PROFINITE COMPLETIONS AND REPRESENTATIONS OF FINITELY GENERATED GROUPS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Ryan F. Spitler

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2019

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. David Ben McReynolds, Chair
School of Mathematics
Dr. Donu Arapura
School of Mathematics
Dr. Deepam Patel
School of Mathematics
Dr. Thomas Sinclair
School of Mathematics

Approved by:

Dr. Plamen Stefanov
Head of the School Graduate Program

This dissertation is dedicated to my wife and children, and to everyone who has helped and guided me throughout my mathematics education.

ACKNOWLEDGMENTS

I want to thank my advisor Ben McReynolds for everything he has done for me. I would also like to thank Alan Reid and Martin Bridson for their inspiring questions which served to bring this work into its current form.

TABLE OF CONTENTS

ABSTRACT

Spitler, Ryan F. PhD, Purdue University, August 2019. Profinite Completions and Representations of Finitely Generated Groups. Major Professor: David Ben McReynolds.

In previous work, the author and his collaborators developed a relationship in the $\mathrm{SL}(2,\mathbb{C})$ representation theories of two finitely generated groups with isomorphic profinite completions assuming a certain strong representation rigidity for one of the groups. This was then exploited as one part of producing examples of lattices in $\mathrm{SL}(2,\mathbb{C})$ which are profinitely rigid. In this article, the relationship is extended to representations in any connected reductive algebraic groups under a weaker representation rigidity hypothesis. The results are applied to lattices in higher rank Lie groups where we show that for some such groups, including $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$, they are either profinitely rigid, or they contain a proper Grothendieck subgroup.

1. INTRODUCTION

1.1 Motivation

Whether or not a finitely generated group Γ is determined by the isomorphism classes of its finite quotients has been an important problem for several decades. Via the correspondence between conjugacy classes of finite index subgroups of the fundamental groups of a manifold and the finite coverings of the manifold, this problem for fundamental groups of manifolds is equivalent to a manifold being determined by the isomorphism classes of the deck/Galois groups of its finite covers.

The isomorphism classes of finite quotients of Γ are conveniently encoded by the profinite completion $\widehat{\Gamma}$ of Γ. This compact, totally disconnected group can be viewed as a pseudo-metric completion of the group or as a limit of an inverse system of finite groups endowed with the discrete topology. It is necessary to restrict to groups Γ that embed into their associated profinite completions $\widehat{\Gamma}$, as the image of Γ will also have profinite completion isomorphic to $\widehat{\Gamma}$; such groups are called groups residually finite.

We say that a group Γ is profinitely rigid when Γ is determined up to isomorphism by its profinite completion (see §3.4 for a precise definition). There are few examples of finitely generated groups that are known to be profinitely rigid. Finitely generated abelian groups are profinitely rigid. However, this fails in general for virtually abelian groups and nilpotent groups by work of Baumslag [3], Pickel [15], and Serre [19]. In particular Bieberbach groups (i.e. the fundamental groups of closed flat n–manifolds) are not in general profinitely rigid. Some examples of profinitely rigid groups include those satisfying certain (verbal) laws. Unfortunately, more conjectures exist on this topic than theorems. Finitely generated free groups are conjectured to be profinitely rigid as are fundamental groups of closed hyperbolic 2- and 3-manifolds (see [18]).

The class of fundamental groups of closed, hyperbolic 2- and 3-manifolds is part of a broader class of finitely presented groups, the class of irreducible lattices in noncompact semisimple Lie groups which are fundamental groups of complete, finite volume, locally symmetric orbifolds. Examples of pairs of non-profinitely rigid lattices have been constructed by Aka [1], Milne–Suh [14], and Stover [21]. The first examples of lattices which *are* profinitely rigid were given in [5]. These were the first examples of finitely generated, residually finite groups that contain a non-abelian free subgroup which are profinitely rigid. The examples in [5] were the smallest covolume uniform and non-uniform arithmetic lattices in $\mathrm{SL}(2,\mathbb{C})$ and finitely many other associated lattices. Most notable is the fundamental group of the Weeks manifold, the unique smallest volume closed, orientable hyperbolic 3-manifold. Additional examples of profinitely rigid lattices, 16 of the arithmetic hyperbolic triangle groups, are established in [6].

In [7] it was shown that if $\Delta < \Gamma$ is finitely generated with the inclusion inducing an isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$, a condition where we call Δ a Grothendieck subgroup of Γ, then Δ and Γ have the same representation theory in a precise sense. It was asked there whether it is possible for a group to have a proper Grothendieck subgroup. Examples of groups having a proper Grothendieck subgroup have since been found ([16], [4]), but they remain difficult to find. It is an open question whether higher-rank arithmetic groups with the congruence subgroup property can have a proper Grothendieck subgroup ([17, p. 434]).

1.2 Main results

In this work, we examine how the representations of Γ and Δ are related when only assuming that we have an isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$. This work generalizes parts of [5] and [6] in both broadening the connections between the representation varieties of profinitely equivalent groups and expanding some of the main technical results needed for the profinite rigidity results established in [5] and [6].

In [5] and [6], we were concerned just with relating the representations of Γ and Δ in $\mathrm{SL}(2,\mathbb{C})$ under the hypothesis that Γ has only a single Zariski-dense representation up to conjugation and automorphisms of $\mathbb{C}$. We applied this to certain groups $\Gamma < \mathrm{SL}(2,\mathbb{C})$ whose defining arithmetic was such that we could ensure Δ had a representation into Γ. Using 3-manifold techniques we could eventually show that Δ must be isomorphic to Γ. In this paper, we extend to considering representations into any fixed reductive group $\mathbf{G}(\mathbb{C})$ and only assume that Γ has finitely many Zariski dense representations up to conjugation. We are able to produce universal representations of Γ and Δ which parametrize the Zariski-dense representations of each, and then using the isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$ we show that these universal representations are locally equivalent, that is, they give the same information for almost all p-adic representations. This result is analogous to Theorem 4.8 in [5]. We then apply this result in the case when Γ is a higher rank arithmetic group, and in some cases show that Δ has an injection into Γ. However, in this setting there is nothing like the powerful 3-manifold technology, and we are unable to show that Δ and Γ are isomorphic.

When Γ is a higher rank arithmetic group, in general there can be other higher rank arithmetic groups, say Λ, with $\widehat{\Lambda} \cong \widehat{\Gamma}$. The range of possibilities for these Λ is well understood, in fact there are only finitely many of them [1]. For some Γ, we can show that this is almost the only obstruction to profinite rigidity. Any Δ with $\widehat{\Delta} \cong \widehat{\Gamma}$ will have an injection into such a Λ where it will live as a Grothendieck subgroup, so either $\Delta = \Lambda$ or Λ contains a proper Grothendieck subgroup. As a specific example we show that for $n \geq 3$, either $\mathrm{SL}(n,\mathbb{Z})$ is profinitely rigid or it contains a proper Grothendieck subgroup.

The main theorems of this work have further applications which will be the subject of a forthcoming joint paper, [13]. One of these regards the question of which profinite groups can be a profinite completion, that is which profinite groups are isomorphic to the profinite completion of some finitely generated, residually finite group. When $n \geq 3$, the profinite completion of $\mathrm{SL}(n,\mathbb{Z})$ is $\prod_p \mathrm{SL}(n,\mathbb{Z}_p)$, so $\prod_p \mathrm{SL}(n,\mathbb{Z}_p)$ is a profinite completion. On the other hand, one can show that $\prod_p \mathrm{SL}(2,\mathbb{Z}_p)$ is not a profinite completion. Similar to the $\mathrm{SL}(n,\mathbb{Z})$ case above, one can show that any group Δ with $\widehat{\Delta} \cong \prod_p \mathrm{SL}(2,\mathbb{Z}_p)$ must embed into $\mathrm{SL}(2,\mathbb{Z})$. However, since $\mathrm{SL}(2,\mathbb{Z})$ is virtually free, this leads to a contradiction. More generally we address which 'adelic' profinite groups, closed subgroups of $\prod_p \mathrm{GL}(n,\mathbb{Z}_p)$, can be profinite completions, and obtain some results in that area.

2. SUMMARY

We start in §3 by introducing the necessary background and conventions used in the paper. In §4, we construct a universal representation for a finitely generated group with a certain representation rigidity in a connected reductive group. In §5, we construct a universal representation for p-adic representations of a profinite group in a connected reductive group, again assuming some representation rigidity. In §6, we show how the universal representation we constructed for a finitely generated group is related to that of its profinite completion. This section culminates in the main theorems 6.1 and 6.2 which relate the universal representations of groups with isomorphic profinite completions. Finally in §7, we apply these theorems to higher rank arithmetic groups to get the main theorem 7.1 and the interesting special case 7.2.

3. BACKGROUND AND PRELIMINARIES

3.1 Fields, Adele Rings, and Étale Algebras

All fields will be assumed to have characteristic zero. We will frequently use the fields $\mathbb{C}$ and $\overline{\mathbb{Q}_p}$ for varying p. These are all abstractly isomorphic as they are algebraically closed, have characteristic zero, and their transcendence bases over $\mathbb{Q}$ have the same cardinality. For each p we choose a specific isomorphism of $\mathbb{C}$ and $\overline{\mathbb{Q}_p}$, which we will leave implicit, and use these to identify objects defined over each of them. Note that these isomorphisms are very far from continuous in the usual topologies on these fields. The topology on $\mathbb{C}$ does not play a role in any of our results, while the topologies of the $\overline{\mathbb{Q}_p}$ are very important. We choose therefore to use the field $\mathbb{C}$ for results and statements that only rely on its abstract field structure. For statements where the topology is significant, we will use $\overline{\mathbb{Q}_p}$ instead, usually with p being understood to be an arbitrary prime.

We use $\mathbb{A}$ to denote the ring of *rational adeles*, $\mathbb{A}^f$ is the ring of finite adeles, and $\mathbb{A}_S$ is the ring of S-*adeles* for S some finite subset of places of $\mathbb{Q}$. For a number field K, we similarly use $\mathbb{A}_K$ to denote the ring of K-*adeles*, $\mathbb{A}_K^f$ for the finite K-adeles. For S some finite subset of places of K, we will call the pair (K, S) a *restricted number field* and will write $\mathbb{A}_{K,S}$ for the ring of S-*adeles of* K. An étale algebra over $\mathbb{Q}$, which we will just call an *étale algebra*, is a finite product of number fields $E = \prod K_i$, and its set of places is the disjoint union of the sets of places of the K_i. We will define the ring of E-*adeles* as $\mathbb{A}_E = \prod \mathbb{A}_{K_i}$ and similarly the ring of finite E-adeles, $\mathbb{A}_E^f$. For S a finite set of places of E, we have $S = \sqcup S_i$ where S_i is a finite set of places of K_i. We will call (E, S) a *restricted étale algebra*, and define the (E, S)-*adeles* as $\mathbb{A}_{E,S} = \prod \mathbb{A}_{K_i,S_i}$. A number field is an étale algebra with a single factor, so the previous definitions are just special cases of this one.

3.2 Local Equivalence

We will say two number fields K and K' are *locally equivalent* when $\mathbb{A}_K \cong \mathbb{A}_{K'}$, and the restricted number fields (K, S) and (K', T) are locally equivalent when $\mathbb{A}_{K,S} \cong \mathbb{A}_{K',T}$. Similarly, two étale algebras E and E' are locally equivalent when $\mathbb{A}_E \cong \mathbb{A}_{E'}$, and the restricted étale algebras (E, S) and (E', T) are locally equivalent when $\mathbb{A}_{E,S} \cong \mathbb{A}_{E',T}$.

Proposition 3.1 *Let E and E' be étale algebras, $K/\mathbb{Q}$ be a finite Galois extension containing each of the subfield factors of E and E', and $H = \mathrm{Gal}(K/\mathbb{Q})$. Further, let S and T be the places of E and E' respectively which are over the rational primes which ramify in K.*

1. *E is determined up to isomorphism by the H-set $X_E = \mathrm{Hom}(E, K)$. In particular, the number of factors of E is the number of orbits in X_E and the dimension of E over $\mathbb{Q}$ is the cardinality of X_E.*

2. *(E, S) and (E', T) are locally equivalent if and only if $\chi(X_E) = \chi(X_{E'})$ as permutation characters of H-sets.*

Proof The first part is well known. The second part follows from [8, p. 101 Prop 2.8] for example. ■

There are many known examples of locally equivalent number fields, but we want to demonstrate that there are locally equivalent étale algebras where the local equivalence does not come from a pairwise local equivalence of the number field factors. To that end we let $K = \mathbb{Q}(x)$ where $x^6 - 2x^5 - 14x^4 + 14x^2 - 2x - 1 = 0$, which is Galois with Galois group isomorphic to S_3, totally real, and ramifies only at the prime 229 [9]. Let K_2 be the subfield with degree 2 and K_3 a subfield of degree 3 (they are all isomorphic). Finally let $E_1 = K_2 \oplus K_3 \oplus K_3$ and $E_2 = K \oplus \mathbb{Q} \oplus \mathbb{Q}$.

Lemma 3.1 *E_1 and E_2 are locally equivalent, but their factor fields are not locally equivalent.*

Proof Let $H = \mathrm{Gal}(K/\mathbb{Q}) \cong S_3$ and let H_2 and H_3 be the subgroups fixing K_2 and K_3 respectively. Then we have $X_{E_1} = H/H_2 \sqcup H/H_3 \sqcup H/H_3$ and $X_{E_2} = H \sqcup 1 \sqcup 1$ as H-sets where 1 represents the trivial H-set. It can be checked that $\chi(X_{E_1}) = \chi(X_{E_2})$, so Proposition 3.1 implies E_1 and E_2 have the same prime splittings at all unramified primes of K. It can also be checked [10] that the ramification behavior of E_1 and E_2 at 229 are the same. Thus they are locally equivalent étale algebras which apparently do not contain factors that are pairwise locally equivalent number fields. ■

3.3 Linear Algebraic Groups

For more details regarding algebraic groups consult [17] and [12].

A *linear algebraic group* $\mathbf{G}$, which we will just call an algebraic group, is an algebraic variety whose points $\mathbf{G}(\mathbb{C})$ are a Zariski closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some n. $\mathbf{G}$ inherits from $\mathrm{GL}(n)$ a multiplication and inversion which are morphisms of algebraic varieties. The coordinate ring of $\mathrm{GL}(n)$ is

$$A = \mathbb{C}[x_{11}, x_{12}, ..., x_{nn}, \det(x_{ij})^{(-1)}],$$

so the coordinate ring of $\mathbf{G}$ is $A/\mathfrak{a}$ where $\mathfrak{a}$ is the ideal of polynomials vanishing on $\mathbf{G}(\mathbb{C})$. If k is a subfield of $\mathbb{C}$ and the ideal $\mathfrak{a}$ is generated in A by $\mathfrak{a}_k = \mathfrak{a} \cap A_k$ where

$$A_k = k[x_{11}, x_{12}, ..., x_{nn}, \det(x_{ij})^{(-1)}],$$

then we say $\mathbf{G}$ is defined over k and for any intermediate field $k < K < \mathbb{C}$ we denote its K-points by

$$\mathbf{G}(K) = \mathbf{G}(\mathbb{C}) \cap \mathrm{GL}(n, K).$$

In this case we will call $\mathbf{G}$ a k-group or a group over k. A homomorphism of algebraic groups $\mathbf{G}$ and $\mathbf{H}$, which we may assume are in the same $\mathrm{GL}(n)$, is a morphism f of algebraic varieties which respects the multiplication and inversion morphisms. This morphism f is given by a collection of n^2 polynomials in A, and if $\mathbf{G}$ and $\mathbf{H}$ are further assumed to be k-groups, we say that f is a k-morphism if these polynomials are all in A_k. We state here useful conditions to determine when a group and a group morphism is defined over k.

Proposition 3.2

1. *If a subset $B \subset \mathbf{G}(\mathbb{C}) \cap \mathrm{GL}(n, k)$ is Zariski dense in $\mathbf{G}(\mathbb{C})$, then $\mathbf{G}$ is defined over k.*

2. *Let $\mathbf{G}$ and $\mathbf{H}$ be two k-groups and $f : \mathbf{G} \to \mathbf{H}$ be a morphism of algebraic groups. If $B \subset \mathbf{G}(k)$ is Zariski dense and $f(B) \subset \mathbf{H}(k)$, then f is a k-morphism.*

Proof See [12, pg. 12] and the references there. ∎

Let $k < \mathbb{C}$ be a field and $\mathbf{G}$ be an algebraic group over k. Given $\sigma : k \to \mathbb{C}$ a possibly different embedding of k, we can produce ${}^{\sigma}\mathbf{G}$, a group over $\sigma(k)$, by applying σ to the polynomials defining $\mathbf{G}$. There is an abstract group homomorphism $\sigma_* : \mathbf{G}(k) \to {}^{\sigma}\mathbf{G}(\sigma(k))$ given by applying σ componentwise, and this agrees with the trace map in that $\mathrm{tr}(\sigma_*(g)) = \sigma(\mathrm{tr}(g))$ for $g \in \mathbf{G}(k)$. When the polynomials defining $\mathbf{G}$ are fixed by σ, we have $\mathbf{G} = {}^{\sigma}\mathbf{G}$, and further when σ is an automorphism of k, σ_* is an automorphism of $\mathbf{G}(k)$. Similarly, when $\mathbf{G}$ and $\mathbf{H}$ are k-groups and $f : \mathbf{G} \to \mathbf{H}$ is a k-morphism, we can apply σ to the polynomials defining f to get ${}^{\sigma}f : {}^{\sigma}\mathbf{G} \to {}^{\sigma}\mathbf{H}$.

For an algebraic group $\mathbf{G}$ (only assumed to be defined over $\mathbb{C}$) and a subfield $k < \mathbb{C}$, we define a *k-form of* $\mathbf{G}$ to be an algebraic group $\mathbf{G}_k$ defined over k along with an algebraic isomorphism $f : \mathbf{G}_k \to \mathbf{G}$ (also only assumed to be defined over $\mathbb{C}$). We say two k-forms $(\mathbf{G}_k, f)$ and $(\mathbf{G}'_k, f')$ are isomorphic if there is a k-defined isomorphism of $\mathbf{G}_k$ and $\mathbf{G}'_k$ which commutes with f and f'.

Now let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}$, $E = \oplus K_i$ be an étale algebra, and for each i let $\mathbf{G}_i$ be a K_i-form of $\mathbf{G}$. We will call the collection $\{\mathbf{G}_i\}_i$ an *E-form of* $\mathbf{G}$, denoted $\mathbf{G}_E$, and define

$$\mathbf{G}_E(E) = \prod \mathbf{G}_i(K_i).$$

For each homomorphism $v : E \to \mathbb{C}$, we get a homomorphism $v_* : \mathbf{G}_E(E) \to \mathbf{G}(\mathbb{C})$ by the following procedure. The homomorphism v factors through the projection to

K_i for some i, and so corresponds to some embedding $w : K_i \to \mathbb{C}$. We then obtain v_* by first projecting onto $\mathbf{G}_i(K_i)$, then applying w_* to get to ${}^w\mathbf{G}_i(w(K_i))$, and finally following ${}^w f_i$ to land in ${}^w\mathbf{G}(\mathbb{C}) = \mathbf{G}(\mathbb{C})$.

Using our chosen field isomorphisms, all of the definitions and statements above are also true when using $\overline{\mathbb{Q}_p}$ in place of $\mathbb{C}$. When $\mathbf{G}$ is a group over $\mathbb{C}$, we will also use $\mathbf{G}$ to denote the corresponding group over $\overline{\mathbb{Q}_p}$. When F is a local field, it is a closed subfield of a unique one of $\mathbb{C}$ or a $\overline{\mathbb{Q}_p}$. An F-group will then be understood as being a subgroup of $\mathrm{GL}(n)$ over the appropriate algebraically closed field. When $\mathbf{G}$ is a group over F, $\mathbf{G}(F)$ is a closed subset of F^{n^2} and the topology it inherits gives it the structure of a locally compact topological group. If F is a p-adic local field, $\mathcal{O}_F$, its ring of integers, is a compact open subring of F. Similarly,

$$\mathbf{G}(\mathcal{O}_F) = \mathbf{G}(F) \cap \mathrm{GL}(n, \mathcal{O}_F)$$

is a compact open subgroup of $\mathbf{G}(F)$.

Let F_i be a possibly infinite collection of local fields where, for each p (including $p = \infty$) the number of p-adic fields (archimedean fields) in the collection is finite and the sum of the degrees of those fields over $\mathbb{Q}_p$ (over $\mathbb{R}$) is universally bounded over all p. Let R be the restricted product of the F_i with respect to the open subrings $\mathcal{O}_{F_i}$. We will call such an R an *adele-type ring*. It can be made into a locally compact topological ring, and has a continuous projection to F_i for each i. When K is a number field and we let $F_i = K_v$ for all places v of K, we recover the K-adeles. Now let $\mathbf{G}$ be an algebraic group and $R = \prod' F_i$ be an adele-type ring. For each i let $\mathbf{G}_i$ be an F_i-form of $\mathbf{G}$. We will call the collection $\{\mathbf{G}_i\}_i$ an R-form of $\mathbf{G}$, denoted $\mathbf{G}_R$, and define $\mathbf{G}_R(R)$ as the restricted product of the $\mathbf{G}_i(F_i)$ with respect to the compact open subgroups $\mathbf{G}_i(\mathcal{O}_{F_i})$. Then $\mathbf{G}_R(R)$ can be made into a locally compact topological group with continuous projections to $\mathbf{G}_i(F_i)$ for each i. Each continuous homomorphism $v : R \to F$ where F is one of $\mathbb{C}$ or a $\overline{\mathbb{Q}_p}$ corresponds to an embedding of a unique F_i into F. In the same way as the étale algebra case, we get a continuous homomorphism $v_* : \mathbf{G}_R(R) \to \mathbf{G}(F)$. When K is a number field, $\mathbf{G}$ is defined over K, and for each K_v, we let $\mathbf{G}_{K_v}$ be the base change of $\mathbf{G}$ along the embedding $K \to K_v$, we recover the usual definition of an adelic group $\mathbf{G}(\mathbb{A}_K)$.

The algebraic group $\mathbf{G}$ is *connected* if $\mathbf{G}(\mathbb{C})$ is connected in the Zariski topology. $\mathbf{G}$ is called *simply connected* if for any connected algebraic group $\mathbf{H}$, any surjection $\mathbf{H} \to \mathbf{G}$ with finite kernel is actually an isomorphism. $\mathbf{G}$ is called *simple* if it is connected, noncommutative, and has no proper connected normal algebraic subgroups.

We call an algebraic group $\mathbf{T}$ a torus if it is isomorphic to the group $\mathbf{D}_n$ of diagonal matrices in $\mathrm{GL}(n, \mathbb{C})$ for some n. Notice that $\mathbf{D}_n$ is defined over $\mathbb{Q}$, so it is defined over any subfield of $\mathbb{C}$. If $\mathbf{T}$ is defined over K, we call it a K-split torus if it is isomorphic over K to $\mathbf{D}_n$. If $\mathbf{G}$ is an algebraic group defined over K, we define $\mathrm{rank}_K\ \mathbf{G}$ to be the maximum dimension of $\mathbf{T}$, a subgroup of $\mathbf{G}$ which is K-defined and a K-split torus. Finally, if K is also a number field and S is a finite set of places of K containing all the archimedean places, we define the *S-rank* of $\mathbf{G}$ to be

$$\mathrm{rank}_S\ \mathbf{G} = \sum_{v \in S} \mathrm{rank}_{K_v}\ \mathbf{G}_{K_v}.$$

The group $\mathrm{GL}(n, \mathbb{C})$ is a subgroup of the algebra $\mathrm{M}(n, \mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$, so the same is true of any algebraic group in a canonical way. $\mathbf{G}(\mathbb{C})$ acts on the subalgebra $\mathbb{C}[\mathbf{G}(\mathbb{C})]$, and a choice of a basis of $\mathbb{C}[\mathbf{G}(\mathbb{C})]$ induces a representation of $\mathbf{G}$ into $\mathrm{GL}(m)$ which is isomorphic onto its image where m is the dimension of $\mathbb{C}[\mathbf{G}(\mathbb{C})]$ over $\mathbb{C}$. We say a connected algebraic group $\mathbf{G}$ is *reductive* if the subalgebra $\mathbb{C}[\mathbf{G}(\mathbb{C})]$ of $\mathrm{M}(n, \mathbb{C})$ is semisimple. The trace map on $\mathrm{M}(n, \mathbb{C})$ restricts to give $\mathrm{tr}(g) \in \mathbb{C}$ for any $g \in \mathbf{G}(\mathbb{C})$. For a reductive algebraic group, the trace form $\mathrm{T}(X, Y) := \mathrm{tr}(XY)$ on $\mathbb{C}[\mathbf{G}(\mathbb{C})]$ is a nondegenerate bilinear form.

Every reductive algebraic group $\mathbf{G}$ is isomorphic (over $\mathbb{C}$) to a group $\mathbf{H}$ defined over $\mathbb{Q}$. We will sometimes implicitly replace $\mathbf{G}$ with such an $\mathbf{H}$ in order to allow us to define an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathbf{G}(\mathbb{C})$. Notice that if $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ is nontrivial, the automorphism of $\mathbf{G}(\mathbb{C})$ induced by σ is not algebraic because it fixes the set $\mathbf{G}(\mathbb{Q})$ which is Zariski dense [17, p. 58 Thm 2.2].

For any element $g \in \mathbf{G}(\mathbb{C})$, the conjugation action induces an inner algebraic automorphism of $\mathbf{G}$. Any algebraic automorphism which is not inner is called outer. The inner automorphisms form a normal subgroup of $\mathrm{Aut}(\mathbf{G})$, so it makes sense to talk about the class of an outer automorphism σ, the set of automorphisms which differ from σ by an inner automorphism.

3.4 Profinite Groups

A *profinite group* is an inverse limit of a directed system of finite groups. They are topological groups which are compact, totally disconnected, and Hausdorff. In fact, these properties characterize profinite groups among topological groups. If $\mathbf{G}$ is an algebraic group over $\mathbb{Q}_p$, $\mathbf{G}(\mathbb{Q}_p)$ is a locally compact, totally disconnected, Hausdorff topological group, so any compact subgroup is profinite. $\overline{\mathbb{Q}_p}$ is the direct limit of the local fields $F/\mathbb{Q}_p$, so any finitely generated subgroup of $\mathbf{G}(\overline{\mathbb{Q}_p})$ is contained in $\mathbf{G}(F)$ for some finite extension $F/\mathbb{Q}_p$. It is also the case that any compact subgroup of $\mathbf{G}(\overline{\mathbb{Q}_p})$ is contained in $\mathbf{G}(F)$ for some such F (a proof appears in [20, p.244] for example). Thus any compact subgroup of $\mathbf{G}(\overline{\mathbb{Q}_p})$ is profinite.

When Γ is a finitely generated group, we can form the profinite group $\widehat{\Gamma} = \varprojlim \Gamma/N$ where N ranges over all finite index normal subgroups of Γ and the connecting homomorphisms are the natural ones. This is called the *profinite completion* of Γ. There is a map $i : \Gamma \to \widehat{\Gamma}$ with dense image, which is injective when Γ is residually finite, that is when

$$\bigcap_{\substack{N<G \\ [G:N]<\infty}} N = \{1\}.$$

$\widehat{\Gamma}$ has a universal property where any $f : \Gamma \to G$ for a profinite group G extends to a unique $\widehat{f} : \widehat{\Gamma} \to G$ with $f = \widehat{f} \circ i$ and $\widehat{f}(\widehat{\Gamma}) = \overline{f(\Gamma)}$.

We will say a finitely generated, residually finite group, Γ, is *profinitely rigid*, if, for any other finitely generated, residually finite group Δ, $\widehat{\Delta} \cong \widehat{\Gamma}$ implies $\Delta \cong \Gamma$. If Γ

is finitely generated and residually finite, and $\Delta < \Gamma$ is a finitely generated subgroup, we will say Δ is a *Grothendieck subgroup* when $\Delta < \Gamma$ induces an isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$.

3.5 Spaces of Representations

For a finitely generated group Γ and an integer n, one can construct an algebra R_0 over $\mathbb{Q}$ and a representation $f_0 : \Gamma \to \mathrm{GL}(n, R_0)$ which represents the space of representations of Γ in $\mathrm{GL}(n)$. What we mean by this is that, for any field K, there is a bijection between $\mathrm{Hom}_{\mathrm{Group}}(\Gamma, \mathrm{GL}(n, K))$ and $\mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(R_0, K)$ which is given by sending $\phi : R_0 \to K$ to $\phi_* \circ f_0$ where $\phi_* : \mathrm{GL}(n, R_0) \to \mathrm{GL}(n, K)$ is the map applying ϕ entrywise. The construction can be found in [11] for instance, but we will sketch it here for convenience.

Let $\langle g_1, ..., g_m | r_1, ... \rangle$ be a presentation for Γ (possibly with infinitely many relations) with a symmetric generating set. Let X^l for l from 1 to m be the n by n matrix whose (i, j)th entry is the variable x^l_{ij}. We associate X^l to g_l so that each r_k gives a product of the X_l, and setting this product equal to I_n gives a collection of n^2 equations. Let I be the ideal in $\mathbb{Q}[x^l_{ij}, \det(X^l)^{-1}]$ generated by all of these equations as r_k ranges among all of the relations. Let

$$R_0 = \mathbb{Q}[x^l_{ij}, \det(X^l)^{-1}]/I,$$

and $f_0 : \Gamma \to \mathrm{GL}(n, R_0)$ be the representation sending g_l to X^l. It is then clear that any representation of Γ in $\mathrm{GL}(n, K)$ for a field K comes by evaluating the entries of the X^l with the appropriate values in K and vice versa.

For any given algebraic group, a similar algebra and universal representation can be constructed to parametrize the representations of Γ. Our goal in this work is to construct algebras and universal representations which represent conjugacy classes of Zariski dense representations in connected reductive algebraic groups in the case where there are only finitely many such classes. We will do this for both finitely generated and profinite groups, and then show how the universal representation of a finitely generated group is related to that of its profinite completion.

4. REPRESENTATIONS OF FINITELY GENERATED GROUPS

We first prove our main technical tool in finding the universal representations we seek, then we apply it to finitely generated groups having a certain representation rigidity property. So let $\mathbf{G} < \mathrm{GL}(n)$ be a reductive algebraic group. Given a Zariski dense subgroup H of $\mathbf{G}(\mathbb{C})$, we want to find a smaller field of definition of the subgroup H, say k, and produce a k-form of $\mathbf{G}_k$ with $H < \mathbf{G}_k(k)$. We further want this k-form to only depend on the conjugacy class of H as a subgroup of $\mathrm{GL}(n, \mathbb{C})$. We will choose k to be the trace field, the significance of which is that a representation of H in $\mathbf{G}(\mathbb{C})$ gotten by base change along a different embedding of k in $\mathbb{C}$ cannot be conjugate to H in $\mathbf{G}(\mathbb{C})$ because they have distinct characters. The construction is based on the proof of [12, p. 283 Prop 3.20].

To this end we let $k = \mathbb{Q}(\mathrm{Tr}(h) : h \in H)$ be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$. Now because H is Zariski dense in $\mathbf{G}(\mathbb{C})$, H generates $A := \mathbb{C}[\mathbf{G}(\mathbb{C})]$ over $\mathbb{C}$, so let $x_1, ..., x_m \in H$ be a basis. The bilinear form T gives a dual basis $x_1^*, ..., x_m^*$ of A. Any $h \in H \subset A$ can be written as

$$h = a_1 x_1^* + ... + a_m x_m^*$$

for $a_i \in \mathbb{C}$. Then $T(h, x_i) = \mathrm{tr}(hx_i) = a_i$ and because $hx_i \in H$, $a_i \in k$ for each i. Thus

$$k[x_1, ..., x_m] \subset k[H] \subset k[x_1^*, ..., x_m^*]$$

and we see the algebra $k[H]$ is a vector space of dimension m over k, so $k[x_1, ..., x_m] = k[H]$ and the action of an element $h \in H$ on an x_i takes it to a k-linear combination of the $x_1, ..., x_m$. Thus the representation of $\mathbf{G}$ into $\mathrm{GL}(m)$ given by using the basis $x_1, ..., x_m$ takes H into $\mathrm{GL}(m, k)$. We now let $\mathbf{G}_k$ be the image of this representation, and since H is Zariski dense, Proposition 3.2 shows $\mathbf{G}_k$ is defined over k. Finally, if

$g \in \mathrm{GL}(n, \mathbb{C})$ and we apply this procedure to $g^{-1}Hg$ we get the same k since traces are preserved by conjugation. Moreover, if $\mathbf{G}'_k$ is the k-form produced from $g^{-1}Hg$, conjugation by g gives an isomorphism, α, of $\mathbf{G}'_k$ and $\mathbf{G}_k$, but because α takes H to $g^{-1}Hg < \mathbf{G}'_k(k)$, Proposition 3.2 shows that α is a k-morphism. We have now proved the following:

Proposition 4.1 *Let $H < \mathbf{G}(\mathbb{C})$ be a Zariski dense subgroup and $k = \mathbb{Q}(\mathrm{Tr}(h) : h \in H)$. Then*

$$\mathbf{G}_k(k) := \mathbf{G}(\mathbb{C}) \cap k[H] < \mathrm{M}(n, \mathbb{C})$$

gives a k-form of $\mathbf{G}$ so that $H < \mathbf{G}_k(k)$, which, up to isomorphism, depends only on the conjugacy class of H in $\mathrm{GL}(n, \mathbb{C})$.

Now let Γ be a finitely generated group. We will say that Γ is $\mathbf{G}$-*representation rigid* when Γ has only finitely many Zariski-dense representations to $\mathbf{G}(\mathbb{C})$ up to conjugation. In this setting, we will construct an étale algebra E whose homomorphisms to $\mathbb{C}$ parametrize the conjugacy classes of representations of Γ in $\mathbf{G}(\mathbb{C})$. We do this by producing an E-form of $\mathbf{G}$, $\mathbf{G}_E$, and a representation of $f : \Gamma \to \mathbf{G}_E(E)$ so that any Zariski dense representation of Γ into $\mathbf{G}(\mathbb{C})$ is conjugate to $v_* \circ f$ for a unique homomorphism $v : E \to \mathbb{C}$.

Corollary 4.1 *Let $\mathbf{G}$ be a reductive algebraic group and Γ be a finitely generated group which is $\mathbf{G}$-representation rigid. Then there is an étale algebra E, an E-form of $\mathbf{G}$, $\mathbf{G}_E$, and a representation $f : \Gamma \to \mathbf{G}_E(E)$ so that any Zariski dense representation of Γ in $\mathbf{G}$ is conjugate to $f_v : \Gamma \to \mathbf{G}_E(\mathbb{C})$ for a unique homomorphism $v : E \to \mathbb{C}$.*

Proof We can first show that there are number fields K_i, $\mathbf{G}_{K_i}$ K_i forms of $\mathbf{G}$, and representations $f_i : \Gamma \to \mathbf{G}_{K_i}(K_i)$ so that any Zariski dense representation of Γ in $\mathbf{G}$ is conjugate to $f_{i,v} : \Gamma \to \mathbf{G}_{K_i}(\mathbb{C})$ for a unique embedding $v : K_i \to \mathbb{C}$. Then we can take $E = \oplus K_i$, $\mathbf{G}_E = \prod \mathbf{G}_{K_i}$, and f to be the product of the f_i.

The group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the finite set $X(\Gamma, \mathbf{G}(\mathbb{C}))$ of conjugacy classes of Zariski dense representations of Γ in $\mathbf{G}(\mathbb{C})$. Choose a representative from each orbit,

f_i, and apply Proposition 4.1 to the subgroups $f_i(\Gamma)$ to get number fields K_i and K_i-forms $\mathbf{G}_{K_i}$ with the f_i taking Γ to $\mathbf{G}_{K_i}(K_i)$. Then any representation $\phi : \Gamma \to \mathbf{G}(\mathbb{C})$ is in the same orbit as a unique f_i. The other representations in the orbit of f_i are those coming from different embeddings of K_i in $\mathbb{C}$. Since distinct embeddings of K_i induce representations of Γ which are not conjugate, there is a unique embedding $v : K_i \to \mathbb{C}$ with $\phi = v_* \circ f_i$. ∎

We will call the triple $(E, \mathbf{G}_E, f)$ constructed in this particular way the *universal representation of* Γ *in* $\mathbf{G}$. It is not always an invariant of the isomorphism class of $\mathbf{G}$, but depends on the specific way that $\mathbf{G}$ is embedded into $\mathrm{GL}(n, \mathbb{C})$. The only ambiguity in the construction of $(E, \mathbf{G}_E, f)$ is the order of the factors of E and the specific representations chosen for the f_i. The representations could differ by an embedding of K_i in $\mathbb{C}$ and by conjugation in $\mathbf{G}(\mathbb{C})$.

Suppose now that every algebraic automorphism, α, of $\mathbf{G}$ is induced by conjugation by some element of $\mathrm{GL}(n, \mathbb{C})$. Then for every Zariski-dense representation, ϕ, of Γ and every algebraic automorphism, α, of $\mathbf{G}$, the trace fields and K-forms of $\mathbf{G}$ produced by Proposition 4.1 from $\phi(\Gamma)$ and $\alpha \circ \phi(\Gamma)$ are the same. Then we have that the outer automorphism group of $\mathbf{G}$ acts on the universal representation $(E, \mathbf{G}_E, f)$ of Γ by permuting these isomorphic factors of E and $\mathbf{G}_E$. Further, the representation f in these permuted factors clearly differs only by an automorphism of $\mathbf{G}$. We can then quotient the universal representation by this action to obtain the *Aut-universal representation* $(\overline{E}, \mathbf{G}_{\overline{E}}, \overline{f})$. Notice that representations coming from distinct homomorphisms $E \to \mathbb{C}$ cannot be related by an algebraic automorphism because they have distinct characters.

Corollary 4.2 *Assume the hypotheses of Corollary 4.1 and that that every algebraic automorphism, α, of* $\mathbf{G}$ *is induced by conjugation by some element of* $\mathrm{GL}(n, \mathbb{C})$. *Then there is an étale algebra $\overline{E}$, an $\overline{E}$-form of* $\mathbf{G}$, $\mathbf{G}_{\overline{E}}$, *and a representation* $\overline{f} : \Gamma \to \mathbf{G}_{\overline{E}}(\overline{E})$ *so that any Zariski dense representation of* Γ *in* $\mathbf{G}$ *agrees, up to an algebraic automorphism of* $\mathbf{G}$, *with* $f_v : \Gamma \to \mathbf{G}_{\overline{E}}(\mathbb{C})$ *for a unique homomorphism* $v : \overline{E} \to \mathbb{C}$.

5. REPRESENTATIONS OF PROFINITE GROUPS

Now let G be a profinite group and continue to let $\mathbf{G} < \mathrm{GL}(n)$ be a reductive algebraic group. We first state a version of Proposition 4.1 specific to a continuous representation of G in $\mathbf{G}(\overline{\mathbb{Q}_p})$

Proposition 5.1 *Let $f : G \to \mathbf{G}(\overline{\mathbb{Q}_p})$ be a continuous representation with Zariski dense image and let $K = \mathbb{Q}_p(\mathrm{Tr}(g) : g \in G)$ be the closed subfield of $\overline{\mathbb{Q}_p}$. Then K is a finite extension of $\mathbb{Q}_p$ and*

$$\mathbf{G}_K(K) := \mathbf{G}(\overline{\mathbb{Q}_p}) \cap K[f(G)] < \mathrm{M}(n, \overline{\mathbb{Q}_p})$$

gives a K-form of $\mathbf{G}$ with $G < \mathbf{G}_K(K)$, which, up to isomorphism, depends only on the conjugacy class of f as a representation into $\mathrm{GL}(n, \overline{\mathbb{Q}_p})$. Further, distinct continuous embeddings $K \to \overline{\mathbb{Q}_p}$ induce representations of G in $\mathbf{G}$ that are not conjugate.

Proof We know that $f(G)$ is a compact subgroup of $\mathbf{G}(\overline{\mathbb{Q}_p})$, so it is contained in $\mathrm{GL}(n, F)$ for some F a finite extension of $\mathbb{Q}_p$. Thus K is also a finite extension of $\mathbb{Q}_p$. We now use Proposition 4.1 with this K to produce our desired $\mathbf{G}_K$. Lastly, since K is the closure of the trace field of $f(G)$, distinct continuous embeddings of K induce representations of G with distinct characters and so they cannot be conjugate. ∎

We will say that G is $\mathbf{G}$-*representation rigid* if there is a constant c such that for all p, the number of conjugacy classes of continuous representations of G in $\mathbf{G}(\overline{\mathbb{Q}_p})$ is less than c.

Corollary 5.1 *Suppose that G is $\mathbf{G}$-representation rigid. Then there is an adele-type ring A, an A-form of $\mathbf{G}$, $\mathbf{G}_A$, and a representation $f : G \to \mathbf{G}_A(A)$ so that any continuous Zariski dense representation of G in $\mathbf{G}(\overline{\mathbb{Q}_p})$ is conjugate to $f_v : G \to \mathbf{G}_A(\overline{\mathbb{Q}_p})$ for a unique continuous homomorphism $v : A \to \overline{\mathbb{Q}_p}$.*

Proof We can first show that for each p, there are finite extensions $K_{p,i}$ of $\mathbb{Q}_p$, $\mathbf{G}_{K_{p,i}}$ $K_{p,i}$ forms of $\mathbf{G}$, and representations $f_{p,i} : G \to \mathbf{G}_{K_{p,i}}(K_{p,i})$ so that any Zariski dense continuous representation of G in $\mathbf{G}(\overline{\mathbb{Q}_p})$ is conjugate to $f_{i,v} : \Gamma \to \mathbf{G}_{K_i}(\overline{\mathbb{Q}_p})$ for a unique continuous embedding $v : K_{p,i} \to \overline{\mathbb{Q}_p}$. Then we will take A to be the restricted product of all of the $K_{p,i}$ over all p and $\mathbf{G}_A$ to be the restricted product of all of the $\mathbf{G}_{K_{p,i}}$ over all p. The fact that G is $\mathbf{G}$-representation rigid ensures that A will be an adele-type ring.

For each p, the group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts on the finite set $X_{cont}(G, \mathbf{G}(\overline{\mathbb{Q}_p}))$ of conjugacy classes of continuous Zariski dense representations of G in $\mathbf{G}(\overline{\mathbb{Q}_p})$. Choose a representative from each orbit, $f_{p,i}$ and apply Proposition 5.1 to the subgroups $f_{p,i}(G)$ to get finite extensions $K_{p,i}$ and $K_{p,i}$-forms $\mathbf{G}_{K_{p,i}}$ with the $f_{p,i}$ taking G to $\mathbf{G}_{K_{p,i}}(K_{p,i})$. Then any representation $f_p : G \to \mathbf{G}(\overline{\mathbb{Q}_p})$ is in the same orbit as a unique $f_{p,i}$. The other representations in the orbit of $f_{p,i}$ are those coming from different continuous embeddings of $K_{p,i}$ in $\overline{\mathbb{Q}_p}$. Since distinct embeddings of $K_{p,i}$ induce representations of G which are not conjugate, there is a unique embedding $v : K_{p,i} \to \overline{\mathbb{Q}_p}$ with $f_p = v_* \circ f_{p,i}$. ■

We will call the triple $(A, \mathbf{G}_A, f)$ constructed in this particular way the *universal representation of* G *in* $\mathbf{G}$. Again, it is not always an invariant of the isomorphism class of $\mathbf{G}$ and the only ambiguity in its construction is the order of the factors of A and the specific representations chosen for the f_i.

The same argument as for Corollary 4.2 gives the following since algebraic automorphisms of $\mathbf{G}(\overline{\mathbb{Q}_p})$ are continuous.

Corollary 5.2 *Assume the hypotheses of Corollary 5.1 and that every algebraic automorphism, α, of $\mathbf{G}$ is induced by conjugation within $\mathrm{GL}(n)$. Then there is an adele-type ring $\overline{A}$, an $\overline{A}$-form of $\mathbf{G}$, $\mathbf{G}_{\overline{A}}$, and a representation $\overline{f} : G \to \mathbf{G}_{\overline{A}}(\overline{A})$ so that any continuous Zariski dense representation of G in $\mathbf{G}(\overline{\mathbb{Q}_p})$ agrees, up to an algebraic automorphism of $\mathbf{G}$, with $f_v : G \to \mathbf{G}_{\overline{A}}(\overline{\mathbb{Q}_p})$ for a unique continuous homomorphism $v : \overline{A} \to \overline{\mathbb{Q}_p}$.*

We will call the triple $(\overline{A}, \mathbf{G}_{\overline{A}}, f')$ constructed in this particular way the *Aut-universal representation of* G *in* $\mathbf{G}$.

6. REPRESENTATIONS OF PROFINITE COMPLETIONS

Let $\mathbf{G}$ be an algebraic group over $\mathbb{C}$ and Γ be a finitely generated group. We will call a representation $f : \Gamma \to \mathbf{G}(\overline{\mathbb{Q}_p})$ *bounded* if the closure $\overline{f(\Gamma)}$ is compact, and hence profinite. The following proposition shows that the bounded representations of Γ can be related to the continuous representations of $\widehat{\Gamma}$.

Proposition 6.1 *For each prime p, there is a bijection between representations of Γ in $\mathbf{G}(\overline{\mathbb{Q}_p})$ with bounded image and continuous representations of $\widehat{\Gamma}$ in $\mathbf{G}(\overline{\mathbb{Q}_p})$. A representation of Γ has Zariski-dense image if and only if the corresponding representation of $\widehat{\Gamma}$ does. Finally, this bijection is equivariant under the action of $\mathbf{G}(\overline{\mathbb{Q}_p})$ by conjugation, and under the action of the algebraic automorphisms of $\mathbf{G}$.*

Proof Fix a prime p and suppose $f : \Gamma \to \mathbf{G}(\overline{\mathbb{Q}_p})$ is a representation with bounded image. This means $\overline{f(\Gamma)}$ is compact and thus profinite. So f extends to a continuous representation $\widehat{f} : \widehat{\Gamma} \to \mathbf{G}(\overline{\mathbb{Q}_p})$ with $\widehat{f}(\widehat{\Gamma}) = \overline{f(\Gamma)}$. Conversely, given a continuous representation $\widehat{f} : \widehat{\Gamma} \to \mathbf{G}(\overline{\mathbb{Q}_p})$, $\widehat{f}(\widehat{\Gamma}) = \overline{f(\Gamma)}$ is compact so f is bounded.

Clearly, if a representation of Γ has Zariski-dense image, then the corresponding representation of $\widehat{\Gamma}$ will as well. Conversely, $f(\Gamma)$ is dense in $\widehat{f}(\widehat{\Gamma})$ in the analytic topology of $\mathbf{G}(\overline{\mathbb{Q}_p})$ which is finer than the Zariski topology, so if $\widehat{f}(\widehat{\Gamma})$ is Zariski-dense then $f(\Gamma)$ is as well.

Algebraic automorphisms of $\mathbf{G}$ give continuous automorphisms of $\mathbf{G}(\overline{\mathbb{Q}_p})$, so if two representations of Γ with bounded image, f and g, are related by an algebraic automorphism, their extensions, $\widehat{f}$ and $\widehat{g}$ are related by the same automorphism. In particular, f and g are conjugate if and only if $\widehat{f}$ and $\widehat{g}$ are. ■

Whether or not Γ is representation rigid can also be detected by $\widehat{\Gamma}$.

Lemma 6.1 Γ *is* $\mathbf{G}$*-representation rigid if and only if* $\widehat{\Gamma}$ *is.*

Proof If Γ is $\mathbf{G}$-representation rigid, it has only finitely many, say c, Zariski-dense representations in $\mathbf{G}(\mathbb{C})$. So for each p, Γ has no more than c bounded Zariski-dense representations in $\mathbf{G}(\overline{\mathbb{Q}_p})$. By Proposition 6.1, $\widehat{\Gamma}$ has no more than c continuous Zariski-dense representations in $\mathbf{G}(\overline{\mathbb{Q}_p})$ for each p. So $\widehat{\Gamma}$ is $\mathbf{G}$-representation rigid.

Conversely, if $\widehat{\Gamma}$ is $\mathbf{G}$-representation rigid, there is a c so that for each p, Γ has no more than c bounded Zariski-dense representations in $\mathbf{G}(\overline{\mathbb{Q}_p})$. Now if Γ has a collection of d distinct Zariski-dense representations, the subring of $\mathbb{C}$ generated by the coefficients of all of the representations is finitely generated, so by Noether normalization its localization at (b) is a finite extension of $\mathbb{Z}_{(b)}[x_1, ..., x_m]$ for some m and some b. For any p that does not divide b, this ring can be embedded into a finite extension of $\mathbb{Z}_p$, a bounded subring of $\overline{\mathbb{Q}_p}$. Thus for each p which does not divide b, Γ has d distinct Zariski-dense bounded representations in $\mathbf{G}(\overline{\mathbb{Q}_p})$. Now by 6.1, $d \leq c$, and Γ is $\mathbf{G}$-representation rigid. ∎

Suppose $\mathbf{G}$ is a reductive algebraic group and Γ is $\mathbf{G}$-representation rigid. Let $(E, \mathbf{G}_E, f)$ be the universal representation of Γ in $\mathbf{G}$. Recall that through our implicit identifications of $\mathbb{C}$ and $\overline{\mathbb{Q}_p}$, the homomorphisms of E into $\overline{\mathbb{Q}_p}$ also parametrize the representations of Γ into $\mathbf{G}(\overline{\mathbb{Q}_p})$.

Lemma 6.2 *Let S be the collection of places of E which are archimedean, or where the corresponding representation of Γ in $\overline{\mathbb{Q}_p}$ is unbounded. Then S is finite.*

Proof First notice that for a homomorphism $v : E \to \overline{\mathbb{Q}_p}$, the property of f_v being a bounded representation is actually a property of the place of v (that is the orbit of v under postcomposition by $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$) so this statement makes sense. This is because elements of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ are continuous automorphisms of $\overline{\mathbb{Q}_p}$, so they induce continuous automorphisms of $\mathbf{G}(\overline{\mathbb{Q}_p})$. Then homomorphisms of E which belong to the same place give representations which are either both bounded or both unbounded.

Now Γ is finitely generated so the matrix entries of a set of generators of $f(\Gamma) < \mathbf{G}_E(E)$ will be a finite collection of elements of E, which will be integral at all but finitely many places of E. Any place where all of these elements are integral will give a bounded representation, so S will be contained in the finite set of places where the generators are not all integral. ■

For the specific S from Lemma 6.2, we will call the quadruple $(E, S, \mathbf{G}_E, f)$ the *bounded universal representation of* Γ *in* $\mathbf{G}$.

Let $A = \mathbb{A}_{(E,S)}$, and $\mathbf{G}_A$ be the group over A coming from $\mathbf{G}_E$. Then the inclusion $\mathbf{G}_E(E) \to \mathbf{G}_A(A)$ gives the representation $f : \Gamma \to \mathbf{G}_A(A)$. Because each place of (E, S) corresponds to a bounded representation of Γ, the closure of $f(\Gamma)$ in $\mathbf{G}_A(A)$ is compact, so profinite. Then let $\widehat{f} : \widehat{\Gamma} \to \mathbf{G}_A(A)$ be the extension of f.

Proposition 6.2 *The universal representation of* $\widehat{\Gamma}$ *is* $(A, \mathbf{G}_A, \widehat{f})$.

Proof First, by Lemma 6.1 $\widehat{\Gamma}$ is also $\mathbf{G}$-representation rigid, so it does have a universal representation. The claim then follows just by comparing the construction of $(A, \mathbf{G}_A, \widehat{f})$ to the construction of the universal representation of $\widehat{\Gamma}$. By Proposition 6.1, the places of (E, S) correspond to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ orbits of continuous representations of $\widehat{\Gamma}$. Further, a factor E_v of A is the closure of $v(E)$ in $\overline{\mathbb{Q}_p}$. Since $v(E)$ is the trace field of $f_v(\Gamma)$, Γ is dense in $\widehat{\Gamma}$, and the trace map is continuous, E_v is also the closure of the trace field of $\widehat{f}_v(\widehat{\Gamma})$. So we see A is in the correct form to be part of the universal representation of $\widehat{\Gamma}$.

A factor of $\mathbf{G}_A$ is the group $\mathbf{G}_E(E_v)$ for some factor E_v of A. By construction, this is given by $\mathrm{G}(\overline{\mathbb{Q}_p}) \cap E_v[f_v(\Gamma)]$ in $\mathrm{M}(n, \overline{\mathbb{Q}_p})$. Again since Γ is dense in $\widehat{\Gamma}$ and E_v is also the closure of the trace field of $\widehat{f}_v(\widehat{\Gamma})$, we have $E_v[f_v(\Gamma)] = E_v[\widehat{f}_v(\widehat{\Gamma})]$ in $\mathrm{M}(n, \overline{\mathbb{Q}_p})$. So we have

$$\mathrm{G}(\overline{\mathbb{Q}_p}) \cap E_v[f_v(\Gamma)] = \mathrm{G}(\overline{\mathbb{Q}_p}) \cap E_v[\widehat{f}_v(\widehat{\Gamma})],$$

and $\mathbf{G}_E(E_v)$ is constructed the correct way to form the universal representation of $\widehat{\Gamma}$. ■

Suppose $\mathbf{G}$ is a reductive algebraic group, Γ is a finitely generated group which is $\mathbf{G}$-representation rigid, and Δ is another finitely generated group with a fixed isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$. It follows from Lemma 6.1 that Δ is $\mathbf{G}$-representation rigid as well. So let $(E, S, \mathbf{G}_E, f)$ and $(E', T, \mathbf{G}_{E'}, g)$ be the bounded universal representations of Γ and Δ respectively. Also let $A = \mathbb{A}_{(E,S)}$ and $B = \mathbb{A}_{(E',T)}$.

Theorem 6.1 *The isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$ gives a bijection of the places of (E, S) with those of (E', T). This bijection induces an isomorphism $A \cong B$ and an isomorphism $\mathbf{G}_A(A) \cong \mathbf{G}_B(B)$ so that the following diagram commutes.*

$$\begin{array}{ccc} \widehat{\Gamma} & \xrightarrow{\sim} & \widehat{\Delta} \\ {\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle \widehat{g}} \\ \mathbf{G}_A(A) & \xrightarrow{\sim} & \mathbf{G}_B(B). \end{array}$$

Proof This follows from applying Proposition 6.2 to both Γ and Δ. We see that $(A, \mathbf{G}_A, \widehat{f})$ and $(B, \mathbf{G}_B, \widehat{g})$ are both the universal representation of $\widehat{\Delta} \cong \widehat{\Gamma}$. Since the universal representation is unique up to reordering the factors, the result follows. ■

Suppose the hypotheses of Theorem 6.1, but also suppose that every algebraic automorphism, α, of $\mathbf{G}$ is induced by conjugation within $\mathrm{GL}(n)$. Then we are able to get the same relationship between the Aut-universal representations of Γ and Δ, $(\overline{E}, \mathbf{G}_{\overline{E}}, \overline{f})$ and $(\overline{E}', \mathbf{G}_{\overline{E}'}, \overline{g})$ respectively. Let $\overline{S}$ and $\overline{T}$ be the places of $\overline{E}$ and $\overline{E}'$ respectively which either are archimedean or correspond to representations of Γ with unbounded image. Let $\overline{A} = \mathbb{A}_{(\overline{E},\overline{S})}$ and $\overline{B} = \mathbb{A}_{(\overline{E}',\overline{T})}$.

Theorem 6.2 *The isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$ gives a bijection of the places of $(\overline{E}, \overline{S})$ with those of $(\overline{E}', \overline{T})$. This bijection induces an isomorphism $\overline{A} \cong \overline{B}$ and an isomorphism $\mathbf{G}_{\overline{A}}(\overline{A}) \cong \mathbf{G}_{\overline{B}}(\overline{B})$ so that the following diagram commutes.*

$$\begin{array}{ccc} \widehat{\Gamma} & \xrightarrow{\sim} & \widehat{\Delta} \\ {\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle \widehat{g}} \\ \mathbf{G}_{\overline{A}}(\overline{A}) & \xrightarrow{\sim} & \mathbf{G}_{\overline{B}}(\overline{B}). \end{array}$$

Proof This follows from the bijection and isomorphisms from Theorem 6.1, and the construction of the Aut-universal representation as a quotient of the universal representation by the action of the group of outer automorphisms. The bijection of places in Theorem 6.1 is equivariant under the action of outer automorphisms by Proposition 6.1, so it descends to the quotient. ■

Example 6.3 We now provide an example which demonstrates that the result in 6.2 is optimal in a certain sense. That although we can show there is a local equivalence between $(\overline{E}, \overline{S})$ and $(\overline{E}', \overline{T})$, it need not come from a collection of local equivalences of the factor number fields. To that end, recall the fields K, K_1, and K_2 and the étale algebras E_1 and E_2 from 3.2, which were shown to be locally equivalent in Lemma 3.1. For $n \geq 3$, let

$$\Gamma = \mathrm{SL}(n, \mathcal{O}_{K_2}) \times \mathrm{SL}(n, \mathcal{O}_{K_3}) \times \mathrm{SL}(n, \mathcal{O}_{K_3})$$

and

$$\Delta = \mathrm{SL}(n, \mathcal{O}_K) \times \mathrm{SL}(n, \mathbb{Z}) \times \mathrm{SL}(n, \mathbb{Z}).$$

These groups each have CSP (see §7 below) because each of their factors do by [2]. So because $\mathbb{A}_{E_1} \cong \mathbb{A}_{E_2}$ we see $\widehat{\Gamma} \cong \widehat{\Delta}$. However the arguments from the proof of Lemma 7.1 below can show that the Aut-universal representations of Γ and Δ into $\mathrm{SL}(n, \mathbb{C})$ are $(E_1, \mathrm{SL}(n, E_1), i_1)$ and $(E_2, \mathrm{SL}(n, E_2), i_2)$ respectively where i_1 and i_2 are the inclusions. Again from Lemma 3.1, we have seen that these étale algebras are locally equivalent, but the equivalence does not come from equivalences of their factor fields.

One special case of the results in [7] is that if Δ is a Grothendieck subgroup of Γ, then for any field k and any algebraic group $\mathbf{G}$ over k, the map $\mathrm{Hom}(\Gamma, \mathbf{G}(k)) \to \mathrm{Hom}(\Delta, \mathbf{G}(k))$ induced by the inclusion is actually a bijection. It is interesting to compare this with the results obtained above. Here, we have only assumed that Δ and Γ have some isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$, which is not assumed to be induced by any map between Δ and Γ themselves. However, we have assumed that the target group $\mathbf{G}$ is a reductive group over $\mathbb{C}$ and the set $X(\Gamma, \mathbf{G}(\mathbb{C}))$ of conjugacy classes of Zariski-dense representations of Γ in $\mathbf{G}(\mathbb{C})$ is finite. We are able to show that $X(\Delta, \mathbf{G}(\mathbb{C}))$ is also finite and has the same cardinality as $X(\Gamma, \mathbf{G}(\mathbb{C}))$, however there is no preferred bijection of these sets. Instead, for each p we do have specific bijections of the sets $X_{\mathrm{Bdd}}(\Gamma, \mathbf{G}(\overline{\mathbb{Q}_p}))$ and $X_{\mathrm{Bdd}}(\Delta, \mathbf{G}(\overline{\mathbb{Q}_p}))$ of classes of representations with bounded images by Lemma 6.1. These bijections varying over all p are what allow us to see the relationships between the local behavior of the global representations of Γ and Δ in $\mathbf{G}(\mathbb{C})$ in Theorem 6.1.

7. PROFINITE COMPLETIONS OF HIGHER RANK ARITHMETIC GROUPS

Let $\mathbf{G}$ be a connected, simply connected, simple algebraic group. Further let K be a number field, $\mathbf{G}_K$ be a K-form of $\mathbf{G}$, and S be a finite set of places of K, containing all the archimedean places, but containing no places v where $\mathbf{G}_K(K_v)$ is compact. Let Γ be a subgroup of $\mathbf{G}_K(\mathbb{C})$ which is commensurable with $\mathbf{G}_K(\mathcal{O}_{K,S})$ (commensurable means that the intersection of Γ with $\mathbf{G}(\mathcal{O}_{K,S})$ has finite index in both). Such a group Γ is called an *S-arithmetic group*. Let $A = \mathbb{A}_{K,S}$, so the closure of Γ in $\mathbf{G}_A(A)$ is compact, so it is profinite and called the congruence completion. The kernel of the induced map $\widehat{i} : \widehat{\Gamma} \to \mathbf{G}_A(A)$ is called the congruence kernel. When the congruence kernel is trivial, we will say that Γ has the *congruence subgroup property* which we will abbreviate CSP.

Suppose now that the S-rank of $\mathbf{G}_K$ is greater than or equal to 2, that Γ has CSP, and that Γ has no nontrivial homomorphisms to the the center of $\mathbf{G}(\mathbb{C})$. Suppose also that every algebraic automorphism, α, of $\mathbf{G}$ is induced by conjugation by some element of $\mathrm{GL}(n, \mathbb{C})$.

Lemma 7.1 *The Aut-universal representation of Γ in $\mathbf{G}$ is $(K, \mathbf{G}_K, i)$ where $i : \Gamma \to \mathbf{G}_K(K)$ is the inclusion. The Aut-universal representation of $\widehat{\Gamma}$ in $\mathbf{G}$ is $(A, \mathbf{G}_A, \widehat{i})$ and $\widehat{i}$ is injective.*

Proof By superrigidity [12, p. 259 Thm. C], each representation of Γ in $\mathbf{G}(\mathbb{C})$ with Zariski dense image comes from a unique embedding $K \to \mathbb{C}$ followed possibly by an automorphism of $\mathbf{G}$. Because K is a number field and $\mathbf{G}$ has only finitely many classes of outer automorphisms [17, p. 64 Thm. 2.8], Γ is $\mathbf{G}$-representation rigid. The trace field of Γ contains K, but cannot be larger than K; if the trace field were larger, its extra embeddings would give extra representations of Γ, which we know cannot exist.

Finally, the K form for $\mathbf{G}$ produced by Proposition 4.1 from Γ must be isomorphic to $\mathbf{G}_K$ by Proposition 3.2. Then because every algebraic automorphism of $\mathbf{G}$ is induced by conjugation in $\mathrm{GL}(n)$, the construction from Corollary 4.2 shows $(K, \mathbf{G}_K, i)$ is the Aut-universal representation.

The only thing more we must show to get the statement about the universal representation for $\widehat{\Gamma}$ is that the representations of Γ in $\mathbf{G}_K(K_v)$ for p-adic places v are bounded exactly when v is not in S. For each $v \in S$, the group $\mathbf{G}_K(\mathcal{O}_{(K,S)})$ is dense in $\mathbf{G}_K(K_v)$, which we have assumed is not compact, so $\mathbf{G}_K(\mathcal{O}_{(K,S)})$ is not bounded. Conversely, when $v \notin S$, $\mathbf{G}_K(\mathcal{O}_{(K,S)})$ has compact closure in $\mathbf{G}_K(K_v)$. Since Γ is commensurable with $\mathbf{G}_K(\mathcal{O}_{(K,S)})$, the same conclusions hold for Γ. That $\widehat{i}$ is injective is the definition of CSP. ■

Let Δ be a finitely generated, residually finite group with a fixed isomorphism $\widehat{\Gamma} \cong \widehat{\Delta}$.

Theorem 7.1 *There is a number field L, an L-form $\mathbf{G}_L$, a finite set of places of L including all the archimedean places T, and Λ a T-arithmetic subgroup of $\mathbf{G}_L$ with an injection $g : \Delta \to \Lambda$ such that $A \cong B$ and $\mathbf{G}_A(A) \cong \mathbf{G}_B(B)$ where $B = \mathbb{A}_{L,T}$. If $\mathbf{G}_L(L_v)$ is not compact for any p-adic place v in T and Λ also has CSP, then $\widehat{\Gamma} \cong \widehat{\Lambda}$ and either $g(\Delta) = \Lambda$ or $g(\Delta)$ is a proper Grothendieck subgroup of Λ.*

Proof Apply Theorem 6.2 and Lemma 7.1 to $\widehat{\Delta} \cong \widehat{\Gamma}$ to get an étale algebra L, L-form $\mathbf{G}_L$, places T, and representation $g : \Delta \to \mathbf{G}_L(L)$ with $A \cong B$ and $\mathbf{G}_A(A) \cong \mathbf{G}_B(B)$. We can see that L is a number field because $A \cong B$ implies (K, S) and (L, T) are locally equivalent étale algebras, and so have the same number of factors. Since $\widehat{i}$ is an injection $\widehat{g}$ must be as well, so g is injective. We take Λ to be $\widehat{g}(\widehat{\Delta}) \cap \mathbf{G}_L(L)$ in $\mathbf{G}_B(B)$. It is a T-arithmetic group because its closure in $\mathbf{G}_B(B) \cong \mathbf{G}_A(A)$ is equal to that of Γ, and so for each factor L_v of B, the closure of Λ is commensurable with $\mathbf{G}_L(\mathcal{O}_{L_v})$ and they are equal for all but finitely many factors.

If $\mathbf{G}_L(L_v)$ is not compact for any p-adic place v in T and Λ also has CSP, then $\widehat{\Lambda} \to \mathbf{G}_B(B)$ is an isomorphism onto its image which is also isomorphic to $\widehat{\Gamma}$. Since

$g(\Delta)$ is dense in the image of $\widehat{\Lambda}$ in $\mathbf{G}_B(B)$, $g : \Delta \to \Lambda$ induces a surjection $\widehat{g} : \widehat{\Delta} \to \widehat{\Lambda}$. But since $\widehat{\Delta} \cong \widehat{\Lambda}$ and finitely generated profinite groups are Hopfian, $\widehat{g}$ must then be an isomorphism. So $g(\Delta)$ is a Grothendieck subgroup of Λ.

■

Corollary 7.2 *For $n \geq 3$, either* $\mathrm{SL}(n, \mathbb{Z})$ *is profinitely rigid, or it contains a proper Grothendieck subgroup.*

Proof We will apply Theorem 7.1 to $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and Δ any finitely generated, residually finite group with an isomorphism $\widehat{\Delta} \cong \widehat{\Gamma}$. Let $\mathbf{G} = \mathrm{SL}(n, \mathbb{C})$, which is connected, simply connected, and simple. Let $\mathbf{G}_{\mathbb{Q}}$ be the $\mathbb{Q}$-form with $\mathbf{G}_{\mathbb{Q}}(\mathbb{Q}) = \mathrm{SL}(n, \mathbb{Q})$ and S be the set containing only the real place of $\mathbb{Q}$. Then Γ is an S-arithmetic lattice in the group $\mathbf{G}_{\mathbb{Q}}$, which has S rank $n - 1 \geq 2$. Γ has CSP and Γ is perfect, so it has no nontrivial homomorphisms to the center of $\mathbf{G}$. The only class of nontrivial outer automorphisms of G is represented by $X \mapsto (X^{-1})^T$. We now construct an embedding of $\mathbf{G}$ in $\mathrm{GL}(2n, \mathbb{C})$ by placing it in diagonally in two blocks, one by the identity and the other by the inverse-transpose map. Replace $\mathbf{G}$ by its image under this embedding. Then the outer automorphism of $\mathbf{G}$ is induced by conjugation in $\mathrm{GL}(2n, \mathbb{C})$.

Now Theorem 7.1 provides a number field L and a finite set of places T of L including all the archimedean places such that $\mathbb{A}^f \cong \mathbb{A}_{(L,T)}$, which clearly shows $L = \mathbb{Q}$ and T the infinite place. We also get an L-form $\mathbf{G}_L$ so that and $\mathrm{SL}(n, \mathbb{A}^f) \cong \mathbf{G}_{\mathbb{A}^f}(\mathbb{A}^f)$. Now the $\mathbb{Q}$-forms of $\mathrm{SL}(n, \mathbb{C})$ are of the form either $\mathrm{SL}(m, D)$ for D a central skew field over $\mathbb{Q}$, or a unitary group $\mathrm{SU}(m, D, f)$ where f is a Hermitian form on D^m [17, pp. 87-88]. However, a unitary group will have infinitely places where the local form will be of the type $\mathrm{SU}(m, D, f)$ over $\mathbb{Q}_p$, which $\mathrm{SL}(n, \mathbb{A}^f)$ does not. So $\mathbf{G}_L$ must be of the form $\mathrm{SL}(m, D)$, but again because the local forms of $\mathbf{G}_L$ are all $\mathrm{SL}(n, \mathbb{Q}_p)$, the classification of central simple algebras implies $D = \mathbb{Q}$ and $\mathbf{G}_L$ is isomorphic to $\mathrm{SL}(n, \mathbb{Q})$.

Next we see that because it has the same congruence completion as $\mathrm{SL}(n, \mathbb{Z})$, the Λ constructed in Theorem 7.1 is actually isomorphic to $\mathrm{SL}(n, \mathbb{Z})$ as well. So Λ has

CSP as well, and $\mathbf{G}_L$ has no extra compact factors. So we produce an injection $g : \Delta \to \mathrm{SL}(n, \mathbb{Z})$ with either $\Delta \cong \mathrm{SL}(n, \mathbb{Z})$ or Δ is isomorphic to a proper Grothendieck subgroup of $\mathrm{SL}(n, \mathbb{Z})$. ■

Another interesting example which demonstrates the difficulty in determining a lattice from just its local information is given by $\Gamma = \mathrm{SL}(4, \mathcal{O}_K)$ where K is any totally real cubic number field. These groups are higher rank and also have CSP by [2]. If we have a finitely generated, residually finite Δ with $\widehat{\Delta} \cong \widehat{\Gamma}$, we obtain an injection of Δ into an arithmetic group Λ, but now Λ may be distinct from Γ. There are K-forms of SL(4) whose local information is the same as $\mathrm{SL}(4, K)$ at all p-adic places, and which only differ at the archimedean places. In this case we cannot rule out the possibility of Λ being an arithmetic subgroup in one of these groups instead.

REFERENCES

REFERENCES

[1] M. Aka, *Arithmetic groups with isomorphic finite quotients*, J. Algebra **352**, (2012), 322–340.

[2] H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n *(*$n \geq 3$*) and* Sp_{2n} *(*$n \geq 2$*)*, Publ. Math. IHES **33** (1967), 59–137.

[3] G. Baumslag, *Residually finite groups with the same finite images*, Compositio Math. **29** (1974), 249–252.

[4] M. R. Bridson and F. Grunewald, *Grothendieck's problems concerning profinite completions and representations of groups*, Annals of Math. **160** (2004), 359–373.

[5] M. R. Bridson, D. B. McReynolds, R. Spitler and A. W. Reid, *Absolute Profinite Rigidity and Hyperbolic Geometry*, arXiv:1811.04394.

[6] M. R. Bridson, D. B. McReynolds, R. Spitler and A. W. Reid, *Profinite rigidity of certain triangle groups and profinite completions of groups*, in preparation.

[7] A. Grothendieck, *Representations lineaires et compactification profinie des groupes discrets*, Manuscripta Math. **2** (1970) 375–396.

[8] G. J. Janusz, *Algebraic Number Fields, Second Edition*, Graduate Studies in Math. **7** A.M.S. (1996).

[9] J. Jones and D. Roberts, *A database of number fields*, LMS J. Comput. Math. 17 (1) (2014) 595-618.

[10] J. Jones and D. Roberts, *Database of Local Fields*, https://math.la.asu.edu/ jj/localfields/, July 2019.

[11] A. Lubotzky and A. Magid, *Varieties of Representations of Finitely Generated Groups*, Memoirs of the American Math. Society **336**, 1985.

[12] G. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17. Springer-Verlag, Berlin, 1991.

[13] D. B. McReynolds and R. Spitler, *Arithmetic lattices, profinite equivalence, and linear representations*, in preparation.

[14] J. Milne and J. Suh, *Nonhomeomorphic conjugates of connected Shimura varieties*. Amer. J. Math., **132**(3):731–750, 2010.

[15] P. F. Pickel, *Finitely generated nilpotent groups with isomorphic finite quotients*, Bull. Am. Math. Soc. **77** (1971), 216–219.

[16] V. Platonov and O. I. Tavgen, *Grothendieck's problem on profinite completions and representations of groups*, K-Theory **4** (1990), 89–101.

[17] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Pure and Applied Mathematics, **139** Academic Press (1994).

[18] A. W. Reid, *Profinite rigidity*, In Proc. Int. Cong. of Math. **2** (2018), 1211–1234.

[19] J.-P. Serre, *Exemples de variétés projectives conjuguées non homéomorphes*, C. R. Acad. Sci. Paris, **258** (1964), 4194–4196.

[20] C. Skinner, *A note on the p-adic Galois representations attached to Hilbert modular forms*, Documenta Math. **14** (2009) 241–258.

[21] M. Stover, *Lattices in* $\mathrm{PU}(n,1)$ *that are not profinitely rigid*, arXiv:1808.07344.