

REES ALGEBRAS AND FIBER CONES OF MODULES

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ABSTRACT

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In the first part of this thesis, we study Rees algebras of modules. We investigate their Cohen-Macaulay property and their defining ideal, using *generic Bourbaki ideals*. These were introduced by Simis, Ulrich and Vasconcelos in [65], in order to characterize the Cohen-Macaulayness of Rees algebras of modules. Thanks to this technique, the problem is reduced to the case of Rees algebras of ideals. Our main results are the following.

In Chapters 3 and 4 we consider a finite module E over a Gorenstein local ring R . In Theorem 3.2.4 and Theorem 4.3.2, we give sufficient conditions for E to be of linear type, while Theorem 4.2.4 provides a sufficient condition for the Rees algebra $\mathcal{R}(E)$ of E to be Cohen-Macaulay. These results rely on properties of the residual intersections of a generic Bourbaki ideal I of E , and generalize previous work of Lin (see [46, 3.1 and 3.4]). In the case when E is an ideal, Theorem 4.2.4 had been previously proved independently by Johnson and Ulrich (see [39, 3.1]) and Goto, Nakamura and Nishida (see [20, 1.1 and 6.3]).

In Chapter 5, we consider a finite module E of projective dimension one over $k[X_1, \dots, X_n]$. Our main result, Theorem 5.2.6, describes the defining ideal of $\mathcal{R}(E)$, under the assumption that the presentation matrix φ of E is *almost linear*, i.e. the entries of all but one column of φ are linear. This theorem extends to modules a known result of Boswell and Mukundan on the Rees algebra of almost linearly presented perfect ideals of height 2 (see [5, 5.3 and 5.7]).

The second part of this thesis studies the Cohen-Macaulay property of the special fiber ring $\mathcal{F}(E)$ of a module E . In Theorem 6.2.14, we prove that the generic Bourbaki ideals of Simis, Ulrich and Vasconcelos allow to reduce the problem to the case of fiber cones of ideals, similarly as for Rees algebras. We then provide sufficient conditions for $\mathcal{F}(E)$ to be Cohen-Macaulay. Our Theorems 6.2.15, 6.1.3 and 6.2.18 are module versions of results proved for the fiber cone of an ideal by Corso, Ghezzi, Polini and Ulrich (see [10, 3.1] and [10, 3.4]) and by Montaña (see [47, 4.8]), respectively.

1. INTRODUCTION

The broad goal of this thesis is to study properties of the Rees algebra and of the special fiber ring of a module.

Let R be a Noetherian ring, and E a finitely generated R -module with a rank. The *Rees algebra* $\mathcal{R}(E)$ of E is defined as the symmetric algebra $\mathcal{S}(E)$ modulo its R -torsion submodule. It can be presented in terms of generators and relations as a quotient of a polynomial ring $R[T_1, \dots, T_n]$ modulo an ideal \mathcal{J} , called the defining ideal of $\mathcal{R}(E)$. If (R, \mathfrak{m}, k) is local, the *special fiber ring*, or *fiber cone*, $\mathcal{F}(E)$ of E is defined as the Rees algebra tensored over R with the residue field k .

We are mostly interested in the Cohen-Macaulay property of both $\mathcal{R}(E)$ (see Chapter 4) and $\mathcal{F}(E)$ (see Chapter 6). The two are usually not related with each other, and different techniques may be needed to study the problem in each case. In the case of Rees algebras, the Cohen-Macaulay property is often deeply interconnected with the shape of its defining ideal. This motivated us to also investigate the defining ideal of the Rees algebra, which we do in Chapters 3 and 5.

Although the methods we use are purely algebraic, some of the motivation to study Rees algebras and fiber cones of ideals and modules is of geometric nature. In fact, Rees algebras arise in Algebraic Geometry as homogeneous coordinate rings of blowups of schemes along a subscheme, and have been studied intensively in connection with resolution of singularities. The fiber cone is the homogeneous coordinate ring of the special fiber of the blowup at the unique closed point.

Furthermore, Rees algebras and fiber cones appear in the study of rational maps. Let ϕ be a rational map from a subvariety $X \subseteq \mathbb{P}_k^{d-1}$ into a projective space \mathbb{P}_k^{n-1} , defined by homogeneous forms f_1, \dots, f_n . Then, $\text{biProj}(\text{graph}(\phi))$ is the the Rees algebra $\mathcal{R}(I)$, where I is the ideal generated by the f_i in $k[X_1, \dots, X_d]$. The the

projective spectrum of the image of ϕ the special fiber ring $\mathcal{R}(I)$. The study of these kind of maps has recently found applications in model theory as well.

In many geometric situations, one is interested in Rees algebras and fiber cones of modules which are not ideals. For instance, the homogeneous coordinate ring of a sequence of successive blow-ups of a scheme along two or more subschemes is the Rees algebra of a direct sum of two or more ideals. Moreover, given a subvariety $X \subseteq \mathbb{A}_k^n$, the conormal variety of X and the graph of the Gauss map from X are projective spectra of Rees algebras of modules which are not ideals. In fact, if R is the homogeneous coordinate ring of X , the conormal variety of X is $\text{Proj}(\mathcal{R}(\mathcal{E}))$, where \mathcal{E} is the Jacobian module of R , while the graph of the Gauss map is $\text{Proj}(\mathcal{R}(M))$ where M is the d -th exterior power of the module of differentials $\Omega_k(R)$ of R . Finally, given a subvariety $X \subseteq \mathbb{P}_k^n$ with homogeneous coordinate ring R , then the tangential variety of X is $\text{Proj}(\mathcal{F}(\Omega_k(R)))$, while the secant variety of X is $\text{Proj}(\mathcal{F}(\mathbb{D}))$, where \mathbb{D} is the diagonal ideal.

Geometrically, the Cohen-Macaulay property of Rees algebras or of fiber cones encodes information on the vanishing of their cohomology modules. Although there is no topological characterization of Cohen-Macaulayness, Cohen-Macaulay schemes are always equidimensional. Moreover, a theorem of Hartshorne proves that they are locally connected in codimension one (see [15, Theorem 18.12]). In fact, many interesting singular varieties are locally Cohen-Macaulay. However, in general this property is not preserved after blowing up. Hence, it is interesting to understand what assumptions are needed in order for Rees algebras to be Cohen-Macaulay, and how the Cohen-Macaulay property relates with the equations defining the blowup.

The literature on the algebraic study of Rees algebras is extremely rich. When $E = I$ is an R -ideal, the Rees algebra $\mathcal{R}(I)$ coincides with the subalgebra

$$R[It] = \bigoplus_{j \geq 0} I^j t^j$$

of the polynomial ring $R[t]$. These rings have been studied by several authors since the late 1950's, in connection with integral dependence of ideals, multiplicity theory,

and asymptotic growth of powers of an ideal. In fact, a good amount of Rees's original work was motivated by the study of integral dependence. His contribution includes the famous Rees' Theorem (see [34, Theorem 11.3.1]), stating that integral dependence of \mathfrak{m} -primary ideals is captured by their Hilbert-Samuel multiplicities. This result gave, and keeps giving rise to a considerable amount of literature, in search of possible generalizations to ideals which are not \mathfrak{m} -primary or to modules (see, for instance, [4, 19, 41, 42, 55, 64], and more recently, [17, 69, 70]).

The study of depth properties was the main focus of the research on Rees rings of an ideal from the late 1970's until the late 1990's, with contributions from Valla, Herzog, Simis, Vasconcelos, Huneke, Sally, Rossi (see [22, 23, 30, 32, 57, 58, 60, 61, 72, 73]) and several more authors. At the same time, a systematic approach to the search for defining equations of Rees rings via homological algebra methods was initiated, mostly by the school of Simis and Vasconcelos [48, 49, 62, 66, 71, 74], and still continues to stimulate intense research in the subject (see, for instance [5, 43, 45]). Another trend, initiated by work of Villarreal [75], is to study the defining ideal of the Rees algebras associated with squarefree monomial ideals, exploiting the combinatorial properties of the given ideals (see also [18]). Much less is known about the Cohen-Macaulay property and the defining ideal of Rees algebras of modules (see, for instance, [46, 65]).

When R is Cohen-Macaulay, the first step in order to study the Cohen-Macaulayness of $\mathcal{R}(I)$ is to investigate the Cohen-Macaulayness of the *associated graded ring*

$$\mathcal{G}(I) = \bigoplus_{j \geq 0} I^j / I^{j+1}.$$

In fact, in this case, Huneke proved that $\mathcal{G}(I)$ is Cohen-Macaulay whenever $\mathcal{R}(I)$ is [32]. Although the converse is not true in general, it holds if furthermore some numerical conditions are satisfied [37, 40, 53, 63, 71]. Investigating these numerical conditions is then the second, and last, step in order to understand whether $\mathcal{R}(I)$ is Cohen-Macaulay (see Section 4.1 for a detailed treatment of the topic).

Now, by analogy with the case of ideals, one defines the powers of an R -module E as

$$E^j := [\mathcal{R}(E)]_j,$$

where $[\mathcal{R}(E)]_j$ denotes the degree- j component of $\mathcal{R}(E)$. However, since E^{j+1} is not contained in E^j , there is no module analogue for the associated graded ring. Hence, the study of the Cohen-Macaulayness of Rees algebras of modules requires a completely different approach than in the case of ideals, and little is known about it (see [24, 46, 62, 65]).

In [65] Simis, Ulrich and Vasconcelos reduced the problem to the case of ideals, thanks to the notion of generic Bourbaki ideals. Given a module E , an ideal I is called a *Bourbaki ideal* of E if there exists an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow I \rightarrow 0,$$

where F is a free submodule of E [6]. These ideals were previously used, for instance, to study modules over normal domains. One would hope that the induced epimorphism $\mathcal{R}(E) \twoheadrightarrow \mathcal{R}(I)$ would allow to transfer properties back and forth between the two Rees algebras. This is too optimistic, however it becomes effective if F is generated by generic elements. An ideal constructed this way is called a *generic Bourbaki ideal* of E (see Definition 2.6.3). The advantage of this adjustment to the construction is that now I is essentially unique, and the map between the Rees algebras creates a much deeper connection. In fact, $\mathcal{R}(E)$ is Cohen-Macaulay if and only if $\mathcal{R}(I)$ is, and in this case $\mathcal{R}(I)$ is a deformation of $\mathcal{R}(E)$ (see Theorem 2.6.4).

In the first part of this thesis, our main goal is to provide a sufficient condition in order for $\mathcal{R}(E)$ to be Cohen-Macaulay, under suitable assumptions on the depth of finitely many powers of E . More specifically, we aim to generalize a well-known result proved independently by Johnson and Ulrich [39, 3.1 and 3.4] and by Goto, Nakamura and Nishida [20, 1.1 and 6.3]. Under assumptions on the residual intersections of I , they show that $\mathcal{R}(I)$ is Cohen-Macaulay if the reduction number of I satisfies a

suitable upper bound while the depths of finitely many powers of I satisfy suitable lower bounds (see Theorem 4.1.5 for a precise statement).

Using generic Bourbaki ideals, the problem is reduced to that of crafting assumptions on the given module E , so that its generic Bourbaki ideal I satisfies the requirements in the result by Johnson and Ulrich and by Goto, Nakamura and Nishida. Despite the simplicity of the idea, in practice transferring assumptions from E to I is not an easy task. Most notably, there is no clear module analogue for the main assumption for the ideals under consideration, namely the so called *Artin-Nagata property*. This is a requirement that certain residual intersections of I are Cohen-Macaulay (see Definition 2.7.4).

Residual intersections of an ideal generalize the notion of algebraic liaison [36]. However, there is no satisfactory module analogue of residual intersections (see the discussions on this topic in Sections 3.2 and 4.2). In fact, it is usually difficult to even identify which properties of E guarantee that I satisfies the necessary Artin-Nagata condition. This latter issue may impose serious restrictions on the class of modules one can consider (see [46, 4.3]). Our main result, Theorem 4.2.4, generalizes previous work of Lin (see [46, 3.4]).

From a technical point of view, the key ingredient in the proof of Theorem 4.2.4 is a deep investigation of the linear type property of E . A module E is said to be of *linear type* if its Rees algebra $\mathcal{R}(E)$ coincides with the symmetric algebra $\mathcal{S}(E)$, since in this case its defining ideal \mathcal{J} consists of linear forms. Moreover, E is of linear type whenever a generic Bourbaki ideal I of E is.

In [46, 3.1], Lin provides a sufficient condition for a module E to be of linear type by showing that its generic Bourbaki ideal I is not only of linear type, but also satisfies the stronger *sliding depth* condition (which we will recall in Definition 2.7.3).

Our Theorem 4.3.2 gives a slightly less restrictive condition for I to satisfy sliding depth. Moreover, in Theorem 3.2.4 we provide a new sufficient condition for a module E to be of linear type, in a situation when a generic Bourbaki ideal I of E

is of linear type without necessarily satisfying sliding depth. This improvement is obtained thanks to a deep investigation of the properties of the residual intersections of I , which allows to replace the Artin-Nagata condition with the less restrictive requirement that I is *residually* S_2 (see Definition 2.7.10).

In Chapter 5, we focus on the problem of determining the defining ideal \mathcal{J} of $\mathcal{R}(E)$, for modules which are not necessarily of linear type. This is usually a difficult task, and the problem in its full generality is wide open. However, given a presentation $R^s \xrightarrow{\varphi} R^n \twoheadrightarrow E$ of E , information about \mathcal{J} can be deduced by exploiting the connection between the Rees algebra $\mathcal{R}(E)$ and the symmetric algebra $\mathcal{S}(E)$.

In fact, the defining ideal \mathcal{L} of $\mathcal{S}(E)$ can be easily determined from the presentation φ , and by construction $\mathcal{L} \subseteq \mathcal{J}$ (see Sections 2.3 and 2.4). The inclusion is strict if E is not of linear type, however it is still possible to determine \mathcal{J} in the case when φ has a particularly rich structure. For instance, if I is a perfect ideal of height two, or a perfect Gorenstein ideal of height three, the structure of φ is dictated by the Hilbert-Burch Theorem (see [7, Theorem 1.4.16]) or by the Buchsbaum-Eisenbud Theorem (see [7, Theorem 3.4.1]), respectively. In these cases, the defining ideal of $\mathcal{R}(I)$ is often determined by means of the so called *Jacobian dual* of φ , as we recall in Section 5.1.

In certain situations, the Cohen-Macaulayness of the Rees algebra turns out to be helpful in order to describe its defining ideal, as it often provides bounds on the degrees of its generators or even an explicit generating set (see [49, 53, 65, 68]). However, less is known on the defining ideal of Rees algebras which are not Cohen-Macaulay [5, 43, 45].

This is, for instance, the case of the Rees algebra of a perfect ideal I of height two which is *almost linearly presented*, i.e. all the entries of a presentation matrix φ of I are linear, except possibly those in one column, homogeneous of degree $m \geq 1$. In fact, in [5, 5.3 and 5.7] Boswell and Mukundan describe the defining ideal of $\mathcal{R}(I)$ using *iterated Jacobian duals* of φ (see Definition 5.1.8), and prove that $\mathcal{R}(I)$ is almost Cohen-Macaulay (i.e. its depth is at least one less than its dimension). However,

$\mathcal{R}(I)$ usually fails to be Cohen-Macaulay (see Theorem 5.1.10 for a precise statement).

Our main goal in this matter is to provide a generalization of the result by Boswell and Mukundan to the case of Rees algebras of modules. More precisely, we aim to describe the defining ideal of the Rees algebra of almost linearly presented modules projective dimension one.

The case when φ is linear was discussed by Simis, Ulrich and Vasconcelos in [65, 4.11], using generic Bourbaki ideals. In fact, in this situation, a generic Bourbaki ideal I of E is a linearly presented perfect ideal of height two, hence the defining ideal of $\mathcal{R}(I)$ is well-understood by a result of Morey and Ulrich [49] (see Theorem 5.1.4). Moreover, $\mathcal{R}(I)$ is Cohen-Macaulay, hence it is a deformation of $\mathcal{R}(E)$, and this makes it possible to ‘lift’ the shape of the defining ideal of $\mathcal{R}(I)$ back to $\mathcal{R}(E)$.

With a similar approach, in Theorem 5.2.6, we describe the defining ideal assuming that the presentation matrix of E is only almost linear. In this case, a generic Bourbaki ideal I satisfies the assumptions in the result of Boswell and Mukundan. In particular, its Rees algebra $\mathcal{R}(I)$ is only almost Cohen-Macaulay. Hence, in order to be able to ‘lift’ the shape of the defining ideal of $\mathcal{R}(I)$ back to $\mathcal{R}(E)$ also in this case, the argument used in the linearly presented case requires a modification. This technical aspect is discussed in Theorem 5.2.5, which guarantees that $\mathcal{R}(I)$ is a deformation of $\mathcal{R}(E)$ also in this case.

In Chapter 6 we discuss the Cohen-Macaulay property of the fiber cone of modules. In general, one should not expect that the Cohen-Macaulayness of $\mathcal{F}(E)$ is related to that of $\mathcal{R}(E)$, as counterexamples are known in the case when E is isomorphic to an ideal (see for instance [14]).

However, the problem can be approached in a similar way in the two cases, in the sense that in both situations one can reduce to the case where E is isomorphic to an ideal. More precisely, our main result, Theorem 6.2.14 shows that, if a module E admits a generic Bourbaki ideal I , then the special fiber ring $\mathcal{F}(E)$ is Cohen

Macaulay if and only if the special fiber ring $\mathcal{F}(I)$ and the Rees algebra $\mathcal{R}(I)$ are both Cohen-Macaulay.

The proof arises from a close examination of the proof of the analogous result on Rees algebras, and provides a technical tool for the search of sufficient conditions for the fiber cone of a module to be Cohen-Macaulay. The latter is the goal of Section 6.2.2, where we extend to fiber cones of modules some results that were previously known for the fiber cone of an ideal (see Theorems 6.2.15, 6.2.18 and 6.1.3).

We now describe how this thesis is organized.

In Chapter 2 we include the background material that will be used throughout the thesis. In particular, we review the main properties of Rees algebras of modules in Section 2.4, and the main results on generic Bourbaki ideals in Section 2.6. In Section 2.7, we recall the basic definitions and results on residual intersections of ideals, including a discussion on Artin-Nagata properties and residually S_2 ideals.

Chapter 3 explores the linear type property of Rees algebras. After recalling some results known for Rees algebras of ideals in Section 3.1, in Section 3.2 we provide a sufficient condition for a module E to be of linear type, Theorem 3.2.4, and detect specific classes of modules which satisfy this result (see Section 3.2.2). Another sufficient condition for a module E to be of linear type is given later in Theorem 4.3.2. In this case, we also show that the Rees algebra $\mathcal{R}(E)$ is Cohen-Macaulay. We postpone this result to Chapter 4, where some of the tools needed for the proof are introduced.

Chapter 4 is dedicated to the Cohen-Macaulay property of Rees algebras. We discuss the case of Rees algebras of ideals in Section 4.1. Our main result is Theorem 4.2.4 (see Section 4.2.1), which provides a sufficient condition for $\mathcal{R}(E)$ to be Cohen-Macaulay when E is a torsion-free, orientable R -module. Another sufficient condition is given in Section 4.4, for a different class of modules, the so called *ideal modules*.

In Chapter 5 we focus on the problem of determining the defining ideal of Rees algebras. Section 5.1 gives an overview on known result for Rees algebras of ideals,

while we treat the case of modules in Section 5.2. Our main result, Theorem 5.2.6, describes the defining ideal of the Rees algebra of almost linearly presented modules of projective dimension one.

Finally, we discuss the Cohen-Macaulay property of the fiber cone of modules in Chapter 6. Our main result is Theorem 6.2.14, which reduces the problem to the case of ideals. Sufficient conditions for the fiber cone of a module to be Cohen-Macaulay are included in Section 6.2.2.

2. PRELIMINARIES

This chapter includes the background notions that will be needed in this thesis. In Section 2.1, we set up the notation and recall the definitions of some homological invariants which we will be using throughout this work. In Sections 2.2 through 2.4, we review Rees algebras of ideals and modules, and describe other graded algebras which play a crucial role in the study of Rees rings. Section 2.5 is dedicated to minimal reductions of ideals and modules. In Section 2.6 we recall the construction of generic Bourbaki ideals, the fundamental technical tool we use throughout this work. Finally, in Section 2.7 we review residual intersections of ideals and some of their properties, which are needed for the proofs of several results in Chapter 3 and Chapter 4.

2.1 Notation

Throughout this work, all rings will be assumed to be commutative with multiplicative unit.

Let R be a Noetherian ring, I an R -ideal, M a finite R -module. We denote with $\text{grade } I$ the *grade* of I , that is, the length of a maximal R -regular sequence in I . We adopt the convention that $\text{grade } I = \infty$ if $I = R$. Similarly, $\text{depth}_I M$ denotes the length of a maximal M -regular sequence in I , and we set $\text{depth}_I M = \infty$ if $IM = M$. If R is local with maximal ideal \mathfrak{m} , $\text{depth}_{\mathfrak{m}} M$ is usually denoted by $\text{depth } M$ and called the *depth* of M . The *grade* of M is defined as the grade of the ideal $\text{ann}_R M$, and denoted by $\text{grade } M$.

In many proofs of this thesis we will often make use of the following homological characterizations of depth, grade and dimension of a module.

Proposition 2.1.1 *Let R be a Noetherian ring, I an R -ideal, M a finite R -module.*

- (a) $\text{depth}_I M = \min \{i \mid \text{Ext}^i(R/I, M) \neq 0\} = \min \{i \mid H_I^i(M) \neq 0\}$.
- (b) $\text{grade } M = \min \{i \mid \text{Ext}^i(M, R) \neq 0\}$.
- (c) $\dim M = \max \{i \mid H_I^i(M) \neq 0\}$.
- (d) *Assume R is a Gorenstein local ring of dimension d . Then,*
 - (i) $d - \text{depth } M = \max \{i \mid \text{Ext}^i(M, R) \neq 0\}$.
 - (ii) $d - \dim M = \min \{i \mid \text{Ext}^i(M, R) \neq 0\} = \text{grade } M$.

The statement in (d) follows from the Local Duality Theorem (see [7, 3.5.11]). We recall that a Noetherian local ring R is *Gorenstein* if it is Cohen-Macaulay (i.e. $\text{depth } R = \dim R$) and coincides with its *canonical module* ω_R . We refer the reader to the book by Bruns and Herzog [7] for properties of canonical modules, and in general for a broad treatment of the homological algebra of commutative Noetherian rings.

2.2 Blowup algebras of an ideal

Let R be a Noetherian ring, I an R -ideal.

Definition 2.2.1 The *Rees algebra* of I is the subalgebra

$$\mathcal{R}(I) := R[It] = \bigoplus_{j \geq 0} I^j t^j \cong \bigoplus_{j \geq 0} I^j$$

of the polynomial ring $R[t]$. More precisely, if $I = (a_1, \dots, a_n)$, then

$$\begin{aligned} \mathcal{R}(I) &= R[a_1 t, \dots, a_n t] \\ &= R \oplus (R a_1 t + \dots + R a_n t) \oplus (R a_1^2 t^2 + R a_1 a_2 t^2 \dots + R a_n^2 t^2) \oplus \dots \end{aligned}$$

This is a Noetherian standard graded R -algebra, with grading inherited by the natural grading of the polynomial ring $R[t]$. Geometrically, $\text{Proj}(\mathcal{R}(I))$ is the blowup of $\text{Spec}(R)$ along the subscheme $V(I)$. The algebraic properties of $\mathcal{R}(I)$ are often studied through its connection with several other graded algebras associated with I . We are mostly interested in the *associated graded ring* $\mathcal{G}(I)$ and the *special fiber ring* $\mathcal{F}(I)$ of I , defined as follows.

Definition 2.2.2 The *associated graded ring* of I is

$$\mathcal{G}(I) = \text{gr}_I(R) := \frac{\mathcal{R}(I)}{I\mathcal{R}(I)} \cong \bigoplus_{j \geq 0} \frac{I^j}{I^{j+1}}$$

Definition 2.2.3 Assume that (R, \mathfrak{m}) is local. The *special fiber ring*, or *fiber cone*, of I is

$$\mathcal{F}(I) := \frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} \cong \frac{\mathcal{G}(I)}{\mathfrak{m}\mathcal{G}(I)}.$$

The associated graded ring and the special fiber ring of I are Noetherian standard graded R -algebras, with grading inherited from the grading of $\mathcal{R}(I)$. From a geometric point of view, $\text{Proj}(\mathcal{G}(I))$ is the exceptional set of the blow up of $\text{Spec}(R)$ along $V(I)$. When R is local, $\text{Proj}(\mathcal{F}(I))$ is the fiber of the unique closed point of the blowup of $\text{Spec}(R)$ along $V(I)$.

The next proposition, whose proof can be found in to [34, 5.1.4 and 5.1.6], discusses the dimension of the Rees algebra and of the associated graded ring.

Proposition 2.2.4 *Let R be a Noetherian ring, I an R -ideal. Then:*

$$\begin{aligned} \dim \mathcal{R}(I) &= \min \{ \dim R, \dim(R/\mathfrak{p}) + 1 \mid \mathfrak{p} \in \text{Min}(R) \setminus V(I) \} \\ &= \dim R + 1 \text{ whenever } \text{ht} I > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \dim \mathcal{G}(I) &= \max \{ \dim R_{\mathfrak{m}} \mid \mathfrak{m} \in \mathfrak{m} - \text{Spec}(R) \cap V(I) \} \\ &= \dim R \text{ if } R \text{ is local and } I \neq R. \end{aligned}$$

The Krull dimension of $\mathcal{F}(I)$ is called the *analytic spread* of I , and denoted by $\ell(I)$. Although there is no general formula to compute $\ell(I)$, one can prove that

$$\text{ht } I \leq \ell(I) \leq \min \{ \mu(I), \dim R \},$$

where $\mu(I)$ denotes the minimal number of generators of I (see [34, 8.4.3]). The analytic spread of I encodes information about the asymptotic growth of the powers of I . More precisely, one has:

1. For $i \gg 0$, $\mu(I^j)$ is a polynomial function of j , of degree $\ell(I) - 1$.
2. $\ell(I) \leq \dim R - \inf \{\text{depth}(R/I^j) \mid j \geq 1\}$ (*Burch's inequality*, see [8])

In order to fully understand the growth of powers of an ideal, one often makes use of the notion of *minimal reduction*, which we recall in Section 2.5 below. In fact, the analytic spread of I also encodes relevant information on the minimal reductions of I (see Proposition 2.5.2).

2.3 The symmetric algebra

Let R be any commutative ring, E a finite R -module.

Definition 2.3.1 The *symmetric algebra* of E is

$$\mathcal{S}(E) := \frac{\otimes E}{\mathcal{H}},$$

where $\otimes E$ is the *tensor algebra* of E and \mathcal{H} is the two-sided ideal generated by elements of the form $x \otimes y - y \otimes x$, with $x, y \in E$.

The symmetric algebra is characterized by a universal property, inherited by the universal property of the tensor algebra (see [7, section 1.6]). Namely, for every commutative R -algebra S and for every R -linear map $\eta: E \rightarrow S$, there exists a unique homomorphism of graded R -algebras $\bar{\eta}: \mathcal{S}(E) \rightarrow S$, which coincides with η when restricted to E .

Notice that if $F \cong R^n$ is a free R -module, then $\mathcal{S}(F) \cong R[T_1, \dots, T_n]$, a polynomial ring in n variables. Now assume that R is Noetherian, so that every finite R -module admits a free resolution consisting of finite free modules. $\mathcal{S}(E)$ can then be equivalently described in terms of a presentation matrix of E as follows.

Remark and Definition 2.3.2 Let R be a Noetherian ring, E a finite R -module, and let

$$R^s \xrightarrow{\varphi} R^n \twoheadrightarrow E$$

be a presentation of E . Then, the columns of φ determine the syzygies of $\mathcal{S}(E)$. More precisely, if $E = Ra_1 + \dots + Ra_n$, there is a natural homogeneous epimorphism

$$R[T_1, \dots, T_n] \twoheadrightarrow \mathcal{S}(E),$$

which maps the variable T_i to the generator $a_i \in [\mathcal{S}(E)]_1$ for each i . Therefore,

$$\mathcal{S}(E) \cong R[T_1, \dots, T_n]/\mathcal{L},$$

where $\mathcal{L} := (\ell_1, \dots, \ell_s)$ and ℓ_1, \dots, ℓ_s are linear forms in $R[T_1, \dots, T_n]$, satisfying

$$[T_1, \dots, T_n] \cdot \varphi = [\ell_1, \dots, \ell_s].$$

The ideal \mathcal{L} is called the *defining ideal* of the symmetric algebra $\mathcal{S}(E)$.

2.4 The Rees algebra and the fiber cone of a module

Let R be a Noetherian ring, $\text{Quot}(R)$ its total ring of quotients, $\text{Ass}(R)$ the set of associated primes of R , and let E be finite R -module. Recall that E has a *rank*, $\text{rank } E = e$, if $E \otimes_R \text{Quot}(R) \cong (\text{Quot}(R))^e$. Equivalently, E has a rank if and only if $E_{\mathfrak{p}} \cong R_{\mathfrak{p}}^e$ for all $\mathfrak{p} \in \text{Ass}(R)$.

Notice that a module may not necessarily have a rank. However, several interesting classes of modules have a rank. In particular:

1. Every free module $F \cong R^r$ has rank r .
2. If R is a domain, every R -module has a rank. (This follows immediately from the definition, since in this case $\text{Quot}(R)$ is a field).
3. If E admits a finite free resolution, then E has a rank. In fact, if

$$0 \rightarrow R^{b_n} \rightarrow \dots \rightarrow R^{b_1} \rightarrow R^{b_0}$$

is a free resolution of E , then $\text{rank}(E) = \sum_{i=0}^n (-1)^i b_i$ (see [7, 1.4.5]).

Definition 2.4.1 Let R be a Noetherian ring, E a finite R -module. If E has a rank, the *Rees algebra* of E is

$$\mathcal{R}(E) := \frac{\mathcal{S}(E)}{\tau_R(\mathcal{S}(E))},$$

where $\tau_R(-)$ denotes the R -torsion.

In particular, thinking of the symmetric algebra of E as in Definition 2.3.2, if $E = Ra_1 + \dots + Ra_n$, there is a natural homogeneous epimorphism

$$R[T_1, \dots, T_n] \twoheadrightarrow \mathcal{R}(E),$$

mapping each T_i to the generator $a_i \in [\mathcal{R}(E)]_1$. The Rees algebra can then be rewritten as

$$\mathcal{R}(E) = \frac{R[T_1, \dots, T_n]}{\mathcal{J}},$$

for some ideal \mathcal{J} that contains the defining ideal \mathcal{L} of the symmetric algebra. \mathcal{J} is called the *defining ideal* of $\mathcal{R}(E)$. Unlike for the defining ideal \mathcal{L} of the symmetric algebra, determining \mathcal{J} is usually not an easy task. The best-case scenario is when $\mathcal{J} = \mathcal{L} = (\ell_1, \dots, \ell_s)$, an ideal of linear forms. This special situation is formalized in the following definition.

Definition 2.4.2 The module E is said to be of *linear type* if $\mathcal{R}(E)$ is naturally isomorphic to $\mathcal{S}(E)$.

We will discuss the problem of determining \mathcal{J} in the case when E is not necessarily of linear type in Chapter 5.

Remark 2.4.3 Let R be a Noetherian ring, E a finite R -module with $\text{rank}(E) = e \geq 0$. If E is torsion-free, then E embeds naturally in a free module of rank e . This induces a natural homomorphism

$$\mathcal{S}(E) \xrightarrow{\phi} \mathcal{S}(R^e) \cong R[t_1, \dots, t_e].$$

In this case, $\mathcal{R}(E)$ coincides with the R -subalgebra of $R[t_1, \dots, t_e]$ generated by $\text{im}(\phi)$.

The description of the Rees algebras given in Remark 2.4.3 is extremely useful in order to study integral dependence of modules (see for instance [34, 16.2.3]). In fact, the Rees algebra of torsionless modules without a rank is often defined this way. However, if we relax the assumption that the module E has a rank, this definition of $\mathcal{R}(E)$ depends on the choice of the embedding of E into a given free module. In order to fix the issue, a more general definition of the Rees algebra of a module was given by Eisenbud, Huneke and Ulrich in [16]. Their definition is independent of the embedding, and makes sense also for modules that are not torsion-free or do not have a rank. Nevertheless, in this work we only treat modules with a rank, as in particular modules of rank one play a crucial role in the theory.

In fact, E is a torsion-free module of rank one if and only if it is isomorphic to an R -ideal I of positive grade. In this case, Remark 2.4.3 implies that

$$\mathcal{R}(E) \cong \frac{\mathcal{S}(E)}{\tau_R(\mathcal{S}(E))} \cong R[It] = \mathcal{R}(I).$$

In other words, the theory of Rees algebras of ideals can be thought of as the rank one case of the more general theory of Rees algebras of torsion-free modules with a rank.

Definition 2.4.4 Let R be a Noetherian ring, E a finite R -module with rank. By analogy with the case of ideals, the powers of E are defined as

$$E^j := [\mathcal{R}(E)]_j.$$

Unlike in the case of ideals, $E^j \not\cong E^{j+1}$. As a consequence, there is no module analogue for the associated graded ring of an ideal. Hence, as we will see in Chapter 4, many techniques used to study Rees algebras of ideals cannot be applied in order to study Rees algebras of modules. On the other hand, we can still define the special fiber ring.

Definition 2.4.5 Assume that (R, \mathfrak{m}, k) is local. Let E be a finite R -module with rank. The *special fiber ring*, or *fiber cone*, of E is

$$\mathcal{F}(E) := \frac{\mathcal{R}(E)}{\mathfrak{m} \mathcal{R}(E)} \cong \mathcal{R}(E) \otimes_R k.$$

The Krull dimension of $\mathcal{F}(E)$ is called the *analytic spread* of E , and is denoted by $\ell(E)$.

The following proposition explains the relationships between the dimension of the Rees algebra $\mathcal{R}(E)$, the analytic spread $\ell(E)$, the rank of E , and the dimension of the ring. We refer the reader to [65, 2.2 and 2.3] for a proof.

Proposition 2.4.6 *Let R be a Noetherian ring of dimension d E a finite R -module with $\text{rank}(E) = e$. Then:*

- (a) $\dim \mathcal{R}(E) = d + e = d + \text{ht } \mathcal{R}(E)_+$.
- (b) *If R is local and $d > 0$, then $e \leq \ell(E) \leq d + e - 1$.*

2.5 Minimal reductions

Many properties of Rees algebras can be studied through the notion of *minimal reductions*.

Definition 2.5.1 Let R be a Noetherian ring, E a finite R -module.

- (a) A *reduction* of E is a submodule $U \subseteq E$ so that $E^{r+1} = UE^r$ for some $r \geq 0$. The least such r is called the *reduction number of E with respect to U* and is denoted by $r_U(E)$.
- (b) A *minimal reduction* of E is a reduction U of E which is minimal with respect to set inclusion. The *reduction number of E* is

$$r(E) = \min \{r_U(E) \mid U \text{ is a minimal reduction of } E\}.$$

Reductions of ideals were first introduced by Northcott and Rees in [51] in order to study integral dependence of ideals, and turn out to be extremely useful in order to study the asymptotic growth of the powers of a given ideal. In fact, a subideal J of I is a reducton of I if and only if $\mathcal{R}(I)$ is integral over the subalgebra generated

by J . Moreover, $I^{j+1} = JI^j$ for all $j \geq r_J(I)$. The notion of reduction was later extended to modules by Rees [56], and several authors contributed to extend many results known for reductions of ideals to the case of modules. We refer the interested reader to the book by Swanson and Huneke [34] for a detailed treatment of the topic, as well as for applications to multiplicity theory (see also [16, 64, 70]).

In order to study blowup algebras, one is interested mostly in reductions that are minimal, essentially because of the following result. A proof in the case of ideals can be found in [34, 8.3.7]. The proof for modules is similar, *mutatis mutandis*.

Proposition 2.5.2 *Let (R, \mathfrak{m}, k) be a Noetherian local ring, E a finite R -module. Then:*

- (a) *For any reduction U of E , $\mu(U) \geq \ell(E)$.*
- (b) *If k is infinite, minimal reductions of E always exist, and all have the same minimal number of generators, equal to $\ell(E)$.*
- (c) *If k is infinite, any submodule U generated by $\ell(E)$ general elements in E is a minimal reduction of E , with $r_U(E) = r(E)$.*

2.6 Generic Bourbaki ideals

In this section we recall the construction of *generic Bourbaki ideals*, and review their main properties. Generic Bourbaki ideals were introduced by Simis, Ulrich and Vasconcelos in [65], in order to study the Cohen-Macaulay property of the Rees algebra of a module. Recall that a standard graded ring S , whose degree 0 component S_0 is a Noetherian local ring, is Cohen-Macaulay if and only if $\text{depth } S_{\mathfrak{M}} = \dim S_{\mathfrak{M}}$, where \mathfrak{M} is the unique homogeneous maximal ideal of S . It is called almost Cohen-Macaulay if $\text{depth } S_{\mathfrak{M}} \geq \dim S_{\mathfrak{M}} - 1$.

In the case when S is the Rees algebra of an ideal, its Cohen-Macaulay property is usually studied by means of the associated graded ring, as we will explain in Section 4.1.1. However, since it is not possible to define the associated graded ring of a

module, a completely different strategy is needed in order to investigate the Cohen-Macaulayness of Rees algebras of modules. Generic Bourbaki ideals allow to reduce the problem to the case of ideals.

The intuitive idea behind the construction is the following. Starting from an R -module E , one wishes to find a sufficiently general sequence of elements x_1, \dots, x_s inside E , so that $E/(x_1, \dots, x_s)$ is a torsion-free module of rank one, hence isomorphic to an ideal I (of positive grade). Furthermore, one would like the Rees algebras of E and I to share similar properties, and in particular $\mathcal{R}(E)$ should be Cohen-Macaulay if and only if $\mathcal{R}(I)$ is. As it turns out, this idea becomes effective only if one first passes to a so called *generic extension* of the ring.

We next proceed to state the formal definition and main properties of generic Bourbaki ideals. We refer the reader to [65] for the proofs of the results mentioned in the rest of this section.

Notation 2.6.1 ([65, 3.3]). Let R be a Noetherian ring, $E = Ra_1 + \dots + Ra_n$ a finite R -module with $\text{rank } E = e > 0$. Let $Z = \{Z_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq e-1\}$ be a set of indeterminates. Let $R' := R[Z]$ and $E' := E \otimes_R R'$. For $1 \leq j \leq e-1$, denote

$$x_j = \sum_{i=1}^n Z_{ij} a_i \in E' \text{ and } F' = \sum_{j=1}^{e-1} R' x_j.$$

If R is local with maximal ideal \mathfrak{m} , let $R'' = R(Z) = R[Z]_{\mathfrak{m}R[Z]}$ and similarly denote $E'' = E \otimes_R R''$, $F'' = F' \otimes_{R'} R''$.

Theorem 2.6.2 ([65, 3.2]). Let R be a Noetherian ring, E a finite R -module with $\text{rank } E = e > 0$. Assume that E is torsion-free and that $E_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth } R_{\mathfrak{p}} \leq 1$. Then, for R' , E' and F' as in Notation 2.6.1, F' is a free R' -module of rank $e-1$ and E'/F' is isomorphic to an R' -ideal J with $\text{grade } J > 0$.

The construction just described associates to a given R -module E an ideal J in a polynomial ring R' over R . However, the dimension of R' is larger than that of R , and if R was local R' certainly is not. This may impose some limitations to the

applicability of the result. For this reason, one takes a further step, and constructs an ideal which lives in a local ring.

Theorem and Definition 2.6.3 ([65, 3.2 and 3.3]). *Let R be a Noetherian local ring, and E a finite R -module with $\text{rank } E = e > 0$. With the assumptions of Theorem 2.6.2, E''/F'' is isomorphic to an R'' -ideal I , called a generic Bourbaki ideal of E . Moreover, if K is another ideal constructed this way using variables Y , then the ideals generated by I and K in $S = R(Z, Y)$ coincide up to multiplication by a unit in $\text{Quot}(S)$, and are equal whenever I and K have grade at least 2.*

Notice that formation of Rees algebras of finite modules commutes with flat extensions (see [16, 1.3]). In particular, $\mathcal{R}(E') \cong \mathcal{R}(E) \otimes_R R'$ and $\mathcal{R}(E'') \cong \mathcal{R}(E) \otimes_R R''$.

We are now ready to state the main theorem about generic Bourbaki ideals.

Theorem 2.6.4 ([65, 3.5 and 3.8]). *In the setting of Notation 2.6.1 and with the assumptions of Theorems 2.6.2 and 2.6.3, the following statements hold.*

- (a) $\mathcal{R}(E)$ is Cohen-Macaulay if and only if $\mathcal{R}(I)$ is Cohen-Macaulay.
- (b) If $\text{grade } \mathcal{R}(E)_+ \geq e$, then $\mathcal{R}(J) \cong \mathcal{R}(E)/(F')$.
- (c) E is of linear type and $\text{grade } \mathcal{R}(E)_+ \geq e$ if and only if I is of linear type, if and only if J is of linear type.

Moreover, if any of the conditions (a) or (c) hold, then $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$ and x_1, \dots, x_{e-1} of F form a regular sequence on $\mathcal{R}(E)$.

Notice that the condition $\text{grade } \mathcal{R}(E)_+ \geq e$ in 2.6.4(b) or (c) is always satisfied if $\mathcal{R}(E)$ is Cohen-Macaulay, by Proposition 2.4.6(a).

Thanks to Theorem 2.6.4, questions about the linear type property of E or the Cohen-Macaulayness of $\mathcal{R}(E)$ can be reduced to the case of ideals. However, this often becomes effective only as long as the equivalent conditions of the following technical result are satisfied. Condition (c) means that $\mathcal{R}(E)$ is a *deformation* of $\mathcal{R}(I)$, or equivalently that $\mathcal{R}(I)$ is a *specialization* of $\mathcal{R}(E)$.

Theorem 2.6.5 ([65, 3.11]). *Let R be a Noetherian ring, E a finite R -module with $\text{rank } E = e > 0$. Let $0 \rightarrow F \rightarrow E \rightarrow I \rightarrow 0$ be an exact sequence where F is a free R -module with basis x_1, \dots, x_{e-1} and I is an R -ideal. The following are equivalent.*

- (a) $\mathcal{R}(E)/(F)$ is R -torsion free.
- (b) $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$.
- (c) $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$ and x_1, \dots, x_{e-1} of F form a regular sequence on $\mathcal{R}(E)$.

Moreover, if I is of linear type, then so is E and the equivalent conditions above hold.

We recall that a finite module E is called *orientable* if E has a rank, $\text{rank}(E) = e > 0$, and $(\bigwedge^e E)^{**} \cong R$, where $\bigwedge^e E$ is the e -th exterior power of E and $(-)^*$ denotes the functor $\text{Hom}_R(-, R)$. This condition is satisfied, for instance, by free modules of finite rank, or modules of finite projective dimension. Moreover, if R is a UFD, then every finite R -module is orientable.

Orientable modules play a distinguished role in the theory of generic Bourbaki ideal. In fact, one has the following result (see [65, Proposition 3.2, Remark 3.4 and their proofs]).

Proposition 2.6.6 *Let R be a Noetherian local ring. In the situation of Theorem 2.6.2, the ideal J can be chosen to have grade at least 2 if and only if J is orientable, if and only if E is orientable. Moreover, if $\text{grade } J \geq 3$, then $E \cong R^{e-1} \oplus L$ for some R -ideal L , in which case $LR' \cong J$.*

The next proposition examines the connection between the analytic spread and reduction number of E and those of its generic Bourbaki ideal I .

Proposition 2.6.7 ([65, 3.10]). *Let R be a Noetherian local ring with $\dim R = d$, E a finite R -module with $\text{rank } E = e$. If E admits a generic Bourbaki ideal I , then $\ell(I) = \ell(E) - e + 1$. Moreover, $r(I) \leq r(E)$ whenever the residue field of R is infinite.*

In the rest of this work, we often consider modules satisfying certain constraints on the minimal number of generators of their localizations. These technical conditions were first introduced by Artin and Nagata [3], and allow to exploit information from the residual intersections of a generic Bourbaki ideal. We will give the necessary background on residual intersections in the next section.

Definition 2.6.8 Let R be a Noetherian ring. A module E of rank e is said to satisfy *condition G_s* if $\mu(E_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} - e + 1$ for every $\mathfrak{p} \in \text{Spec}(R)$ with $1 \leq \dim R_{\mathfrak{p}} \leq s - 1$. If the same condition holds for all s , then E is said to satisfy G_{∞} .

Equivalently, E is G_s if and only if $\text{ht Fitt}_i(E) \geq i - e + 2$ for $e \leq i \leq e + s - 2$, where $\text{Fitt}_i(E)$ is the i -th Fitting ideal of E . This makes this condition easy to check on a computer algebra system like Macaulay2 [21], once a presentation of E is known.

Notice that for an ideal I , the G_s condition can be restated as $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq s - 1$. Moreover, if R has dimension d , then I is G_{∞} if and only if it is G_{d+1} .

Notice also that, if E satisfies G_2 , then $E_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth } R_{\mathfrak{p}} \leq 1$. In particular, by Theorem 2.6.2 it follows that every torsion-free R -module satisfying G_s for some $s \geq 2$ admits a generic Bourbaki ideal. Moreover, the G_s property transfers from a module to its generic Bourbaki ideal.

Proposition 2.6.9 ([65, 3.1]). *Let R be a Noetherian local ring, and E a finite R -module of positive rank e . Under the assumptions of Theorem 2.6.2, if E satisfies G_s then so does J . In particular, a generic Bourbaki ideal I of E is G_s whenever E is.*

We conclude this section by recalling that most of the properties of generic Bourbaki ideals are proved by induction on the rank. In fact, the construction described in Notation 2.6.1 and Definition 2.6.3 can be thought of as the $(e - 1)$ -st step of an iterative construction, where at each step the ring is extended by adjoining n generic variables. This will be relevant for the proofs of some technical results of Chapter 4

and Chapter 5, and even more in Chapter 6. In particular, we will need the following theorem (see [65, 3.6 and 3.8]).

Theorem 2.6.10 *Let R be a Noetherian ring, $E = Ra_1 + \cdots + Ra_n$ a finite R -module with positive rank. Let Z_1, \dots, Z_n be indeterminates, $\tilde{R} := R[Z_1, \dots, Z_n]$, $\tilde{E} := E \otimes_R \tilde{R}$ and $x := \sum_{i=1}^n Z_i a_i \in \tilde{E}$. Assume that $\text{rank } E = e \geq 2$. Then*

- (a) *x is regular on $\mathcal{R}(\tilde{E})$.*
- (b) *Let $\overline{\mathcal{R}} := \mathcal{R}(\tilde{E})/(x)$. The kernel of the natural epimorphism $\pi: \overline{\mathcal{R}} \twoheadrightarrow \mathcal{R}(\tilde{E}/\tilde{R}x)$ is $\ker(\pi) = H_{\overline{\mathcal{R}}_+}^0(\overline{\mathcal{R}})$.*
- (c) *If $\text{grade } \mathcal{R}(E)_+ \geq 2$, then π is an isomorphism.*

2.7 Residual intersections

The most general definition of residual intersections of an ideal was formalized by Huneke and Ulrich in [36]. The geometric idea behind this concept can be found already in previous work by Artin and Nagata [3], and can be informally explained as follows. Let X be closed subscheme of a given algebraic variety V . A *geometric residual intersection* of X is a closed subscheme Z of the same variety V , such that $\text{codim } Z \geq \text{codim } X$ and $X \cup Z = Y$, for some closed subscheme Y defined by $\text{codim } Z$ equations and such that $X \subseteq Y$. In other words, Z is ‘how much one needs to add to X ’ so that $X \cup Z$ is the (scheme theoretic) intersection of $\text{codim } Z$ hyper-surfaces contained in X , which explains the terminology. The algebraic definition is the following.

Definition 2.7.1 Let R be a Cohen-Macaulay local ring, I an R -ideal with $\text{ht } I = g$, and $s \geq g$ an integer. Let K be a proper ideal.

- (a) K is an *s -residual intersection* of I if there exists an ideal $J \subseteq I$ such that $K = J :_R I$, $\mu(J) \leq s$ and $\text{ht } K \geq s$.

- (b) An s -residual intersection K of I is called a *geometric s -residual intersection* if in addition $\text{ht}(I + K) \geq s + 1$.

Equivalently, if $J \subseteq I$ satisfies $\mu(J) \leq s$, then $J: {}_R I$ is an s -residual intersection of I if and only if $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} with $\dim R_{\mathfrak{p}} \leq s - 1$. It is a geometric residual intersection if in addition $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ also for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} = s$. That is, an s -residual intersection of I can be thought of as an approximation of I locally up to codimension $s - 1$, and up to codimension s if it is a geometric s -residual intersection.

Notice that, if $J: {}_R I$ is an s -residual intersection of I , then $\mu(I_{\mathfrak{p}}) \leq s$ for any prime \mathfrak{p} with $\dim R_{\mathfrak{p}} \leq s - 1$. Conversely, assume that the slightly stronger G_s condition holds, that is, that $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq s - 1$ (see Definition 2.6.8). Then, s -residual intersections of I always exist, as the following result shows.

Theorem 2.7.2 ([67, 1.5 and 1.6(a)]) *Let R be a Noetherian local ring, s an integer, and let $J \subseteq I$ be R -ideals with $\text{ht}(J: {}_R I) \geq s$. Assume that I satisfies G_s , and that J is generated (possibly not minimally) by s elements. Then:*

- (a) *There exists a generating set b_1, \dots, b_s of J such that $\text{ht}((b_1, \dots, b_i): {}_R I) \geq i$ for $0 \leq i \leq s$ and $\text{ht}(I + (b_1, \dots, b_i): {}_R I) \geq i + 1$ for $0 \leq i \leq s + 1$.*
- (b) *Assume that the residue field of I is infinite. Then, J admits a generating sequence b_1, \dots, b_s so that the properties in (a) are satisfied for any subsequence b_{k_1}, \dots, b_{k_i} of b_1, \dots, b_s .*

Theorem 2.7.2 also shows that, when I satisfies G_s , every s -residual intersection $J: {}_R I$ of I can be filtered via i -residual intersections $J_i: {}_R I$ for all $0 \leq i \leq s$, for some subideals $J_i \subseteq J$. Moreover, these i -residual intersections are geometric for all $0 \leq i \leq s - 1$. In other words, ideals satisfying G_s admit s -residual intersection that have a ‘nice’ generating set. Theorem 2.7.7 in the next subsection gives a condition for which the generating set b_1, \dots, b_s for J of Theorem 2.7.2 allows to write K iteratively.

2.7.1 Artin-Nagata properties

When R is a Gorenstein local ring and I is an unmixed ideal of height g , a g -residual intersection of I is an ideal $K = J :_R I$ such that J is a complete intersection. In other words, K is a so called *link* of I , or is said to be *linked* to I . In fact, the theory of residual intersections is a generalization of *linkage*, that was first introduced in geometric terms by Peskine and Szpiro in [52], and later formalized in algebraic terms by Huneke and Ulrich in [35].

One of the main themes in linkage theory is to investigate what properties of I are preserved when passing to a link K of I . Peskine and Szpiro proved that, if R is Gorenstein and R/I is Cohen-Macaulay, then R/K is Cohen-Macaulay [52]. This was incorrectly ‘proved’ by Artin and Nagata in [3, 2.1] also for an s -residual intersection K of I with $s > g$. Later, Huneke found a counterexample [33, 3.3], and showed that R/K is Cohen-Macaulay for every s -residual intersection K of I in the case when I is *strongly Cohen-Macaulay* [33, 3.1].

Definition 2.7.3 Let R be a Cohen-Macaulay local ring, $I = (x_1, \dots, x_n)$ a proper ideal. For each $j \geq 0$, let $H_j := H_j(x_1, \dots, x_n; R)$ denote the j -th Koszul homology module.

- (a) I is *strongly Cohen-Macaulay* if all the H_j are Cohen-Macaulay R -modules.
- (b) I satisfies *sliding depth* if the Koszul homology modules H_j satisfy $\text{depth } H_j \geq \dim R - n + j$ for all j .

Although the Koszul homologies a priori depend on the choice of the generators for I , the strongly Cohen-Macaulay and sliding depth properties only depend on the ideal I . Also, a strongly Cohen-Macaulay ideal I clearly satisfies sliding depth, and is such that $R/I \cong H_0$ is Cohen-Macaulay. Moreover, a result of Herzog, Vasconcelos and Villarreal [25, 3.3] shows that if I satisfies G_∞ and sliding depth, then, for every s -residual intersection K of I , R/K is Cohen-Macaulay.

The search for a minimal sufficient condition for the Cohen-Macaulayness of residual intersections motivated the following definition.

Definition 2.7.4 ([67, 1.1]) Let R be a Cohen-Macaulay local ring, I an R -ideal with $\text{ht } I = g$, and $s \geq g$ an integer. I is said to satisfy the *Artin-Nagata property* AN_s if for all $g \leq i \leq s$ and every i -residual intersection K of I , R/K is Cohen-Macaulay. If the same property holds for every geometric i -residual intersection, then I is said to satisfy AN_s^- .

The Artin-Nagata condition AN_s is satisfied (at least when R is Gorenstein) by a large class of ideals, including perfect ideals of height 2, perfect Gorenstein ideals of height 3, complete intersections, and in general any *licci* ideal that satisfies G_s . We recall that I is *licci* if it is in the *linkage class* of a complete intersection ideal K , i.e. there exist finitely many ideals $K_0 = I, K_1, \dots, K_n = K$, such that K is a complete intersection and K_i is linked to K_{i+1} for all i . Notice that *licci* ideals in a Cohen-Macaulay local ring are always strongly Cohen-Macaulay [31, 1.14].

In fact, a weaker condition in order for I to satisfy AN_s is the following.

Theorem 2.7.5 ([67, 2.9]). *Let R be a local Gorenstein ring with $\dim(R) = d$, I an R -ideal with $\text{ht } I = g$. Assume that I satisfies G_s for some $s \geq g$. If $\text{depth } I^j \geq d - g - j + 2$ for $1 \leq j \leq s - g + 1$, then I satisfies AN_s .*

The assumption on the depths of powers of I in Theorem 2.7.5 are always satisfied if the Koszul homologies H_j of I are Cohen-Macaulay for $0 \leq j \leq s - g$ [67, 2.10], and in particular if I is strongly Cohen-Macaulay. The two conditions are not equivalent [67, 2.11]. However, if I is a Cohen-Macaulay ideal that satisfies G_∞ and the depth assumptions of Theorem 2.7.5 with $s = \mu(I) - 2$, then I is strongly Cohen-Macaulay (see [67, 2.13]).

The following result explains the connection between the sliding depth and Artin-Nagata properties.

Theorem 2.7.6 *Let R be a Cohen-Macaulay local ring of dimension d , I an R -ideal with $\text{ht } I = g$. Assume that I satisfies G_s for some $s \geq g$.*

- (a) ([25, 3.4] and [67, 1.7]). If I is sliding depth, then I satisfies AN_s .
- (b) ([67, 1.8]). If $s = \mu(I)$ and I is AN_{s-1}^- , then I satisfies sliding depth.
- (c) ([67, 1.9]) If $s = d - 1$ and I is AN_{s-1}^- , then I satisfies AN_k for all k . In this case, I satisfies sliding depth whenever it is G_∞ .

We will return to the relationship between the Artin-Nagata properties and the sliding depth condition in Section 3.1.1, where we will connect both of them with the linear type property of an ideal.

A very interesting feature of ideals satisfying the Artin-Nagata condition is that their residual intersections can be conveniently expressed iteratively, as successive links by one element.

Theorem 2.7.7 ([67, 1.5 and 1.6(a)]) *Let R be a Cohen-Macaulay local ring, $I \neq 0$ a proper R -ideal. Assume that I satisfies G_s and AN_{s-1} for some integer $s \geq \text{ht}(I) > 0$. Let $K = J :_R I$ be an s -residual intersection of I , and let b_1, \dots, b_s be a generating set of J as in Theorem 2.7.2. For $1 \leq i \leq s$, let $K_i := (b_1, \dots, b_i) :_R I$. Then:*

- (a) For all $i \leq s$, K_i is unmixed of height i .
- (b) For $1 \leq i \leq s - 1$, $K_i = (b_1, \dots, b_i) :_R (b_{i+1})$ and $I \cap K_i = (b_1, \dots, b_i)$.
- (c) (**Reduction to linkage**) For $1 \leq i \leq s - 1$, let $'$ denote images in the ring $R' := R/K_i$. Then, $K'_{i+1} = (b'_{i+1}) : I'$ and b'_{i+1} is regular on R' .

This iteration property allows to prove statements about residual intersections by induction. This turns out to be particularly useful in order to study the Cohen-Macaulay property of Rees algebras of ideals (see Section 4.1.1), when combined with the following result (see [67, 1.6], [39, 2.7], or [54, 3.1]).

Proposition 2.7.8 *Let R be a Cohen-Macaulay local ring with infinite residue field. Let I an ideal of positive height with $\ell(I) = \ell$, and let J be an ideal generated by*

general elements b_1, \dots, b_ℓ in I . Assume that I is G_ℓ . Then, $K = J :_R I$ is an ℓ -residual intersection of I . Moreover, for $1 \leq i \leq \ell$, $(b_1, \dots, b_i) :_R I$ is an i -residual intersection of I , which is geometric for $1 \leq i \leq \ell - 1$.

We conclude this subsection by recalling the behavior of Artin-Nagata properties under localization.

Proposition 2.7.9 (see [67, 1.10]). *Let R be a Cohen-Macaulay local ring, I an R -ideal and $\mathfrak{p} \in V(I)$.*

- (a) *If I satisfies G_{s+1} and AN_s^- , then $I_{\mathfrak{p}}$ satisfies G_{s+1} and AN_s^- .*
- (b) *I satisfies G_s and AN_s , then $I_{\mathfrak{p}}$ satisfies G_s and AN_s .*

2.7.2 Residually S_2 ideals

Although many classes of ideals satisfy the Artin-Nagata property AN_s , the depth conditions of Theorem 2.7.5 are difficult to check. For several applications, it is then convenient to have a weaker notion, which still guarantees good properties of residual intersections.

We recall that a finite R -module M satisfies *Serre's condition S_2* if

$$\text{depth } M_{\mathfrak{p}} \geq \min \{ 2, \dim M_{\mathfrak{p}} \}.$$

Definition 2.7.10 Let R be a Noetherian ring, I an R -ideal with $\text{ht}(I) = g$, and $s \geq g$ an integer.

- (a) I is said to be *s -residually S_2* if for all $g \leq i \leq s$ and every i -residual intersection K of I , R/K satisfies S_2 .
- (b) I is said to be *weakly s -residually S_2* if the same condition holds for every geometric i -residual intersection of I .

The notion of residually S_2 ideals was introduced by Chardin, Eisenbud and Ulrich in [9], and used in order to study the Hilbert function of certain homogeneous ideals.

In fact, many results known for ideals satisfying the Artin-Nagata condition can be generalized to residually S_2 ideals.

Theorem 2.7.11 (see [9, 3.4], [12, 2.4]) *Let R be a Cohen-Macaulay local ring of dimension d , I an R -ideal with $\text{ht } I = g > 0$, and let s be an integer with $s \geq g$. Assume that I is G_s and weakly $(s-1)$ -residually S_2 . Let $K = J :_R I$ be an s -residual intersection of I , and let b_1, \dots, b_s be a generating set of J as in Theorem 2.7.2. Then*

- (a) *For all $i \leq s$, K_i is unmixed of height i .*
- (b) *For $1 \leq i \leq s-1$, $K_i = (b_1, \dots, b_i) :_R (b_{i+1})$.*
- (c) *For $1 \leq i \leq s-1$, let $'$ denote images in the ring $R' := R/K_i$. Then, $K'_{i+1} = (b'_{i+1}) : I'$ and b'_{i+1} is regular on R' .*

The main idea underlying the proof of Theorem 2.7.5 is that the Cohen-Macaulay property of a residual intersection K of I is encoded in the canonical module of R/K (see [67, 2.1 and 2.3] for details). A deeper investigation on the canonical module of residual intersections of I led to the following sufficient condition for an ideal to be s -residually S_2 .

Theorem 2.7.12 ([9, 4.1]) *Let R be a local Gorenstein ring with $\dim(R) = d$, I an R -ideal with $\text{ht}(I) = g$. Assume that I satisfies G_s for some $s \geq g$. If $\text{Ext}^{g+j-1}(I^j, R) = 0$ for $1 \leq j \leq s-g+1$, then I is s -residually S_2 .*

Notice that, since R is Gorenstein, the depth assumptions in the statement of Theorem 2.7.5 imply that $\text{Ext}^i(I^j, R) = 0$ for all $1 \leq j \leq s-g+1$ and all $i \leq g-j+1$. In particular, Theorem 2.7.12 is satisfied in the most restrictive setting of Theorem 2.7.5.

3. MODULES OF LINEAR TYPE

In this chapter, we aim to provide sufficient conditions for a module to be of linear type. Using generic Bourbaki ideals, we reduce the problem to that of finding conditions for an ideal to be of linear type. Our main result is Theorem 3.2.4 and will be crucial for most of the results of Chapter 4. Another sufficient condition will be given later in Theorem 4.3.2.

The techniques we use at the level of ideals are based on the theory of residual intersections, as recalled in Section 2.7. We refer the reader to Chapter 2 for the necessary background on Rees algebras of ideals and modules and generic Bourbaki ideals.

3.1 Background

The problem of determining the defining ideal of Rees algebras has been studied for decades by several authors. In general, this is a difficult problem, which only becomes treatable if enough information can be extracted from the presentation matrix of the given ideal or module. We will discuss some general techniques in Chapter 5. Here instead, we focus on ideals and modules of linear type, that is, such that their Rees algebra coincides with their symmetric algebra. We start with the following observation.

Remark 3.1.1 *Let (R, \mathfrak{m}, k) be a Noetherian local ring, E a finite R -module with $\text{rank } E = e$. If E is of linear type, then $\mu(E) = \ell(E)$.*

Indeed, if E is of linear type, then

$$\mathcal{F}(E) = \mathcal{R}(E) \otimes_R k \cong \mathcal{S}(E) \otimes_R k \cong \mathcal{S}(E \otimes_R k).$$

But $E \otimes_R k \cong k^n$, where $n = \mu(E)$, so $\mathcal{F}(E) \cong \mathcal{S}(E \otimes_R k) \cong k[Y_1, \dots, Y_n]$. Hence, by computing dimensions, it follows that $n = \mu(E) = \ell(E)$, as claimed.

In particular, this means that linear type modules have no proper reductions. Moreover, since by construction the linear type property localizes, the argument above implies that $\mu(E_{\mathfrak{p}}) = \ell(E_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} + e - 1$ for all $\mathfrak{p} \in \text{Spec}(R)$, where the latter inequality holds by Proposition 2.4.6. In other words, recalling the G_{∞} condition of Definition 2.6.8, we have just proved the following statement.

Remark 3.1.2 *Let R be a Noetherian ring, E a finite R -module with $\text{rank} E = e$. If E is of linear type, then E satisfies G_{∞} .*

The converse is not true. In fact, if $R = k[x, y, z, w]$ and $I = (xy, xw, yz, zw)$, then I is G_{∞} but not of linear type. Indeed, the defining ideal $\mathcal{J} \subseteq R[T_1, \dots, T_4]$ of $\mathcal{R}(I)$ includes a non-linear generator $T_1T_4 - T_2T_3$, so it properly contains the defining ideal \mathcal{L} of $\mathcal{S}(I)$.

Nevertheless, one has the following important result, due to Herzog, Simis and Vasconcelos (see [23, 6.1]). We refer the reader to Definition 2.7.3 for the sliding depth property of an ideal.

Theorem 3.1.3 (Herzog-Simis-Vasconcelos) *Let R be a Cohen-Macaulay local ring, I an R -ideal with $\text{ht}(I) > 0$. If I satisfies G_{∞} and sliding depth, then I is of linear type. Moreover, $\mathcal{R}(I)$ is Cohen-Macaulay.*

The result was obtained by means of the so called *approximation complexes*, which approximate a free resolution of the symmetric algebra. A detailed discussion of the topic would be out of the scope of this thesis. We refer the interested reader to the original papers [22, 23]. What will be relevant to us is that the assumptions of Theorem 3.1.3 can be weakened if one is only interested to the linear type property of I . This is possible thanks to the notion of *d-sequence*.

Definition 3.1.4 A sequence of elements a_1, \dots, a_s in I is called a *d-sequence relative to I* if $[(a_1, \dots, a_i) : (a_{i+1})] \cap I = (a_1, \dots, a_{i+1})$ for $0 \leq i \leq s - 1$. The sequence is called a *d-sequence* if furthermore the a_i generate I .

In fact, if R is a Cohen-Macaulay local ring with infinite residue field and I satisfies G_∞ and sliding depth, then I is generated by a d-sequence (see [22, 2.5] and [23, 4.1]). This condition guarantees that I is of linear type.

Theorem 3.1.5 (Huneke) ([29, 3.1]). *Let R be a Noetherian ring, and let I be an R -ideal generated by a d-sequence. Then, I is of linear type.*

We next discuss the linear type property of an ideal by examining its residual intersections.

3.1.1 Residual intersections and ideals of linear type

We saw in Section 2.7.1 that the Artin-Nagata properties of an ideal I are connected with its sliding depth property. Theorem 3.1.3 then suggests that, if we also assume that I is G_∞ , the Artin-Nagata properties could provide a sufficient condition for I to be of linear type, while guaranteeing at the same time that the Rees algebra $\mathcal{R}(I)$ is Cohen-Macaulay. In fact, we show that the G_∞ condition, although necessary in order for I to be of linear type, is not required in order to prove that I is linear type.

Our first result in this direction is a slight improvement of Theorem 2.7.6(b). The proof really follows from the paper of Ulrich [67], but we write it here for lack of a specific reference.

Theorem 3.1.6 *Let R be a Gorenstein local ring of dimension d . Let I be an R -ideal with $\ell(I) = \ell$. If I is $G_{\ell+1}$ and $AN_{\ell-1}^-$, then I satisfies sliding depth.*

Proof. Without loss of generality, we may assume that the residue field of R is infinite. Let J be a minimal reduction of I generated by ℓ general elements. Since I

satisfies $G_{\ell+1}$, then $\text{ht}(J: {}_R I) \geq \ell + 1$. This is clear if $J = I$. If $J \subsetneq I$, then $J: {}_R I$ is a geometric ℓ -residual intersection of I by Theorem 2.7.2. Hence, since I and J have the same radical, $\text{ht}(J: {}_R I) \geq \ell + 1$, as claimed.

Now, if $J: {}_R I$ is a proper ideal, it is an ℓ -residual intersection of I . But then, since by assumption I is $G_{\ell+1}$ and AN_{ℓ}^- , by Theorem 2.7.7 it follows that $\text{ht}(J: {}_R I) = \ell$, a contradiction. So, it must be that $I = J$. Hence, $\ell = \mu(I)$, so that I satisfies sliding depth by Theorem 2.7.6(b). ■

From the proof it follows that, if I satisfies the assumptions of Theorem 3.1.6, then I has no proper reductions. Hence, for all $\mathfrak{p} \in \text{Spec}(R)$ one has $\mu(I_{\mathfrak{p}}) = \ell(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$. That is, I satisfies G_{∞} . As a consequence, we have the following result.

Theorem 3.1.7 *Let R be a Gorenstein local ring of dimension d . Let I be an R -ideal with $\text{ht } I = g \geq 1$ and $\ell(I) = \ell$. Assume that I is $G_{\ell+1}$ and that $\text{depth } I^j \geq d - g - j + 2$ for $1 \leq j \leq \ell - g$. Then, I is of linear type and $\mathcal{R}(I)$ is Cohen-Macaulay.*

Proof. By Theorem 2.7.5, the assumption on the depths of powers of I imply that I satisfies $AN_{\ell-1}$. Hence, by Theorem 3.1.6 and its proof, we deduce that I is sliding depth and G_{∞} . Hence, by Theorem 3.1.3 it follows that I is of linear type and $\mathcal{R}(I)$ is Cohen-Macaulay. ■

If one is only interested to the linear type property of I , by Theorem 3.1.5 it is enough to provide conditions so that I is generated by a d-sequence. The residually S_2 property turns out to be helpful to this scope. This follows from work of Chardin, Eisenbud and Ulrich [9], but we include the proof here for the reader's convenience, due to lack of a specific reference. We start by recalling a sufficient condition for I to be generated by a d-sequence.

Theorem 3.1.8 (*[9, 3.6]*) *Let R be a Cohen-Macaulay local ring of dimension d , I an R -ideal with $\text{ht } I = g > 0$. Assume that I is G_s for some integer s with $s \geq g$. Let $J \subseteq I$ be an R -ideal such that $K = J: {}_R I$ and $\text{ht } K \geq s \geq \mu(J)$, and let b_1, \dots, b_s*

be generators for J as in Theorem 2.7.2. If I is weakly $(s - 2)$ -residually S_2 , then b_1, \dots, b_s is a d -sequence relative to I .

As a consequence, we can then identify a class of ideals of linear type which do not necessarily satisfy sliding depth, nor any Artin-Nagata condition.

Theorem 3.1.9 *Let R be a local Gorenstein ring, $d = \dim R$, I an R -ideal with $\text{ht } I = g \geq 1$ and $\ell(I) = \ell$. Assume that I is $G_{\ell+1}$ and $(\ell - 1)$ -residually S_2 . Then, I is generated by a d -sequence.*

Proof. Without loss of generality, we may assume that the residue field of R is infinite. Let J be a minimal reduction of I generated by ℓ general elements. As in the proof of Theorem 3.1.6, the assumption that I is $G_{\ell+1}$ implies that $\text{ht}(J: {}_R I) \geq \ell + 1$, and that hence $I = J$. Also, by assumption I is $G_{\ell+1}$ and $(\ell - 1)$ -residually S_2 . Hence, by Theorem 3.1.8, I is generated by a d -sequence. ■

In particular, we have the following result.

Theorem 3.1.10 *Let R be a local Gorenstein ring, $d = \dim R$, I an R -ideal with $\text{ht } I = g \geq 1$ and $\ell(I) = \ell$. Assume that I is $G_{\ell+1}$ and that $\text{Ext}_R^{g+j-1}(I^j, R) = 0$ for $1 \leq j \leq \min\{\ell - g, d - g - 1\}$. Then, I is of linear type.*

Proof. The given vanishing conditions on the Ext modules in the statement imply that I is $(\ell - 1)$ -residually S_2 , thanks to Theorem 2.7.12. Hence, by Theorem 3.1.9 and its proof, I is generated by a d -sequence. Therefore, I is of linear type by Theorem 3.1.5. ■

Notice that, since R is Gorenstein, the assumptions of Theorem 3.1.10 are satisfied if the depth conditions of Theorem 3.1.7 hold. However, unlike Theorem 3.1.7, Theorem 3.1.10 does not give any information about the Cohen-Macaulay property of the Rees algebra $\mathcal{R}(I)$.

3.2 Residual intersections and modules of linear type

Our main goal in the rest of this chapter is to provide sufficient conditions for a module to be of linear type. In particular, we wish to recover the results of the previous section for ideals.

The task is challenging from a technical point of view, since some of the notions involved in the theorems above do not seem to have a well-understood module analogue. For instance, if one replaced ideals with modules in the definition of d -sequence, then for elements a_1, \dots, a_n in E the intersections $[(Ra_1 + \dots + Ra_i) :_R Ra_{i+1}] \cap E$ would not make sense anymore, since $E \cap R$ is not well-defined. Moreover, should one reasonably define an s -residual intersection of E as a colon ideal $(Ra_1 + \dots + Ra_s) :_R E$ for some $a_1, \dots, a_s \in E$, it is not clear whether good residual intersection properties would transfer from E to a generic Bourbaki ideal I of E , since a priori factoring out generic elements may not preserve these properties.

Nevertheless, one could hope to be able to generalize the statements of Theorem 3.1.7 and Theorem 3.1.10, as these only involve powers of the module E , which are well-defined as graded components of its Rees algebra $\mathcal{R}(E)$. In fact, under the stronger assumption that E satisfies G_∞ , a module version of Theorem 3.1.7 was given by Lin, using generic Bourbaki ideals.

Theorem 3.2.1 (*[46, 3.1]*). *Let R be a Gorenstein local ring of dimension d . Let E be a finite, orientable R -module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that E is G_∞ and that $\text{depth } E^j \geq d - j$ for $1 \leq j \leq \ell - g + 1$. Then, E is of linear type and $\mathcal{R}(E)$ is Cohen-Macaulay.*

Still using generic Bourbaki ideals as our main tool, in Theorem 4.3.2 we will be able to obtain the same conclusion while weakening the G_∞ condition to the less restrictive $G_{\ell-e+2}$. The proof is postponed to the next chapter, as it uses some technical results from Chapter 4. In Theorem 3.2.4 below we will instead extend Theorem 3.1.10 to modules.

3.2.1 Main result

Throughout the rest of this chapter, we consider the following situation.

Setting 3.2.2 Let R be a Noetherian local ring of dimension d . Let E be a finite R -module with $\text{rank}(E) = e > 0$ and analytic spread $\ell(E) = \ell$. For a fixed generating set a_1, \dots, a_n of E , let E' , E'' , x_1, \dots, x_{e-1} , F' and F'' be constructed as in Notation 2.6.1. Assume that E'/F' is isomorphic to an R' -ideal J as an R' -module, let $I = JR''$ be a generic Bourbaki ideal of E , and let $g = \text{ht } I$.

In the situation of Setting 3.2.2, our goal in this section is to generalize Theorem 3.1.10 to modules. That is, we wish to be able to deduce the linear type property of E from the vanishing of finitely many modules of the form $\text{Ext}_R^i(E^j, R)$, for finitely many given values of i and j .

Thanks to Theorem 2.6.4(c), it suffices to prove that a generic Bourbaki ideal I of such a module E satisfies the assumptions of Theorem 3.1.10. In particular, we aim to show that if $\text{Ext}_R^i(E^j, R) = 0$ for given values of i and j , then $\text{Ext}_R^i(I^j, R) = 0$ for the same values of i and j .

From the definition of generic Bourbaki ideal, one has the exact sequences

$$0 \rightarrow F' \rightarrow E' \rightarrow J \rightarrow 0$$

and

$$0 \rightarrow F'' \rightarrow E'' \rightarrow I \rightarrow 0.$$

These induce R -epimorphisms $(E')^j \twoheadrightarrow J^j$ and $(E'')^j \twoheadrightarrow I^j$, obtained as degree j components of the homogeneous epimorphisms $\mathcal{R}(E') \twoheadrightarrow \mathcal{R}(J)$ and $\mathcal{R}(E'') \twoheadrightarrow \mathcal{R}(I)$. Hence, for each j the augmented Koszul complexes $\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E'))$ and $\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E''))$ induce complexes

$$\mathbb{C}'_j: [\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E'))]_j \xrightarrow{\partial'_0} J^j \rightarrow 0$$

and

$$\mathbb{C}_j'': [\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E''))]_j \xrightarrow{\partial_0''} I^j \rightarrow 0.$$

The exactness of these complexes plays a crucial role in transferring assumptions from E^j to J^j or I^j respectively, as shown by the following lemma (see Lemma 4.2.2 for another instance where a similar phenomenon occurs).

Lemma 3.2.3 *Under the assumptions of Setting 3.2.2, let k and s be positive integers with $k \leq s - 2$. Assume that*

$$(i) \quad \text{Ext}_R^{j+1}(E^j, R) = 0 \text{ for } 1 \leq j \leq k.$$

(ii) *The complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all $\mathfrak{q} \in \text{Spec}(R')$ with $\text{depth } R'_{\mathfrak{q}} \leq s - 1$ and all $1 \leq j \leq k$.*

Then, $\text{Ext}_{R'}^{j+1}(J^j, R') = 0$ for $1 \leq j \leq k$.

Proof. Fix j with $1 \leq j \leq k$. For $i \leq j$, let C'_i , Z_i , B_i and H_i be the i th module, cycle, boundary and homology of the complex

$$\mathbb{C}'_j: [\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E'))]_j \xrightarrow{\partial_0'} J^j \rightarrow 0$$

respectively. By assumption (i) we know that $\text{Ext}_{R'}^{j+1-i}(C'_i, R') = 0$ for $0 \leq i \leq j$. Also, assumption (ii) implies that $\text{grade } H_i \geq s \geq k + 2$. Hence, $\text{Ext}_{R'}^n(H_i, R') = 0$ for all $n \leq k + 1$ and all $0 \leq i \leq j$. Now, by decreasing induction on $i \leq j$, we prove that $\text{Ext}_{R'}^{j+1-i}(B_{i-1}, R') = 0$ for $0 \leq i \leq j$. The assertion will then follow by the case $i = 0$.

Assume that $i = j$. Since $\text{Ext}_{R'}^n(H_j, R') = 0$ for $n = 0$ and $n = 1$, and $B_j = \text{im}(\partial_{j+1}') = 0$, it follows that $\text{Hom}_{R'}(Z_j, R') \cong \text{Hom}_{R'}(B_j, R') = 0$. Therefore, the long exact sequence of $\text{Ext}_{R'}^\bullet(-, R')$ induced by $0 \rightarrow Z_j \rightarrow C'_j \rightarrow B_{j-1} \rightarrow 0$ shows that $\text{Ext}_{R'}^1(B_{j-1}, R') = 0$ as well. Now, assume that $j \geq i + 1$ and $\text{Ext}_{R'}^{j-i}(B_i, R') = 0$. Since $\text{Ext}_{R'}^n(H_i, R') = 0$ for $n = j - i$ and $n = j - i + 1$, it follows that $\text{Ext}_{R'}^{j-i}(Z_i, R') \cong \text{Ext}_{R'}^{j-i}(B_i, R') = 0$. So, the long exact sequence of $\text{Ext}_{R'}^\bullet(-, R')$ induced by $0 \rightarrow Z_i \rightarrow C'_i \rightarrow B_{i-1} \rightarrow 0$ shows that $\text{Ext}_{R'}^{j+1-i}(B_{i-1}, R') = 0$, and the

proof is complete. ■

We can now state and prove our module version of Theorem 3.1.10.

Theorem 3.2.4 *Let R be a local Gorenstein ring of dimension d . Let E be a finite, torsion-free and orientable R -module with $\text{rank } E = e > 0$ and $\ell(E) = \ell$. Assume that E is $G_{\ell-e+2}$ and that $\text{Ext}_R^{j+1}(E^j, R) = 0$ for $1 \leq j \leq \min\{\ell - e - 1, d - 3\}$. Then, E is of linear type and E'/F' is isomorphic to an R' -ideal of linear type.*

Proof. Without loss of generality, we may assume that E is not free. Let $E = Ra_1 + \dots + Ra_n$. Since E is torsion-free, orientable and satisfies $G_{\ell-e+2}$, by Theorem 2.6.2, $E'/F' \cong J$ and $E''/F'' \cong I$, where I and J are ideals of height at least 2, satisfying $G_{\ell-e+2}$, i.e. $G_{\ell(I)+1}$ (see Proposition 2.6.9). Moreover, we may assume that $g \leq d - 1$. In fact, if $g = d$, then $\ell(I) = d$, and the G_∞ condition implies that I is a complete intersection. Therefore I satisfies G_∞ and sliding depth, whence it is of linear type by Theorem 3.1.3. Then, by Theorem 2.6.4(c) E and J are of linear type, and $\text{grade}(\mathcal{R}(E)_+) \geq e$.

If $e = 1$, then $R'' = R$ and $E \cong I$, an R -ideal of height $2 \leq g \leq d - 1$. In fact, it must be that $g = 2$. Otherwise, since R is Gorenstein, by assumption we would have $\text{Ext}_R^{g-1}(I^{g-2}, R) = 0$, contradicting the fact that $\text{grade } I^{g-2} = \text{grade } I = g$. Hence $g = 2$, and so I is of linear type by Theorem 3.1.10.

If $e \geq 2$, we proceed by induction on $d = \dim R' \geq g \geq 2$. If $d = 2$, then $g = d$ and we have already proved that E and J are of linear type in this case. So, we may assume that $d > 2$.

We claim that $E'_\mathfrak{q}$ and $J_\mathfrak{q}$ are of linear type for all $\mathfrak{q} \in \text{Spec}(R')$ with $\dim R'_\mathfrak{q} \leq d - 1$. Indeed, for any such \mathfrak{q} let $\mathfrak{p} = \mathfrak{q} \cap R$. Then, $E_\mathfrak{p}$ is a finite, torsion-free and orientable $R_\mathfrak{p}$ -module with $\ell(E_\mathfrak{p}) \leq \ell$, satisfying $G_{\ell(E_\mathfrak{p})-e+2}$ and such that $\text{Ext}_{R_\mathfrak{p}}^{j+1}(E_\mathfrak{p}^j, R_\mathfrak{p}) = 0$ for $1 \leq j \leq \min\{\ell(E_\mathfrak{p}) - e - 1, \dim_{R_\mathfrak{p}} - 3\}$. Hence, by the induction hypothesis, $E_\mathfrak{p}$ and $J_\mathfrak{p} \cong E'_\mathfrak{p}/F'_\mathfrak{p}$ are of linear type, where $E'_\mathfrak{p}$ and $F'_\mathfrak{p}$ are constructed as in Notation 2.6.1 by choosing the images of a_1, \dots, a_n in $E_\mathfrak{p}$ as generators for $E_\mathfrak{p}$. Hence, their respective localizations $E'_\mathfrak{q}$ and $J_\mathfrak{q}$ are of linear type, as claimed.

In particular, by Theorem 2.6.5, the complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all j and for all $\mathfrak{q} \in \text{Spec}(R')$ with $\text{depth } R'_{\mathfrak{q}} \leq d - 1$. Therefore, by Lemma 3.2.3, it follows that $\text{Ext}_{R'}^{j+1}(J^j, R') = 0$ for $1 \leq j \leq \min\{\ell - e - 1, d - 3\}$. Hence, for j in the same range, also $\text{Ext}_{R''}^{j+1}(I^j, R'') = 0$. Now, if $3 \leq g \leq d - 1$, we could choose $j = g - 2$ to obtain $\text{Ext}_{R''}^{g-1}(I^{g-2}, R'') = 0$. But this would contradict the fact that $\text{grade } I^{g-2} = \text{grade } I = g$. So, it must be that $g = 2$, whence I is of linear type by Theorem 3.1.10. ■

Corollary 3.2.5 *Let R be a local Gorenstein ring with $\dim_R = d$. Let E be a finite, torsion-free and orientable R -module with $\text{rank } E = e > 0$ and $\ell(E) = \ell$. Assume that E is $G_{\ell-e+2}$ and that $\text{depth } E^j \geq d - j$ for $1 \leq j \leq \ell - e - 1$. Then, E is of linear type and E'/F' is isomorphic to an R' -ideal of linear type.*

Proof. Since R is Gorenstein, the assumption on the depths of powers of E implies that $\text{Ext}_R^{j+1}(E^j, R) = 0$ for $1 \leq j \leq \ell - e - 1$, hence also for $1 \leq j \leq \min\{\ell - e - 1, d - 3\}$. Hence, the conclusion follows from Theorem 3.2.4. ■

Notice that Corollary 3.2.5 recovers part of the statement of Theorem 3.1.7 for ideals, though it gives no information on the Cohen-Macaulay property of $\mathcal{R}(E)$. In order to provide a full generalization of Theorem 3.1.7 to modules, we need to develop some more technical machinery. We postpone the discussion to Section 4.3. In the next subsection, we describe instead specific situations where the assumptions of Theorem 3.2.4 are satisfied.

3.2.2 Applications

The first class of modules we are interested in are modules of projective dimension one. We start by recalling a well-known result on their Rees algebras (see [2, Proposition 3], [30, 1.1], [66, 3.4]).

Theorem 3.2.6 *Let R be a Cohen-Macaulay local ring and E a finite R -module with $\text{projdim}(E) \leq 1$. Assume that $\mu(E_{\mathfrak{p}}) \leq \max\{\dim R_{\mathfrak{p}} + e - 1, e\}$ for every $\mathfrak{p} \in \text{Spec}(R)$. Then, $\mathcal{S}(E)$ is torsion-free, and is a complete intersection.*

As a consequence, we show that modules of projective dimension one with $G_{\ell-e+2}$ satisfy the assumptions of Theorem 3.2.4.

Proposition 3.2.7 *Let R be a local Cohen-Macaulay ring, E a finite, torsion-free R -module with $\text{rank } E = e > 0$ and $\ell(E) = \ell$. Assume that $\text{projdim } E = 1$ and that E satisfies $G_{\ell-e+2}$. Then, $\text{Ext}_R^{j+1}(E^j, R) = 0$ for all $j \geq 1$. In particular, E is of linear type.*

Proof. Without loss of generality, we may assume that the residue field of R is infinite. Let $\mathfrak{p} \in \text{Spec}(R)$ be such that $\dim R_{\mathfrak{p}} \leq \ell - e + 1$. Then, $E_{\mathfrak{p}}$ satisfies G_{∞} and has projective dimension one. Hence, by Theorem 3.2.6, $\mathcal{R}(E_{\mathfrak{p}}) \cong \mathcal{S}(E_{\mathfrak{p}})$ is a complete intersection. In particular, for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq \ell - e + 1$, $E_{\mathfrak{p}}$ is of linear type, and the Koszul complex on the generators of the defining ideal of $\mathcal{S}(E_{\mathfrak{p}})$ gives a free resolution of $\mathcal{S}(E_{\mathfrak{p}}) \cong \mathcal{R}(E_{\mathfrak{p}})$. Hence, $\text{projdim } E_{\mathfrak{p}}^j \geq j$ for all $j \geq 1$. Therefore, it must be that $\text{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ for all $j \geq 1$ and all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq \ell - e + 1$.

Now, let U be a minimal reduction of E , generated by ℓ general elements. Since $E_{\mathfrak{p}}$ is of linear type for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq \ell - e + 1$, then for any such \mathfrak{p} one has $E_{\mathfrak{p}} = U_{\mathfrak{p}}$, so that $(E/U)_{\mathfrak{p}} = 0$. Hence, $\text{ht}(F_0(E/U)) \geq \ell - e + 2$, where $F_0(E/U)$ is the 0th Fitting ideal of E/U . On the other hand, since E/U is generated by $\mu(E) - \ell$ elements, by Eagon-Northcott's Theorem [13, Theorem 3], if $F_0(E/U) \neq R$ we must have $\text{ht}(F_0(E/U)) \leq \ell - e + 1$. This is a contradiction, hence it must be $E = U$, in which case $\mu(E) = \ell \leq \dim R + e - 1$. In particular, E satisfies G_{∞} , and the argument above shows that then $\text{Ext}_R^{j+1}(E^j, R) = 0$ for all $j \geq 1$.

Finally, notice that E is orientable, since $\text{projdim } E = 1$. Hence, E is of linear type by Theorem 3.2.4. ■

Next, we prove that strongly Cohen-Macaulay ideals of height two can be used in order to produce modules that satisfy the assumptions of Theorem 3.2.4.

Proposition 3.2.8 *Let R be a local Gorenstein ring of dimension d . Let I be a strongly Cohen-Macaulay ideal of height two with $\ell(I) = \ell$, satisfying $G_{\ell+1}$. Let F be a free R -module of rank $e - 1 > 0$, and let $E = I \oplus F$. Then, $\ell(E) = \ell + e - 1$ and E satisfies $G_{\ell(E)+e-2}$. Moreover, E is of linear type and $\text{Ext}_R^{j+1}(E^j, R) = 0$ for $1 \leq j \leq \ell(E) - e + 1$.*

Proof. Notice that $E = I \oplus F$ is a torsion-free R -module with $\text{rank } E = e$ and satisfies $G_{\ell+1}$ since I does. Moreover, since $\mathcal{S}(F)$ is a polynomial ring in $e-1$ variables over R , if k denotes the residue field of R , one has

$$\mathcal{F}(E) = \mathcal{R}(E) \otimes_R k \cong (\mathcal{R}(I) \otimes_R \mathcal{S}(F)) \otimes_R k \cong \mathcal{F}(I) \otimes_R \mathcal{S}(F).$$

Hence, computing dimensions, we obtain $\ell(E) = \ell + e - 1$. In particular, E satisfies $G_{\ell(E)-e+2}$.

Now, since I is strongly Cohen-Macaulay of height 2 and satisfies $G_{\ell+1}$, by [67, 2.10] we have $\text{depth } I^i \geq d - i$ for $1 \leq i \leq \ell$. Hence, since R is Gorenstein, it follows that $\text{Ext}_R^{j+1}(I^i, R) = 0$ whenever $1 \leq i \leq j \leq \ell$. Therefore, I is of linear type by Theorem 3.1.10. In particular,

$$\mathcal{S}(E) \cong \mathcal{S}(I) \otimes_R \mathcal{S}(F) \cong \mathcal{R}(I) \otimes_R \mathcal{S}(F)$$

is a polynomial ring over $\mathcal{R}(I)$, hence a torsion free R -module. Thus, $\mathcal{R}(E) \cong \mathcal{S}(E)$, whence for all j $E^j \cong [\mathcal{S}(E)]_j \cong [\mathcal{R}(I) \otimes_R \mathcal{S}(F)]_j = \bigoplus_{0 \leq i \leq j} (I^i \otimes_R F^{j-i})$. The discussion above then implies that $\text{Ext}_R^{j+1}(E^j, R) = 0$ for $1 \leq j \leq \ell = \ell(E) - e + 1$. ■

We now give a general technique to construct modules of linear type, as submodules of modules that satisfy Theorem 3.2.4.

Theorem 3.2.9 *Let R be a local Gorenstein ring. Let M be a finite, torsion-free R -module with $\text{rank } M = e > 0$ and $\ell(M) = \ell$. Assume that $\text{Ext}_R^{j+1}(M^j, R) = 0$ for*

$1 \leq j \leq \min \{ \ell - e - 1, d - 3 \}$ and that M satisfies $G_{\ell-e+2}$. Let E be an orientable submodule of M such that $\dim(M/E) \leq \max \{ d - \ell + e - 2, 0 \}$. If $\ell = d + e - 1$, assume also that E is generated by ℓ elements. Then, E is of linear type.

Proof. We may assume that $d > 0$. We show that E satisfies the assumptions of Theorem 3.2.4. First, notice that since $\dim(M/E) \leq \max \{ d - \ell + e - 2, 0 \}$, it follows that $E_{\mathfrak{p}} = M_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} with $\dim R_{\mathfrak{p}} < \min \{ \ell - e + 2, d \}$. In particular, E has a rank, and $\text{rank } E = e$.

Moreover, E is $G_{\ell-e+2}$. This is clear from the discussion above if $\ell \leq d + e - 2$. Otherwise, by Proposition 2.4.6 one must have that $\ell = d + e - 1$, in which case E satisfies G_d . Since by assumption $\mu(E) \leq \ell$, it follows that E is $G_{\ell-e+2}$. Since $\ell(E) \leq \mu(E) \leq \ell$, this then implies that E satisfies $G_{\ell(E)-e+2}$.

Therefore, it only remains to show that our assumptions imply the vanishing of the Ext modules in the statement of Theorem 3.2.4. Since $E_{\mathfrak{p}} = M_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} with $\dim R_{\mathfrak{p}} < \min \{ \ell - e + 2, d \}$, it follows that $\text{grade}(M^j/E^j) \geq \min \{ \ell - e + 2, d \}$ for all $j \geq 1$. In particular, $\text{Ext}_R^{j+2}(M^j/E^j, R) = 0$ for $1 \leq j \leq \min \{ \ell - e - 1, d - 3 \}$. The long exact sequence of $\text{Ext}_R^\bullet(-, R)$ induced by the exact sequence

$$0 \rightarrow E^j \rightarrow M^j \rightarrow M^j/E^j \rightarrow 0$$

then shows that $\text{Ext}_R^{j+1}(E^j, R) = 0$ for $1 \leq j \leq \min \{ \ell - e - 1, d - 3 \}$. Hence, E is of linear type by Theorem 3.2.4. ■

The next lemma will allow us to provide explicit examples of modules E as in Theorem 3.2.9.

Lemma 3.2.10 *Let (R, \mathfrak{m}) be a local Cohen-Macaulay ring with infinite residue field, let M be a finitely generated, torsion-free R -module with $\text{rank } M = e > 0$, satisfying G_s for some $s \geq 2$. Let E be a submodule generated by $s + e - 1$ general elements in $\mathfrak{m}M$. Then, $\dim(M/E) \leq d - s$.*

Proof. Since E is generated by $s + e - 1$ general elements x_1, \dots, x_{s+e-1} in $\mathfrak{m}M$, it makes sense to define the module $\overline{M} := M/(Rx_1 + \dots + Rx_{s+e-1})$. Since M is torsion-free and satisfies G_s with $s \geq 2$, by [11, 2.2] it follows that \overline{M} is a torsion-free module of rank 1 and satisfies G_s . That is, $\overline{M} \cong I$, an R -ideal satisfying G_s .

Let J be the ideal generated by the images of x_e, \dots, x_{s+e-1} in I . Since the x_e, \dots, x_{s+e-1} are general elements, by Proposition 2.7.8 it follows that $J: {}_R I$ is an s -residual intersection of I , whence $\text{ht}(J: {}_R I) \geq s$. Hence,

$$\dim(M/E) = \dim(I/J) = \dim R - \text{ht}(J: {}_R I) \leq \dim R - s,$$

as we wanted to prove. ■

Recalling that every module over a UFD is orientable, we have the following corollary.

Corollary 3.2.11 *Let R be a local Gorenstein UFD, of dimension d and infinite residue field. Let M be a finite, torsion-free R -module with $\text{rank } M = e > 0$ and $\ell(M) = \ell$. Assume that $\text{Ext}_R^{j+1}(M^j, R) = 0$ for $1 \leq j \leq \min\{\ell - e - 1, d - 3\}$ and that M satisfies $G_{\ell-e+2}$.*

If $\ell \leq d + e - 2$, let E be a submodule generated by $\ell + 1$ general elements inside $\mathfrak{m}M$, while if $\ell = d + e - 1$, let E be a submodule generated by ℓ general elements inside $\mathfrak{m}M$. Then, E is of linear type.

Proof. Since M satisfies $G_{\ell-e+2}$, by Lemma 3.2.10 it follows that

$$\dim(M/E) \leq \begin{cases} d - \ell - e + 2 & \text{if } \ell \leq d + e - 2 \\ 0 & \text{if } \ell = d + e - 1 \end{cases}$$

Moreover, E is orientable since R is assumed to be a UFD. Therefore, the conclusion follows from Theorem 3.2.9. ■

In particular, for a module M satisfying the assumptions of Theorem 3.2.4, if M has large enough analytic spread, we can always find a submodule $E \subseteq \mathfrak{m}M$ with $\dim(M/E)$ small enough, that satisfies the same assumptions.

Combining this result with Theorem 3.2.7 and Theorem 3.2.8 respectively, we then have the following corollaries.

Corollary 3.2.12 *Let R be a local Gorenstein UFD, of dimension d and infinite residue field. Let M be a finite R -module with $\text{rank } M = e > 0$, $\ell(M) = \ell$ and $\text{projdim } M = 1$, satisfying $G_{\ell-e+2}$.*

If $\ell \leq d + e - 2$, let E be a submodule generated by $\ell + 1$ general elements inside $\mathfrak{m}M$, while if $\ell = d + e - 1$, let E be a submodule generated by ℓ general elements inside $\mathfrak{m}M$. Then, E is of linear type.

Corollary 3.2.13 *Let R be a local Gorenstein UFD, of dimension d and infinite residue field. Let I be a strongly Cohen-Macaulay ideal of height two with $\ell(I) = \ell$ satisfying $G_{\ell+1}$, F a free R -module of rank $e - 1 > 0$, and let $M = I \oplus F$.*

If $\ell \leq d + e - 2$, let E be a submodule generated by $\ell + 1$ general elements inside $\mathfrak{m}M$, while if $\ell = d + e - 1$, let E be a submodule generated by ℓ general elements inside $\mathfrak{m}M$. Then, E is of linear type.

4. MODULES WITH COHEN-MACAULAY REES ALGEBRA

The goal of this chapter is to provide a sufficient condition for the Rees algebra of a module E to be Cohen-Macaulay. In particular, we wish to recover a well-known result proved independently by Johnson and Ulrich, and by Goto, Nakamura and Nishida for Rees algebras of ideals (see Theorem 4.1.5 below). This theorem requires that the ideals under consideration satisfy the Artin-Nagata conditions of Definition 2.7.4, which do not seem to have a well-understood counterpart for modules. This makes the problem challenging from a technical point of view. Our main result, Theorem 4.2.4, is obtained using generic Bourbaki ideals, and requires a deep investigation of the residual intersection properties of a generic Bourbaki ideal of the given module E .

As a consequence of our Lemma 4.2.2, we are able to provide a condition for which a module is of linear type and also has Cohen-Macaulay Rees algebra (see Theorem 4.3.2). This extends Theorem 3.1.7 to modules, and generalizes previous work of Lin which we recalled in Theorem 3.2.1.

We refer the reader to Chapter 2 for the necessary background on generic Bourbaki ideals and residual intersections. Due to the crucial role of Rees algebras of ideals in our approach, for our and the reader's convenience, we start by recalling some of the literature on their Cohen-Macaulay property.

4.1 Cohen-Macaulay property of the Rees algebra of an ideal

The Cohen-Macaulay property of Rees algebra of an ideal is deeply interconnected with that of its associated graded ring. One of the first results to highlight this connection is the following, due to Huneke (see [32, Proposition 1.1]).

Theorem 4.1.1 (Huneke) *Let R be a Cohen-Macaulay local ring, I an ideal with positive height. If $\mathcal{R}(I)$ is Cohen-Macaulay, then $\mathcal{G}(I)$ is Cohen-Macaulay.*

The converse is not true, however the Cohen-Macaulayness of the associated graded ring implies that of the Rees ring when some extra numerical conditions are satisfied. Recall that for a d -dimensional graded ring S with a unique homogeneous maximal ideal \mathfrak{m} , the a -invariant of S is

$$a(S) := \sup\{i \mid [H_{\mathfrak{m}}^d(S)]_i \neq 0\} = -\inf\{i \mid [\omega_S]_i \neq 0\},$$

where ω_S is the *graded canonical module* of S , and the last equality holds by the graded version of the Local Duality Theorem.

Theorem 4.1.2 (Ikeda-Trung) (*[37, 1.1]*) *Let R be a Cohen-Macaulay local ring, I an ideal with positive height. Then, $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $\mathcal{G}(I)$ is Cohen-Macaulay and $a(\mathcal{G}(I)) < 0$.*

Alternatively, one could deduce information on the Cohen-Macaulayness of $\mathcal{R}(I)$ by inspecting the local reduction numbers of I .

Theorem 4.1.3 (Johnston-Katz) (*[40, 2.3]*) *Let R be a Cohen-Macaulay local ring, I an ideal with positive height. Then, $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $\mathcal{G}(I)$ is Cohen-Macaulay and $r_{J_{\mathfrak{p}}}(I_{\mathfrak{p}}) \leq \ell(I_{\mathfrak{p}}) - 1$ for all $\mathfrak{p} \in V(I)$ and all minimal reductions $J_{\mathfrak{p}}$ of $I_{\mathfrak{p}}$.*

In particular, if $\mathcal{R}(I)$ is Cohen-Macaulay, it follows that $r(I) \leq \ell(I) - 1$. We refer the reader to [1] and [63] for similar results.

With these results in mind, studying the Cohen-Macaulay property of $\mathcal{R}(I)$ then becomes a two-step process. First, one determines a sufficient condition for the associated graded ring $\mathcal{G}(I)$ to be Cohen-Macaulay. Next, one investigates the canonical module of $\mathcal{G}(I)$ in order to estimate the a -invariant of $\mathcal{G}(I)$, or computes the local reduction numbers of I .

Most of the literature on the Cohen-Macaulayness of the associated graded ring relies heavily on the following fundamental result of Valabrega and Valla (see [72, 2.6 and 2.7]).

Theorem 4.1.4 (Valabrega-Valla) *Let R be a Noetherian local ring with $\dim R = d$ and infinite residue field. Let I be an R -ideal with $\text{ht } I > 0$ and $\ell(I) = \ell$. For a minimal reduction $J = (b_1, \dots, b_\ell)$ of I , denote $b_i^* = b_i + I^2 \in I/I^2 = [\text{gr}_1(R)]_1$, for $1 \leq i \leq \ell$. The following statements are true.*

- (i) *If the b_i form an R -regular sequence and $I^j \cap (b_1, \dots, b_i) = I^{j-1}(b_1, \dots, b_i)$ for all $j \geq 1$ and all $1 \leq i \leq \ell$, then the b_i^* form a regular sequence on $\mathcal{G}(I)$.*
- (ii) *The b_i^* form a regular sequence on $\mathcal{G}(I)$ if and only if the b_i form an R -regular sequence and $I^j \cap J = I^{j-1}J$ for all $j \geq 1$.*

In the case when I is \mathfrak{m} -primary, the Hilbert-Samuel function and the Hilbert-Samuel multiplicity turn out to provide effective tools in order to check whether the intersection condition in Theorem 4.1.4(ii) is satisfied, exploiting also the fact that the generators of J can be chosen to be a *superficial sequence* for I . In fact, thanks to a result of Sally known as *Sally's machine* (see [59, Lemma 1.4]), superficial sequences allow to reduce the problem to the case of ideals in a one-dimensional ring. We refer the interested reader to the monograph by Rossi and Valla [59] for a discussion on the depth of associated graded rings of \mathfrak{m} -primary ideals.

If I is not \mathfrak{m} -primary, Theorem 4.1.4(ii) is not useful in order to determine whether $\mathcal{G}(I)$ is Cohen-Macaulay, since it provides a too short regular sequence on $\mathcal{G}(I)$, unless the $\ell(I) = \dim R$ (recall that $\dim \mathcal{G}(I) = \dim R$). However, one can still hope to be able to exploit Theorem 4.1.4(i). This happens to be the case if one consider ideals with good residual intersection properties.

4.1.1 Residual intersections and Cohen-Macaulay Rees algebras

Let R be a Noetherian local ring with infinite residue field, and let I be an ideal with analytic spread $\ell(I) = \ell$. Recall from Proposition 2.5.2 that ℓ general elements in I form a minimal reduction J of I with $r_J(I) = r(I)$. Moreover, by Proposition 2.7.8 it follows that, if I satisfies G_ℓ , then $J:_R I$ is an ℓ -residual intersection of I . Finally, by Theorem 2.7.7, these residual intersections can be written in an iterative way, as long as I satisfy good enough Artin-Nagata properties.

These observations together suggest that one could try to exploit this iterative property of the residual intersections of I and Theorem 4.1.4(i), in order to study the Cohen-Macaulayness of the associated graded ring of an ideal.

In fact, it turns out that the intersection conditions in Theorem 4.1.4(i) hold if I satisfies Theorem 2.7.7(b) and (c). This is the core observation behind Johnson and Ulrich's proof of the following fundamental result (see [39, Theorem 3.1 and proof of Lemma 2.8], as well as [27, 28] for some preliminary work in the case of ideals with small analytic deviation). We give here a refined statement, which incorporates a result proved independently by Goto, Nakamura and Nishida using different techniques (see [20, 1.1 and 6.3]).

Theorem 4.1.5 (Johnson-Ulrich, Goto-Nakamura-Nishida) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field. Let I be an R -ideal with analytic spread ℓ , height g and reduction number r , and let k be an integer. Assume that $g \geq 1$, $1 \leq k \leq \ell - 1$, $r \leq k$ and I satisfies G_ℓ and $AN_{\ell-k-1}$. If $\text{depth } I^j \geq d - \ell + k - j + 1$ for $1 \leq j \leq k$, then $\mathcal{R}(I)$ is Cohen-Macaulay.*

Notice that this assumption on the depth of powers of I implies that $k \leq \ell - g + 1$. In the rest of this chapter, our goal will be to extend this result to modules. A generalization corresponding to the case when $k = \ell - g + 1$ appears in [46, 3.4] (see Corollary 4.2.10 below). Our main result, Theorem 4.2.4, provides a module version of Theorem 4.1.5 for every admissible value of k .

4.2 Cohen-Macaulay property of Rees algebras of modules

As discussed in Chapter 2, the fact that for a module E the powers $E^j := [\mathcal{R}(E)]_j$ do not form a filtration implies that there is no module analogue for the associated graded ring. As a consequence, in order to study the Cohen-Macaulay property of Rees algebras of modules, the approach described in the previous section for Rees algebras of ideals must be discarded. Nevertheless, the notion of generic Bourbaki ideals indicates a clear strategy to approach the problem in the case of modules.

In fact, as recalled in Theorem 2.6.4, the Rees algebra $\mathcal{R}(E)$ of a module E is Cohen-Macaulay if and only if the Rees algebra $\mathcal{R}(I)$ of a generic Bourbaki ideal I of E is Cohen-Macaulay. Hence, the problem of determining a sufficient condition for $\mathcal{R}(E)$ to be Cohen-Macaulay is ultimately reduced to that of crafting assumptions on E so that I has a Cohen-Macaulay Rees algebra.

This simple idea is often not easy to put in practice, for a couple of reasons. First of all, some of the notions that make sense for ideals may not have a well-understood module counterpart. For instance, this is the case of the Artin-Nagata properties, in relation with the intersection condition of Valabrega-Valla's Theorem 4.1.4(i). In fact, although one could reasonably define an s -residual intersection of E as a colon ideal $(Ra_1 + \dots + Ra_s):_R E$ for some elements a_1, \dots, a_s in E , the intersections $(Ra_1 + \dots + Ra_i) \cap E^j$ would not make sense for $j \neq 1$. At the same time, it is not even clear how a possible module analogue of the Artin-Nagata condition would behave under the passage to a generic Bourbaki ideal, since factoring out generic elements may not preserve the structure of these colon ideals.

The second source of difficulty is the fact that very often transferring properties from E to I and backwards is only possible if $\mathcal{R}(E)$ is a deformation of $\mathcal{R}(I)$.

In the remaining part of this chapter, we aim to provide a sufficient condition for $\mathcal{R}(E)$ to be Cohen-Macaulay, that recovers Theorem 4.1.5 in the case when E is an

ideal. More precisely, we wish to deduce the Cohen-Macaulayness of $\mathcal{R}(E)$ under the assumption that the depths of finitely many powers of E are large enough.

Partial results were given by Lin for orientable modules in [46, 3.4], and for ideal modules in [46, 4.3] (we refer the reader to Section 4.4 for the definition of ideal modules). In both cases, a generic Bourbaki ideal is shown to satisfy Theorem 4.1.5 under more restrictive assumptions. More precisely, in [46, 3.4] the depth conditions on the powers of I turn out to be much stronger than in Theorem 4.1.5, and at the same time the reduction number of I is forced to be the largest possible. In [46, 4.3], I turns out to satisfy stronger Artin-Nagata properties.

Our main result, Theorem 4.2.4, gives a more accurate generalization of Theorem 4.1.5 to orientable modules over a Gorenstein local ring. In fact, we recover [46, 3.4] in Corollary 4.2.10. Although the result in [46, 4.3] does not follow from Theorem 4.2.4, nevertheless our techniques allow us to give a simplified proof (see Theorem 4.4.4).

4.2.1 Main result

Throughout the rest of this chapter, we adopt the following assumptions and notation.

Setting 4.2.1 Let R be a Noetherian local ring of dimension d . Let E be a finite R -module with $\text{rank}(E) = e > 0$ and analytic spread $\ell(E) = \ell$. For a fixed generating set a_1, \dots, a_n of E , let E' , E'' , x_1, \dots, x_{e-1} , F' and F'' be constructed as in Notation 2.6.1. Assume that E'/F' is isomorphic to an R' -ideal J as an R' -module, let $I = JR''$ be a generic Bourbaki ideal of E , and let $g = \text{ht } I$.

With Theorem 4.1.5 in mind, our goal is to deduce that the Rees algebra $\mathcal{R}(E)$ of a module E is Cohen-Macaulay by imposing that the depths of the powers E^j are sufficiently large, for j in a suitable finite range. Using generic Bourbaki ideals, our first goal is to transfer these depth conditions from each E^j to the corresponding I^j .

Consider the complexes

$$\mathbb{C}'_j: [\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E'))]_j \xrightarrow{\partial'_0} J^j \rightarrow 0$$

and

$$\mathbb{C}''_j: [\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E''))]_j \xrightarrow{\partial''_0} I^j \rightarrow 0$$

described in Section 3.2. The following lemma shows that the exactness of the complexes \mathbb{C}''_j allows to transfer assumptions on the depths of powers of E to the depths of the corresponding powers of I .

Lemma 4.2.2 *Under the assumptions of Setting 4.2.1, let k and s be integers such that $\text{depth } E^j \geq s - j$ for $0 \leq j \leq k$. If the complexes \mathbb{C}''_j are exact for $0 \leq j \leq k$, then $\text{depth } I^j \geq s - j$ for $0 \leq j \leq k$.*

Proof. Fix j with $0 \leq j \leq k$. For $0 \leq i \leq j$, let C''_i and B_i be the i th module and boundary of the complex

$$\mathbb{C}''_j: [\mathbb{K}.(x_1, \dots, x_{e-1}; \mathcal{R}(E''))]_j \xrightarrow{\partial''_0} I^j \rightarrow 0,$$

respectively. By assumption, $\text{depth } (E'')^j \geq s - j$ for $0 \leq j \leq k$. Hence, for $0 \leq i \leq j$, one has that $\text{depth } C''_i \geq s - j + i$. This implies that $\text{depth } B_{i-1} \geq s - j + i$ for all $0 \leq i \leq j$, which can be proved by decreasing induction on i , using the Depth Lemma. ■

Notice that it suffices to prove that the complexes \mathbb{C}'_j are exact for $0 \leq j \leq k$, since \mathbb{C}''_j is just a localization of \mathbb{C}'_j . In fact, our next lemma shows that it suffices to check the exactness of the \mathbb{C}'_j only locally in small codimension.

Lemma 4.2.3 *Under the assumptions of Setting 4.2.1, let k and s be integers such that $0 \leq k \leq s$. Assume that $\text{depth } E^j \geq d - s + k - j$ for $0 \leq j \leq k$. Then, for all $0 \leq j \leq k$ and every minimal prime \mathfrak{q} in $\text{Supp}\left(\bigoplus_{i=0}^j H_i(\mathbb{C}'_j)\right)$, one has that $\dim R'_\mathfrak{q} \leq s$.*

Proof. Fix j with $0 \leq j \leq k$ and suppose that $\dim R'_q \geq s + 1$ for some prime ideal \mathfrak{q} as in the statement. We will prove that $(\mathbb{C}'_j)_q$ is exact, and this will contradict the fact that $H_i(\mathbb{C}'_j)_q \neq 0$ for some i with $0 \leq i \leq j$.

First, notice that $H_i(\mathbb{C}'_j)_q$ is zero or has depth 0 for all $0 \leq i \leq j$, since \mathfrak{q} is minimal in $\text{Supp}\left(\bigoplus_{i=0}^j H_i(\mathbb{C}'_j)\right)$. Now, let $\mathfrak{p} = \mathfrak{q} \cap R$. Since $R_{\mathfrak{p}} \rightarrow R'_q$ is a flat local map, one has that

$$\dim R'_q - \text{depth}(E'_q)^j = \dim R_{\mathfrak{p}} - \text{depth}(E_{\mathfrak{p}})^j \leq d - \text{depth} E^j$$

(see [50, Theorem 15.1, Theorem 23.3 and Exercise 17.5]) for all $0 \leq i \leq j$. Hence, from our assumption on $\text{depth} E^j$ it follows that $\text{depth}(E'_q)^{j-i} \geq \dim R'_q - s + k - j + i \geq i + 1$. Hence, $(\mathbb{C}'_j)_q$ is acyclic by the Acyclicity Lemma (see [15, Lemma 20.11]), and therefore it is exact, since $H_{-1}(\mathbb{C}'_j) = 0$ by construction. ■

We are now ready to state and prove the desired generalization of Theorem 4.1.5.

Theorem 4.2.4 *Let R be a local Gorenstein ring of dimension d with infinite residue field. Let E be a finite, torsion-free, orientable R -module, with $\text{rank} E = e > 0$ and $\ell(E) = \ell$. Let g be the height of a generic Bourbaki ideal of E , and assume that the following conditions hold.*

(a) E satisfies $G_{\ell-e+1}$.

(b) $r(E) \leq k$ for some integer $1 \leq k \leq \ell - e$.

(c) $\text{depth} E^j \geq \begin{cases} d - g - j + 2 & \text{for } 1 \leq j \leq \ell - e - k - g + 1 \\ d - \ell + e + k - j & \text{for } \ell - e - k - g + 2 \leq j \leq k \end{cases}$

(d) If $g = 2$, $\text{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}^j, R_{\mathfrak{p}}) = 0$ for $\ell - e - k \leq j \leq \ell - e - 3$ and for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} = \ell - e$ such that $E_{\mathfrak{p}}$ is not free.

Then, $\mathcal{R}(E)$ is Cohen-Macaulay.

Proof. Let $E = Ra_1 + \dots + Ra_n$. By Theorem 2.6.2, E admits a generic Bourbaki ideal I with $\text{ht} I = g \geq 2$ and $r(I) \leq r(E) \leq k$, which satisfies $G_{\ell-e+1}$, i.e. $G_{\ell(I)}$

(see Proposition 2.6.9). We will prove that $\mathcal{R}(I)$ is Cohen-Macaulay, so that $\mathcal{R}(E)$ is Cohen-Macaulay by Theorem 2.6.4.

If $e = 1$, then $R'' = R$ and $E \cong I$, an R -ideal with $\ell(I) = \ell$ and such that

$$\text{depth } E^j \geq \begin{cases} d - g - j + 2 & \text{for } 1 \leq j \leq \ell - k - g \\ d - \ell + k - j + 1 & \text{for } \ell - k - g + 1 \leq j \leq k \end{cases}$$

Since R is Gorenstein, by Theorem 2.7.5 it follows that I satisfies $AN_{\ell-k-1}$. Moreover, all assumptions of Theorem 4.1.5 are satisfied, hence $\mathcal{R}(I)$ is Cohen-Macaulay.

Now, assume $e \geq 2$. It suffices to prove that

$$\text{depth } I^j \geq \begin{cases} d - g - j + 2 & \text{for } 1 \leq j \leq \ell - e - k - g + 1 \\ d - \ell + e + k - j & \text{for } \ell - e - k - g + 2 \leq j \leq k \end{cases}$$

In fact, this would imply that $\text{depth } I^j \geq d - \ell(I) + k - j + 1$ for $1 \leq j \leq k$, and by Theorem 2.7.5, I would satisfy $AN_{\ell-k-e}$, i.e. $AN_{\ell(I)-k-1}$. Hence, $\mathcal{R}(I)$ would be Cohen-Macaulay by Theorem 4.1.5.

Notice that $\text{depth } (E'')^j \geq \text{depth } E^j$, since R'' is flat over R . Also, if $g \geq 3$, then by Theorem 2.6.6 I is a free direct summand of E'' , so for every j , I^j satisfies the same depth condition as $(E'')^j$, hence as E^j . Hence, we may assume that $g = 2$, so that assumption (c) becomes

$$\text{depth } E^j \geq \begin{cases} d - j & \text{for } 1 \leq j \leq \ell - e - k - 1 \\ d - \ell + e + k - j & \text{for } \ell - e - k \leq j \leq k \end{cases}$$

In particular, $\text{depth } E^j \geq d - \ell + e + k - j$ for $1 \leq j \leq k$, and also for $j = 0$ since R is Cohen-Macaulay. Therefore, by Lemma 4.2.2 (with $s = d - \ell + e + k$) and Lemma 4.2.3 (with $s = \ell - e$), it suffices to show that the complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all $\mathfrak{q} \in \text{Spec}(R')$ with $\dim R'_{\mathfrak{q}} \leq \ell - e$ and all $0 \leq j \leq k$.

For each such \mathfrak{q} , let $\mathfrak{p} = \mathfrak{q} \cap R$. If $E_{\mathfrak{p}}$ is free, then also its localization $E'_{\mathfrak{q}}$ is free. Hence, $E'_{\mathfrak{q}}$ is of linear type and $\mathcal{R}(E'_{\mathfrak{q}})$ is Cohen-Macaulay. In particular, $\text{grade } \mathcal{R}(E'_{\mathfrak{q}})_+ \geq e$, so by Theorems 2.6.4(b) and 2.6.5, the complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all j . If $E_{\mathfrak{p}}$ is not free, then by assumption $\text{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}^j, R_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \text{Spec}(R)$

with $\dim R_{\mathfrak{p}} \leq \ell - e$ and all j with $\ell - e - k \leq j \leq \ell - e - 3$. In particular, this is true for $\ell - e - k \leq j \leq \min \{\ell(E_{\mathfrak{p}}) - e - 1, \dim R_{\mathfrak{p}} - 3\}$, since $\dim R_{\mathfrak{p}} = \ell - e$. Moreover, since $R_{\mathfrak{p}}$ is Gorenstein, by assumption (c) the same vanishing holds if $1 \leq j \leq \ell - e - k - 1$ (see Proposition 2.1.1). Hence,

$$\mathrm{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}^j, R_{\mathfrak{p}}) = 0 \text{ for all } 1 \leq j \leq \min \{\ell(E_{\mathfrak{p}}) - e - 1, \dim R_{\mathfrak{p}} - 3\}.$$

Therefore, by Theorem 3.2.4, $E'_{\mathfrak{p}}/F'_{\mathfrak{p}}$ is isomorphic to an $R'_{\mathfrak{p}}$ -ideal of linear type, where $E'_{\mathfrak{p}}$ and $F'_{\mathfrak{p}}$ are constructed as in Notation 2.6.1 by choosing the images of a_1, \dots, a_n in $E_{\mathfrak{p}}$ as generators for $E_{\mathfrak{p}}$. Hence, its localization $(E'/F')_{\mathfrak{q}}$ is isomorphic to an $R'_{\mathfrak{q}}$ -ideal of linear type. Thus, all the $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact by Theorem 2.6.5. ■

Depending on the values of the various parameters involved in the statement of Theorem 4.2.4, some of the assumptions may be redundant. We will examine several possible cases in the next subsection, where we also recover Lin's result [46, 3.4].

We notice here that our assumptions are usually not satisfied for an important class of modules, namely the so called *ideal modules* (see Definition 4.4.1 in Section 4.4). Nevertheless, thanks to Lemma 4.2.2, in Theorem 4.4.4 we are able to provide a sufficient condition for the Rees algebra of an ideal module to be Cohen-Macaulay. This recovers Lin's [46, 4.3], with a simplified proof.

4.2.2 Applications

Comparing the statements of Theorem 4.1.5 and of Theorem 4.2.4, it is clear that assumption (d) in Theorem 4.2.4 is redundant in case E is of rank one. Moreover, in this case assumption (c) simplifies to

$$(c') \quad \mathrm{depth} E^j \geq d - \ell + e + k - j \quad \text{for } 1 \leq j \leq k.$$

Hence, it is natural to ask what other situations produce similar simplified statements for Theorem 4.2.4.

Notice that assumption (d) in Theorem 4.2.4 is vacuously satisfied when $k \leq 2$, or when $\ell - e \leq 3$. Also, if $\ell - e \leq 2$, assumption (c) can be replaced by assumption (c') above. In particular, we have the following two corollaries, which correspond to the cases when a generic Bourbaki ideal of E has analytic deviation at most 1 or at most 2, respectively.

Corollary 4.2.5 *Let R be a local Gorenstein ring of dimension d with infinite residue field. Let E be a finite, torsion-free, orientable R -module, with $\text{rank}E = e > 0$ and $\ell(E) = \ell$. Let g be the height of a generic Bourbaki ideal of E , and assume that the following conditions hold.*

- (a) E satisfies $G_{\ell-e+1}$.
- (b) $r(E) \leq k$ for some integer $1 \leq k \leq \ell - e$ and $\ell - e \leq 2$.
- (c) $\text{depth} E^j \geq d - \ell + e + k - j$ for $1 \leq j \leq k$.

Then, $\mathcal{R}(E)$ is Cohen-Macaulay.

Corollary 4.2.6 *Let R be a local Gorenstein ring of dimension d with infinite residue field. Let E be a finite, torsion-free, orientable R -module, with $\text{rank}E = e > 0$ and $\ell(E) = \ell$. Let g be the height of a generic Bourbaki ideal of E , and assume that the following conditions hold.*

- (a) E satisfies $G_{\ell-e+1}$.
- (b) $r(E) \leq k$ for some integer $1 \leq k \leq \ell - e$ and $\ell - e \leq 3$.
- (c) $\text{depth} E \geq d - g + 1$ and $\text{depth} E^j \geq d - \ell + e + k - j$ for $2 \leq j \leq k$.

Then, $\mathcal{R}(E)$ is Cohen-Macaulay.

Observe also that assumption (b) of Theorem 4.2.4 implies that $r(E) \leq \ell(E) - e$. This condition is necessary in order for $\mathcal{R}(E)$ to be Cohen-Macaulay [65, 4.2], and extends the result of Theorem 4.1.3 to modules. Since $\ell(E) - e \leq d - 1$, it then

makes sense to investigate the Cohen-Macaulay property of $\mathcal{R}(E)$ for rings of small dimensions. In particular, in [65, 4.4 and 4.6], sufficient conditions were given in the case when $d \leq 5$, under the assumption that $r(E) \leq 2$ and E satisfies G_d . As a consequence of Theorem 4.2.4, we can give new results, assuming that E satisfies the weaker condition $G_{\ell-e+1}$, for a wider range of possible reduction numbers.

Corollary 4.2.7 *Let R be a Gorenstein local ring of dimension 4 with infinite residue field. Let E be a finite, torsion-free, orientable R -module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that E satisfies $G_{\ell-e+1}$ and $r(E) \leq \ell - e$. Then, $\mathcal{R}(E)$ is Cohen-Macaulay if one of the following conditions holds.*

- (a) $r(E) = 1 < \ell - e$ and $\text{depth } E \geq 2$, or $r(E) = 1 = \ell - e$ and $\text{depth } E \geq 3$.
- (b) $r(E) = 2 < \ell - e$ and $\text{depth } E \geq 2$, or $r(E) = 2 = \ell - e$ and $\text{depth } E^j \geq 4 - j$ for $1 \leq j \leq 2$.
- (c) $r(E) = 3$ and $\text{depth } E^j \geq 4 - j$ for $1 \leq j \leq 3$.

Proof. Since $\ell - e \leq d - 1 = 3$, if $\ell - e = 3$ it follows that E satisfies G_d . Hence, if $r(E) \leq 2$, by [65, 4.6(b)] it suffices to assume that $\text{depth } E \geq 2$. In all the remaining cases, our assumptions imply the assumptions of Theorem 4.2.4 with $d = 4$ and $k = r(E)$. In fact, since $r(E) \leq \ell - e \leq 3$, assumption (d) in Theorem 4.2.4 is vacuously satisfied, and if $r(E) = 3$ then it must be $\ell - e = 3$. ■

Corollary 4.2.8 *Let R be a Gorenstein local ring of dimension 5 with infinite residue field. Let E be a finite, torsion-free, orientable R -module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that E satisfies $G_{\ell-e+1}$. Let g be the height of a generic Bourbaki ideal of E . Then, $\mathcal{R}(E)$ is Cohen-Macaulay if one of the following conditions holds.*

- (a) $\ell - e = 4$, $r(E) \leq 2$ and $\text{depth } E \geq 4$.
- (b) $\ell - e = 4$, $r(E) \geq \ell - e - 1 = 3$, and $\text{depth } E^j \geq 5 - \ell + e + k - j$ for $1 \leq j \leq r(E)$. If $g = 2$, assume also that $\text{Ext}_{R_{\mathfrak{p}}}^2(E_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} = 4$ such that $E_{\mathfrak{p}}$ is not free.

(c) $r(E) = \ell - e \leq 3$, and $\text{depth } E^j \geq 5 - j$ for $1 \leq j \leq r(E)$.

(d) $r(E) = \ell - e - 1 \leq 2$, and $\text{depth } E^j \geq 4 - j$ for $1 \leq j \leq r(E)$.

(e) $\ell - e = 3$, $r(E) = 1$, and $\text{depth } E \geq \begin{cases} 4 & \text{if } g = 2 \\ 3 & \text{if } g \geq 3 \end{cases}$

Proof. In the situation of (a), E must satisfy G_d , and the result was proved in [65, 4.6(c)]. The remaining claims follow from Theorem 4.2.4 with $d = 5$ and $k = r(E)$, by noticing that assumption (d) is non-vacuous only when $\ell - e = 4$. ■

As another application of Theorem 4.2.4, we now examine the case of modules with large reduction number. In fact, from the proof of Theorem 4.2.4 it follows that assumption (c) implies that $k \leq \ell - e - g + 2$ (see also the discussion in [39, page 14]) and that assumption (d) is automatically satisfied when $k = \ell - e - g + 2$. Moreover, if $k \geq \ell - e - g + 1$, we only need R to be Gorenstein locally in codimension $\ell - e$, since the Artin-Nagata property $AN_{\ell - e - k}$ is automatically satisfied when $\ell - e - k < g$. In this case, the depth condition in (c) can be simplified to assumption (c'). These observations prove the following corollaries.

Corollary 4.2.9 (see also [20, 6.5]) *Let R be a local Cohen-Macaulay ring with infinite residue field, and assume that R is Gorenstein locally in codimension $\ell - e$. Let E be a finite, torsion-free, orientable R -module, $\text{rank } E = e > 0$, $\ell(E) = \ell$, $\ell - e + 1 \geq 2$. Let g be the height of a generic Bourbaki ideal of E . Then, $\mathcal{R}(E)$ is Cohen-Macaulay if the following conditions hold:*

(a) E satisfies $G_{\ell - e + 1}$.

(b) $r(E) \leq \ell - e - g + 1$.

(c) $\text{depth } E^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - e - g + 1$.

(d) If $g = 2$, $\text{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}^j, R_{\mathfrak{p}}) = 0$ for $1 \leq j \leq \ell - e - 3$, for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} = \ell - e$ such that $E_{\mathfrak{p}}$ is not free.

Corollary 4.2.10 (see also [46, 3.4] and [20, 6.4]). *Let R be a local Cohen-Macaulay ring with infinite residue field, and assume that R is Gorenstein locally in codimension $\ell - e$. Let E be a finite, torsion-free, orientable R -module, $\text{rank} E = e > 0$, $\ell(E) = \ell$, $\ell - e + 1 \geq 2$. Let g be the height of a generic Bourbaki ideal of E . Then, $\mathcal{R}(E)$ is Cohen-Macaulay if the following conditions hold:*

- (a) E satisfies $G_{\ell-e+1}$.
- (b) $r(E) \leq \ell - e - g + 2$.
- (c) $\text{depth} E^j \geq d - g - j + 2$ for $1 \leq j \leq \ell - e - g + 2$.

In Lin's proof of [46, 3.4], a generic Bourbaki ideal I of E is shown to satisfy the sliding depth property locally in codimension $\ell - e$. This is obtained by applying Lin's Theorem 3.2.1 locally in codimension at most $\ell - e$. With our proof, I is only required to be of linear type. The next section discusses an improvement of Theorem 3.2.1.

4.3 Modules of linear type with Cohen-Macaulay Rees algebra

In this section we complete the study of the linear type property of the Rees algebra of a module we initiated in Chapter 3. In particular, we aim to provide a full generalization of Theorem 3.1.7 to modules. That is, we want to be able to deduce both the linear type property of E and the Cohen-Macaulay property of $\mathcal{R}(E)$ from the assumption that finitely many powers E^j of E have sufficiently large depth.

Using generic Bourbaki ideals, the problem reduces to being able to show that the assumptions on $\text{depth} E^j$ imply similar assumptions on $\text{depth} I^j$. The discussion in Section 4.2.1 suggests that Lemma 4.2.2 could be useful. We will also need the following preliminary result.

Lemma 4.3.1 *Let R be a Cohen-Macaulay local ring of dimension d , and $E = Ra_1 + \dots + Ra_n$ a finite R -module. As in Theorem 2.6.10, let $\tilde{R} := R[Z_1, \dots, Z_n]$ be a polynomial ring, $\tilde{E} := E \otimes_R \tilde{R}$ and $x := \sum_{i=1}^n Z_i a_i \in \tilde{E}$. Let s be a positive integer,*

and assume that $\text{depth } E^j \geq d - j$ for $1 \leq j \leq s$. Then, the following statements hold.

(a) For all $\mathfrak{q} \in \text{Spec } \tilde{R}$ and for all $1 \leq j \leq s$, $\text{depth}\left(\frac{\tilde{E}_{\mathfrak{q}}^j}{x\tilde{E}_{\mathfrak{q}}^{j-1}}\right) \geq \dim \tilde{R}_{\mathfrak{q}} - j$.

(b) Assume furthermore that $s \leq d - 1$ and that $(\tilde{E}/\tilde{R}x)_{\mathfrak{q}}$ is of linear type for all $\mathfrak{q} \in \text{Spec } \tilde{R}$ such that $\dim R_{\mathfrak{q} \cap R} \leq d - 1$. Then, for all $1 \leq j \leq s$

$$\frac{\tilde{E}^j}{x\tilde{E}^{j-1}} \cong \left(\frac{\tilde{E}}{\tilde{R}x}\right)^j.$$

Proof. For $\mathfrak{q} \in \text{Spec}(\tilde{R})$, let $\mathfrak{p} = \mathfrak{q} \cap R$. Since $R_{\mathfrak{p}} \rightarrow \tilde{R}_{\mathfrak{q}}$ is a flat local map, for all $1 \leq j \leq s$ we have that

$$\dim \tilde{R}_{\mathfrak{q}} - \text{depth} \tilde{E}_{\mathfrak{q}}^j = \dim R_{\mathfrak{p}} - \text{depth } E_{\mathfrak{p}}^j \leq d - \text{depth } E^j \leq j$$

(see [50, Theorem 15.1, Theorem 23.3 and Exercise 17.5]). Moreover, by Theorem 2.6.10 x is a regular element on $\mathcal{R}(\tilde{E})$, hence on $\mathcal{R}(\tilde{E}_{\mathfrak{q}})$. Therefore, for all $j \geq 1$ we have $x\tilde{E}_{\mathfrak{q}}^{j-1} \cong \tilde{E}_{\mathfrak{q}}^{j-1}$. Hence, the conclusion in (a) follows from the Depth Lemma applied to the short exact sequence

$$0 \rightarrow x\tilde{E}_{\mathfrak{q}}^{j-1} \rightarrow \tilde{E}_{\mathfrak{q}}^j \rightarrow \frac{\tilde{E}_{\mathfrak{q}}^j}{x\tilde{E}_{\mathfrak{q}}^{j-1}} \rightarrow 0.$$

Now assume that $s \leq d - 1$. In order to prove (b), by Theorem 2.6.5 it suffices to show that $\tilde{E}_{\mathfrak{q}}^j/x\tilde{E}_{\mathfrak{q}}^{j-1}$ is a torsion-free \tilde{R} -module for all $1 \leq j \leq s$. Equivalently, we need to show that $\text{depth}\left(\frac{\tilde{E}_{\mathfrak{q}}^j}{x\tilde{E}_{\mathfrak{q}}^{j-1}}\right) \geq 1$ for all $\mathfrak{q} \in \text{Spec}(\tilde{R})$ with $\dim \tilde{R}_{\mathfrak{q}} \geq 1$.

Let $\mathfrak{p} = \mathfrak{q} \cap R$. If $\dim R_{\mathfrak{p}} \leq d - 1$, then by assumption $(\tilde{E}/\tilde{R}x)_{\mathfrak{q}}$ is of linear type. Hence, by Theorem 2.6.5 it follows that $\tilde{E}_{\mathfrak{q}}^j/x\tilde{E}_{\mathfrak{q}}^{j-1}$ is torsion-free for all $1 \leq j \leq s$. If $\dim R_{\mathfrak{p}} = d$, by part (a) we have that

$$\text{depth}\left(\frac{\tilde{E}_{\mathfrak{q}}^j}{x\tilde{E}_{\mathfrak{q}}^{j-1}}\right) \geq \dim \tilde{R}_{\mathfrak{q}} - j \geq d - j,$$

and the latter is at least 1 since $d \leq s + 1$. This completes the proof. ■

We are now ready to state and prove our module version of Theorem 3.1.7.

Theorem 4.3.2 *Let R be a Gorenstein local ring of dimension d . Let E be a finite, torsion-free, orientable R -module with $\text{rank } E = e > 0$ and $\ell(E) = \ell$. Assume that E satisfies $G_{\ell-e+2}$ and that $\text{depth } E^j \geq d - j$ for $1 \leq j \leq \ell - e - 1$. Then, E is of linear type and $\mathcal{R}(E)$ is Cohen-Macaulay.*

Proof. Without loss of generality, we may assume that E is not free. Let $E = Ra_1 + \dots + Ra_n$. Since E is torsion-free, orientable and satisfies $G_{\ell-e+2}$, by Theorem 2.6.2, $E'/F' \cong J$ and $E''/F'' \cong I$, where I and J are ideals of height at least 2, satisfying $G_{\ell-e+2}$, i.e. $G_{\ell(I)+1}$ (see Proposition 2.6.9). Moreover, since E is orientable, we may assume that $g \geq 2$.

By Theorem 2.6.4, it suffices to show that I is of linear type and $\mathcal{R}(I)$ is Cohen-Macaulay. If $e = 1$, then $R = R''$ and $E \cong I$, so by assumption $\text{depth } I^j \geq d - j \geq d - j - g + 2$ for $1 \leq j \leq \ell - e - 1 = \ell(I) - 2$, hence for $1 \leq j \leq \ell(I) - g$. Then, Theorem 3.1.7 implies that I is of linear type and $\mathcal{R}(I)$ is Cohen-Macaulay.

So, assume that $e \geq 2$. We induct on $d \geq 2$. If $d = 2$, then $\ell(I) = d = 2 = g$, and I is G_∞ . This implies that I is a complete intersection, whence I satisfies sliding depth. Therefore, Theorem 3.1.3 implies that I is of linear type and $\mathcal{R}(I)$ is Cohen-Macaulay. So, we may assume that $d > 2$.

Notice that, by Theorem 3.1.7, it suffices to show that

$$\text{depth } I^j \geq d - j \geq d - j - g + 2 \quad \text{for } 1 \leq j \leq \ell - e - 1 = \ell(I) - 2.$$

By Lemma 4.2.2 and Lemma 4.2.3, this follows once we show that the complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all $\mathfrak{q} \in \text{Spec}(R')$ with $\dim R'_{\mathfrak{q}} \leq \ell - e$ and all $0 \leq j \leq \ell - e - 1$. In turn, by Theorem 2.6.5 we only need to show that $(E'/F')_{\mathfrak{q}} \cong J_{\mathfrak{q}}$ is of linear type for any such prime \mathfrak{q} .

For any such \mathfrak{q} , let $\mathfrak{p} = \mathfrak{q} \cap R$. Then, notice that $E_{\mathfrak{p}}$ is a finite, torsion-free, orientable $R_{\mathfrak{p}}$ -module of rank e and analytic spread $\ell(E_{\mathfrak{p}}) \leq \ell$. Moreover, since $R_{\mathfrak{p}} \rightarrow \tilde{R}_{\mathfrak{q}}$ is a flat local map, we have that

$$\text{depth } E_{\mathfrak{p}}^j \geq \dim R_{\mathfrak{p}} - j \quad \text{for } 1 \leq j \leq \ell - e - 1,$$

hence for $1 \leq j \leq \ell(E_{\mathfrak{p}}) - e - 1$. Since $\dim R_{\mathfrak{p}} \leq \dim R'_{\mathfrak{q}} \leq \ell - e - 1 \leq d - 1$, by the induction hypothesis $E_{\mathfrak{p}}$ is of linear type and $\mathcal{R}(E_{\mathfrak{p}})$ is Cohen-Macaulay.

Now, we apply Lemma 4.3.1 to $E_{\mathfrak{p}}$. Let $\widetilde{R}_{\mathfrak{p}} := R_{\mathfrak{p}}[Z_1, \dots, Z_n]$, $\widetilde{E}_{\mathfrak{p}} := E_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widetilde{R}_{\mathfrak{p}}$ and $x := \sum_{i=1}^n Z_i \frac{a_i}{1} \in \widetilde{E}_{\mathfrak{p}}$ be as in Lemma 4.3.1. Since $E_{\mathfrak{p}}$ is of linear type and $\mathcal{R}(E_{\mathfrak{p}})$ is Cohen-Macaulay, by Theorem 2.6.4(c) it follows that $(\widetilde{E}_{\mathfrak{p}}/\widetilde{R}_{\mathfrak{p}}x)$ is of linear type. Hence, by Lemma 4.3.1 we have that

$$\frac{\widetilde{E}_{\mathfrak{p}}^j}{x\widetilde{E}_{\mathfrak{p}}^{j-1}} \cong \left(\frac{\widetilde{E}_{\mathfrak{p}}}{\widetilde{R}_{\mathfrak{p}}x} \right)^j$$

and $\text{depth}(\widetilde{E}_{\mathfrak{p}}/\widetilde{R}_{\mathfrak{p}}x) \geq \dim \widetilde{R}_{\mathfrak{p}} - j$, for all $1 \leq j \leq \ell - e - 1$. Therefore, the same conclusions hold after tensoring with the ring $S := \widetilde{R}_{\mathfrak{p}}_{\mathfrak{m}\widetilde{R}_{\mathfrak{p}}}$, where \mathfrak{m} is the maximal ideal of R .

In particular, the S -module $(SE_{\mathfrak{p}}/Sx)$ satisfies the same assumptions as the $R_{\mathfrak{p}}$ -module $E_{\mathfrak{p}}$. Hence, we can iterate Lemma 4.3.1 $e - 1$ times, to obtain that a generic Bourbaki ideal of $E_{\mathfrak{p}}$, constructed as in Definition 2.6.3 with respect to the generators $\frac{a_1}{1}, \dots, \frac{a_n}{1}$, satisfies $\text{depth}(E_{\mathfrak{p}}''/F_{\mathfrak{p}}'')^j \geq d - j$ for $1 \leq j \leq \ell - e - 1$. The same ideal also satisfies $G_{\ell-e+2}$, hence $G_{\ell(E_{\mathfrak{p}})-e+2}$, since $E_{\mathfrak{p}}$ does. Thus, by Theorem 3.1.7 it is of linear type, with Cohen-Macaulay Rees algebra.

This implies that $E_{\mathfrak{p}}$ is of linear type, whence finally $J_{\mathfrak{q}}$ is of linear type by Theorem 2.6.4(c). ■

4.4 The case of ideal modules

In this section we discuss the Cohen-Macaulay property of Rees algebras of ideal modules.

Definition 4.4.1 Let R be a Noetherian ring. An R -module $E \neq 0$ is called an *ideal module* if E is finitely generated and torsion-free, and moreover E^{**} is free, where $-^*$ denotes the functor $\text{Hom}_R(-, R)$.

Examples of ideal modules include R -ideals of grade at least 2, and finite direct sums of such ideals. Moreover, the Jacobian module of a normal complete intersection ring $R = k[Y_1, \dots, Y_m]/J$ over a perfect field k is an ideal module.

Ideal modules behave similarly to ideals of positive grade. In fact, we have the following result, due to Simis, Ulrich and Vasconcelos (see [65, 5.1 and 5.2]).

Proposition 4.4.2 *Let R be a Noetherian ring, $E \neq 0$ an R -module.*

- (a) *E is an ideal module if and only if E embeds into a finite free module G with $\text{grade}(G/E) \geq 2$. In particular, if E is an ideal module, then E has a rank, and $\text{rank } E = \text{rank } G$.*
- (b) *Assume that E is an ideal module with $\text{rank } E = e$. Then, $E_{\mathfrak{p}}$ is not free if and only if $\mathfrak{p} \in V(\text{Fitt}_e(E)) = \text{Supp}(G/E)$.*
- (c) *Assume that R is local and E is an ideal module of rank e that is not free. Then, the analytic spread of E satisfies $\ell(E) \geq c - e + 1$, where $c := \text{ht}(\text{Fitt}_e(E))$.*

In the case when $e = 1$, an ideal module E is isomorphic to an R -ideal I with $\text{grade}(I) \geq 2$. So in (a) one can choose $G = R$, while (b) and (c) simply mean that $\ell(I) \geq \text{ht}(I) = c$.

Generic Bourbaki ideals of ideal modules have good residual intersection properties, as shown by the following result.

Theorem 4.4.3 ([65, 5.3]). *Let R be a Cohen-Macaulay local ring. Every ideal module E admits a generic Bourbaki ideal I of height at least two. Moreover, if E is free locally in codimension $s - 1$, then I satisfies G_s and AN_{s-1} .*

Due to the relevance of the Artin-Nagata condition in our discussion on the Cohen-Macaulay property of Rees algebras of modules, it is natural to ask whether Theorem 4.4.3 can be exploited in order to extend Theorem 4.1.5 to ideal modules. The next result is a minor variation of a result of Lin [46, 4.3], with a different - and technically simpler - proof.

Theorem 4.4.4 *Let R be a local Cohen-Macaulay ring, and let E be an ideal module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that the following conditions hold.*

- (a) $r(E) \leq k$, where k is an integer such that $1 \leq k \leq \ell - e$.
- (b) E is free locally in codimension $\ell - e - \min\{2, k\}$, and satisfies $G_{\ell-e+1}$.
- (c) $\text{depth } E^j \geq d - \ell + e + k - j$ for $1 \leq j \leq k$.

Then, $\mathcal{R}(E)$ is Cohen-Macaulay.

Proof. Since E is an ideal module, then E admits a generic Bourbaki ideal I with $\text{ht } I = g \geq 2$ and $r(I) \leq r(E) \leq k$, which satisfies $G_{\ell-e+1}$, that is, $G_{\ell(I)}$ (see Proposition 2.6.9). Moreover, since E is free locally in codimension $\ell - e - \min\{2, k\}$, by Theorem 4.4.3 I satisfies $AN_{\ell-e-\min\{2,k\}}$. We next prove that

$$\text{depth } I^j \geq d - \ell(I) + k - j + 1 = d - \ell + k - j + e \quad \text{for } 1 \leq j \leq k.$$

Hence, by Theorem 4.1.5 it would follow that $\mathcal{R}(I)$ is Cohen-Macaulay, whence the proof would be complete thanks to Theorem 2.6.4.

The depth condition above is clearly satisfied if $e = 1$, so assume that $e \geq 2$. By Lemma 4.2.2 (with $s = d - \ell + e + k$) and Lemma 4.2.3 (with $s = \ell - e$), it suffices to show that the complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all $\mathfrak{q} \in \text{Spec}(R')$ with $\dim R'_{\mathfrak{q}} \leq \ell - e$ and all $0 \leq j \leq k$.

For each such \mathfrak{q} , let $\mathfrak{p} = \mathfrak{q} \cap R$. Then, $E_{\mathfrak{p}}$ is an ideal module, which is free locally in codimension $\ell - e - \min\{2, k\}$, and satisfies $G_{\ell-e+1}$, i.e. G_{∞} . Hence, $E'_{\mathfrak{p}}/F'_{\mathfrak{p}}$ is isomorphic to an $R'_{\mathfrak{p}}$ ideal which satisfies G_{∞} and $AN_{\ell-e-\min\{2,k\}}$. Then, by Proposition 2.7.9, its localization $(E'/F')_{\mathfrak{q}}$ satisfies G_{∞} and $AN_{\ell-e-\min\{2,k\}}$, whence also $AN_{\dim R'_{\mathfrak{q}}-2}$, since $\dim R'_{\mathfrak{q}} \leq \ell - e$. By Theorem 2.7.6(c) it then follows that $(E'/F')_{\mathfrak{q}}$ satisfies sliding depth. Hence, it is of linear type by Theorem 3.1.3, so that the complexes $(\mathbb{C}'_j)_{\mathfrak{q}}$ are exact for all $0 \leq j \leq k$, thanks to Theorem 2.6.5. ■

5. REES ALGEBRAS OF MODULES OF PROJECTIVE DIMENSION ONE

In this chapter we are interested in determining the defining ideal of Rees algebras of modules. The problem in its full generality is wide open, however we are able to describe the defining ideal of $\mathcal{R}(E)$ in the case when E has projective dimension one and is *almost linearly presented*, i.e. the entries of a presentation matrix of E are linear, except possibly for those of one column, homogeneous of degree $m \geq 1$. Our main result, Theorem 5.2.6, extends to modules a previously-known result of Boswell and Mukundan for almost linearly presented perfect ideals of height two (see Theorem 5.1.10 below).

5.1 Background

Let $R^s \xrightarrow{\varphi} R^n \twoheadrightarrow E$ be a presentation of E . With abuse of notation, we also let φ denote the $n \times s$ matrix associated with φ . Recall from Chapter 2 that the symmetric algebra of E is

$$\mathcal{S}(E) = R[T_1, \dots, T_n]/(\ell_1, \dots, \ell_s),$$

where ℓ_1, \dots, ℓ_s are linear forms in $R[T_1, \dots, T_n]$ so that $[T_1, \dots, T_n] \cdot \varphi = [\ell_1, \dots, \ell_s]$. This description is independent of the choice of φ .

Let $\mathcal{L} = (\ell_1, \dots, \ell_s)$. Then, whenever E has a rank, one can write

$$\mathcal{R}(E) = R[T_1, \dots, T_n]/\mathcal{J},$$

for some ideal \mathcal{J} containing \mathcal{L} . \mathcal{J} is called the *defining ideal* of $\mathcal{R}(E)$, and its generators are called the *defining equations* of $\mathcal{R}(E)$.

The inclusion $\mathcal{L} \subseteq \mathcal{J}$ is strict whenever E is not of linear type, and the problem of determining the remaining generators of \mathcal{J} is then usually approached by exploiting the structure of the presentation matrix φ .

In particular, a good amount of the literature on the subject focuses on perfect ideals of grade two and perfect Gorenstein ideals of grade three (see for instance [5, 38, 45, 48, 49, 53, 71]), thanks to the fact that the presentation matrices of ideals of these kinds must obey to prescribed structures. In fact, perfect ideals of grade two can be characterized in terms of properties of the ideal of maximal minors of their presentation matrix (see [7, Theorem 1.4.16] for a proof).

Theorem 5.1.1 (Hilbert-Burch) *Let R be a Noetherian ring, I an ideal with a presentation of the form*

$$0 \rightarrow R^{n-1} \xrightarrow{\varphi} R^n \rightarrow I \rightarrow 0$$

Then, $I = aI_{n-1}(\varphi)$ for some nonzerodivisor a of R , and $\text{ht}(I_{n-1}(\varphi)) = 2$ whenever $I_{n-1}(\varphi) \neq R$. Conversely, let φ be any $n \times (n-1)$ matrix with coefficients in R , and let a be a nonzerodivisor of R . If $\text{ht}(I_{n-1}(\varphi)) \geq 2$, then $I = aI_{n-1}(\varphi)$ has a presentation of the form $0 \rightarrow R^{n-1} \xrightarrow{\varphi} R^n \rightarrow I \rightarrow 0$.

A structure theorem for perfect Gorenstein ideals of grade three was given by Buchsbaum and Eisenbud, in terms of Pfaffians of their presentation matrix. We omit the statement here, as it will not be relevant for the rest of this thesis. However, we refer the interested readers to [7, Theorem 3.4.1].

In many cases of interest, when $\mathcal{L} \subsetneq \mathcal{J}$, at least part of the information on the structure of \mathcal{J} is encoded in some auxiliary matrix associated with φ , namely the so called *Jacobian dual* of φ , introduced by Vasconcelos in [74]. Although the notion makes sense over any Noetherian ring, for simplicity we recall the construction in the case when R is a polynomial ring over a field, as this will be the case for the rest of this chapter.

Setting 5.1.2 Let $R = k[Y_1, \dots, Y_d]$ be a standard graded polynomial ring over a field k and let T_1, \dots, T_n be indeterminates. On $S = R[T_1, \dots, T_n]$ define a

bigrading by setting $\deg(Y_i) = (1, 0)$ and $\deg(T_i) = (0, 1)$, and set $\underline{Y} = Y_1, \dots, Y_d$, $\underline{T} = T_1, \dots, T_n$. Let

$$R^s \xrightarrow{\varphi} R^n$$

be an $n \times s$ matrix whose entries are homogeneous of constant \underline{Y} -degrees $\delta_1, \dots, \delta_s$ along each column and assume that $I_1(\varphi) \subseteq (\underline{Y})$.

Definition 5.1.3 ([74]). With R, S and φ as in Setting 5.1.2, let ℓ_1, \dots, ℓ_s be linear forms in the T_i variables, satisfying the equation

$$[\ell_1, \dots, \ell_s] = [\underline{T}] \cdot \varphi, \quad (5.1)$$

that is, ℓ_1, \dots, ℓ_s generate the defining ideal of the symmetric algebra $\mathcal{S}(\text{coker}(\varphi))$. Since the entries of φ are contained in (\underline{Y}) , the equation above can be rewritten as

$$[\ell_1, \dots, \ell_s] = [\underline{T}] \cdot \varphi = [\underline{Y}] \cdot B(\varphi) \quad (5.2)$$

where $B(\varphi)$ is a $d \times s$ matrix whose entries are linear in the T_i variables, and homogeneous of constant \underline{Y} -degrees $\delta_1 - 1, \dots, \delta_s - 1$ along each column. $B(\varphi)$ is called a *Jacobian dual* of φ .

Notice that $B(\varphi)$ is not necessarily unique, unless the entries of φ are all linear, in which case $B(\varphi)$ has entries in the polynomial ring $T = k[T_1, \dots, T_n]$. Moreover, by Cramer's rule one always has $\mathcal{L} + I_d(B(\varphi)) \subseteq \mathcal{J}$. This inclusion is often strict, however equality holds if $\text{coker}(\varphi)$ is a perfect ideal of grade 2 and the entries of φ are all linear, as described by the following result (see [49, 1.3]).

Theorem 5.1.4 (Morey-Ulrich) *Let $R = k[Y_1, \dots, Y_d]$ be a standard graded polynomial ring over an infinite field k . Let I be a perfect ideal of grade 2 satisfying G_d , with $n = \mu(I) > d$. Assume that I admits a linear presentation*

$$R^{n-1} \xrightarrow{\varphi} R^n \twoheadrightarrow I.$$

Then, $\ell(I) = d$, $r(I) = \ell(I) - 1$, $\mathcal{R}(I)$ is Cohen-Macaulay and its defining ideal is $\mathcal{J} = \mathcal{L} + I_d(B(\varphi))$.

The key point in the proof of this theorem is that, under these assumptions, the ideal $\mathcal{L} + I_d(B(\varphi)) = ([\underline{Y}] \cdot B(\varphi)) + I_d(B(\varphi))$ is a geometric d -residual intersection of (\underline{Y}) (see [49, proof of Theorem 1.2]). The assumption that I is G_d is crucial in the proof, since, together with the fact that I satisfies sliding depth, it implies that I is of linear type on the punctured spectrum of R , thanks to Theorem 3.1.3. In this case, one can use local cohomology to study the defining ideal of $\mathcal{R}(I)$, thanks to the following observation.

Remark 5.1.5 Let (R, \mathfrak{m}) be a Noetherian local ring, and I an R -ideal of positive grade. Assume that I is of linear type on the punctured spectrum of R , and let \mathcal{A} be the kernel of the natural map $\mathcal{S}(I) \twoheadrightarrow \mathcal{R}(I)$. Then,

$$\mathcal{A} = H_{\mathfrak{m}}^0(\mathcal{S}(I)) = \bigcup_{n \geq 0} (0 :_{\mathcal{S}(I)} \mathfrak{m}^n).$$

In fact, an investigation of the local cohomology modules $H_{\mathfrak{m}}^i(\mathcal{S}(I))$ via approximate resolutions of the symmetric algebra made it possible to describe the defining ideal of $\mathcal{R}(I)$ in the case when I is linearly presented perfect Gorenstein ideal of grade three (see [44, 45]).

Theorem 5.1.6 (Kustin-Polini-Ulrich) ([45, 9.1]). *Let $R = k[Y_1, \dots, Y_d]$ be a standard graded polynomial ring over a field k , let I be a perfect Gorenstein ideal of grade three with $\mu(I) = n$, and assume that I has a minimal linear presentation φ . Let $S = R[T_1, \dots, T_n]$, and $T = k[T_1, \dots, T_n]$. Then, the defining ideal of $\mathcal{R}(I)$ is*

$$\mathcal{J} = \mathcal{L} + I_d(B(\varphi))S + C(\varphi)S,$$

where $C(\varphi)$ is the content ideal of a submaximal Pfaffian of the matrix

$$\begin{bmatrix} \varphi & -B(\varphi)^t \\ B(\varphi) & 0 \end{bmatrix}.$$

Notice that in this case $\mathcal{L} + I_d(B(\varphi)) \subsetneq \mathcal{J}$. In fact, as we will discuss in the following subsection, the equality occurs very rarely even in the case of perfect ideals of grade two.

5.1.1 Perfect ideals of grade two

For perfect ideals of grade two which are not necessarily linearly presented, the defining ideal of the Rees algebra can be described in terms of the degrees of the columns of the presentation matrix.

Theorem 5.1.7 (Kustin-Polini-Ulrich) ([45, 6.1]). *Let $R = k[Y_1, \dots, Y_d]$ be a standard graded polynomial ring over a field k , and let I be a perfect ideal of grade two with $\mu(I) = n$. For $1 \leq i \leq n - 1$, let $\epsilon_1 \geq \epsilon_2 \geq \dots \geq \epsilon_{n-1}$ be the degrees of the columns of a minimal presentation φ of I . Then, the defining ideal of $\mathcal{R}(I)$ is*

$$\mathcal{J} = \mathcal{L}: (Y_1, \dots, Y_d)^N,$$

where $N = 1 + \sum_{i=1}^d (\epsilon_i - 1)$.

This formula recovers the result of Morey and Ulrich when φ is linear, since in this case $\mathcal{J} = \mathcal{L}: (Y_1, \dots, Y_d) = \mathcal{L} + I_d(B(\varphi))$ (see proof of [49, 1.2]). However, if at least one of the ϵ_i is at least 2, then \mathcal{J} is strictly larger than $\mathcal{L} + I_d(B(\varphi))$.

Now, suppose that φ is *almost linear*, i.e. the entries in all columns of φ are linear, except possibly for those of one column, homogeneous of degree $m \geq 1$. In this case, the defining ideal of the Rees algebra can be described by means of the *iterated Jacobian duals* of φ , introduced by Boswell and Mukundan in [5, 4.1].

Theorem and Definition 5.1.8 *Let $R = k[Y_1, \dots, Y_d]$ and $S = R[T_1, \dots, T_n]$ be polynomial rings with gradings as in Setting 5.1.2, and let $\varphi: R^s \rightarrow R^n$. For a Jacobian dual $B(\varphi)$ of φ , set $B_1(\varphi) := B(\varphi)$. Assume that matrices $B_j(\varphi)$ with d rows have been inductively constructed for $1 \leq j \leq i$, such that each $B_j(\varphi)$ has homogeneous entries of constant \underline{Y} -degrees and \underline{T} -degrees along each column.*

Then, there exists a matrix C_i , with entries in S which are homogeneous of constant \underline{Y} -degrees and \underline{T} -degrees in each column, such that $B_{i+1}(\varphi) := [B_i(\varphi) | C_i]$ satisfies

$$(\underline{Y} \cdot B_i(\varphi)) + (I_d(B_i(\varphi)) \cap (\underline{Y})) = (\underline{Y} \cdot B_i(\varphi)) + (\underline{Y} \cdot C_i). \quad (5.3)$$

where, for a matrix A , $(\underline{Y} \cdot A)$ denotes the ideal generated by the entries of the row vector $[\underline{Y}] \cdot A$.

A matrix $B_i(\varphi)$ as above is called an i -th iterated Jacobian dual of φ .

Iterated Jacobian duals satisfy the following properties (see [5, 4.2 and 4.5]).

Theorem 5.1.9 *With R , S and φ as in Definition 5.1.8, for all $i \geq 1$ let $B_i(\varphi)$ be iterated Jacobian duals of φ . Then, the following are satisfied.*

(a) *The ideal $(\underline{Y} \cdot B(\varphi)) + I_d(B_i(\varphi))$ only depends on φ .*

(b) $(\underline{Y} \cdot B_i(\varphi)) + I_d(B_i(\varphi)) = (\underline{Y} \cdot B(\varphi)) + I_d(B_i(\varphi)) \subseteq (\underline{Y} \cdot B(\varphi)) + I_d(B_{i+1}(\varphi))$.

In particular, the algorithm that constructs iterated Jacobian duals terminates after finitely many iterations.

(c) $(\underline{Y} \cdot B(\varphi)) + I_d(B_i(\varphi)) \subseteq ((\underline{Y} \cdot B(\varphi)) : (\underline{Y})^i)$.

In the case when φ is the minimal presentation of a perfect ideal of grade two and it is almost linear, the iterated Jacobian duals of φ help describe the defining ideal of the Rees algebra.

Theorem 5.1.10 ([5, 5.3 and 5.6]). *Let $R = k[Y_1, \dots, Y_d]$ be a standard graded polynomial ring over a field k . Let I be a perfect ideal of height 2 admitting an almost linear presentation. Assume that I is G_d and $\mu(I) = d + 1$. Then, the defining ideal of the Rees algebra $\mathcal{R}(I)$ is*

$$\mathcal{J} = \mathcal{L} + I_d(B_m(\varphi)) = \mathcal{L} : (Y_1, \dots, Y_d)^m.$$

Moreover, $\mathcal{R}(I)$ is almost Cohen-Macaulay, i.e. $\text{depth } \mathcal{R}(I) \geq d - 1$, and is not Cohen-Macaulay if $m > 1$.

Notice that this result recovers Theorem 5.1.4 if $m = 1$. The main goal of this chapter is to extend Theorem 5.1.10 to modules, which we do in Theorem 5.2.6.

5.2 Modules of projective dimension one

Notice that perfect ideals of grade two are R -modules of projective dimension one and rank one. Our goal in this section will be to extend this result to modules of projective dimension one with arbitrary rank.

Inspired by [65, 4.11], we approach the problem using generic Bourbaki ideals. In fact, it is always possible to relate a presentation matrix of E with a presentation matrix of a generic Bourbaki ideal I of E , as observed in the following remark.

Remark 5.2.1 Let $R = k[Y_1, \dots, Y_d]_{(Y_1, \dots, Y_d)}$, where k is a field. A minimal presentation

$$R^s \xrightarrow{\varphi} R^n \rightarrow E \rightarrow 0$$

of E induces a minimal presentation of a generic Bourbaki ideal I of E as follows. With Z and x_j as in Notation 2.6.1, by possibly multiplying φ from the left by an invertible matrix with coefficients in $k(Z)$, we may assume that φ presents E'' with respect to a minimal generating set of the form $x_1, \dots, x_{e-1}, a_e, \dots, a_n$. Then, $\varphi = \begin{bmatrix} A \\ \psi \end{bmatrix}$, where A and ψ are submatrices of size $(e-1) \times s$ and $(n-e+1) \times s$, respectively. By construction, ψ is a presentation of I , and is minimal since $\mu(I) = \mu(E) - e + 1 = n - e + 1$. Also, if the entries of φ are homogeneous polynomials of constant degrees $\delta_1, \dots, \delta_s$ along each column, then the entries of ψ are homogeneous polynomials of constant \underline{Y} -degrees $\delta_1, \dots, \delta_s$ along each column.

Notice that, if E is a module of projective dimension one and satisfies G_2 , then E is torsion-free, whence by Theorem 2.6.2 it admits a generic Bourbaki ideal I , which is perfect of grade 2. Hence, it is natural to ask whether in this case the defining ideal of $\mathcal{R}(E)$ has the same shape as the defining ideal of $\mathcal{R}(I)$. This was shown to be the case when the presentation matrix of E is linear in [65, 4.11].

Theorem 5.2.2 (Simis-Ulrich-Vasconcelos) *Let $R = k[Y_1, \dots, Y_d]$ be a polynomial ring over a field k , where $d \geq 2$. Let E be a finite R -module with projective dimension one satisfying G_d . Assume that E admits a minimal free resolution*

$$0 \rightarrow R^{n-e} \xrightarrow{\varphi} R^n \rightarrow E \rightarrow 0$$

where the entries of φ are linear. Then, $\mathcal{R}(E)$ is Cohen-Macaulay, with defining ideal

$$\mathcal{J} = (\underline{Y} \cdot B(\varphi)) : (\underline{Y}) = (\underline{Y} \cdot B(\varphi)) + I_d(B(\varphi)).$$

In the next subsection we discuss the case when φ is almost linear. Our main result, Theorem 5.2.6, extends Theorem 5.1.10 to modules.

5.2.1 Almost linearly presented modules of projective dimension one

For the remaining part of this section, we will assume the following.

Setting 5.2.3 Let $R = k[Y_1, \dots, Y_d]$ be a polynomial ring over a field k , where $d \geq 2$. Let E be a finite R -module with projective dimension one, satisfying G_d . Let $n = \mu(E)$ and let

$$0 \rightarrow R^{n-e} \xrightarrow{\varphi} R^n \rightarrow E \rightarrow 0$$

be a minimal free resolution of E . Assume that φ is *almost linear*, i.e. has linear entries, except possibly for those in the last column, which are homogeneous of degree $m \geq 1$.

Remark 5.2.4 In the situation of Setting 5.2.3, one has the following.

- (i) Since E is G_d and $d \geq 2$, then E is torsion-free. Hence, after localizing at the unique homogeneous maximal ideal, by Theorem 2.6.2 E admits a generic Bourbaki ideal I , which is perfect of grade 2. Now, let ψ be a minimal presentation of I obtained from φ as in Remark 5.2.1. Since by construction ψ is almost linear, it follows that I satisfies Theorem 5.1.10.

- (ii) Since E satisfies G_d , by Theorem 3.2.6 it follows that $E_{\mathfrak{p}}$ is of linear type for all $\mathfrak{p} \in \text{Spec}(R) \setminus V(R_+)$. As a consequence, if \mathcal{L} is the defining ideal of $\mathcal{S}(E)$, then the defining ideal \mathcal{J} of $\mathcal{R}(E)$ satisfies $\mathcal{J} \supseteq \mathcal{L} : (Y_1, \dots, Y_d)^i$ for all i .

In the linearly presented case, the key ingredient in the proof of Theorem 5.2.2 is that, in this situation, $\mathcal{R}(I)$ is Cohen-Macaulay (thanks to Theorem 5.1.4). This implies that $\mathcal{R}(I)$ is a deformation of $\mathcal{R}(E)$ (see Theorem 2.6.4), and the latter is crucial in order to ‘lift’ the shape of the defining ideal of $\mathcal{R}(I)$ back to $\mathcal{R}(E)$.

However, in the case when the presentation matrix is almost linear, by Theorem 5.1.10, $\mathcal{R}(I)$ is only almost Cohen-Macaulay, and in general not Cohen-Macaulay. In this case, it is not obvious that $\mathcal{R}(I)$ is a deformation of $\mathcal{R}(E)$. However, this will follow from the following technical result, which will also allow to transfer the almost Cohen-Macaulay property from $\mathcal{R}(I)$ to $\mathcal{R}(E)$. The proof will make use of [65, 3.6] (stated later as Theorem 6.2.6), which is a slightly more general version of Theorem 2.6.10.

Theorem 5.2.5 *Let R be a Noetherian local ring, E a finite R -module with $\text{rank } E = e \geq 2$. Let $U = Ra_1 + \dots + Ra_n$ be a reduction of E , Z_1, \dots, Z_n indeterminates. Let $R'' := R(Z_1, \dots, Z_n)$, $E'' := E \otimes_R R''$, $x := \sum_{i=1}^n Z_i a_i$, $\overline{E}'' := E''/(x)$.*

Assume that $\text{depth } \mathcal{R}(\overline{E}''_{\mathfrak{q}}) \geq 2$ for all $\mathfrak{q} \in \text{Spec}(R'')$ such that $\overline{E}''_{\mathfrak{q}}$ is not of linear type. Then, the natural epimorphism $\pi: \mathcal{R}(E'')/(x) \twoheadrightarrow \mathcal{R}(\overline{E}'')$ is an isomorphism, and x is regular on $\mathcal{R}(E'')$.

Proof. We modify the proof of [65, 3.7] (stated later as Theorem 6.2.7). Since x is regular on $\mathcal{R}(E'')$ by [65, 3.6], we only need to show that $K = \ker(\pi)$ is zero. In fact, we only need to prove this locally at primes $\mathfrak{q} \in \text{Spec}(R'')$ such that $\overline{E}''_{\mathfrak{q}}$ is not of linear type. Indeed, if $\overline{E}''_{\mathfrak{q}}$ is of linear type, then $\mathcal{R}(\overline{E}''_{\mathfrak{q}}) \cong \mathcal{S}(\overline{E}''_{\mathfrak{q}})$ is isomorphic to $\mathcal{S}(E''_{\mathfrak{q}})/(x)$ by construction, whence $K_{\mathfrak{q}} = 0$.

Let $\overline{\mathcal{R}}''$ denote $\mathcal{R}(E'')/(x)$ and let $M = (\mathfrak{m}, \overline{\mathcal{R}}''_+)$ be the unique homogeneous maximal ideal of $\mathcal{R}(E'')$. Notice that $K \subseteq H_M^0(\overline{\mathcal{R}}'')$. In fact, after localizing R'' if needed, we may assume that K vanishes locally on the punctured spectrum of R'' .

Hence, K is annihilated by a power of \mathfrak{m} . Also, by [65, 3.6] it follows that K is annihilated by a power of $U\overline{\mathcal{R}''}$, and hence by a power of $E\overline{\mathcal{R}''} = (\overline{\mathcal{R}''})_+$, since E is integral over U .

Thus, it suffices to show that $H_{M_{\mathfrak{q}}}^0(\overline{\mathcal{R}''}_{\mathfrak{q}}) = 0$ for all $\mathfrak{q} \in \text{Spec}(R'')$ such that $\overline{E''}_{\mathfrak{q}}$ is not of linear type. Consider the long exact sequence of local cohomology induced by the exact sequence

$$0 \rightarrow K_{\mathfrak{q}} \rightarrow \overline{\mathcal{R}''}_{\mathfrak{q}} \rightarrow \mathcal{R}(\overline{E''}_{\mathfrak{q}}) \rightarrow 0.$$

Since by assumption $\text{depth } \mathcal{R}(\overline{E''}_{\mathfrak{q}}) \geq 2$, then $H_{M_{\mathfrak{q}}}^i(\overline{\mathcal{R}''}_{\mathfrak{q}}) \cong H_{M_{\mathfrak{q}}}^i(K_{\mathfrak{q}})$ for $i = 0, 1$. In particular, since $K_{\mathfrak{q}} \subseteq H_{M_{\mathfrak{q}}}^0(\overline{\mathcal{R}''}_{\mathfrak{q}})$, it follows that $0 = H_{M_{\mathfrak{q}}}^1(K_{\mathfrak{q}}) \cong H_{M_{\mathfrak{q}}}^1(\overline{\mathcal{R}''}_{\mathfrak{q}})$. Therefore, the exact sequence

$$0 \rightarrow \mathcal{R}(E''_{\mathfrak{q}})(-1) \xrightarrow{x} \mathcal{R}(E''_{\mathfrak{q}}) \longrightarrow \overline{\mathcal{R}''}_{\mathfrak{q}} \rightarrow 0$$

induces the exact sequence

$$0 \rightarrow H_{M_{\mathfrak{q}}}^0(\overline{\mathcal{R}''}_{\mathfrak{q}}) \longrightarrow H_{M_{\mathfrak{q}}}^1(\mathcal{R}(E''_{\mathfrak{q}}))(-1) \xrightarrow{x} H_{M_{\mathfrak{q}}}^1(\mathcal{R}(E''_{\mathfrak{q}})) \rightarrow 0.$$

Now, similarly as in [65, 3.7], one can show that $H_{M_{\mathfrak{q}}}^1(\mathcal{R}(E''_{\mathfrak{q}}))$ is finitely generated, as a consequence of the graded version of the Local Duality Theorem. Thus, by the graded version of Nakayama's Lemma, it follows that $H_{M_{\mathfrak{q}}}^1(\mathcal{R}(E''_{\mathfrak{q}})) = 0$, whence also $H_{M_{\mathfrak{q}}}^0(\overline{\mathcal{R}''}_{\mathfrak{q}}) = 0$. ■

We are finally ready to state and prove the main result of this chapter.

Theorem 5.2.6 *Under the assumptions of Setting 5.2.3, set $\underline{Y} = Y_1, \dots, Y_d$ and assume that $n = d + e$. Then, the defining ideal of $\mathcal{R}(E)$ is*

$$\mathcal{J} = ((\underline{Y} \cdot B_m(\varphi)) : (\underline{Y})^m) = (\underline{Y} \cdot B_m(\varphi)) + I_d(B_m(\varphi)),$$

where $B_m(\varphi)$ denotes an m -th iterated Jacobian dual of φ as in Definition 5.1.8.

Moreover, $\mathcal{R}(E)$ is almost Cohen-Macaulay, and it is Cohen-Macaulay if and only if $m = 1$.

Proof. We modify the proof of [65, 4.11]. Let a_1, \dots, a_n be a minimal generating set for E corresponding to the presentation φ , and let $R[T_1, \dots, T_n] \twoheadrightarrow \mathcal{R}(E)$ be the natural epimorphism, mapping T_i to a_i for all i . Localizing at the unique homogeneous maximal ideal, we may assume that R is local and that E admits a generic Bourbaki ideal I , which is perfect of grade 2 and such that $\mu(I) = n - e + 1 = d + 1$. If $e = 1$, then $E \cong I$ and the statement follows from Theorem 5.1.10.

So, assume that $e \geq 2$. With x_j as in Notation 2.6.1, for $1 \leq j \leq e - 1$ set $X_j = \sum_{i=1}^n Z_{ij}T_i$, and note that X_j is mapped to x_j under the epimorphism $R''[T_1, \dots, T_n] \twoheadrightarrow \mathcal{R}(E'')$. Set $\underline{T} = [T_1, \dots, T_n]$. As in Remark 5.2.1, we can construct a minimal, almost linear presentation ψ of I , such that

$$[\underline{Y}] \cdot B(\varphi) \equiv [\underline{T}] \cdot \begin{bmatrix} 0 \\ \psi \end{bmatrix} \text{ modulo } (X_1, \dots, X_{e-1}).$$

Let $B(\psi)$ be a Jacobian dual of ψ defined by $[\underline{T}] \cdot \begin{bmatrix} 0 \\ \psi \end{bmatrix} = [\underline{Y}] \cdot B(\psi)$. Then, by Theorem 5.1.10, the defining ideal of $\mathcal{R}(I)$ is

$$(\underline{Y} \cdot B(\psi)) + I_d(B_m(\psi)) = ((\underline{Y} \cdot B(\psi)) : (\underline{Y})^m),$$

where m is the degree of the non-linear column of φ . Moreover, $\mathcal{R}(I)$ is almost Cohen-Macaulay, and Cohen-Macaulay if and only if ψ is linear.

In particular, $\text{depth } \mathcal{R}(I) \geq \dim \mathcal{R}(I) - 1 = d \geq 2$. Also, since E satisfies G_d , by Theorem 3.2.6 it follows that $E''_{\mathfrak{q}}$ is of linear type for all \mathfrak{q} in the punctured spectrum of R'' . Hence, inducting on e and using Theorem 5.2.5, we obtain that $\mathcal{R}(I) \cong \mathcal{R}(E'')/(F'')$ and x_1, \dots, x_{e-1} form a regular sequence on $\mathcal{R}(E'')$. Thus, if \mathcal{J} is the defining ideal of $\mathcal{R}(E)$, it follows that X_1, \dots, X_{e-1} form a regular sequence modulo $\mathcal{J}R''$ and that $\mathcal{R}(E'')$ is almost Cohen-Macaulay. Hence, $\mathcal{R}(E)$ is almost Cohen-Macaulay, and, by Theorem 2.6.4, it is Cohen-Macaulay if and only if $\mathcal{R}(I)$ is. The discussion above shows that this happens if and only if φ is linear.

Now, notice that the defining ideal of $\mathcal{R}(I)$ is $\mathcal{J}R'' + (X_1, \dots, X_{e-1})$, and equals $(\underline{Y} \cdot B(\psi)) + I_d(B_m(\psi))$ by the discussion above. Hence, by Lemma 5.2.7 below,

$$\mathcal{J}R'' + (X_1, \dots, X_{e-1}) = (\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi)) + (X_1, \dots, X_{e-1}).$$

Also, Remark 5.2.4(ii) and Theorem 5.1.8(c) imply that

$$\mathcal{J} \supseteq ((\underline{Y} \cdot B(\varphi)) : (\underline{Y})^m) \supseteq (\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi))$$

Therefore, since X_1, \dots, X_{e-1} form a regular sequence modulo $\mathcal{J}R''$, we have that

$$\begin{aligned} \mathcal{J}R'' &= ((\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi)) + (X_1, \dots, X_{e-1})) \cap \mathcal{J}R'' \\ &= ((\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi))) + (X_1, \dots, X_{e-1}) \cap \mathcal{J}R'' \\ &= ((\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi))) + (X_1, \dots, X_{e-1}) \mathcal{J}R''. \end{aligned}$$

By the graded version of Nakayama's Lemma, this means that

$$\mathcal{J}R'' = (\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi)) = (\underline{Y} \cdot B(\varphi)) : (\underline{Y})^m,$$

hence

$$\mathcal{J} = (\underline{Y} \cdot B(\varphi)) + I_d(B_m(\varphi)) = (\underline{Y} \cdot B(\varphi)) : (\underline{Y})^m,$$

which completes the proof. ■

Lemma 5.2.7 *Let $R = k[Y_1, \dots, Y_d]_{(Y_1, \dots, Y_d)}$, and denote $\underline{Y} = Y_1, \dots, Y_d$. Let φ, ψ , $B(\psi)$, and X_1, \dots, X_{e-1} be as in the proof of Theorem 5.2.6. Then, for all i and for any Jacobian dual $B(\varphi)$ of φ , in $R''[T_1, \dots, T_n]$ we have that*

$$(\underline{Y} \cdot B(\varphi)) + I_d(B_i(\varphi)) + (X_1, \dots, X_{e-1}) = (\underline{Y} \cdot B(\psi)) + I_d(B_i(\psi)).$$

Proof. Choose $B(\psi)$ such that $[\underline{Y}] \cdot B(\psi) = [\underline{T}] \cdot \begin{bmatrix} 0 \\ \psi \end{bmatrix}$, as in the proof of Theorem 5.2.6. Then, in $R''[T_1, \dots, T_n]$ we have

$$[\underline{Y}] \cdot B(\varphi) \equiv [\underline{Y}] \cdot B(\psi) \text{ modulo } (X_1, \dots, X_{e-1}).$$

So, the statement is proved for $i = 1$. Now, let $i + 1 \geq 2$ and assume that the statement holds for $B_i(\varphi)$. Let C_i be a matrix as in Definition 5.1.8. Since

$$(\underline{Y} \cdot B_i(\varphi)) + I_d(B_i(\varphi)) \cap (\underline{Y}) = (\underline{Y} \cdot B_i(\varphi)) + (\underline{Y} \cdot C_i)$$

and the $B_i(\varphi)$ are bigraded, going modulo (X_1, \dots, X_{e-1}) , in $R''[T_1, \dots, T_n]$ we have

$$(\underline{Y} \cdot B_i(\psi)) + I_d(B_i(\psi)) \cap (\underline{Y}) = (\underline{Y} \cdot B_i(\psi)) + (\underline{Y} \cdot \overline{C_i}),$$

where $\overline{C_i}$ denotes the image of C_i modulo (X_1, \dots, X_{e-1}) .

Hence, defining $B_{i+1}(\psi) = [B_i(\psi) \mid \overline{C_i}]$, it follows that $B_{i+1}(\varphi)$ reduces to $B_{i+1}(\psi)$ modulo (X_1, \dots, X_{e-1}) . Hence, in $R''[T_1, \dots, T_n]$ we have that

$$(\underline{Y} \cdot B(\varphi)) + I_d(B_{i+1}(\varphi)) + (X_1, \dots, X_{e-1}) = (\underline{Y} \cdot B(\psi)) + I_d(B_{i+1}(\psi)),$$

as claimed. ■

6. COHEN-MACAULAY PROPERTY OF THE FIBER CONE OF A MODULE

In this chapter we investigate the Cohen-Macaulay property of the special fiber ring $\mathcal{F}(E)$ of a finite module E . We thank Jonathan Montaña for suggesting this problem, and for fruitful conversations on this topic.

In Section 6.2.1 we show that the construction of generic Bourbaki ideals allows to reduce the problem to the case of ideals, similarly as for the Cohen-Macaulayness of Rees algebras. In fact, the proof of our main result, Theorem 6.2.14, is obtained by modifying the proof of Theorem 2.6.4 for Rees algebras (see [65, proof of 3.5]).

As an application, we then provide sufficient conditions for the fiber cone of a module to be Cohen-Macaulay in Section 6.2.2. We recall the necessary background on fiber cones of ideals in the next section.

6.1 Ideals with Cohen-Macaulay fiber cone

Let R be a Cohen-Macaulay local ring and I an ideal of positive height. Unlike for the Cohen-Macaulay property of the Rees algebra $\mathcal{R}(I)$, there is no clear relation among the Cohen-Macaulayness of the special fiber ring $\mathcal{F}(I)$ and that of the associated graded ring $\mathcal{G}(I)$. In fact, D'Anna, Guerrieri and Heinzer provided an example of an ideal where $\mathcal{R}(I)$ and $\mathcal{G}(I)$ are Cohen-Macaulay but $\mathcal{F}(I)$ is not [14]. At the same time, one can construct examples of perfect ideals I of height 2 so that $\mathcal{F}(I)$ is Cohen-Macaulay while $\mathcal{R}(I)$ and $\mathcal{G}(I)$ are not (see the introduction of [10]).

Nevertheless, in some circumstances the Cohen-Macaulay property of $\mathcal{G}(I)$ implies that of $\mathcal{F}(I)$, as proved, for instance, in the following theorems.

Theorem 6.1.1 (Corso-Ghezzi-Polini-Ulrich) ([10, 3.1]). *Let R be a Noetherian local ring with infinite residue field, and let I be an ideal with $\text{ht}(I) = g$, $\ell(I) = \ell$ and $r(I) = r$. Assume that I satisfies G_ℓ , and that one of the following holds.*

- (i) *If $\mu(I) \geq \ell + 2$, then $\mathcal{F}(I)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - g\}$.*
- (ii) *If $\mu(I) = \ell + 1$, then $\mathcal{F}(I)$ has at most two homogeneous generating relations in degrees $\leq \ell - g$.*

If $\mathcal{G}(I)$ is Cohen-Macaulay, then $\mathcal{F}(I)$ is Cohen-Macaulay.

Theorem 6.1.2 (Montaño) ([47, 4.8]). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d with infinite residue field. Let I be an ideal with $\text{ht}(I) = g \geq 2$, $\ell(I) = \ell$ and $r(I) = r$. Assume that I satisfies G_ℓ and $AN_{\ell-2}$, and that $I\mathfrak{m} = J\mathfrak{m}$ for a minimal reduction J of I . Consider the following statements.*

- (i) *$\mathcal{R}(I)$ is Cohen-Macaulay.*
- (ii) *$\mathcal{G}(I)$ is Cohen-Macaulay.*
- (iii) *$\mathcal{F}(I)$ is Cohen-Macaulay and $a(\mathcal{F}(I)) \leq -g + 1$.*
- (iv) *$r \leq \ell - g + 1$.*

Then, (i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv). Moreover, all the statements are equivalent if in addition $\text{depth } I^j \geq d - g - j + 2$ for $1 \leq j \leq \ell - g + 1$.

Theorem 6.1.3 (Corso-Ghezzi-Polini-Ulrich) ([10, 3.4]). *Let R be a Cohen-Macaulay local ring with infinite residue field, and let I be a perfect ideal of height 2 with $\ell(I) = \ell$. Assume that I satisfies G_ℓ . If $\mathcal{R}(I)$ is Cohen-Macaulay, then $\mathcal{F}(I)$ is Cohen-Macaulay.*

One of the goals of this chapter is to extend these theorems to modules, which will be possible thanks to Theorem 6.2.14.

6.2 Generic Bourbaki ideals and fiber cones of modules

In this section we reexamine the construction of generic Bourbaki ideals, with the ultimate purpose of proving that it allows to reduce the problem of the Cohen-Macaulayness of the special fiber ring of a module to the case of ideals (see Theorem 6.2.14). In order to do so, and in view of the applications of Theorem 6.2.14 discussed in Section 6.2.2, it is convenient to work in a slightly more general setting than that of Notation 2.6.1. Hence, in the remaining part of this thesis we will adopt the following notation.

Notation 6.2.1 ([65, 3.3]). Let R be a Noetherian ring, E a finite R -module with $\text{rank } E = e > 0$. Let $U = Ra_1 + \cdots + Ra_n$ be a submodule of E , and let

$$Z = \{Z_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq e - 1\}$$

be a set of indeterminates. Let $R' := R[Z]$ and $E' := E \otimes_R R'$. For $1 \leq j \leq e - 1$, denote $x_j = \sum_{i=1}^n Z_{ij}a_i \in E'$ and $F' = \sum_{j=1}^{e-1} R'x_j$. If R is local with maximal ideal \mathfrak{m} , let $R'' = R(Z) = R[Z]_{\mathfrak{m}R[Z]}$ and similarly denote $E'' = E \otimes_R R''$, $F'' = F' \otimes_{R'} R''$.

This includes the situation of Notation 2.6.1 as the particular case when $U = E$. In the setting of Notation 6.2.1, the existence of generic Bourbaki ideals is guaranteed by the following result.

Theorem and Definition 6.2.2 ([65, 3.2 and 3.3]). *Let R be a Noetherian local ring, and E a finite R -module with $\text{rank } E = e > 0$. Assume that the following conditions hold.*

(i) E is torsion-free.

(ii) $E_{\mathfrak{p}}$ is free for all $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth } R_{\mathfrak{p}} \leq 1$.

(iii) $\text{grade}(E/U) \geq 2$.

Then, for R' , E' and F' as in Notation 6.2.1, F' is a free R' -module of rank $e - 1$ and E'/F' is isomorphic to an R' -ideal J with $\text{grade } J > 0$. Moreover, E''/F'' is isomorphic to an R'' -ideal I , called a generic Bourbaki ideal of E with respect to U . Moreover, if K is another ideal constructed this way using variables Y , then the ideals generated by I and K in $T = R(Z, Y)$ coincide up to multiplication by a unit in $\text{Quot}(T)$, and are equal whenever I and K have grade at least 2.

In particular, this definition coincides with Definition 2.6.3 in the case when $E = U$. Also, notice that assumption (ii) holds whenever E satisfies G_2 , and assumption (iii) is automatically satisfied if U is a reduction of E . In fact, often U will be assumed to be a minimal reduction of E . This is particularly useful in combination with the following observation.

Remark 6.2.3 Let R be a Noetherian local ring with infinite residue field. Let E be a finite, torsion-free R -module and U a minimal reduction of E . Let $I \cong E''/F''$ be a generic Bourbaki ideal with respect to U . Then, the ideal $K \cong U''/F''$ is a minimal reduction of I .

Indeed, U''/F'' is a reduction of I , since U is a reduction of E and the natural map $\mathcal{R}(E) \twoheadrightarrow \mathcal{R}(I)$ is homogeneous. Moreover, it is minimal since by construction $\mu(U''/F'') = \mu(U) - e + 1 = \ell(E) - e + 1 = \ell(I)$.

In fact, sometimes it is also possible to keep track of the behavior of the reduction number after passing to a generic Bourbaki ideal, as explained in item (d) of the following theorem.

Theorem 6.2.4 ([65, 3.5]). *In the setting of Notation 6.2.1, let U be a reduction of E . Let I be a generic Bourbaki ideal of E with respect to U , and let $K \cong U''/F''$. Then, the following statements hold.*

(a) $\mathcal{R}(E)$ is Cohen-Macaulay if and only if $\mathcal{R}(I)$ is Cohen-Macaulay.

- (b) E is of linear type and $\text{grade } \mathcal{R}(E)_+ \geq e$ if and only if I is of linear type, if and only if J is of linear type.
- (c) If any of condition (a) or (b) hold, then $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$ and x_1, \dots, x_{e-1} of F form a regular sequence on $\mathcal{R}(E)$.
- (d) If $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$, then K is a reduction of I , with $r_K(I) = r_U(E)$. In this case, if in addition the residue field of R is infinite and $U = E$, then $r(E) = r(I)$.

For some technical results, it will often be convenient to think of the rings R' and R'' as the result of an iterative process, where at each step only n variables are adjoined. This is formalized in the following notation.

Notation 6.2.5 Let R be a Noetherian ring, E a finite R -module with positive rank, $U = Ra_1 + \dots + Ra_n$ a submodule of E . Let Z_1, \dots, Z_n be indeterminates, $\tilde{R} := R[Z_1, \dots, Z_n]$, $\tilde{E} := E \otimes_R \tilde{R}$, $\tilde{U} := U \otimes_R \tilde{R}$, and $x := \sum_{i=1}^n Z_i a_i \in \tilde{U}$. If R is local with maximal ideal \mathfrak{m} , let $S := R(Z_1, \dots, Z_n) = \tilde{R}_{\mathfrak{m}\tilde{R}}$.

In fact, R' and R'' as in Notation 6.2.1 are obtained from R by iterating the construction of the rings \tilde{R} and S as in Notation 6.2.5 $e - 1$ times.

The following result is a more general version of Theorem 2.6.10.

Theorem 6.2.6 ([65, 3.6 and 3.8]) *In the setting of Notation 6.2.5, assume that $\text{rank } E = e \geq 2$ and that E/U is a torsion R -module. Let $\overline{E} := \tilde{E}/\tilde{R}x$ and $\overline{\mathcal{R}} := \mathcal{R}(\tilde{E})/(x)$. Then,*

- (a) x is regular on $\mathcal{R}(\tilde{E})$.
- (b) The kernel of the natural epimorphism $\pi: \overline{\mathcal{R}} \rightarrow \mathcal{R}(\overline{E})$ is $K = H_{U\overline{\mathcal{R}}}^0(\overline{\mathcal{R}})$.
- (c) If U is a reduction of E and $\text{grade } \mathcal{R}(E)_+ \geq 2$, then π is an isomorphism.

We will also need to make use of the next theorem (see [65, 3.7]).

Theorem 6.2.7 *In the setting of Notation 6.2.5, assume that R is local, that $\text{rank } E = e \geq 2$ and that U is a reduction of E . Let \overline{E} denote $(E \otimes_R S)/xS$ and $\overline{\mathcal{R}} := \mathcal{R}(E \otimes_R S)/(x)$. If $\mathcal{R}(E)$ satisfies S_2 , then the natural epimorphism $\pi: \overline{\mathcal{R}} \rightarrow \mathcal{R}(\overline{E})$ is an isomorphism, and x is regular on $\mathcal{R}(E \otimes_R S)$.*

Notice that formation of Rees algebras of finite modules commutes with flat extensions (see [16, 1.3]). Hence, one has that $\mathcal{R}(\tilde{E}) \cong \mathcal{R}(E) \otimes_R \tilde{R}$, as well as $\mathcal{R}(E \otimes_R S) \cong \mathcal{R}(E) \otimes_R S$. Therefore, $\mathcal{R}(\tilde{E}) \otimes_R k \cong \mathcal{F}(E) \otimes_R \tilde{R}$, and $\mathcal{F}(E \otimes_R S) \cong \mathcal{F}(E) \otimes_R S$.

6.2.1 Main result

Let E be a finite R -module admitting a generic Bourbaki ideal I . Our main goal is to show that the Cohen-Macaulay property can be transferred from $\mathcal{F}(E)$ to $\mathcal{F}(I)$ and backwards, at least in situations when the Rees algebras $\mathcal{R}(E)$ and $\mathcal{R}(I)$ have good depth properties.

Our proof is inspired by the proof of Theorem 2.6.4 (see [65, 3.5]). The first step is to prove that an analogous statement as that of Theorem 6.2.6 holds for fiber cones. This is done in the next two propositions, which together will make it possible to transfer the Cohen-Macaulay property from $\mathcal{F}(E)$ to $\mathcal{F}(I)$ (without needing to assume any condition on the Rees rings, see Theorem 6.2.14(a)).

Proposition 6.2.8 *Let (R, \mathfrak{m}, k) be a Noetherian local ring, E a finite R -module with $\text{rank } E = e \geq 2$, and let U be a submodule of E such that E/U is torsion. In the setting of Notation 6.2.5, let L be the kernel of the natural epimorphism*

$$\pi: (\mathcal{F}(E) \otimes_R \tilde{R})/(x) \rightarrow \mathcal{R}(\tilde{E}/\tilde{R}x) \otimes_R k.$$

Then,

(a) $L \subseteq H_U^0((\mathcal{F}(E) \otimes_R \tilde{R})/(x)).$

(b) *If in addition U is a reduction of E and $\text{depth } \mathcal{F}(E) > 0$, then x is regular on $\mathcal{F}(E) \otimes_R \tilde{R}$.*

Proof. For simplicity of notation, let $\overline{\mathcal{R}}$ denote $\mathcal{R}(\tilde{E})/(x)$. By Theorem 6.2.6(b), there is an exact sequence

$$0 \rightarrow K \xrightarrow{\iota} \overline{\mathcal{R}} \xrightarrow{\pi} \mathcal{R}(\tilde{E}/\tilde{R}x) \rightarrow 0$$

where $K = H_{U\overline{\mathcal{R}}}^0(\overline{\mathcal{R}})$. Tensoring with the residue field k , it then follows that

$$L = (\iota \otimes k)(H_{U\overline{\mathcal{R}}}^0(\overline{\mathcal{R}}) \otimes_R k) \subseteq H_U^0((\mathcal{F}(E) \otimes_R \tilde{R})/(x))$$

This proves (a).

Part (b) follows from [26], after noticing that $\text{depth } \mathcal{F}(E) = \text{grade } U\mathcal{F}(E)$. ■

Proposition 6.2.9 *Let R be a Noetherian local ring, E a finite R -module with $\text{rank } E = e \geq 2$, and let U be a reduction of E . With the notation of Proposition 6.2.8, assume that $\text{depth } \mathcal{F}(E) \geq 2$. Then,*

$$\pi: (\mathcal{F}(E) \otimes_R \tilde{R})/(x) \rightarrow \mathcal{R}(\tilde{E}/\tilde{R}x) \otimes_R k.$$

is an isomorphism, and x is regular on $\mathcal{F}(E) \otimes_R \tilde{R}$.

Proof. Let $\overline{\mathcal{R}}$ denote $\mathcal{R}(E \otimes_R \tilde{R})/(x)$. By Proposition 6.2.8 it follows that $L = \ker(\pi) \subseteq H_U^0(\mathcal{F}(E) \otimes_R \tilde{R})/(x)$ and that x is regular on $\mathcal{F}(E) \otimes_R \tilde{R} = (\iota \otimes k)(H_{U\overline{\mathcal{R}}}^0(\overline{\mathcal{R}}) \otimes_R k) \subseteq H_U^0((\mathcal{F}(E) \otimes_R \tilde{R})/(x))$. In particular, there is an exact sequence

$$0 \rightarrow (\mathcal{F}(E) \otimes_R \tilde{R})(-1) \xrightarrow{x} \mathcal{F}(E) \otimes_R \tilde{R} \rightarrow (\mathcal{F}(E) \otimes_R \tilde{R})/(x) \rightarrow 0.$$

Now, since U is a reduction of E , it follows that

$$\text{grade } U\mathcal{F}(E) \otimes_R \tilde{R} = \text{grade } E\mathcal{F}(E) \otimes_R \tilde{R} \geq \text{depth } \mathcal{F}(E) \geq 2.$$

Therefore, $H_U^1(\mathcal{F}(E) \otimes_R \tilde{R}) = 0$ and hence the long exact sequence of local cohomology implies that $H_U^0(\mathcal{F}(E) \otimes_R \tilde{R}) = 0$. Thus, $L = 0$ and so π is an isomorphism. ■

We now proceed to set up the technical framework in order for the Cohen-Macaulay property to be transferred from $\mathcal{F}(I)$ back to $\mathcal{F}(E)$. The key result is Theorem 6.2.12 below.

Lemma 6.2.10 *Let R be a Noetherian local ring, E a finite R -module with $\text{rank } E = e \geq 2$, and let U be a reduction of E . Then, in the setting of Notation 6.2.5,*

$$\text{Supp}_{\mathcal{F}(E) \otimes_R \tilde{R}}(0:_{\mathcal{F}(E) \otimes_R \tilde{R}} x) \subseteq V(E(\mathcal{F}(E) \otimes_R \tilde{R})).$$

Proof. It suffices to show that, if $\mathfrak{q} \in \text{Spec}(\mathcal{F}(E) \otimes_R \tilde{R}) \setminus V(E(\mathcal{F}(E) \otimes_R \tilde{R}))$ then $(0:_{\mathcal{F}(E) \otimes_R \tilde{R}} x)_{\mathfrak{q}} = 0$.

Since $U = Ra_1 + \dots + Ra_n$ is a reduction of E , it follows that $U(\mathcal{F}(E) \otimes_R \tilde{R})$ has the same radical as $E(\mathcal{F}(E) \otimes_R \tilde{R})$. Hence, for any such \mathfrak{q} , one has $\mathfrak{q} \not\supseteq U(\mathcal{F}(E) \otimes_R \tilde{R})$. Hence, if we let \mathfrak{p} denote $\mathfrak{q} \cap \mathcal{F}(E)$, we have that $\mathfrak{p} \not\supseteq U\mathcal{F}(E)$. In particular, at least one of the a_i is a unit in $\mathcal{F}(E)_{\mathfrak{p}}$.

Since by definition $x = \sum_{i=1}^n Z_i a_i$ and the Z_i are variables, this implies that x is a nonzerodivisor in $\mathcal{F}(E)_{\mathfrak{p}}[Z_1, \dots, Z_n]$, hence also in its further localization $(\mathcal{F}(E)[Z_1, \dots, Z_n])_{\mathfrak{q}} = (\mathcal{F}(E) \otimes_R \tilde{R})_{\mathfrak{q}}$. Thus, $(0:_{\mathcal{F}(E) \otimes_R \tilde{R}} x)_{\mathfrak{q}} = 0$, as wanted. ■

The conclusion in Lemma 6.2.10 means that x is a *filter-regular element* on $\mathcal{F}(E) \otimes_R \tilde{R}$ with respect to the ideal $E(\mathcal{F}(E) \otimes_R \tilde{R})$. We refer the reader to [59, p. 13] for the notion of filter-regular sequences, and their connection with superficial sequences.

Lemma 6.2.11 *Let R be a positively graded Noetherian ring with R_0 local, and let x be a homogeneous non-unit element of R . Let M be a finite graded R -module, and assume that $\dim(0:_M x) < \text{depth}(M/xM)$. Then, x is a nonzerodivisor on M .*

Proof. Notice that x is a nonzerodivisor on M if and only if $0:_M x = 0$, or equivalently, if and only if $H_{(x)}^0(M) = 0$. To prove that $H_{(x)}^0(M) = 0$, consider the short exact sequences

$$0 \rightarrow 0:_M x \rightarrow M \rightarrow M/(0:_M x) \rightarrow 0 \tag{6.1}$$

and

$$0 \rightarrow M/(0:_M x) \xrightarrow{x} M \rightarrow M/xM \rightarrow 0. \tag{6.2}$$

Since $0 :_M x = H_{(x)}^0(0 :_M x)$, it follows that $H_{(x)}^1(0 :_M x) = 0$. Hence, the long exact sequence of local cohomology induced by 6.1 implies that $H_{(x)}^0(M)$ surjects onto $H_{(x)}^0(M/(0 :_M x))$. Hence, the long exact sequence of local cohomology induced by 6.2

$$0 \rightarrow H_{(x)}^0(M/(0 :_M x)) \xrightarrow{x} H_{(x)}^0(M) \rightarrow H_{(x)}^0(M/xM)$$

in turn induces an exact sequence

$$H_{(x)}^0(M) \xrightarrow{x} H_{(x)}^0(M) \rightarrow H_{(x)}^0(M/xM) \subseteq M/xM.$$

In particular, $H_{(x)}^0(M)/xH_{(x)}^0(M)$ embeds into M/xM . We claim that

$$\text{Ass}(H_{(x)}^0(M)/xH_{(x)}^0(M)) = \emptyset.$$

From the claim it follows that $H_{(x)}^0(M)/xH_{(x)}^0(M) = 0$, whence $H_{(x)}^0(M) = 0$ by Nakayama's Lemma.

To prove our claim, suppose that there exists some $\mathfrak{p} \in \text{Ass}(H_{(x)}^0(M)/xH_{(x)}^0(M))$. Then, $\mathfrak{p} \in \text{Supp}(H_{(x)}^0(M)/xH_{(x)}^0(M)) = \text{Supp}(0 :_M x)$, whence

$$\dim(R/\mathfrak{p}) \leq \dim(0 :_M x).$$

On the other hand, since $\text{Ass}(H_{(x)}^0(M)/xH_{(x)}^0(M)) \subseteq \text{Ass}(M/xM)$, we also have that $\dim(R/\mathfrak{p}) \geq \text{depth}(M/xM)$. But then, $\dim(H_{(x)}^0(M)/xH_{(x)}^0(M)) \geq \text{depth}(M/xM)$, which contradicts the assumption. So, it must be that $\text{Ass}(H_{(x)}^0(M)/xH_{(x)}^0(M)) = \emptyset$, as claimed. ■

Theorem 6.2.12 *In the setting of Notation 6.2.5, assume that R is local and that $\text{rank } E = e \geq 2$. Let U be a reduction of E , and denote $\overline{E} := (E \otimes_R S)/Sx$ and $\overline{\mathcal{F}} := \mathcal{F}(E \otimes_R S)/(x)$. Assume that $\text{depth } \mathcal{F}(\overline{E}) > 0$, and that one of the two following conditions hold:*

(i) $\mathcal{R}(E)$ satisfies S_2 , or

(ii) $\text{depth } \mathcal{R}(\overline{E}_{\mathfrak{q}}) \geq 2$ for all $\mathfrak{q} \in \text{Spec}(S)$ such that $\overline{E}_{\mathfrak{q}}$ is not of linear type.

Then, the natural epimorphism $\pi: \overline{\mathcal{F}} \twoheadrightarrow \mathcal{F}(\overline{E})$ is an isomorphism, and x is regular on $\mathcal{F}(E \otimes_R S)$.

Proof. For simplicity of notation, write \mathcal{R} for $\mathcal{R}(E \otimes_R S)$. Assumption (i) and Theorem 6.2.7 together imply that the natural epimorphism

$$\pi: \overline{\mathcal{R}} \twoheadrightarrow \mathcal{R}(\overline{E})$$

is an isomorphism. The same conclusion holds if assumption (ii) is satisfied, thanks to Theorem 5.2.5. Hence, $\pi: \overline{\mathcal{F}} \twoheadrightarrow \mathcal{F}(\overline{E})$ is an isomorphism as well. In particular, by assumption it then follows that $\text{depth} \overline{\mathcal{F}} > 0$. Moreover, by Lemma 6.2.10 we know that $(0:_{\mathcal{F}(E \otimes_R S)} x)$ is an Artinian $\mathcal{F}(E \otimes_R S)$ -module. Hence, x is regular thanks to Lemma 6.2.11. ■

Notice that from the proof of Theorem 6.2.12 it follows that the assumption that $\text{depth} \mathcal{F}(\overline{E}) > 0$ is not needed in order to prove that π is an isomorphism. In fact, one has the following useful corollary.

Corollary 6.2.13 *Let R be a Noetherian local ring, and let E be a finite R -module with $\text{rank } E = e$. Let I be a generic Bourbaki ideal of E with respect to a reduction U of E .*

- (a) *Assume that either $\mathcal{R}(I)$ is S_2 , or $\text{depth} \mathcal{R}(I_{\mathfrak{q}}) \geq 2$ for all $\mathfrak{q} \in \text{Spec}(R'')$ so that $I_{\mathfrak{q}}$ is not of linear type. Then, the natural epimorphism $\pi: \mathcal{F}(E'')/(F'') \twoheadrightarrow \mathcal{F}(I)$ is an isomorphism.*
- (b) *If in addition $\mathcal{F}(I)$ is Cohen-Macaulay, then $F''\mathcal{F}(E'')$ is generated by a regular sequence of linear forms.*

Proof. Notice that the assumptions in (a) imply that the assumptions (i) or (ii) in Theorem 6.2.12 are satisfied at each iteration, thanks to Theorem 6.2.7 or Theorem 5.2.5 respectively. Hence, $\mathcal{R}(E'')/(F'') \cong \mathcal{R}(I)$, and $F''\mathcal{R}(E'')$ is generated by a regular sequence on $\mathcal{R}(E'')$. Hence, by iteration of Theorem 6.2.12, it follows

that $\mathcal{F}(E'')/(F'') \cong \mathcal{F}(I)$. In fact, from the proof of Theorem 6.2.12 it follows that the assumption that $\text{depth } \mathcal{F}(\overline{E}) > 0$ is not needed in order to prove that π is an isomorphism.

Now, if furthermore $\mathcal{F}(I)$ is Cohen-Macaulay, then the proof of Theorem 6.2.12 implies that also $F''\mathcal{F}(E'')$ is generated by a regular sequence on $\mathcal{F}(E'')$. That the generators of $F''\mathcal{F}(E'')$ are linear forms in $\mathcal{F}(E'')$ is clear by construction. ■

We are now ready to state and prove our main result.

Theorem 6.2.14 *Let R be a Noetherian local ring, E a finite R -module with $\text{rank } E = e$, U a reduction of E . Let I be a generic Bourbaki ideal of E with respect to U .*

- (a) *If $\mathcal{F}(E)$ is Cohen-Macaulay, then $\mathcal{F}(I)$ is Cohen-Macaulay.*
- (b) *Assume that either $\mathcal{R}(I)$ is S_2 , or $\text{depth } \mathcal{R}(I_{\mathfrak{q}}) \geq 2$ for all $\mathfrak{q} \in \text{Spec}(R'')$ so that $I_{\mathfrak{q}}$ is not of linear type. If $\mathcal{F}(I)$ is Cohen-Macaulay, then $\mathcal{F}(E)$ is Cohen-Macaulay.*

Proof. We may assume that $e \geq 2$. If $\mathcal{F}(E)$ is Cohen-Macaulay, then $\text{depth } \mathcal{F}(E) = \ell(E)$. Also, $\ell(E) \geq e$ by Proposition 2.6.7(a). Hence, by iteration of Proposition 6.2.9, we get that $\text{depth } \mathcal{F}(I) = \ell(E) - e + 1$. The latter equals $\ell(I)$, hence $\mathcal{F}(I)$ is Cohen-Macaulay.

Conversely, if the assumptions in (b) hold, then, by Theorem 6.2.7 and Theorem 5.2.5 respectively, assumptions (i) and (ii) of Theorem 6.2.12 are preserved after iterating the construction of the ring S . Hence, by iteration of Theorem 6.2.12, we conclude that $\text{depth } \mathcal{F}(E) = \ell(I) + e - 1 = \ell(E)$. Therefore, $\mathcal{F}(E)$ is Cohen-Macaulay. ■

From Theorem 6.2.14 it follows that, in particular, $\mathcal{F}(E)$ is Cohen-Macaulay whenever both $\mathcal{F}(I)$ and $\mathcal{R}(I)$ are Cohen-Macaulay. Using this fact, in the following subsection we list some sufficient conditions so that $\mathcal{F}(E)$ is Cohen-Macaulay.

6.2.2 Modules with Cohen-Macaulay fiber cone

Our first results relates the Cohen-Macaulay property of the Rees algebra and the fiber cone of a module E , once information is provided about the degrees of the generating relations of $\mathcal{F}(E)$. This extends Theorem 6.1.1 to modules.

Theorem 6.2.15 *Let R be a Cohen-Macaulay local ring with infinite residue field. Let E be a finite, torsion-free R -module with $\text{rank } E = e > 0$, $\ell(E) = \ell$ and $r(E) = r$. Assume that E satisfies $G_{\ell-e+1}$ and $\ell - e + 1 \geq 2$. Let I be a generic Bourbaki ideal of E , and $g = \text{ht}(I)$. Suppose that one of the following conditions holds.*

- (i) *If $\mu(E) \geq \ell + 2$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - e - g + 1\}$.*
- (ii) *If $\mu(E) = \ell + 1$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \ell - e - g + 1$.*

If $\mathcal{R}(E)$ is Cohen-Macaulay, then $\mathcal{F}(E)$ is Cohen-Macaulay.

Proof. By Theorem 2.6.2, E admits a generic Bourbaki ideal I with $\text{ht } I = g \geq 1$, $\mu(I) = \mu(E) - e + 1$ and $r(I) \leq r$, which satisfies $G_{\ell-e+1}$, i.e. $G_{\ell(I)}$ (see Proposition 2.6.9). If $e = 1$, then the conclusion follows from Theorem 6.1.1, since $\mathcal{G}(I)$ is Cohen-Macaulay whenever $\mathcal{R}(I)$ is, by Theorem 4.1.1. So we may assume that $e \geq 2$. We show that $\mathcal{F}(I)$ is Cohen-Macaulay, whence $\mathcal{F}(E)$ is Cohen-Macaulay by Theorem 6.2.14.

Since by assumption $\mathcal{R}(E)$ is Cohen-Macaulay, then, by Theorem 2.6.4, also $\mathcal{R}(I)$ is Cohen-Macaulay, hence the associated graded ring $\mathcal{G}(I)$ is Cohen-Macaulay by Theorem 4.1.1. Therefore, by Theorem 6.1.1, in order to prove that $\mathcal{F}(I)$ is Cohen-Macaulay, it only remains to show that the degrees of the homogeneous generating relations of $\mathcal{F}(I)$ satisfy the assumptions of Theorem 6.1.1.

Now, since $\mathcal{R}(I)$ is Cohen-Macaulay, by Corollary 6.2.13, we have a homogeneous isomorphism $\mathcal{F}(E'')/(F'') \cong \mathcal{F}(I)$. Therefore, if condition (i) holds, whenever

$\mu(I) \geq \ell + 2 - e + 1 = \ell(I) + 2$, then $\mathcal{F}(I)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - e - g + 1\}$, hence in degrees $\leq \max\{r(I), \ell(I) - g\}$. Similarly, in the situation of assumption (ii), whenever $\mu(I) = \ell(I) + 1$, then $\mathcal{F}(I)$ has at most two homogeneous generating relations in degrees $\leq \ell - e - g + 1 = \ell(I) - g$. Hence, $\mathcal{F}(I)$ is Cohen-Macaulay by Theorem 6.1.1. ■

In particular, combining the previous theorem with the results of Chapter 4, we deduce the following sufficient conditions for $\mathcal{F}(E)$ to be Cohen-Macaulay. These both recover [10, 2.9] in the case when E is an ideal of grade at least two.

Corollary 6.2.16 *Let R be a local Gorenstein ring of dimension d with infinite residue field. Let E be a finite, torsion-free, orientable R -module, with $\text{rank} E = e > 0$ and $\ell(E) = \ell$. Let g be the height of a generic Bourbaki ideal of E , and assume that the following conditions hold.*

(a) E satisfies $G_{\ell-e+1}$.

(b) $r(E) \leq k$ for some integer $1 \leq k \leq \ell - e$.

(c) $\text{depth } E^j \geq \begin{cases} d - g - j + 2 & \text{for } 1 \leq j \leq \ell - e - k - g + 1 \\ d - \ell + e + k - j & \text{for } \ell - e - k - g + 2 \leq j \leq k \end{cases}$

(d) If $g = 2$, $\text{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}^j, R_{\mathfrak{p}}) = 0$ for $\ell - e - k \leq j \leq \ell - e - 3$ and for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} = \ell - e$ such that $E_{\mathfrak{p}}$ is not free.

Assume furthermore that one of the following two conditions holds.

(i) If $\mu(E) \geq \ell + 2$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - e - g + 1\}$.

(ii) If $\mu(E) = \ell + 1$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - e - g + 1\}$.

Then, $\mathcal{F}(E)$ is Cohen-Macaulay.

Proof. Assumptions (a)-(d) together imply that $\mathcal{R}(E)$ is Cohen-Macaulay, thanks to Theorem 4.2.4. Hence, if either condition (i) or (ii) hold, then $\mathcal{F}(E)$ is Cohen-Macaulay by Theorem 6.2.15. ■

Corollary 6.2.17 *Let R be a local Cohen-Macaulay ring, and let E be an ideal module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that the following conditions hold.*

- (a) $r(E) \leq k$, where k is an integer such that $1 \leq k \leq \ell - e$.
- (b) E is free locally in codimension $\ell - e - \min\{2, k\}$, and satisfies $G_{\ell-e+1}$.
- (c) $\text{depth}(E^j) \geq d - \ell + e + k - j$ for $1 \leq j \leq k$.

Assume furthermore that one of the following two conditions holds.

- (i) If $\mu(E) \geq \ell + 2$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - e - g + 1\}$.
- (ii) If $\mu(E) = \ell + 1$, then $\mathcal{F}(E)$ has at most two homogeneous generating relations in degrees $\leq \max\{r, \ell - e - g + 1\}$.

Then, $\mathcal{F}(E)$ is Cohen-Macaulay.

Proof. From assumptions (a)-(c) it follows that $\mathcal{R}(E)$ is Cohen-Macaulay, thanks to Theorem 4.4.4. Hence, if either condition (i) or (ii) hold, then $\mathcal{F}(E)$ is Cohen-Macaulay by Theorem 6.2.15. ■

We next describe module versions of Theorem 6.1.2. We start by the case when E is an ideal module.

Theorem 6.2.18 *Let R be a local Cohen-Macaulay ring with infinite residue field, and let E be an ideal module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that the following conditions hold.*

- (a) E is free locally in codimension $\ell - e - 1$, and satisfies $G_{\ell-e+1}$.
- (b) $U\mathfrak{m} = E\mathfrak{m}$ for a minimal reduction U of E .

Let g be the height of a generic Bourbaki ideal I of E , and assume that $\mathcal{R}(E)$ is Cohen-Macaulay. Then, $\mathcal{F}(E)$ is Cohen-Macaulay, $a(\mathcal{F}(E)) \leq -e - g + 2$, and $r(E) \leq \ell - e - g + 2$.

Proof. Since E is an ideal module, then E admits a generic Bourbaki ideal I with respect to U , with $\text{ht } I = g \geq 2$, satisfying $G_{\ell-e+1}$, that is, $G_{\ell(I)}$ (see Proposition 2.6.9). Moreover, since E is free locally in codimension $\ell - e - 1$, by Theorem 4.4.3 it follows that I satisfies $AN_{\ell-e-1}$, i.e. $AN_{\ell(I)-2}$. Also, by Remark 6.2.3, the R'' -ideal $K \cong U''/F''$ is a minimal reduction of I such that $K\mathfrak{m} = I\mathfrak{m}$. Now, if $e = 1$, the conclusion follows from Theorem 6.1.2. So, assume that $e \geq 2$.

If $\mathcal{R}(E)$ is Cohen-Macaulay, by Theorem 6.2.4 it follows that $\mathcal{R}(I)$ is Cohen-Macaulay, and that $r(I) = r(E)$. Hence, Theorem 6.1.2 implies that $\mathcal{F}(I)$ is Cohen-Macaulay, $a(\mathcal{F}(I)) \leq -g + 1$, and $r(E) = r(I) \leq \ell(I) - g + 1 = \ell - e - g + 2$. Furthermore, by Theorem 6.2.14 it follows that $\mathcal{F}(E)$ is Cohen-Macaulay, while Corollary 6.2.13 implies that $\mathcal{F}(I)$ is isomorphic to a quotient of $\mathcal{F}(E'')$ modulo a regular sequence of $e - 1$ linear forms. Hence, since $a(\mathcal{F}(I)) \leq -g + 1$, it must be that $a(\mathcal{F}(E'')) \leq -e - g + 2$, hence $a(\mathcal{F}(E)) \leq -e - g + 2$. ■

Corollary 6.2.19 *Let R be a local Cohen-Macaulay ring with infinite residue field, and let E be an ideal module with $\text{rank } E = e$ and $\ell(E) = \ell$. Assume that the following conditions hold.*

- (a) $r(E) \leq k$, where k is an integer such that $1 \leq k \leq \ell - e$.
- (b) E is free locally in codimension $\ell - e - 1$, and satisfies $G_{\ell-e+1}$.
- (c) $\text{depth}(E^j) \geq d - \ell + e + k - j$ for $1 \leq j \leq k$.
- (d) $U\mathfrak{m} = E\mathfrak{m}$ for a minimal reduction U of E .

Then, $\mathcal{F}(E)$ is Cohen-Macaulay and $a(\mathcal{F}(E)) \leq -e - g + 2$.

Proof. As in the proof of Theorem 4.4.4, E admits a generic Bourbaki ideal I with respect to U with $\text{ht } I = g \geq 2$, $r(I) \leq r(E) \leq k$ and satisfying $G_{\ell-e+1}$, that is, $G_{\ell(I)}$. Moreover, since E is free locally in codimension $\ell - e - 1$, by Theorem 4.4.3 it follows that I satisfies $AN_{\ell-e-1}$. In particular, I satisfies $AN_{\ell-e-\min\{2,k\}}$, so, as in the proof of Theorem 4.4.4, one concludes that $\mathcal{R}(E)$ is Cohen-Macaulay. Now the conclusion follows from Theorem 6.2.18. ■

For a torsion-free, orientable R -module E which is not necessarily an ideal module, we can still give conditions so that $\mathcal{F}(E)$ is Cohen-Macaulay, thanks to Theorem 4.2.4.

Theorem 6.2.20 *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d , with infinite residue field. Let E be a finite, torsion-free, orientable R -module, with $\text{rank}(E) = e > 0$ and $\ell(E) = \ell$. Assume that the following conditions hold:*

(a) E satisfies $G_{\ell-e+1}$

(b) $r(E) \leq k$ for some integer $1 \leq k \leq \ell - e$.

(c) $\text{depth } E^j \geq \begin{cases} d - g - j + 2 & \text{for } 1 \leq j \leq \ell - e - 1 \\ d - \ell + e + k - j & \text{for } \ell - e \leq j \leq k \end{cases}$

(d) If $g = 2$, $\text{Ext}_{R_{\mathfrak{p}}}^{j+1}(E_{\mathfrak{p}}^j, R_{\mathfrak{p}}) = 0$ for $\ell - e - k \leq j \leq \ell - e - 3$ and for all $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} = \ell - e$ such that $E_{\mathfrak{p}}$ is not free.

(e) $U\mathfrak{m} = E\mathfrak{m}$ for a minimal reduction U of E .

Then, $\mathcal{F}(E)$ is Cohen-Macaulay and $a(\mathcal{F}(E)) \leq -e - g + 2$.

Proof. Since E is torsion-free, orientable and satisfies $G_{\ell-e+1}$ with $\ell - e + 1 \geq 2$, then E admits a generic Bourbaki ideal with respect to U , which satisfies $G_{\ell-e+1}$, i.e. $G_{\ell(I)}$ and has height at least 2. Moreover, K is a minimal reduction of I with $K\mathfrak{m} = I\mathfrak{m}$. Now, similarly as in the proof of Theorem 4.2.4, assumptions (a)-(d) imply that $\mathcal{R}(I)$

is Cohen-Macaulay, and that I satisfies $AN_{\ell-e-1}$, that is, $AN_{\ell(I)-2}$. Therefore, by Theorem 6.1.2, it follows that $\mathcal{F}(I)$ is Cohen-Macaulay, with $a(\mathcal{F}(I)) \leq -g + 1$. Now, by Theorem 6.2.14 it follows that $\mathcal{F}(E)$ is Cohen-Macaulay, and Corollary 6.2.13 implies that $\mathcal{F}(I)$ is a specialization of $\mathcal{F}(E'')$. Hence, since $a(\mathcal{F}(I)) \leq -g + 1$, it must be that $a(\mathcal{F}(E'')) \leq -e - g + 2$, whence $a(\mathcal{F}(E)) \leq -e - g + 2$. ■

We conclude this section by examining the case of modules of projective dimension one. The following result extends Theorem 6.1.3 to modules.

Theorem 6.2.21 *Let R be a local Cohen-Macaulay ring, and let E a finite, torsion-free R -module with $\text{projdim}(E) = 1$, with $\ell(E) = \ell$. Assume that E satisfies $G_{\ell-e+1}$. If $\mathcal{R}(E)$ is Cohen-Macaulay, then $\mathcal{F}(E)$ is Cohen-Macaulay.*

Proof. Since E is a torsion-free module of projective dimension one which satisfies $G_{\ell-e+1}$, then E admits a generic Bourbaki ideal I , which is perfect of height 2. If $e = 1$, then the conclusion follows from Theorem 6.1.3. Otherwise, notice that $\mathcal{R}(I)$ is Cohen-Macaulay by Theorem 2.6.4, whence $\mathcal{F}(I)$ is Cohen-Macaulay by Theorem 6.1.3. Hence, $\mathcal{F}(E)$ is Cohen-Macaulay by Theorem 6.2.14. ■

Recall that for a module E of projective dimension one the Cohen-Macaulay property of the Rees algebra $\mathcal{R}(E)$ is controlled by the presentation matrix of E or by the reduction number of E (see [65, 4.7]). Hence, we have the following corollary.

Corollary 6.2.22 *Let R be a local Cohen-Macaulay ring, and let E a finite, torsion-free R -module with $\text{projdim}(E) = 1$, with $\ell(E) = \ell$. Let $n = \mu(E)$ and let*

$$0 \rightarrow R^{n-e} \xrightarrow{\varphi} R^n \rightarrow E \rightarrow 0$$

be a minimal free resolution of E . Assume that E satisfies $G_{\ell-e+1}$ and that one of the following equivalent conditions hold.

(i) $r(E) \leq \ell - e$.

(ii) $r(E_{\mathfrak{p}}) \leq \ell - e$ for every prime \mathfrak{p} with $\dim R_{\mathfrak{p}} = \ell(E_{\mathfrak{p}}) - e + 1 = \ell - e + 1$.

(iii) After elementary row operations, $I_{n-\ell}(\varphi)$ is generated by the maximal minors of the last $n - \ell$ rows of φ .

Then, $\mathcal{F}(E)$ is Cohen-Macaulay.

Proof. By [65, 4.7], each of the conditions (i)-(iii) is equivalent to $\mathcal{R}(E)$ being Cohen-Macaulay. Hence, the conclusion follows from Theorem 6.2.21. ■

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VITA

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