs neighborhood $X' \to X$. There exists a sharp and optimized rational chart Q (i.e. $Q = \overline{M_x}$), and a Kummer extension $Q \subset Q'$, where $Q' = \overline{M_{s(x)}}$, such that the morphism $X' \to X$ factors through $X' \to (X \times_Q Q')^{n-log}$.*

Proof Consider the following diagram whose rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & P^* \arrow[r] \arrow[d, equal] & P^{gp} \arrow[r] \arrow[d] & \overline{P}^{gp} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'^* \arrow[r] & (P')^{gp} \arrow[r] & \overline{P'}^{gp} \arrow[r] & 0
\end{tikzcd}$$

Since P' is fs, then it has a splitting $P' = P'^* \oplus \overline{P'}$, induced from a retraction $P'^{gp} \to P'^*$. Given that $P \subset P'$, $P'^* = P^*$, then any retraction $P' \to P'^*$ induces a retraction $P \to P^*$. Hence we have compatible splittings

$$\begin{tikzcd}
P' \arrow[r] & \overline{P} \oplus P'^* \\
P \arrow[r] \arrow[u] & \overline{P} \oplus P^* \arrow[u]
\end{tikzcd}$$

Consider the induced section $\overline{P'} \to \overline{P'} \oplus P'^* \to P'$. From the diagram below, we conclude that P' is isomorphic to the pushout of the induced section $\overline{P} \to P$, and the Kummer extension $\overline{P} \to \overline{P'}$.

$$\begin{tikzcd}
\overline{P} \arrow[r] \arrow[d] & \overline{P} \oplus P^* = P \arrow[d] \arrow[dr] & \\
\overline{P'} \arrow[r] & \overline{P'} \oplus P^* \arrow[r] & P'
\end{tikzcd}$$

Hence, we can assume that P and P' are both sharp and $P = Q = \overline{M_x}$.

Now P' is a sharp fs chart optimized at a point $x' \in X$ above x. Since X' is fs, then $P' = \overline{M_{x'}} = \overline{M_{s(x')}}$. On the other hand, as $X' \to X$ maps strata to strata, we get an induced map $\overline{M_{s(x)}} \to \overline{M_{s(x')}}$. Thus we have a factorization of Kummer extensions $P \to Q' \to P'$ (see the diagram below).

$$\begin{array}{ccc} \overline{M_x} & \longrightarrow & \overline{M_{s(x)}} \\ \downarrow & & \downarrow \\ \overline{M_{x'}} & \longrightarrow & \overline{M_{s(x')}} \end{array}$$

With this we obtain a morphism $X' = (X \times_P P')^{n-log} \to (X \times_Q Q')^{n-log}$. ■

Definition 3.6.2 *Let (X, M) be a Kummer fs variety, $P \to M$ be a sharp rational chart optimized at $x \in X$ and $P' := \overline{M_{s(x)}}$. We call $X' = (X \times_P P')^{n-log}$ a minimal Kummer cover of X at x.*

Whenever we have an optimized chart $R \to M$ which is not necessarily sharp, making choice of splitting $\overline{R} \to R$, we get a minimal Kummer cover X' associated to $\overline{R}$. Then, $X' \to X$ is a minimal Kummer cover induced from R and the Kummer extension $R \subset (R \oplus_{\overline{R}} \overline{R}')^{sat}$. We refer to X' as a minimal Kummer cover associated to R.

Lemma 3.6.3 *Given a quasi log smooth morphism $f : X \to Y$, a point $y \in Y$, and a minimal Kummer cover $Y' \to Y$ at y, there exists an Kummer neighborhood $X' \to X$ and a morphism $X' \to Y'$ which is normally log smooth.*

Proof Let $X' \to X$ be a Kummer cover, such that $X' \to Y$ is normally log smooth. Let $X'' := (X' \times_Y Y'')^{n-log}$. By base change $X'' \to X'$ is a Kummer neighborhood. On the other hand, by base change, $X'' \to Y'$ must be normally log smooth. ■

Lemma 3.6.4 *Let $f : X \to Y$ be a quasi log smooth morphism with a sharp and optimized rational chart $(P \to Q)$. For any $x \in X$, $y = f(x) \in Y$, there is a lifting of f on minimal Kummer covers $X' \to Y'$.*

Proof Since f must map strata to strata, we get the following diagram.

$$\begin{array}{ccccc} P & = & \overline{M_y} & \longrightarrow & \overline{M_{s(y)}} \\ \downarrow & & \downarrow & & \downarrow \\ Q & = & \overline{M_x} & \longrightarrow & \overline{M_{s(x)}} \end{array}$$

Hence, we get an induced lifting of minimal Kummer covers $(X \times_P \overline{M_{s(x)}})^n \to (Y \times_Q \overline{M_{s(y)}})^n$. ■

If we work étale locally, we can strengthen 3.6.4 and 3.6.1.

Proposition 3.6.5 *Let $f : X \to Y$ be a quasi log smooth morphism of Kummer fs varieties and let $x \in X$. Let $Y' \to Y$ a minimal Kummer neighborhood at $y = f(x) \in Y$. Then (étale locally) there exists a minimal Kummer cover $X' \to X$, and a morphism $f' : X' \to Y'$ lifting f.*

Proof Suppose Y' is given by the rational chart $Y \to A_P$ (where $P = \overline{M_y}$) and the Kummer extension $P' := \overline{M_{s(y)}} \supset P$. After étale localization, we can find an optimized rational chart $(X \to A_Q, f_\# : P \to Q)$ of f extending $Y \to A_P$.

Since Q is saturated we can find a splitting $Q = \overline{Q} \oplus Q^*$, and two projections $p_1 : Q \to \overline{Q}$, $p_2 : Q \to Q^*$. Let $\alpha : P \to \overline{Q}$, $\beta : P \to Q^*$ be the morphisms induced from the composition of $f_\#$ with the projections.

Consider the morphism $f_\# \circ \beta : P \to Q^*$, and let H be the following pushout.

$$\begin{array}{ccc} P & \longrightarrow & Q^* \\ \downarrow & & \downarrow \\ P' & \longrightarrow & H \end{array}$$

Let $\beta' : P \to H$ be the induced composition. Note that $X \times_{Q^*} H \to X$ is étale. Hence, after an étale extension, we can assume there is a morphism $\beta' : P' \to Q^*$ which lifts $f_\# \circ \beta : P \to Q^*$.

Let $T := \overline{M_{s(x)}} \supset \overline{Q}$ be the minimal Kummer extension, and let Q' be the pushout $(T \leftarrow \overline{Q} \to Q)$. Then $Q \to Q'$ is Kummer and $Q'^* = Q^*$. In particular, $Q' = Q^* \oplus T$.

Since f is quasi log smooth, it must preserve generic points of strata, so we get a morphism $P' = \overline{M_{s(y)}} \to \overline{M_{s(x)}} = T$, making the following diagram commute.

$$\begin{array}{ccc} P & \longrightarrow & \overline{Q} \\ \downarrow & & \downarrow \\ P' & \longrightarrow & T \end{array}$$

Let Q' be the pushout $(T \leftarrow \overline{Q} \to Q)$. In particular, $X' := (X \times_Q Q')^{n-log}$ is a minimal Kummer cover at $x \in X$. Note also that, from the splitting of Q, we get $Q' = T \oplus Q^*$. Consider the induced morphism $(\alpha', \beta') : P' \to Q'$. Since $f_\# : P \to Q$ coincides with $(f_\# \circ \alpha, f_\# \circ \beta) : P \to \overline{Q} \oplus Q^*$, we get the following diagram.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ P' \arrow[r] & Q' \end{tikzcd}$$

Therefore, since $X \to Y$ is dominant, we get an induced morphism $X' = (X \times_Q Q')^{n-log} \to (Y \times_P P')^{n-log} = Y'$ lifting f. ■

Remark. The lifting $f' : X' \to Y'$ is not canonical, it depends upon the splitting $\overline{Q} \oplus Q^* = Q$.

Corollary 3.6.6 *Let $f : X'' \to X$ be an fs Kummer cover and let $X' \to X$ be a minimal Kummer cover. Then, étale locally, there exists a Kummer normally log étale morphism $X'' \to X'$ factoring f. In particular, any two minimal Kummer covers at the same point must be étale locally isomorphic.*

Proof Since X'' is fs, then it is its own minimal Kummer cover. Hence, by 3.6.5, étale locally on X'', we can find a factorization $X'' \to X' \to X$. Finally, by 3.2.15, $X'' \to X'$ must be Kummer normally log étale.

To see the last assertion, note that if both X'' and X' are minimal Kummer covers, then the Kummer normally log étale morphism $X'' \to X'$ locally has a Kummer chart $Q' \to Q''$ where $\overline{Q'} = \overline{Q''}$. Hence, $(X' \times_{Q'} Q'')^{n-log} \to X'$ must actually be étale since the Kummer extension $Q' \to Q''$ coincides with the pushout $Q''^* \leftarrow Q'^* \to Q'$. ■

3.7 Quasi log smooth morphisms of quasi-toroidal varieties.

Lemma 3.7.1 *Let $(X, M_X) \to (Y, M_Y)$ be a quasi log smooth morphism, where (Y, M_Y) is quasi-toroidal, and (X, M_X) is a Kummer fs variety with M_X contained and saturated in $\mathcal{O}_X$. Then X is quasi-toroidal.*

Proof By 3.6.3, for a minimal Kummer cover $Y' \to Y$, there is an Kummer cover $X' \to X$ and a normally log smooth morphism $X' \to Y'$. Since Y' is toroidal, then (by 3.5.4) X' must be toroidal. Hence, X must be quasi log smooth over $spec(k)$. So, by 3.5.3, X is quasi-toroidal. ∎

Lemma 3.7.2 *Let X be a quasi-toroidal variety, and $X' \to X$ be a minimal Kummer cover associated to a sharp optimized chart $X \to X_{\delta^0}$. Let $X \to X_\sigma \to X_{\delta^0}$ be a toric chart associated to $X \to X_{\delta^0}$. Then, X' has an étale map to the toroidal pair $(X_{\delta\times\tau}, X_\tau)$, where τ is a regular face of σ.*

Proof As in the proof of 3.5.3, étale locally on X, the minimal Kummer cover X' is induced from a toric chart $X \to X_\sigma \to X_{\delta^0}$, where $X \to X_\sigma$ is étale, $\sigma = \tau + \delta$ and τ is a regular face of σ . Then, X' coincides with $X \times_{X_\sigma} X_{(\sigma\times_\delta\delta_0)^{sat}} = X \times_{X_\sigma} X_{\delta\times\tau}$. Since $X' \to X$ is a Kummer cover, with smooth chart $X_{\delta_0} \to X_\delta$, we have an induced strict étale morphism $(X', M_{X'}) \to (X_{\delta\times\tau}, X_\tau)$. Note that $(X_{\delta\times\tau}, X_\tau)$ is toroidal since τ is regular. ∎

We recall the statement of Kummer descent from [16]. We only state the of log smooth and étale case of [16, Theorem 0.2.], but the theorem works also in the log flat case which we will not use in this work.

Theorem 3.7.3 *[16, Theorem 0.2.] Let $f : X \to Y$, $g : Y \to Z$ be morphisms of fs log schemes, and suppose that f is surjective and Kummer. If f and $g \circ f$ are log smooth (étale), then g is also log smooth (étale).*

Proposition 3.7.4 *A morphism of quasi-toroidal varieties $f : X \to Y$ is quasi log smooth if and only if (étale locally) for any minimal Kummer cover $Y' \to Y$, there exists a lifting on minimal Kummer covers $X' \to Y'$ which is classically log smooth.*

Proof By 3.6.5 we can, étale locally, find a lifting on minimal Kummer covers $f' : X' \to Y'$. The morphism f' factors through $(X \times_Y Y')^{n-log}$ (since the induced morphism $X' \to X \times_Y Y'$ must dominate a component).

By base change we have that $(X \times_Y Y')^{n-log} \to Y'$ is quasi log smooth.

On the other hand, since $X' \to X$ and $(X \times_Y Y')^{n-log} \to X$ are Kummer normally log étale, then (by 3.2.15) $X' \to (X \times_Y Y')^{n-log}$ is also Kummer normally log étale.

Thus, the composition $X' \to Y'$ is quasi log smooth. Hence, there is a Kummer cover $X'' \to X'$, such that the composition $f'' : X'' \to X' \to Y'$ is normally log smooth.

Since X is quasi-toroidal, then X' is toroidal. Similarly Y' must also be toroidal. Given that $k : X'' \to X'$ is Kummer normally log étale, then X'' is also toroidal by 3.5.4.

Then (by 3.5.4) f'' must be classically log smooth and k is classically Kummer log étale. Finally, from [16, theorem 0.2], we see that $X' \to Y'$ must be classically log smooth. ■

Corollary 3.7.5 *A quasi log smooth morphism of toroidal varieties $f : X \to Y$, must be classically log smooth.*

Proof Since Y and X are toroidal, their minimal Kummer covers are isomorphisms. ■

3.7.6 . Remark. In the proof of 3.7.4 we used Kummer étale descent [16]. However, as the example below shows, there is no (Kummer) normally log étale descent.

3.7.7 . Example. Consider the log smooth morphism

$$(spec(k[x, y]), y \neq 0) \to (spec(k[y^2]), y \neq 0),$$

induced from the inclusion $k[y^2] \to k[x, y]$. This morphism factors through the surjective Kummer normally log étale morphism $spec(k[x, y]) \to spec(k[x^2, xy^2, y^2])$. However, $f : spec(k[x^2, xy^2, y^2]) \to spec(k[y^2])$ is not normally log smooth.

Definition 3.7.8 *Let X be a quasi-toroidal variety. Let $X \to (X_\sigma, X_\tau)$ be an étale toric doubleton chart. We call X with the toroidal structure induced from σ an enriched log structure on X.*

Note that enriched log structures on quasi-toroidal varieties depend upon the choice of toric doubleton charts.

The following lemma can be also found in [3]. We give a proof for completeness.

Proposition 3.7.9 *[3, Proposition 4.7.5.(2)] Normally log smooth morphisms of quasi-toroidal varieties are log smooth with respect to enriched log structure.*

Proof Let $f : X \to Y$ be a normally log smooth morphism of quasi-toroidal varieties and let $Y \to (X_\sigma, X_\delta)$ be a toric chart. Let $X_{\delta_1} \to X_\delta$, $X \to X_{\delta_1}$ be a smooth optimized rational chart of f, including X_δ. Then $X \to X_{(\sigma\times_\delta\delta_1)^{sat}}$ is étale. Letting $\sigma_1 := (\sigma \times_\delta \delta_1)^{sat}$, then we see that $X \to (X_{\sigma_1}, X_{\delta_1})$ is a toric doubleton chart of X. Moreover $X_{\sigma_1} \to X_\sigma$ is log smooth (with enriched log structures), and hence f is smooth with respect to this enriched log structures. ■

Proposition 3.7.10 *Quasi log smooth morphisms of quasi-toroidal varieties are log smooth with respect to enriched fs log structures.*

Proof It is enough to check the result on toric doubletons. Let $f : (X_{\sigma_1}, X_{\tau_1}) \to (X_{\sigma_2}, X_{\tau_2})$ be a quasi log smooth morphism of quasi-toroidal varieties. By 3.7.4 we have an induced log smooth morphism on minimal Kummer covers $f' : (X_{\tau_1\times\delta_1}, X_{\tau_1}) \to (X_{\tau_2\times\delta_2}, X_{\tau_2})$. Making an appropriate choice of local parameters we may suppose that τ_1 maps to τ_2. Hence f' is log smooth with respect to enriched log structures $\tau_1 \times \delta_1$, $\tau_2 \times \delta_2$. Since $(X_{\tau_1\times\delta_1}, X_{\tau_1}) \to (X_{\sigma_1}, X_{\tau_1})$, $(X_{\tau_2\times\delta_2}, X_{\tau_2}) \to (X_{\sigma_2}, X_{\tau_2})$ are both Kummer log étale with respect to enriched log structures, then by [16, theorem 0.2] applied to $(X_{\tau_1\times\delta_1}, X_{\tau_1}) \to (X_{\sigma_1}, X_{\tau_1}) \to (X_{\sigma_2}, X_{\tau_2})$, we see that f is log smooth. ■

3.8 Quasi Kummer étale site for normal rational varieties

Lemma 3.8.1 *Let X be a rational log variety and let $X \to A_P$ be a rational chart optimized at $x \in X$. Let $P \subset P'$ be a Kummer extension. Then $(X \times_P P')^{n-log} \to X$ is surjective.*

Proof Since $P \subset P'$ is Kummer, then $A_{P'} \to A_P$ is finite and surjective. Let $U \subset X$ be the locus where the log structure is trivial. Then $U \times_P P'$ (which is equal to $(U \times_{P^{gp}} P'^{gp})$ in this case) surjects onto U. Hence, by properness of $X \times_P P' \to X$, we get that the $(X \times_P P')^{n-log}$ (which is the closure of $(U \times_P P')$ in $X \times_P P'$) must be surjective onto X. ∎

Lemma 3.8.2 *Let $Y' \to Y$ be a surjective normally Kummer log étale morphism. Given another rational log variety X, and a morphism $f : X \to Y$, then $(X \times_Y Y')^{n-log} \to X$ is a surjective normally Kummer log étale .*

Proof It is sufficient to work étale locally on Y. Let $y \in Y$, $Y \to A_P$ be an optimized rational chart at $y = f(x)$ and let $y' \in Y'$ be a point above y. Let $P \subset P'$ be a Kummer extension, which gives a smooth optimized chart for $Y' \to Y$.

Consider $(Y \times_P P')^{n-log} \to Y$ (this is surjective by 3.8.1). Let $U_X \subset X$ (resp. $U_Y \subset Y$) be the log triviality locus of X (resp. Y). Since $A_{P'} \to A_P$ is surjective, then $U_X \times_P P' \to U_X$ is also surjective. On the other hand, given that U_X maps to U_Y, then $U_X \times_{P^{gp}} (P')^{gp} = U_X \times_P P'$. Given that the log structures considered never map to the zero element in the coordinate ring, then U_X and U_Y are nonempty. Since $X \times_P P' \to X$ is proper (because $A'_P \to A_P$ is), then $(X \times_P P')^{n-log} \to X$ must be surjective.

Hence, $U_X \times_P P' \to U_Y \times_P P'$ is a morphism of normal varieties and we have an induced morphism (by 3.2.1) $(X \times_P P')^{n-log} \to (Y \times_P P')^{n-log}$ giving $(X \times_P P')^{n-log} = (X \times_Y (Y \times_P P')^{n-log})^{n-log}$.

Since $Y' \to (Y \times_P P')^{n-log}$ is étale, then

$$
\begin{aligned}
(X \times_P P')^{n-log} \times_{(Y \times_P P')^{n-log}} Y' &= ((X \times_P P')^{nlog} \times_{(Y \times_P P')^{n-log}} Y')^{n-log} \\
&= (X \times_Y Y')^{n-log}
\end{aligned}
$$

.

Hence, by base change, $(X \times_Y Y')^{n-log} \to (X \times_P P')^{n-log}$ is étale and (since $y = f(x)$) there is a point $x' \in (X \times_Y Y')^{n-log}$ mapping to x. Thus $(X \times_Y Y')^{n-log} \to X$ must then be surjective normally Kummer log étale. ∎

Lemma 3.8.3 *Let $f : X \to Y$, $g : Y \to Z$ be dominant morphisms of Kummer fs varieties. If $g \circ f$ and g are quasi Kummer log étale, then so is f.*

Proof Let $X' \to X$ be a Kummer neighborhood such that $X' \to Z$ is normally log étale. Similarly, let $Y' \to Y$ be a Kummer neighborhood such that $Y' \to Z$ is normally log étale. Then by base change, $(X' \times_Z Y')^{n-log} \to Y'$ is Kummer normally log étale. Hence the composition $(X' \times_Z Y')^{n-log} \to Y' \to Y$ is Kummer normally log étale. On the other hand, also by base change, $(X' \times_Z Y')^{n-log} \to X'$ is Kummer normally log étale and hence so is the composition $(X' \times_Z Y')^{n-log} \to X' \to X$. Hence, $X \to Y$ is quasi Kummer log étale. ∎

Example 3.8.4 *We give an example of a quasi Kummer étale morphism, which is not normally Kummer log étale .*

Let $X := spec(k[x^2, xy, y^2, xz, yz, z^2])$ with $z \neq 0$ log structure. Let $Y := spec(k[x^2, xy, y^2, z^2])$ with $z \neq 0$ log structure, and consider the finite morphism $f : X \to Y$ induced from the inclusion of rings. We claim that f is quasi Kummer étale. To see this, note that $X' := spec(k[x^2, xy, y^2, z]) \to X$ gives a minimal Kummer cover at the origin and the induced morphism $X' \to Y$ is a Kummer étale cover. However $X \to Y$ cannot be normally Kummer log étale at the origin since Y is toroidal and X is not (3.5.4).

3.8.5 . Let X be a Kummer fs variety. We can form the category $X_{\text{két}}$ whose objects are Kummer fs varieties with a quasi Kummer étale morphism $Y \to X$. Morphisms, are morphisms over X. By 3.8.3, all morphisms are quasi Kummer étale.

Lemma 3.8.6 *Given two morphisms $f : Y \to Z$, $g : W \to Z$ in $X_{k\acute{e}t}$, their product $(Y \times_Z W)^{n-log}$ (as defined in 3.1.3) is a categorical fiber product in $X_{k\acute{e}t}$.*

Proof Let $Y \times_Z W$ denote the usual fiber product of schemes. Let $V \to W$, $V \to Y$ be two morphisms in X_{ket}, which coincide after composing with g and f respectively. Then, since all morphisms are Kummer étale, they must be dominant and quasi-finite. So $Y \times_Z W \to W$, is also quasi-finite dominant. Since the composition

$V \to Y \times_Z W \to W$ is also quasi-finite dominant, then $V \to Y \times_Z W$ must be dominant onto a component. So we get a unique induced morphism $V \to (Y \times_Z W)^{n-log}$, factoring $V \to (Y \times_Z W)$. ■

3.8.7 . By the above lemma, we can form a Kummer étale topology on $X_{\text{két}}$, whose coverings are collections of jointly surjective quasi Kummer étale morphisms $\{\cup V_i \to V\}$.

Remark. It is shown in [12, 2.2.2], that surjectivity of log morphisms for fs schemes is preserved for Kummer log étale maps, but not for general morphisms. The main point of this issue happens when we consider so called hollow log schemes (e.g. $spec(k)$ with log structure $\mathbb{N} \to k$ sending every element to 0). In the definition of Kummer fs varieties, we always assume that $M \subset \mathcal{O}_X \setminus \{0\}$, so hollow log schemes don't apply.

4. LOG RESOLUTION OF KUMMER FS VARIETIES

Similarly as in the above sections, k will denote a field of characteristic 0. All Kummer fs varieties are assumed to be generically fs. Given a Kummer fs variety over k, then the assumption that locally $M \subset \mathcal{O}_X \setminus \{0\}$ ensures that (X, M) is generically logarithmically smooth.

4.1 Embeddings into quasi-toroidal varieties

By definition, any variety can locally be embedded into affine space. In the context of logarithmic varieties the analog result, shown in [3] (3.7), is that (étale locally) any log variety may be strictly embedded into a toroidal variety. The goal of this section is to show that, étale locally, any Kummer fs variety can be strictly embedded into a quasi-toroidal variety. Moreover, we can construct such a local closed immersion which lifts to an equivariant closed immersion on minimal Kummer covers.

4.1.1 . Let X be a Kummer fs variety over k and assume that k has enough roots of unity. Given a minimal Kummer fs cover $X' \to X$ at $x \in X$, choose $x' \in X'$ above x. Let s' denote the stratum of x'. Suppose X' is obtained from X by a sharp optimized chart $X \to A_P$ and a sharp Kummer extension $P \subset P'$ coming from a generic chart along the stratum. Recall, from the proof in 3.5.3, that $G_P := Hom((P')^{gp}/P^{gp}, \mathbb{Z})$ acts effectively on X', with quotient X and fixing x'. Since s' is invariant under the G_P action, in a neighborhood of x', we can find semi-invariant global functions $t_1, ...t_n$ generating the cotangent space to s' at x' and a G_P-equivariant morphism $X' \to A_{P'} \times \mathbb{A}^n$.

Lemma 4.1.2 *[3](3.7.2). After restricting X, the morphism $f : X' \to A_{P'} \times \mathbb{A}^n$ is strict and unramified at x'. This morphism factors as a closed immersion $i : X' \to Z$ followed by an étale morphism $Z \to A_{P'} \times \mathbb{A}^n$.*

Proof We repeat the proof found in [3]. By abuse of notation, let $s(x')$ denote the closure of the stratum of x'. Then, locally, P' generates the ideal of $s(x')$ and hence $\mathcal{O}_{s(x')} := \mathcal{O}_{X'}/(P')$, where (P') denotes the ideal generated P'. Let $m_{x'} \subset \mathcal{O}_X$ be the maximal ideal corresponding to x'. Since the functions $t_1, ...t_n \in m_{x'}$, were chosen such that their images form a basis of $m_{s(x'),x'}/m^2_{s(x'),x'}$, then by Nakayamma's lemma we have that $(P', t_1, ...t_n)$ generates $m_{x'} \subset \mathcal{O}_{X,x'}$.

On the other hand, let o denote the origin in $A_{P'} \times \mathbb{A}^n$. Since we are working over characteristic zero, the extension of residue fields $k(o) \to k(x')$ is finite and unramified. Therefore f is unramified at x'.

The last statement follows from [13, Tag 0395]. ■

Proposition 4.1.3 *The quotient morphism $X \to (A_{P'} \times \mathbb{A}^n)/G_P$ is unramified at x.*

Proof The following argument is taken from ideas in the proof of [14, Lemma 4.6.7].

Step 1. Since $G_P = spec(\mathbb{Z}[P'^{gp}/P^{gp}])$, and $x' \in X'$ is fixed under the G_P-action, we get a $G_P^\vee := P'^{gp}/P^{gp}$-grading on $\mathcal{O}_{X',x'}$ and on the local ring of the origin in $A_{P'} \times \mathbb{A}^n$. The induced morphism $\mathcal{O}_{X',x'} \leftarrow k[t_1, ...t_n, P']_{(t_1,...,t_n,P')}$ preserves the G_P-grading.

Step 2. Let $z \in Z$ be the image of $x' \in X'$ under $X' \to Z$ and denote by $k(z)$ the residue field of z. Given that $Z \to A_{P'} \times \mathbb{A}^n$ is étale, then by [14, Theorem 2.2.9] we get a (non-canonical) isomorphism $\widehat{\mathcal{O}}_{Z,z} = k(z)[[t_1, ..., t_n, P']]$. Therefore there is an induced $G_P^\vee$-grading on $\widehat{\mathcal{O}}_{Z,z}$, which is compatible with that on $k[[t_1, ..., t_n, P']]$, trivial on $k(z)$, and makes the morphism $\widehat{\mathcal{O}}_{Z,z} \to k[[t_1, ..., t_n, P']]$ strongly étale.

Step 3. Since G_P acts trivially on the residue field of x' (because P' maps to the maximal ideal m'_x), then the induced grading on $\widehat{\mathcal{O}}_{Z,z}$ is compatible with that on $\widehat{\mathcal{O}}_{X',x'}$. That is, $\widehat{\mathcal{O}}_{Z,z} \to \widehat{\mathcal{O}}_{X',x'}$ preserves the $G_P^\vee$-grading. Hence, we have an induced surjection $\widehat{\mathcal{O}}_{Z,z}^{G_P} \to \widehat{\mathcal{O}}_{X,x}$.

Step 4. By the above, the induced quotient morphism $k[[t_1, ..., t_n, P']]^{G_P} \to \widehat{\mathcal{O}}_{X,x}$ factors as an étale morphism $k[[t_1, ..., t_n, P']]^{G_P} \to \widehat{\mathcal{O}}^{G_P}_{Z,z}$, followed by a surjection $\widehat{\mathcal{O}}^{G_P}_{Z,z} \to \widehat{\mathcal{O}}_{X,x}$. Therefore $X \to (A_{P'} \times \mathbb{A}^n)/G_P$ is unramified at x since it's unramified on completions. ■

Lemma 4.1.4 *The variety $W := (A_{P'} \times A^n)/G_P$, has the structure of a toric doubleton.*

Proof The argument is quite similar to that in 3.5.3.

Step 1. There is an induced finite morphism $A^n \times A_{P'} \to (W \times_P P')^n$. To see this, first note that $A_{P'} \times A^n \to W$ is dominant (because it's a finite quotient morphism). Secondly, $A_{P'} \times A^n \to W$ factors as $A_{P'} \times A^n \to W \times_P P' \to W$ (both morphisms which are quasi-finite). We conclude that $A_{P'} \times A^n \to W \times_P P'$ is dominantes a component, and hence (by the universal property of normalization) we get an induced morphism $A^n \times A_{P'} \to (W \times_P P')^n$. Now this morphism must be finite since $(A^n \times A_{P'})/G_P = W = (W \times_P P')^n/G_P$, and both $(A^n \times A_{P'}) \to W$ and $(W \times_P P')^n \to W$ are finite.

Step 2. Let W_0' denote the component of $(W \times_P P')^n$ which is dominated by $(A^n \times A_{P'})$. From the factorization $(A^n \times A_{P'}) \to (W \times_P P')^n \to W$, we get the following inclusion of function fields $K(W) \subset K(W_0') \subset K(A^n \times A_{P'})$. The inclusions $K(W) \subset K(W_0')$, $\quad K(W) \subset K(A^n \times A_{P'})$ are Galois and both have automorphism group G_P, thus $K(W_0') = K(A^n \times A_{P'})$. Since both $(A^n \times A_{P'})$ and $(W \times_P P')^n$ are normal, then $A^n \times A_{P'}$ must be a component of $(W \times_P P')^n$.

Step 3. Let $U \subset (A_{P'} \times A^n)$ be the open subset there where the log structure is trivial. Then by *step* 2, we get that $(A_{P'} \times A^n) \to W$ is étale on U (since the elements of P and P' are invertible on U).

Step 4. $(A_{P'} \times A^n)$ is toric when it's log structure is enriched with respect to the semi-invariant coordinates in A^n. The open subset U is torus invariant, and corresponds to a regular face of the associated cone. Moreover, the action of G_P factors through the action of the torus. Hence, W has the scheme structure of a toric variety, corresponding to some cone σ, with lattice structure N_σ. By *step* 3, U/G_P must be regular and corresponds to a regular face τ of σ (in the quotient toric structure). From the toric structure on $(A_{P'} \times A^n)$, we see that there exists a face δ of σ for which $\tau \cap \delta = 0$ and $\tau \times \delta = \sigma$ (over $\mathbb{Q}$). Hence, W corresponds to a toric doubleton (X_σ, X_τ). ■

4.1.5 . By [17, V, theoreme 1.], we can locally find a factorization of $X \to W$ into a closed immersion $X \to Y$ followed by an étale morphism $Y \to W$. Therefore, by the above lemma, Y is a quasi-toroidal variety. Hence, étale locally, any Kummer fs variety can be embedded in a quasi-toroidal variety.

We now construct a strict closed embedding of a Kummer fs variety into a quasi-toroidal variety, which lifts to an equivariant closed immersion on Kummer covers.

The following preliminary lemma is a G-equivariant version of [17, V, theoreme 1.] and is probably known to experts. Since we were unable to find an appropriate reference we provide a proof.

Lemma 4.1.6 *Let G be a finite abelian group acting on normal affine varieties X and Y over a field of characteristic 0. Let $f : X \to Y$ be a G-equivariant morphism. Suppose that G fixes a point $x \in X$, G acts trivially on the residue fields $k(x)$, $k(f(x))$, and f is unramified at x. Then, locally at x, f can be factorized as a G-equivariant closed immersion $X \to Y'$, and a G -equivariant étale morphism $Y' \to Y$.*

Proof The proof is almost verbatim to that in [17, V, theoreme 1.]. We follow the proof found there and make the necessary adjustments.

Using a G equivariant version of Zariski's main theorem, see for instance [15, 4.], we can assume, after localizing, that $f : X \to Y$ is finite. In our case, (following the proof in [15, 4.]) G-equivariant Zariski's main theorem follows by taking the integral closure R of $\mathcal{O}_Y$ in $\mathcal{O}_X$. Since $\mathcal{O}_Y$ and $\mathcal{O}_X$, are of finite type over a field, then so is R. Given that X is normal, then so is *spec*(R). Hence as $X \to$ *spec*(R) is quasi-finite birational, it must be an open immersion. It is clear, by construction of

R, that $spec(R) \to Y$ is finite. On the other hand, from the universal property of normalizations, we get that the factorization $X \to spec(R) \to Y$ is G-equivariant.

Denote $y := f(x)$, and let $k(y)$ be it's residue field. Then, since f is unramified, $\mathcal{O}_X \times_{\mathcal{O}_Y} k(y)$ is a product of local rings. One of the factors must be $k(x)$.

Since f is unramified at x, then $k(x) \supset k(y)$ is a finite separable field extension (this is immediate in our case since we are working over characteristic 0). Let $\overline{u}$ be a function in $\mathcal{O}_X \otimes_{\mathcal{O}_Y} k(y)$, which vanishes at all components of the fiber except for the $k(x)$ factor, and which generates $k(x)$ as a $k(y)$-algebra. Take any lift $u \in \mathcal{O}_X$ of $\overline{u}$ and notice that, since x is fixed and G acts trivially on the residue field $k(x)$, then for any $g \in G$, we have that $\overline{gu}$ must also vanishes on every component of $X \times_Y k(y)$ except the one corresponding to $k(x)$. Hence, we may lift $\overline{u}$ to some G-invariant $u \in \mathcal{O}_X$ (for example, we could take $|G|^{-1} \sum_{g \in G} gu$ for any lifting u of $\overline{u}$). Consider the subring $\mathcal{O}_Y[u] \subset \mathcal{O}_X$. Let z be the image of x under the induced G-equivariant morphism $\phi : X \to spec(\mathcal{O}_Y[u]) =: Z$. Then, since ϕ is equivariant, z must also be fixed under the G action.

We claim that X and Z are locally isomorphic on a G-invariant neighborhood of x and z. Since $\mathcal{O}_Y \subset \mathcal{O}_X$ is finite, then so is $\mathcal{O}_Z = \mathcal{O}_Y[u] \subset \mathcal{O}_X$ and thus so is $\mathcal{O}_{Z,z} \subset \mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,z}$. By the construction of u, we have that x is the unique point in the fiber of z. Moreover, since $\overline{u}$ generates $k(x)$ over $k(y)$, then we have that $\mathcal{O}_{Z,z} \otimes_{\mathcal{O}_Y} k(y) \to k(x)$ must be surjective. From Nakayama's lemma, we infer that $\mathcal{O}_{Z,z} \to \mathcal{O}_{X,x}$ must be surjective. Since it is also injective (given that it is a finite extension), then it must be an isomorphism. Hence, as z and x are fixed, we can choose $f \in (\mathcal{O}_{Z,z}^G)^*$, such that $(\mathcal{O}_Z)_f = (\mathcal{O}_X)_f$. Thus X and Z are locally isomorphic.

Then, restricting X and Y, we may suppose that $\mathcal{O}_X = \mathcal{O}_Y[u]$.

Let k be the rank of $\mathcal{O}_X \otimes_{\mathcal{O}_Y} k(y)$ over $k(y)$. Then, by Nakayama's lemma $1, u, u^2, ...u^k$ generate $\mathcal{O}_X$ over $\mathcal{O}_Y$, and u corresponds to a root of a monic polynomial $P(X)$ on $\mathcal{O}_Y[X]$. Now P may not be G invariant, but we can take

$$Q := \frac{1}{|G|} \sum_{g \in G} gP$$

Since u is G invariant, then $gP(u) = \sum (ga_i)u^i = g \sum a_i u^i = g(P(u)) = 0$. Hence, Q is G invariant, monic, of the same degree as P and u is a root of Q. Therefore $\mathcal{O}_X$ is a quotient of $\mathcal{O}_Y[X]/Q$. Let x' be the image of x under $X \to spec(\mathcal{O}_Y[X]/Q)$. Given that $k(y)[X]/\overline{Q} \cong \mathcal{O}_X \otimes_{\mathcal{O}_Y} k(y)$, then $\mathcal{O}_Y[X]/Q$ is unramified over $\mathcal{O}_Y$ at y. This follows since the property of being unramified can be checked on fibers see [17, chap.III prop 10.]. Hence, the derivative of Q is invertible at x'. We may then find a function h, which is G invariant, does not vanish at x' (since x' is fixed), and makes $(\mathcal{O}_Y[X]/Q)_h$ a standard étale G equivariant algebra over $\mathcal{O}_Y$ which, locally at x, surjects G-equivariantly onto $\mathcal{O}_X$. ■

Lemma 4.1.7 *Let X be a Kummer fs variety and $X' \to X$ be a minimal Kummer cover at a point $x \in X$, associated to a minimal Kummer extension $P \subset P'$ of a sharp optimized chart $P \to \mathcal{O}_X$, with associated character group G_P. Let $X' \to A_{P'} \times A^n$ be a G_P-equivariant strict unramified morphism as constructed above. Denote $X_\sigma := (A_{P'} \times A^n)/G_P$. Then, étale locally at x, there exists a factorization $X \to Y \to X_\sigma$, where $X \to Y$ is a closed immersion, $Y \to X_\sigma$ is étale, and such that the induced morphism $X' \to Y \times_{X_\sigma} (A_{P'} \times A^n)$ is a closed immersion.*

Proof Restricting X, we can assume that G_P fixes a point $x' \in X'$ above x. Note that since P is assumed to be a sharp optimized chart, then P maps to the maximal ideal m_x in $\mathcal{O}_{X,x}$. Thus G_P acts trivially on the residue fields $k(x)$, $k(x')$. By 4.1.6, we can locally find a G_P-equivariant factorization $X' \to Y' \to A_{P'} \times A^n$ where the first map is a closed immersion and the second map is étale. Since $X' \to Y'$ is G_P -equivariant, the image of x' is fixed by the G_P action in Y'. Hence, locally the map $Y' \to A_{P'} \times A^n$ preserves stabilizers. Let $Y := Y'/G_P$. By Luna's fundamental lemma we have that $Y \to X_\sigma$ is étale. Since $X' \to Y'$ is a G_P-equivariant closed immersion, then so is $X \to Y$.

Given that $Y \to X_\sigma$ is étale, then $Y' = Y \times_{X_\sigma} (A_{P'} \times A^n)$ (this also follows from Luna's fundamental lemma since $Y' \to A_{P'} \times A^n$ must be strongly étale). ■

Note that, in general, $X' \to Y'$ is not automatically a closed immersion for any closed immersion $X \to Y$. For example, consider the embedding $spec(k[x^2, xy, y^2]) \to spec(k[u, v, w])$, and the finite cover $spec(k[u, v, w^{1/2}]) \to spec(k[u, v, w])$. The normalized fiber product is $spec(k[x, y])$, whose induced morphism $spec(k[x, y]) \to spec(k[u, v, w^{1/2}])$ is not a closed immersion since $u \mapsto x^2$, $v \mapsto xy$, $w^{1/2} \mapsto y$.

4.2 Functoriality of log resolutions

The following result is shown in [3]. This lemma provides functoriality of the resolution algorithm in [3, Theorem 1.3.4]. For completeness, we repeat the argument.

Lemma 4.2.1 *[3, lemma 3.7.6] Let $X \to Y$ be a strict embedding of an fs variety into a toroidal variety Y. Let $f : X' \to X$ be a log smooth morphism. Then, étale locally, we can find a strict embedding $X' \to Y'$, where Y' is a toroidal variety, and a log smooth morphism $Y' \to Y$, such that $X' = X \times_Y Y'$.*

Proof Étale locally, at any $x \in X$, we can find a log smooth chart of $P \to Q$ of f, such that $X' \to X \times_P Q$ is étale. Let $z \in X \times_P Q$, be a point above x. Lifting the chart $P \to M_X$ to Y, we find a strict embedding $X \times_P Q \to Y \times_P Q$. We can assume that X', X, Y are affine varieties. Let $R := \mathcal{O}_{Y \times_P Q}$, and $S := \mathcal{O}_{X \times_P Q}$, by hypothesis the induced morphism $R \to S$ is a surjection. Then, locally we can express $X' \to X \times_P Q$ as a standard étale morphism, so we can assume $\mathcal{O}_{X'} = (S[t]/f(t))_{g(t)}$ where $f'(t)$ in invertible in $(S[t]/f(t))_{g(t)}$ above z. Lifting f and g to R, then f' must still be invertible in a neighborhood above z. Hence, making $Y' := spec((R[t]/f(t))_{g(t)})$, we get the desired result. ■

4.3 Étale independence upon embedding of the same dimension

Also in [3], it is shown that, étale locally, any two strict embedding of an fs log variety into toroidal varieties of the same dimension are étale locally isomorphic. We recall the statement, and repeat the proof for convenience of the reader.

Theorem 4.3.1 *[3, Theorem 3.7.8] Let X be an fs variety and let $i_1 : X \to Y$, $i_2 : X \to Y'$ be two strict embeddings of X into toroidal varieties of the same dimension. Then, étale locally at any $x \in X$, we can find a common étale neighborhood $f_1 : Z \to Y$, $f_2 : Z \to Y'$, which are isomorphisms over $i_1(X)$, $i_2(X)$ (respectively). Moreover the embeddings $f_1^{-1} \circ i_1$ and $f_1^{-1} \circ i_2$ coincide.*

Proof Let $x \in X$, and take a sharp optimized chart $X \to A_P$ at x. Lift the chart to charts $Y \to A_P$, $Y' \to A_P$. Since the statement is local, we may assume that X is affine. Consider global functions $f_1, ...f_n \in k[X]$ which restrict to a basis of the cotangent space to the stratum $s(x)$. Lift $f_1, .., f_n$ to $k[Y]$ and extend them to a functions $f_1, ...f_m$ which restrict to a family of parameters at $i_1(x) \in s(i_1(x))$. Similarly we lift $f_1, .., f_n$ to $k[Y']$ and extend them to a functions $f'_1, ...f'_m$ which restrict to a family of parameters at $i_2(x) \in s(i_2(x))$. The m's are the same since Y and Y' are of the same dimension. Consider the induced morphisms $Y \to A_P \times A^m \leftarrow Y'$. Since Y and Y' are toroidal, then the morphisms $Y \to A_P \times A^m$, $Y' \to A_P \times A^m$ are étale. Let $x_1, ...x_m$ denote coordinates in the A^m factor. Note that the restrictions $X \to Y \to A_P \times A^m$ coincide as the image of X is contained in the locus where $x_{n+1} = ... = x_m = 0$. Let $Z := Y \times_{A_P \times A^{m-n}} Y'$, and note that $Z \to Y$, $Z \to Y'$ are both étale by base change. The induced morphism $X \to Z$ must then be a closed immersion as we have a factorization $X \to Z \to Y$, where $X \to Y$ is a closed immersion and $Z \to Y$ is separated. ■

4.4 Quotient log smooth morphisms

Definition 4.4.1 *Let X be a Kummer fs variety and let $X' \to X$ be a Kummer cover. We say $X' \to X$ is an étale minimal Kummer neighborhood if étale locally for any $x \in X$, we can find a minimal optimized Kummer chart $Q \to Q'$, such that the induced morphism $X' \to (X \times_Q Q')^n$ is étale.*

Lemma 4.4.2 *Let X be a Kummer fs variety and $x \in X$. Let $X' \to X$ be a minimal Kummer cover at x. Then, after restricting X, the morphism $X' \to X$ is an étale minimal Kummer neighborhood.*

Proof Restricting X, we may suppose that $X' = X \times_P P'$, where $P \subset P'$ is a sharp minimal Kummer extension. Let $z \in X$ be a point whose stratum specializes to $s(x)$. Then, after taking an étale neighborhood $U \to X$, we can optimize P to a chart P_1 at z. We may assume P_1 is torsion free.

Let $P_1' := (P' \oplus_P P_1)^{sat}$. Consider the following diagram.

$$\begin{tikzcd} P' \arrow[r] \arrow[rr, bend left] & P_1' \arrow[r] & \overline{M_{s(z)}} \\ P \arrow[u] \arrow[r] & P_1 \arrow[u] \arrow[r] & \overline{P_1} \arrow[u] \end{tikzcd}$$

Since $P' = \overline{M_{s(x)}} \to \overline{M_{s(z)}}$ is surjective, then there is an induced surjective morphism $\overline{P_1} \to \overline{M_{s(z)}}$. On the other hand, since $P_1 \to P_1'$ is Kummer, then so is the induced morphism $\overline{P_1} \to \overline{P_1'}$. Given that $\overline{P_1} = \overline{M_z} \to \overline{M_{s(z)}}$ is Kummer, and $\overline{P_1'}$ is reduced, then from the factorization $\overline{P_1} \to \overline{P_1'} \to \overline{M_{s(z)}}$, we see that $\overline{M_{s(z)}} = \overline{P_1'}$.

Let $P_0 := ((P_1')^* \oplus_{P_1^*} P_1)^{sat}$. Note that as $P_0^* = (P_1')^*$, then a splitting $P_1' = \overline{P_1} \oplus (P_1')^*$ induces a compatible splitting $P_0 = \overline{P_0} \oplus P_0^*$. Denote $P_0' := (P_0 \oplus_P P')$ and note that, by the above, it is already saturated. We replace U by the étale neighborhood obtained from the Kummer extension $P_1 \to P_0$.

Consider the pullback diagram.

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & A_{P_1'} \arrow[r] \arrow[d] & A_{\overline{P_1'}} \arrow[d] \\ U_0 \arrow[r] \arrow[d] & A_{P_0} \arrow[r] \arrow[d] & A_{\overline{P_0}} \\ U \arrow[r] & A_{P_1} & \end{tikzcd}$$

Since $\overline{P_1'} = \overline{M_{s(z)}}$, then $U' \to U_0$ is a minimal Kummer cover. Then we let $U_0' := (U_0 \times_P P')$. Note that, by base change, $U_0' \to U'$ is étale. We get the following diagram.

$$\begin{tikzcd}
U_0' \arrow[r] \arrow[d] & U' \arrow[r] \arrow[d] \arrow[dl] & A_{P'} \arrow[d] \\
U_0 \arrow[r] & U \arrow[r] & A_P
\end{tikzcd}$$

Let $\Gamma := (\overline{P_1'}^{gp}/\overline{P_1}^{gp})^\vee$, $\Gamma_z := (\overline{P_1'}^{gp}/\overline{P_0}^{gp})^\vee$, and $\Gamma^z := (P_0^{gp}/P_1^{gp})^\vee$. Then, $U_0' \to U'$ is the quotient of the induced Γ^z-action. From the inclusion $\Gamma_z \subset \Gamma$, we get a Γ_z action on U_0', which is compatible with the action of Γ_z on U' under $U_0' \to U'$. The morphism $U_0' \to U'$ is obtained via base change of the finite étale extension $k[P_1^*] \subset k[P_0^*]$, which are elements of degree zero with respect to the induced $\overline{P_1'}^{gp}/\overline{P_0}^{gp}$-grading. Hence, $U_0' \to U'$ must be strongly Γ_z- étale. Thus, the quotient morphism $U_0'/\Gamma_z \to U'/\Gamma_z$ is étale. Moreover, since $U' \to U_0$ is minimal and $U_0' \to U'$ is finite strongly equivariant, then Γ_z may be identified with the stabilizer of a point $z' \in U_0'$ above z.

From the factorization $U_0' \to U' \to U_0$, we see that $X' \to X$ is an étale minimal cover at z. ■

Remark. It follows from the definition that an étale minimal Kummer neighborhood $X' \to X$ is normally log smooth.

Definition 4.4.3 *Let $f : X \to Y$ be a morphism of Kummer fs log varieties. We say f is quotient log smooth if, étale locally, for any $y \in Y$, there is a minimal Kummer fs cover $Y' \to Y$, such that*

1. *$X' := (Y' \times_Y X)^{n-log} \to X$ is an étale minimal Kummer neighborhood.*

2. *The induced morphism $X' \to Y'$ is log smooth.*

Remark. A quotient log smooth morphism of fs log varieties is classically log smooth. This follows because the minimal Kummer cover of any fs log variety is the identity. On the other hand, a quotient log smooth morphism is quasi log smooth.

Example 4.4.4 *Let* $X = spec(k[x^2, xy, y^2])$ *and* $X' = spec(k[x, y])$. *Then* $X' \to X$ *is an is a minimal Kummer cover, and it is also quotient log smooth. To see that it is quotient log smooth, note that the normalized fiber product of the minimal Kummer cover with itself gives to copies of* X' *above* X'. *Thus* $(X' \times_X X')^{n-log} \to X'$ *is étale.*

Example 4.4.5 *Let* $X = spec(k[x^2, y^2, z^2, xy, yz, xz])$, *and* $Y = spec(k[x^2, xy, y^2])$ *both with log structure* $y \neq 0$. *The morphism* $X \to Y$ *induced from the inclusion of rings is quotient log smooth as the normalized fiber product of the minimal Kummer cover* $\mathbb{A}^2 \to Y$, *with* $X \to Y$, *gives the minimal Kummer cover* $\mathbb{A}^3 \to X$

Lemma 4.4.6 *Let* $Z \to X$ *be a quotient log smooth morphism. Let* $X' \to X$ *be a minimal Kummer cover, with character group* Γ. *Denote* $Z' := (Z \times_X X')^n$, *the induced étale minimal Kummer neighborhood. Then, étale locally for any* $z \in Z$, *there exists a subquotient* $\Gamma \to \Gamma^z$, *acting on* Z', *for which* $Z' \to Z'/\Gamma^z$ *is étale and* $Z'/\Gamma^z \to Z$ *is a minimal Kummer cover at* z. *Let* Γ_z *be the character group of* $Z'/\Gamma^z \to Z$. *Then,* Γ_z *can be identified with the stabilizer of a point* $z' \in Z'$ *above* z *and* $Z' \to Z'/\Gamma^z$ *is strongly* Γ_z*-equivariant.*

Proof The proof is very similar to 4.4.2. Étale locally, since Z' is obtained as the pullback of a minimal Kummer cover $X' \to X$, we can find an optimized chart $Q \to Q'$ (at z) of $Z' \to Z$, such that $Z' = (Z \times_Q Q')^{n-log}$. Thus, $\Gamma = (Q'^{gp}/Q^{gp})^\vee$. Since Q' is fs, we have a splitting $Q' = \overline{Q'} \oplus Q'^*$. Consider the monoid $Q_0 := ((Q')^* \oplus_{Q^*} Q)^{sat}$. Denoting $Z_0 := (Z \times_Q Q_0)^{n-log}$, then the induced morphism $Z_0 \to Z$ is étale because $\overline{Q_0} = \overline{Q}$.

Since $Q_0^* = Q'^*$, we get compatible splittings $Q_0 = \overline{Q_0} \oplus Q_0^*$, $Q' = \overline{Q'} \oplus Q'^*$. By hypothesis $Z' \to Z$ is an étale minimal Kummer cover, hence $\overline{Q'} = \overline{M_{s(z)}}$. Given that $(Q')^* = Q_0^*$, then $Z' \to Z_0$ is a minimal Kummer cover. Let $\Gamma^z := (Q_0^{gp}/Q^{gp})^\vee$, and note that $\Gamma_z = ((Q')^{gp}/Q_0^{gp})^\vee$. By base change, $(Z_0 \times_Q Q')^{n-log} \to Z'$ is étale. The subquotient Γ^z acts on $(Z_0 \times_Q Q')^{n-log}$ with quotient Z'.

Same as in 4.4.2, Γ_z acts on $(Z_0 \times_Q Q')^{n-log}$, and since $(Z_0 \times_Q Q')^{n-log} \to Z'$ is induced from the finite étale extension $k[Q^*] \subset k[Q_0^*]$, which are elements of degree zero with respect to the $(Q')^{gp}/Q_0^{gp}$-grading, then $(Z_0 \times_Q Q')^{n-log} \to Z'$ étale and Γ_z-strongly equivariant. Hence, the quotient morphism $(Z_0 \times_Q Q')^{n-log}/\Gamma_z \to Z'/\Gamma_z = Z_0$ is étale. By minimality of $Z' \to Z_0$, Γ_z may be identified with the stabilizer of a point $z' \in (Z_0 \times_Q Q')^{n-log}$ above z. ■

Lemma 4.4.7 *(c.f. [3, 5.1.24]) Let $Z' \to Z$, $Z'' \to Z$, be two minimal Kummer covers at $z \in Z$, with character group Γ. Then, étale locally at z, Z' and Z'' are Γ-equivariantly isomorphic.*

Proof Let $(Z \to A_P,\ P \subset P')$, $(Z \to A_Q,\ Q \subset Q')$ be the data inducing Z' and Z'' respectively. Then, we have $P \cong \overline{M_z} \cong Q$ and $P' \cong \overline{M_{s(z)}} \cong Q'$. Consider a chart R containing both P and Q, such that $R = PR^* = QR^*$. Let $R_{P'} := (R \oplus_P P')^{sat}$ and $R'' := ((Q' \oplus_Q R_{P'})^{sat})^{red}$. Consider the induced Kummer extension $R^* \to R''^*$ and let $R' := (R \oplus_{R^*} R''^*)^{sat}$. Note that, in fact, the construction of R' is independent of the order that we take P and Q. Letting $Z_1 := (Z \times_R R')^{n-log}$, then (in a neighborhood of z) the induced morphism $Z_1 \to Z$ is étale since the elements of R^* map to units in $\mathcal{O}_{Z,z}$.

Let $R_{Q'} := (R' \oplus_Q Q')^{sat}$. Given that $R' = QR'^* = PR'^*$, we have the following the exact sequences.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q^{gp} & \longrightarrow & R'^{gp} & \longrightarrow & R'^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q'^{gp} & \longrightarrow & (R_{Q'})^{gp} & \longrightarrow & R'^* & \longrightarrow & 0
\end{array}$$

Given that R'^* and Q'^{gp} are torsion free, we see that $R_{Q'}^{gp}$ is torsion free. Hence, $R_{Q'}$ is reduced. We have a factorization $R' \to R_{Q'} \to R''$, where $R' \to R''$, $R' \to R_{Q'}$ are both Kummer. Since $\overline{Q'} = \overline{R_{Q'}} = \overline{R''}$, $R'^* = R''^*$, and R'' is reduced, we conclude that $R_{Q'} = R''$. Similarly, $(R' \oplus_P P')^{sat} = R''$.

From the factorizations $Z_1 \to A_{R'} \to A_P$, $Z_1 \to A_{R'} \to A_Q$, we have induced isomorphisms $(Z_1 \times_R R'')^{n-log} \to (Z_1 \times_Q Q'')^{n-log}$ and $(Z_1 \times_R R'')^{n-log} \to (Z_1 \times_P P'')^{n-log}$.

As above, we have the following pushout diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P^{gp} & \longrightarrow & P'^{gp} & \longrightarrow & P'^{gp}/P^{gp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & R'^{gp} & \longrightarrow & R''^{gp} & \longrightarrow & R''^{gp}/R'^{gp} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & Q^{gp} & \longrightarrow & Q'^{gp} & \longrightarrow & Q'^{gp}/Q^{gp} & \longrightarrow & 0
\end{array}
$$

We can identify $\Gamma = spec(\mathbb{Z}[R''^{gp}/R'^{gp}])$. Then, $\phi : R''^{gp} \rightarrow R'''^{gp}/R'^{gp}$ gives a R''^{gp}/R'^{gp} grading on $k[R'']$ which induces a Γ-action on $A_{R'''}$.

From the above diagram, we see that the isomorphisms $(Z_1 \times_Q Q'')^{n-log} \leftarrow (Z_1 \times_R R'')^{n-log} \rightarrow (Z_1 \times_P P'')^{n-log}$ are Γ-equivariant. ■

Example 4.4.8 *The following example illustrates the construction in the proof of the above lemma. Let $Z = spec(k[y, u^{\pm 1}])$ with $y \neq 0$ log structure. Consider the two charts $Z \rightarrow spec(k[y]) = A_P$, $Z \rightarrow spec(k[uy]) = A_Q$, with respective Kummer covers $k[y] \rightarrow k[y^{1/2}] = k[P']$, $k[uy] \rightarrow k[(uy)^{1/2}] = k[Q']$. Then, using the above notation, we have $R'' = [y^{1/2}, u^{\pm 1/2}]$ and $Z_1 = spec(k[y, u^{\pm 1/2}])$. We see that $(Z_1 \times_P P')^{n-log} = spec(k[y^{1/2}, u^{\pm 1/2}]) = (Z_1 \times_Q Q')^{n-log}$.*

Lemma 4.4.9 *Let $f : X \rightarrow Y$ be a quotient log smooth morphism. Let $Y_1' \rightarrow Y$ be an étale minimal Kummer neighborhood at a point $y = f(x) \in Y$. Then $(X \times_Y Y_1')^{n-log} \rightarrow Y_1'$ is log smooth. Moreover, $(X \times_Y Y_1')^{n-log} \rightarrow X$ is an étale minimal Kummer neighborhood.*

Proof As $Y_1' \rightarrow Y$ is an étale minimal Kummer cover, étale locally it factors through a minimal Kummer cover $Y' \rightarrow Y$. By 4.4.7, as any two minimal Kummer covers are étale locally isomorphic, then (after replacing Y by an étale neighborhood and taking pullbacks of X and Y_1') we have that $(X \times_Y Y')^{n-log} \rightarrow X$ is an étale minimal Kummer cover and $(X \times_Y Y')^{n-log} \rightarrow Y'$ is log smooth. Since by base change $(X \times_Y Y_1')^{n-log} \rightarrow (X \times_Y Y')^{n-log}$ is étale, then $(X \times_Y Y_1')^{n-log} \rightarrow X$ is an étale minimal Kummer cover.

■

Lemma 4.4.10 *Quotient log smooth morphisms are preserved under composition.*

Proof Let $X \to Y$ and $Y \to Z$ be quotient log smooth morphisms. Then we can find a minimal Kummer cover $Z' \to Z$, such that $Y' := (Y \times_Z Z')^{n-log} \to Z'$ is log smooth and $Y' \to Y$ is an étale minimal Kummer neighborhood. By 4.4.9, we have that $(X \times_Y Y')^{n-log} \to Y'$ is log smooth and $(X \times_Y Y')^{n-log} \to X$ is an étale minimal Kummer neighborhood. ∎

4.5 Central Charts

Definition 4.5.1 *(cf. [18, 2.2.7]). Let X be a Kummer fs variety and let $X \to A_P$ be a rational chart. Let $C(P) := \{x \in X \mid \overline{P} \cong \overline{M}_{X,x}\}$, we call $C(P)$ the center of P. We say the chart is central if $C(P) \neq \emptyset$.*

Remark. Note that if $X \to A_P$ is a rational optimized chart at x, then A_P is a central chart. Moreover, any two points x, x' in the center are in the same rank stratum, and hence any minimal Kummer cover at x is also a minimal Kummer cover at x'. Also, by 2.12.2 and 2.14.5 we see that $C(P)$ is closed.

Lemma 4.5.2 *Let X be a Kummer fs variety. Then, we can cover X with finitely many étale neighborhoods $\cup X_i \to X$, such that for every $x \in X$, there is an X_i, having a central chart $X_i \to A_{P_i}$ with $x_i \in C(P_i)$ (for a point $x_i \in X_i$ above x).*

Proof Any point $x \in X$, has an étale neighborhood $U_x \to X$, with a rational chart $U_x \to A_P$ optimized above x. By quasi-compactness of X, we can find a finite refinement $\{U_i\}_i$ of $\{U_x\}_x$ covering X.

For each U_i, we have a central chart $U_i \to A_{P_i}$. Let s_i denote the rank stratum of $C(P_i)$. By 2.14.5, s_i has a stratification by multiplicity. Therefore, for each multiplicity stratum $s_{i,j} \subset s_i$ we can find an étale cover $\cup_k U_{i,j,k} \to U_i$, such that for every $u \in s_{i,j}$ there is an $U_{i,j,k}$ which has an optimized chart above u. By quasi compactness of $s_{i,j}$ we can find a finite refinement of $\{U_{i,j,k}\}$.

Repeating the argument for every rank stratum of X, we obtain the desired cover.

■

4.6 Functorial non embedded log resolution of Kummer fs varieties

The main ingredients in the proof of 1.0.2 are [3, Theorem 1.3.2, Theorem 1.3.4]. We recall the statements.

Theorem 4.6.1 *([3](1.3.2)) Let I be a nonzero ideal sheaf on a quasi-toroidal variety X over a field of characteristic zero. Then, there exists a sequence of blowings up $X_n \to ... \to X_1 \to X$, at smooth-monomial centers supported on $V(I)$, such that $I\mathcal{O}_{X_n}$ is locally monomial on a Kummer normally log étale covering. Up to trivial blowups, the sequence is functorial with respect to normally log smooth morphisms.*

Theorem 4.6.2 *([3](1.3.4)) Let X be a generically logarithmically smooth, locally equidimensional, fs log variety of finite type over a field of characteristic zero. Then, there exists a projective birational morphism $D(X) \to X$, where $D(X)$ is quasi-toroidal, and which is an isomorphism over the toroidal locus of X. The assignment $X \mapsto D(X)$ is functorial with respect to logarithmically smooth morphisms.*

We emphazise that [3, Theorem 1.3.4] works in the more general context where the fs log varieties are not necessarily normal. Hence, Theorem 1.0.2 is only a partial extension of this result.

Remark. By functoriality in the statement of Theorem 1.0.2, we mean that, given a quotient log smooth morphism $Y \to X$, then $D(Y) = (Y \times_X D(X))^{n-log}$.

Proof (Proof of Theorem 1.0.2)

Step 0. It is enough to construct a resolution étale locally. To see this, we follow the argument in [3, 8.4]. Suppose that $Z \to X$ is étale, and we have a resolution $Z' \to Z$, obtained by the sequence of blowups $(Z', I_n) \to (Z_{n-1}, I_{n-1}) \to ...(Z_1, I_1) \to (Z, I_0)$. Then, by functoriality with respect to étale morphisms, we get that $Z'' :=$

$Z' \times_Z (Z \times_X Z)$ is the resolution of $Z \times_X Z$. Let $p_1, p_2 : (Z \times_X Z) \rightrightarrows Z$ denote the projections. At each step, by functoriality, $\pi_1^*(I_i) = \pi_2^*(I_i)$. Hence, the centers I_i descend and give a sequence of blowups $(X_n, J_n) \to (X_{n-1}, J_{n-1}) \to ... \to (X, J_0)$, where $Z_n \to X_n$ is étale. We get that X_n is quasi-toroidal and gives a log resolution for X which is compatible with that of Z.

Step 1. First we construct the resolution locally by taking a minimal Kummer cover, and show that it is independent of the chosen cover. We do this by following the proof in [3, theorem 1.3.4].

Then we go on to prove functoriality with respect to quotient log smooth morphisms, which in turn gives that the procedure glues and is independent of the chosen point.

Let X be a Kummer fs variety and $x \in X$ a closed point. Since it suffices to work étale locally, we may suppose that X has a global minimal Kummer cover at $x \in X$. By 4.1.7, locally, any minimal Kummer cover $X' \to X$ can be Γ_x equivariantly embedded into a toroidal variety Y'. By [3, Theorem 1.3.2] we get an embedded, Γ_x equivariant, resolution $D(X') \to X'$, where $D(X')$ is quasi-toroidal.

We claim that given any other embedding of X' into a toroidal variety Z', of the same dimension as Y', the resolutions obtained from either embedding coincide. By 4.3.1, we can construct a common étale covering $Z' \leftarrow T' \to Y'$, which are isomorphisms on the images of X', and moreover the induced embeddings $X' \to T'$ coincide. Let $X_1 := (X' \times_{Y'} T')$, $X_2 := (X' \times_{Z'} T')$. Then, as $T' \to Z'$ is an isomorphism on the image of X', similarly for $T' \to Y'$, we get that $X_1 \cong X' \cong X_2$. From the embedding $X' \to T'$, we get sections $X' \to X_1$, $X' \to X_2$, which coincide after composing them with embeddings $X_1 \to T'$ and $X_2 \to T'$ (respectively). Let $D_1(X')$ be the resolution obtained from $X' \to Y'$ (similarly for $D_2(X')$ and $X' \to Z'$) and let $D_T(X')$ be the resolution obtained from $X' \to T'$. By functoriality of [3, Theorem 1.3.2], we get that $D(X_1) \cong (D_1(X') \times_{X'} X_1)^{n-log} \cong D_1(X')$, similarly $D(X_2) \cong (D_2(X') \times_{X'} X_2)^{n-log} \cong D_2(X')$. Moreover, as the embeddings $X' \to X_1 \to$

T', $X' \to X_2' \to T'$ coincide, then the ideals of X_1 and X_2 are the same ideal sheaf in T'. Therefore $D_1(X') \cong D_T(X') \cong D_2(X')$.

By the above, to check that the resolution obtained from embeddings of different dimension coincide, it is enough to check that, given a strict embedding $X' \to Y'$ into a toroidal variety, we get the same resolution if we instead consider the embedding $X' \to Y' \times \mathbb{A}^1$. In this case the coordinate t in the $\mathbb{A}^1$ factor becomes a maximal contact element for the ideal of X' in $Y' \times \mathbb{A}^1$. The algorithm then proceeds by restricting X' to a maximal contact hypersurface, so we can restrict to the original embedding $X' \to Y'$ (see the proof in [3, Theorem 8.2.1]). Hence, these resolutions coincide.

Step 2. Given a log smooth morphism $Z' \to X'$, then by functoriality of [3, Theorem 1.3.4] we have that $D(Z') = (Z' \times_{X'} D(X'))^{n-log}$. Moreover, if $Z' \to X'$ is Γ_x equivariant, then so is $D(Z') \to D(X')$. To see this let $g \in \Gamma_x$ and consider the following diagram (a priori without the dashed arrow) induced from functoriality of [3, Theorem 1.3.4] and Γ_x equivariantness of $Z' \to X$

$$\begin{tikzcd}
D(Z') \arrow[rrr] \arrow[dr] \arrow[ddd, dashed] & & & D(X') \arrow[dl] \arrow[ddd, "g"] \\
& Z' \arrow[r] \arrow[d, "g"] & X' \arrow[d, "g"] & \\
& Z' \arrow[r] & X' & \\
D(Z') \arrow[rrr] \arrow[ur] & & & D(X') \arrow[ul]
\end{tikzcd}$$

As $g : Z' \to Z'$ is log smooth (it is an automorphism preserving the log structure), then by the functoriality in [3, Theorem 1.3.4] we get an induced morphism $D(Z') \to D(Z')$ making the above diagram commute.

Thus, there is a Γ_x equivariant desingularization $D(X') \to X'$, where $D(X')$ is quasi-toroidal, which is functorial with respect to log smooth morphisms. By functoriality with respect to log smooth morphisms, we have that $X' \mapsto D(X')$ preserves the toroidal locus.

Step 3. Define $D(X) := D(X')/\Gamma_x$. The morphism $D(X) \to X$ must also be projective and birational as it is a quotient of a projective birational morphism by a finite group. Moreover since $D(X') \to D(X)$ is Kummer normally log étale, then $D(X)$ must also be quasi-toroidal, as it is quasi log smooth over a field. We note that the definition of $D(X)$ is independent of the minimal Kummer cover at x, since (by 4.4.7) any two minimal Kummer covers are étale locally Γ-equivariantly isomorphic.

Let $X' \to X$ be a global minimal Kummer cover at $x \in X$, such that X' has a Γ equivariant embedding into a toroidal variety. We have defined $D(X) = D(X')/\Gamma$. Let $Z \to X$ be a quotient log smooth morphism. Taking normalized pullback, we get an induced log smooth morphism of fs Kummer neighborhoods $Z' \to X'$. By the above, we find that $D(Z') \to Z'$ is a Γ-equivariant resolution. Then we have the quasi-toroidal resolution $D(Z')/\Gamma \to Z$. Let $z \in Z$ be any point, and let $Z'' \to Z$ be a minimal Kummer cover at z with character group Γ_z. Then, we claim that, étale locally at z, $D(Z'')/\Gamma_z$ and $D(Z')/\Gamma$ coincide. As $D(Z'')/\Gamma_z$ is independent of the minimal Kummer cover at z, then it is sufficient to show the claim for any Z''.

Since $Z \to X$ is quotient log smooth, then by 4.4.6, étale locally at z, we have a factorization $Z' \to Z'' \to Z$, where $Z'' \to Z$ is a minimal Kummer cover at z, and $Z' \to Z''$ is an étale quotient by $\Gamma \to \Gamma^z$. Let Γ_z be the character group induced from the minimal Kummer cover $Z'' \to Z$. Also by 4.4.6, the morphism $Z' \to Z''$, is strongly Γ_z-equivariant and the stabilizer in Γ of a point $z' \in Z'$, above z, can be identified with Γ_z. Hence, the quotient morphism $Z'/\Gamma_z \to Z'/\Gamma = Z$ is étale at the image of z' in Z'/Γ_z and $Z' \to Z'/\Gamma_z$ is minimal. By functoriality of the resolution with respect to log smooth morphisms, we get that $D(Z') \to D(Z'')$ is an étale quotient by Γ^z and is Γ_z-strongly equivariant. Therefore, we get a common finite étale neighborhood $D(Z')/\Gamma \leftarrow D(Z')/(\Gamma_z) \to D(Z'')/\Gamma_z$. On the other hand,

$D(Z')/\Gamma \to Z$ and $D(Z'')/\Gamma_z \to Z$ are proper birational and make the following diagram commute.

$$\begin{array}{ccc} D(Z')/(\Gamma_z) & \longrightarrow & D(Z')/\Gamma \\ \downarrow & & \downarrow \\ D(Z'')/\Gamma_z & \longrightarrow & Z \end{array}$$

By considering the fiber product $D(Z')/\Gamma \times_Z D(Z'')/\Gamma_z$ we see that $D(Z')/(\Gamma_z) \to D(Z')/\Gamma \times_Z D(Z'')/\Gamma_z$ is finite onto a component $W \subset D(Z')/\Gamma \times_Z D(Z'')/\Gamma_z$. Then the restriction $W \to D(Z')/\Gamma$ must be birational, and finite (since $D(Z')/\Gamma \leftarrow D(Z')/(\Gamma_z)$ is finite). Hence as $D(Z')/\Gamma$ is normal, then $W \cong D(Z')/\Gamma$. Similarly $D(Z'')/\Gamma_z \cong W$. Also, note that $D(Z')/(\Gamma_z)$ is the induced resolution on Z'/Γ_z (which is an étale neighborhood of Z).

At this point, we have that $D(Z'')/\Gamma_z$ and $D(Z')/\Gamma$ coincide étale locally. That is, there exists an étale neighborhood $U \to Z$, such that $(D(X') \times_X U)^{n-log}/\Gamma \cong D(U'')/\Gamma_z = D(U)$. The argument below shows that $(D(X') \times_X U)^{n-log}/\Gamma \cong ((D(X')/\Gamma) \times_X U)^{n-log}$, and hence the resolution $X \mapsto D(X')/\Gamma$ is independent of the fs Kummer cover $X' \to X$.

Step 4. Suppose we are given a quotient log smooth morphism $f : Y \to X$, and a global minimal Kummer cover $X' \to X$, with character group Γ. Since f is quotient log smooth, then $Y' := (X \times_Y Y')^{n-log} \to X'$ is classically log smooth and Γ equivariant. As the resolution $X' \mapsto D(X')$ is functorial with respect to Γ equivariant log smooth morphisms, we get that $D(Y') \cong Y' \times_X D(X')$ and that $D(Y') \to D(X')$ is Γ equivariant. Call $D_1(Y) := D(Y')/\Gamma$. We get an induced morphism $D_1(Y) \to D(X)$. Then, we claim that $D_1(Y) \cong (D(X) \times_X Y)^{n-log}$.

Given that $X' \to X$, $D(X') \to D(X)$ are finite and $D(X') \to X'$ is birational, we get that there is an induced morphism $D(X') \to (D(X) \times_X X')^{n-log}$ which must be finite. Note that, as $D(X) \to X$ is birational, and $X' \to X$ is a minimal Kummer cover, then by the definition of $()^{n-log}$ we see that $(D(X) \times_X X')^{n-log}$ has the same number of components as X'. As $D(X') \to X'$ is birational on each component and we have the factorization $D(X') \to (D(X) \times_X X')^{n-log} \to X'$, then $D(X') \to (D(X) \times_X$

$X')^{n-log}$ is birational on every component. Thus $D(X') \cong (D(X) \times_X X')^{n-log}$ since both varieties are normal. Similarly we obtain that $D(Y') \cong (D_1(Y) \times_Y Y')^{n-log}$. Following the diagram,

$$\begin{tikzcd}
D(Y') \arrow[rrr] \arrow[dr] \arrow[ddd] & & & D(X') \arrow[dl] \arrow[ddd] \\
& D_1(Y) \arrow[r] \arrow[d] & D(X) \arrow[d] & \\
& Y \arrow[r] & X & \\
Y' \arrow[ur] \arrow[rrr] & & & X' \arrow[ul]
\end{tikzcd}$$

we see that $D(Y') \cong (D(X) \times_{D_1(Y)} D(Y'))^{n-log}$. Since $D(Y') \cong Y' \times_{X'} D(X')$, then we must also have that $D(Y') \cong ((Y \times_X D(X))^{n-log} \times_Y Y')^{n-log}$. Thus, the induced morphism $D(Y') \to (Y \times_X D(X))^{n-log}$, must be finite. Given that $D(Y') \to D(Y)$ is finite and surjective, then from the factorization $D(Y') \to D_1(Y) \to (D(X) \times_X Y)^{n-log}$, we get that the induced morphism $D_1(Y) \to (D(X) \times_X Y)^{n-log}$ must be finite also [13, Tag 0AH6]. The morphism $D_1(Y) \to (D(X) \times_X Y)^{n-log}$ must be birational, since both have function field $K(Y')^\Gamma = K(Y)$. Hence $D_1(Y) \cong (D(X) \times_X Y)^{n-log}$ since both varieties are normal.

Applying this argument to $X' \to X$ and $U \to Z \to X$ in *Step* 3, we get that $D(U) = (D(X) \times_X U)^{n-log}$. This establishes functoriality with respect to quotient log smooth morphisms.

Step 5. The functoriality with respect to quotient log smooth morphisms, implies the resolution glues on all of X. This is because, from 4.5.2, we can pick an étale covering $\cup X_i \to X$ (finitely many X_i's) where each X_i has a central chart and the images of all of this centers cover X. Hence, each X_i has a global minimal Kummer cover at every point of its center. By the argument in *Step* 0, we get a resolution of X.

Step 6. Finally, we prove that the quasi-toroidal locus is preserved. Since the minimal Kummer cover of a quasi-toroidal variety $X' \to X$ is toroidal, and the desingularization $X' \to D(X')$ preserves the toroidal locus, then $D(X) = X$. Now for a general Kummer fs variety X, the quasi-toroidal locus of is an open sub log scheme $U \subset X$. Thus by functoriality with respect to quotient log smooth morphisms, U is preserved on $D(X)$. ■

Remarks.

- The use of the Γ_x-equivariant embedding of X' into a toroidal variety is not strictly necessary, as the log resolution of X' is already functorial and so it commutes with group actions which preserve the log structure. We chose to use the equivariant embedding in the proof above as to show an application of 4.1.7.

- Whenever $X \to A_P$ is a central chart, with $x \in C(P)$, then the definition of the resolution at x ($D(X')/\Gamma_x$ where $X' \to X$ is a minimal Kummer cover) is simultaneously a resolution for every other point $x' \in C(P)$ as they have the same minimal Kummer cover.

- Since quasi-toroidal varieties are Kummer fs varieties which are quasi log smooth over a field (1.0.1), one would expect the resolution algorithm to be functorial with respect to quasi log smooth morphisms. We do not know if this is true. However, we can show the following result.

Proposition 4.6.3 *Let $X \to Y$ be a morphism of Kummer fs log varieties, with Y an fs log scheme. Suppose that at every $x \in X$, we can find a minimal fs Kummer neighborhood such that $X' \to X \to Y$ is classically log smooth (in particular $X \to Y$ is quasi-log smooth). Then $D(X) \cong (D(Y) \times_Y X)^{n-log}$.*

Proof As above, we only need to build the resolution étale locally, so we can assume that X has a central chart at a given $x \in X$. Then, we can find a minimal Kummer fs neighborhood $X' \to X$ (with character group Γ_x), such that $X' \to Y$ is classically log smooth. Hence $D(X') = (D(Y) \times_{X'} Y)^{n-log}$ by the functoriality of [3, Theorem

1.3.4]. On the other hand, by the same functoriality $D(X') \to D(Y)$ must be Γ_x-equivariant (with trivial action on $D(Y)$). Recall that we defined $D(X) := D(X')/\Gamma_x$. Using similar arguments as above, we find that there is an induced finite birational morphism $D(X) \to (X \times_Y D(Y))^{n-log}$. We conclude that $D(X) \cong (X \times_Y D(Y))^{n-log}$ since both varieties are normal. ■

Example 4.6.4 *Consider the log scheme spec(K) (K a field), with log structure given by $M := \mathbb{N} \oplus K^\times$ ($\mathbb{N}$ maps to zero). We remark that this example is not a Kummer fs variety in out sense, since $M \not\subset \mathcal{O}_X \setminus \{0\}$. We have a strict embedding of $(spec(K), M)$ into the toroidal variety $(spec(K[T]), \mathbb{N})$. Note that $(spec(K), M)$ is not generically log smooth. If there was an embedded resolution of $(spec(K), M)$, then (by the nature of admissible Kummer blowups) it would have to be some monoscheme A_P over $(spec(K), M)$. However, since the generator of $\mathbb{N}$ maps to zero in K, there is no logarithmic morphism $A_P \to (spec(K), M)$. Thus, there is no embedded resolution for such kind of log scheme.*

Example 4.6.5 *Similarly, consider $X := spec(K[T])$ with log structure coming from $\mathbb{N}^2 \to [t]$ sending $(1,0) \mapsto T$, $(0,1) \mapsto T$. Then $(X, (\mathbb{N}^2)^a)$ has a log resolution, namely $(X, \mathbb{N}^a)$, with log structure induced from $\mathbb{N} \to T$. The morphism $(X, (\mathbb{N})^a) \to (X, (\mathbb{N}^2)^a)$ is the identity on underlying schemes and $\mathbb{N}^2 \to \mathbb{N}$ is the sum.*

With a similar argument as above, log resolution works even when $M \not\subset \mathcal{O}_X$, provided no element in M gets sent to zero. That is, on the first step, one just replaces M by its image N in $\mathcal{O}_X$ and then resolves (X, N). Note that $(X, N) \to (X, M)$ preserves the quasi-toroidal locus (by definition if $x \in (X, M)$ is in the quasi-toroidal locus, then $M_x \subset \mathcal{O}_x \setminus \{0\}$).

Example 4.6.6 *For rational log varieties (X, M) which do not satisfy 2.13.1, the algorithm above may not produce a log resolution. Consider for example $X :=$ spec($k[x, y]$) with log structure given by $\mathbb{N} \to [x, y]$ ($1 \mapsto xy$). Then X is quasi-toroidal everywhere but at the origin. X can be embedded strictly into the toroidal* *variety* $Z := spec(k[x, y, z])$ *with* $z = 0$ *log structure. However, since the ideal of* X *in* Z *is already monomial the algorithm stops without modifying* X.

REFERENCES

REFERENCES

[1] K. Kato, "logarithmic structures of fontaine-illusie," *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pp. 191–224, 1989.

[2] A. Ogus, *Lectures on Logarithmic Algebraic Geometry.* Cambridge University Press, 2018.

[3] D. Abramovich, M. Temkin, and J. Włodarczyk, "Principalization of ideals, and canonical desingularization on logarithmic varieties," *preprint*, 2018.

[4] D. Abramovich, "Resolution of singularities of complex algebraic varieties and their families," *arXiv preprint arXiv:1711.09976*, 2017.

[5] K. Kato, "Toric singularities," *American Journal of Mathematics*, vol. 116, no. 5, pp. 1073–1099, 1994.

[6] D. Abramovich, M. Temkin, and J. Włodarczyk, "Principalization of ideals on toroidal orbifolds," *arXiv preprint arXiv:1709.03185*, 2017.

[7] J. Włodarczyk, "Simple hironaka resolution in characteristic zero," *Journal of the American Mathematical Society*, vol. 18, no. 4, pp. 779–822, 2005.

[8] K. Kato, "Logarithmic degeneration and dieudonné theory," *preprint*, 1989.

[9] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, "Toroidal embeddings i, vol. 339 of lecture notes in mathematics," 1973.

[10] T. Tsuji, "Saturated morphisms of logarithmic schemes," *Tunisian Journal of Mathematics*, vol. 1, no. 2, pp. 185–220, 2018.

[11] W. Nizioł, "K-theory of log-schemes i," *Documenta Mathematica*, vol. 13, pp. 505–551, 2008.

[12] C. Nakayama, "Logarithmic étale cohomology," *Mathematische Annalen*, vol. 308, no. 3, pp. 365–404, 1997.

[13] T. Stacks Project Authors, "*Stacks Project*," http://stacks.math.columbia.edu, 2018.

[14] D. Abramovich and M. Temkin, "Luna's fundamental lemma for diagonalizable groups," *arXiv preprint arXiv:1505.00754*, 2015.

[15] D. Luna, "Slices étales," *Bull. Soc. Math. France*, vol. 33, pp. 81–105, 1973.

[16] L. Illusie, C. Nakayama, and T. Tsuji, "On log flat descent," *Proceedings of the Japan Academy*, vol. 89, no. 1, p. 1, 2013.

[17] M. Raynaud, *Anneaux locaux henséliens.* Springer, 2006, vol. 169.

[18] D. Abramovich and M. Temkin, "Torification of diagonalizable group actions on toroidal schemes," *Journal of Algebra*, vol. 472, pp. 279–338, 2017.