A PRIORI ESTIMATES
FOR THE HOMOGENEOUS MONGE-AMPÈRE EQUATION
ON KÄHLER MANIFOLDS

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"More than the help of a friend, we need the assurance of it."

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ABSTRACT

Brubaker, Katherine Ph.D., Purdue University, December 2019. A Priori Estimates for the Homogeneous Monge-Ampère Equation on Kähler Manifolds. Major Professor: László Lempert.

In connection with the question of geodesics in the space of Kähler metrics on a compact Kähler manifold (M^n, ω), in [11] Donaldson studied smooth solutions $v \in C^\infty(\overline{U} \times M, \mathbb{R})$ for the following Dirichlet problem:

$$\begin{cases} (\widetilde{\omega} + i\partial\overline{\partial}v)^{n+1} = 0 & \\ \widetilde{\omega} + i\partial\overline{\partial}v > 0 & \text{on slices } \{z\} \times M \\ v|_{\partial U \times M} = F, & F \in C^\infty(\partial U \times M) \text{ given.} \end{cases} \tag{MA(F)}$$

Here $U \subset \mathbb{C}$ is the unit disc and $\widetilde{\omega}$ denotes the pullback of ω by the projection $U \times M \to M$. Donaldson showed that the space of boundary functions F for which MA(F) admits a smooth solution is open, but that there exist boundary functions with no smooth solution.

This thesis further investigates the existence of smooth solutions to MA(F), proving a priori estimates on the leaves of the foliation that corresponds to smooth solutions. We demonstrate that sequences of leaves of Monge-Ampère foliations converge to holomorphic disks.

1. INTRODUCTION

The complex Monge-Ampère equation has a rich history in several complex variables and complex geometry. Plurisubharmonic solutions to equations of the form $\det(u_{z_j\bar{z}_k}) = 0$ are the most natural analogue of harmonic functions in several complex variables. (See e.g. [4, 2, 18]). The complex Monge-Ampère equation appeared in complex geometry, first in the solution of the Calabi conjecture by Yau [25], then as the geodesic equation in the space of Kähler metrics. This thesis is motivated by the Monge-Ampère problem in Kähler geometry, approaching it from the perspective of similar questions several complex variables.

1.1 The Problem

In a groundbreaking 1987 paper, Mabuchi endowed the space of Kähler metrics on a compact Kähler manifold with a Riemannian structure [21]. More precisely, fixing a Kähler form ω on a compact Kähler manifold M, each form in the cohomology class of ω may be written as $\omega + i\partial\overline{\partial}v$ for some ω-plurisubharmonic function v. These functions v are called Kähler potentials. The space of Kähler potentials is an infinite dimensional Fréchet manifold with a Riemannian metric and connection. It then becomes natural to ask whether any two potentials in this space can be connected by a geodesic.

In [23] Semmes, in search of new insight into the homogeneous complex Monge-Ampère equation (HCMA) in C^n, interpreted it geometrically as the geodesic equation for an infinite dimensional manifold of functions. His construction was equivalent to Mabuchi's space of Kähler metrics. Thus the question of geodesics in the space of Kähler potentials becomes a Monge-Ampère type equation $(\omega + i\partial\overline{\partial}v)^{n+1} = 0$ on an $n+1$ complex dimensional space $R \times M$, for R a Riemann surface with boundary.

This equation has been the subject of intensive study in complex geometry in last 20 years. It had been hoped (see e.g. [10]) that the boundary value problem

$$\begin{cases} (\omega + i\partial\overline{\partial} v)^{n+1} = 0 & \text{on } R \times M \\ \omega + i\partial\overline{\partial} v > 0 & \text{on slices } \{z\} \times M \\ v|_{\partial R \times M} = F, & F \in C^\infty(\partial R \times M) \text{ given} \end{cases} \tag{MA}$$

would admit at least C^2 solutions v. (The second condition ensures that at every $z \in U$, the $(1,1)$-form $\omega + i\partial\overline{\partial} v$ is a Kähler form on M.) But in a series of papers since 2011, Lempert, Vivas, and Darvas exhibited boundary values for which (MA) does not admit even C^2 solutions [20, 8, 7]. Though a priori estimates on the solutions, X.X. Chen, J. Chu, Tosatti, and Weinkove were able to show that (MA) does always have (generalized) $C^{1,1}$ solutions [5, 6]. These "weak" solutions have provided the foundation for a substantial body of work on the space of Kähler metrics and generalizations.

Another avenue of approach to the HCMA equation in the Kähler setting would be to attempt to characterize the boundary values for which the system (MA) admits smooth solutions. This problem was proposed by Donaldson in [11]. He considered a version of the MA system on $\overline{U} \times M$, where $U \subset \mathbb{C}$ is the unit disk:

$$\begin{cases} (\omega + i\partial\overline{\partial} v)^{n+1} = 0 & \\ \omega + i\partial\overline{\partial} v > 0 & \text{on slices } \{z\} \times M \\ v|_{\partial U \times M} = F, & F \in C^\infty(\partial U \times M) \text{ given} \\ v \in C^\infty(\overline{U} \times M), & \end{cases} \tag{1.1}$$

in order to focus on the smoothness of solutions. We will refer to (1.1) as MA(F). Donaldson showed that the set of boundary data for which this system has a smooth solution is open. However, he also provided a example of a boundary function F for which no smooth solution is possible.

Significantly, Donaldson's proof of openness focused on a foliation of $U \times M$ by holomorphic disks, which is equivalent to the existence of a solution to MA(F). This foliation is obtained by integrating the vector field along which $\omega + i\partial\overline{\partial}v$ is degenerate. The leaves may be characterized as the graphs of holomorphic maps $U \to M$, which we call *leaf functions*. The existence of such a foliation (a "MA foliation") which extends smoothly to ∂U is sufficient to guarantee the existence of the solution.

For homogeneous Monge-Ampère problems in $\mathbb{C}^n$, there is a tradition of using an associated foliation to prove the existence of smooth solutions (begun by Bedford and Kalka in 1977 in [1]). The holomorophic nature of the leaves of the MA foliation is key. It can be employed to obtain higher order estimates from C^1 or even $C^{1/2}$ estimates on the leaves of the foliation. (See e.g. [18].)

1.2 Results

In this thesis, we address the question of a priori estimates on the leaves of the MA foliation. We prove a uniform 1/2-Hölder estimate on leaf functions corresponding to solutions of MA(F), for boundary functions in a Banach space neighborhood of a given F_0. From this initial estimate, we show that C^k estimates follow in the case when ω and F_0 are assumed to be analytic.

In particular, we prove the following two theorems for M a compact Kähler manifold with Kähler form ω such that $\{\omega\} \subset H^2_{dR}(M, \mathbb{R})$ is an integral class. We also assume that $T^{1,0}M$ admits a semi-negatively curved hermitian metric.

Theorem 1.2.1 (Theorem 3.0.1) *Given a C^4 boundary function $F_0 : \partial U \times M \to \mathbb{R}$ there exists a neighborhood $\mathcal{F}_a = \{F : \|F_0 - F\|_{C^4} < a\}$ and $C > 0$ such that if $f : U \to M$ is a leaf of a Monge-Ampère foliation corresponding to a solution of MA(F) for $F \in \mathcal{F}_a$, we have*

$$d(f(\zeta), f(z)) \;\leq\; C|\zeta - z|^{1/2}$$

for all $\zeta, z \in \overline{U}$, where d indicates the distance on M determined by ω.

The assumption of semi-negative curvature ensures that we have a maximum principle for holomorphic maps into the manifold.

In order to state higher order estimates on manifold valued maps, we choose a smooth embedding of M into some $\mathbb{R}^q$, which we call Θ. Define the k-size $||f||_k$ of $f \in C^k(U, M)$ as the norm of $\Theta \circ f$ in the Banach space $C^k(U, \mathbb{R}^q)$.

Theorem 1.2.2 *Suppose F is a real analytic boundary function, ω is analytic, and $k \in \mathbb{N}$. There is a positive number $C_k > 0$ such that if $f : \bar{U} \to M$ is a leaf function of the Monge-Ampère foliation corresponding to a solution of MA(tF) for any $t \in [0, 1]$, then $||f||_k < C_k$.*

These theorems open the way for a larger project to analyze the convergence of sequences of MA foliations. The aim is to characterize the boundary values for which sequences of foliations converge to a genuine MA foliation, ensuring the existence of a limiting solution. We hope to achieve such a characterization through a geometric interpretation of the boundary functions F. In our argument, they appear as boundaries of domains in a complex manifold.

1.3 Overview

The heart of the argument is the following uniform estimate on the differential of leaf functions in the MA foliation, for a neighborhood of a fixed boundary function.

Theorem 1.3.1 (Theorem 3.3.1) *Under the same conditions as Theorem 3.0.1, there is a neighborhood $\mathcal{F}_a = \{F : \|F_0 - F\|_{C^4} < a\}$ and $C > 0$ such that if $f : U \to M$ is a leaf of a Monge-Ampère foliation corresponding to a solution of MA(F) for $F \in \mathcal{F}_a$, we have*

$$\left| f_*(z) \frac{\partial}{\partial z} \right| \leq C(1 - |z|)^{-1/2} \text{ for any } z \in U$$

where on the left side, the length $|\cdot|$ is measured in the metric determined by ω.

Theorem 3.0.1 will follow from this theorem via a generalization of a result due to Hardy and Littlewood, of which we give a proof in the next chapter (Theorem 2.3.1).

The proof of Theorem 3.3.1 (Chapter 3) brings together three main ideas. The first is to map the leaves of the foliation into something similar to a convex domain in $\mathbb{C}^n$. This is achieved by constructing a negatively curved line bundle over $U \times M$, with metrics depending on the boundary values F. The maps f may be lifted to maps into a strongly pseudoconvex unit ball bundle. This step depends on $\{\omega\} \subset H^2_{dR}(M, \mathbb{R})$ being an integral class.

The second main idea is to estimate the derivative of these maps using the Kobayashi metric. In a 1973 paper, Ian Graham estimated the Kobayashi metric near the boundary of a strongly pseudoconvex domain in $\mathbb{C}^n$ in terms of the distance to the boundary and the complex Hessian of the defining function [14]. In Chapter 4, we prove a similar estimate that applies uniformly across a family of domains in a complex manifold. The family is obtained from a Banach space neighborhood of a fixed defining function.

Theorem 1.3.2 (Theorem 4.0.2) *Let X be a complex manifold and ρ_0 a smooth exhaustion function on X which is strongly plurisubharmonic outside of $\{\rho_0 < -2\}$. There exist a, δ, and $C > 0$ such that for any $\|\rho - \rho_0\|_{C^3} < a$ and $\xi \in T^{1,0}_w X$ with $-\delta < \rho(w) < 0$, the Kobayashi metric on $\{\rho < 0\}$ satisfies*

$$F_{\{\rho<0\}}(\xi)^2 \, |\rho(w)| \;\geq\; C|\xi|^2.$$

The proof of Theorem 4.0.2 is involves approximating the boundary of each domain locally with ellipsoids, and comparing the Kobayashi metric of $\{\rho < 0\}$ with that of the approximating ellipsoid. The challenge is to preserve uniformity across the infinite dimensional family of defining functions.

After composing with suitable automorphisms of the unit disk, Theorem 4.0.2 provides an estimate on the derivatives of holomorphic maps into the family of domains $\{\rho < 0\} \subset X$:

$$\left| g_*(z) \frac{\partial}{\partial z} \right| \;\leq\; C \, |\rho(g(z))|^{1/2} (1 - |z|)^{-1}.$$

Finally, the maps g which correspond to leaves of the MA foliation can be understood as extremal maps with respect to the line bundle metric determined by solutions to MA(F). This is employed to show that $\rho(g(z))$ is controlled by $1-|z|$ near ∂U, yielding the estimate of Theorem 3.3.1.

In Chapter 5, we use the 1/2-Hölder estimate from Theorem 3.0.1, together with the establised $C^{1,\alpha}$ estimates on solutions of (MA), to obtain higher order estimates (Theorem 3.0.1). This is achieved through a variation on the reflection principle, which holomorphically continues the leaf functions across the boundary of the unit disk. The lower order estimates are used to uniformly control the size of neighborhoods on which the reflection is defined.

2. BACKGROUND

2.1 Kähler geometry background

This section provides a basic introduction to Kähler geometry and the space of Kähler metrics.

Let X be a complex manifold with $\dim_{\mathbb{C}} M = n$. Then X carries a Reimannian metric g and a smooth complex structure $J : TX \to TX$ with $J^2 = -Id$, such that $g(J\zeta, J\zeta) = g(\zeta, \zeta)$. These two elements combine in a natural hermitian metric h, given by $h(\zeta, \eta) = g(\zeta, \eta) + ig(\zeta, J\eta)$. In local coordinates, h looks like

$$h = \sum_{j,k=1}^{n} g_{j,k} dz_j \otimes d\bar{z}_k.$$

Kähler geometry is mostly concerned with an associated real, positive $(1,1)$-form ω, given by $\omega(\zeta, \eta) = \operatorname{Im} h(\zeta, \eta)$. In local coordinates, we have

$$\omega = i \sum_{j,k=1}^{n} g_{j,k} dz_j \wedge d\bar{z}_k.$$

It is easy to compute that ω will be real valued and positive.

When $d\omega = 0$, the manifold X is Kähler and ω is called a Kähler form. An equivalent and slightly more meaningful characterization is that X is Kähler when the connections corresponding to the complex structure (Chern) and the Riemannian structure (Levi-Civita) agree. Stated another way, the complex structure J is parallel with respect to the Levi-Civita connection.

There are two equivalent ways of talking about the positivity of ω. First, "ω is positive" means that for all nonzero tangent vectors $\zeta \in TX$ we have $\omega(\zeta, J\zeta) > 0$. Second, as a $(1,1)$-form we may consider ω as acting on the complexified tangent bundle of X. The $2n-$complex dimensional bundle $T_{\mathbb{C}}X := TX \otimes \mathbb{C}$ splits into two vector bundles, notated as $T^{1,0}$ and $T^{0,1}$. The first bundle is commonly called the "holomorphic" tangent bundle. Its local frames are given by elements of the form $\partial_j - iJ\partial_j$. Thus for $\xi = \zeta - iJ\zeta \in T^{1,0}X$, we have

$$\omega(\xi,\bar{\xi}) \;=\; \omega(\zeta,\zeta) + \omega(-iJ\zeta,\zeta) + \alpha(\zeta,iJ\zeta) + \omega(J\zeta,J\zeta) \;=\; 2i\omega(\zeta,J\zeta). \tag{2.1}$$

Thus the positivity of ω is equivalent to $-i\omega(\xi,\bar{\xi}) > 0$.

As a closed $(1,1)-$form, ω may be written locally as $i\partial\overline{\partial}u$ for some smooth, real function u called a "potential". Similarly, for any strictly plurisubharmonic function $\rho : X \to \mathbb{R}$, its Hessian matrix in local coordinates is positive definite. This corresponds to $i\partial\overline{\partial}\rho$ being a positive $(1,1)$-form.

The Kähler forms within a fixed cohomology class are characterized by potential functions in the following fundamental result (see e.g. Lemma 1.14 in [24]).

Lemma 2.1.1 ($\partial\overline{\partial}$-Lemma) *Let (M,ω_0) be a compact Kähler manifold. Every Kähler form $\omega \in \{\omega_0\} \in H^2_{dR}(M,\mathbb{R})$ is given by $\omega_0 + i\partial\overline{\partial}v$ for some $v \in C^\infty(M,\mathbb{R})$.*

Such functions $v \in C^\infty(X,\mathbb{R})$ are called *Kähler potentials.* On the other hand, a smooth function $v : X \to \mathbb{R}$ satisfying $\omega + i\partial\overline{\partial}v > 0$ is called "strongly ω-plurisubharmonic" (or "strictly" ω-psh). For such a v, it follows that $\omega + i\partial\overline{\partial}v$ is a Kähler form in the same De Rham class as ω. (We refer to the weaker notion $\omega + i\partial\overline{\partial}v \geq 0$ as "ω-psh".)

2.1.1 The Space of Kähler Metrics

Let (M,w) be a compact Kähler manifold and define $\mathcal{H} = \{v : \omega + i\partial\overline{\partial}v > 0\} \subset C^\infty(M,\mathbb{R})$. For convenience, $\omega + i\partial\overline{\partial}v$ is often written ω_v. The set $\mathcal{H}$ is called the space of Kähler potentials.

In [21], Mabuchi introduced a Riemannian metric and connection on the infinite dimensional Fréchet space $\mathcal{H}$. The first item to note is that the tangent space to $\mathcal{H}$ at any fixed potential $v \in \mathcal{H}$ can be canonically identified with the space $C^\infty(M,\mathbb{R})$. Thus the metric on $T\mathcal{H}$ can be defined as

$$\langle\, \varphi,\psi \,\rangle_v \;=\; \int_M \varphi\psi\,(\omega_v)^n\,, \qquad \text{for } \varphi,\psi \in C^\infty(M,\mathbb{R}) \approx T_v\mathcal{H} \tag{2.2}$$

for every $v \in \mathcal{H}$.

Mabuchi's "Riemannian connection" is defined on paths in $\mathcal{H}$. A path is a function $\Phi : [a,b] \to \mathcal{H}$, which is to say, a curve with $\Phi(s) \in \mathcal{H} \subset C^\infty(M, \mathbb{R})$ for every $s \in [a,b]$. At each "point" $\Phi(s)$ in $\mathcal{H}$, we may consider the tangent vector $\Phi' := \frac{\partial \Phi}{\partial s} \in T_{\Phi(s)}\mathcal{H}$.

Let $\eta : [a,b] \to T\mathcal{H}$ be a smooth vector field along Φ. This means that $\eta(s) \in T_{\Phi(s)}\mathcal{H} = C^\infty(M, \mathbb{R})$, and Mabuchi defines an "s" derivative of the vector field η. The covariant derivative of η at $\Phi(s)$ in the direction of Φ' as

$$\nabla_{\Phi'}\eta \; := \; \frac{\partial \eta}{\partial s} \; - \; \frac{1}{2}\, g_\Phi\big(\operatorname{grad}\eta(s), \operatorname{grad}\Phi'\big)$$

where g_Φ is the Riemannian metric on M corresponding to ω_Φ and grad is the gradient according to g_Φ. One can check that this definition yields a connection which is the Levi-Civita connection of the metric on $\mathcal{H}$ defined in (2.2).

Finally, according to this covariant derivative, a path Φ in $\mathcal{H}$ satisfies the geodesic equation if

$$\nabla_{\Phi'}\Phi' \; = \; \Phi'' - \frac{1}{2}\big|\partial \Phi'\big|^2_{\omega_\Phi} \; = \; 0. \tag{2.3}$$

via the isomorphism of TM and $T^{1,0}$.

2.1.2 The Monge-Ampère Problem in the Kähler setting

The relationship between the geodesic equation in the space of Kähler metrics and the Monge-Ampère equation was first established by Semmes in [23], though he was working in the opposite direction. We sketch the relation between equation (2.3) and a PDE of Monge-Ampère type.

In equation (2.3), there are derivatives with respect to two different domains: $s \in [0,1]$ and across M. (Replace $[a,b]$ with $[0,1]$ for simplicity.) Thus it's helpful to think of a path in $\mathcal{H}$ as a smooth function of two variables: $\Phi : [0,1] \times M \to \mathbb{R}$, where $\Phi(s, \cdot) = \Phi_s \in C^\infty(M, \mathbb{R})$.

Further, we can re-write (2.3) in terms of entirely complex derivatives by replacing $s \in [0,1]$ with $z \in S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ and simply ignoring the imaginary direction. Let $v : S \times M \to \mathbb{R}$ be $v(z,x) = \Phi(\mathrm{Re}\, z, x)$ and $\tilde{u}$ be a local potential for $\widetilde{\omega}$, the pullback of ω by the projection map $\pi_2 : S \times M \to M$. Then a little linear algebra shows that for $\rho = \widetilde{u} + v \in C^\infty(S \times M)$, the geodesic equation (2.3) is equivalent to

$$\left(i\partial\overline{\partial}u\right)^{n+1} \;=\; \left(\widetilde{\omega} \,+\, i\partial\overline{\partial}v\right)^{n+1} \;=\; 0\,. \tag{2.4}$$

In [10], Donaldson formulated the following MA boundary problem on $S \times M$, whose smooth solutions are equivalent to the existence of a geodesic between two potentials in $\mathcal{H}$.

$$\begin{cases} (\widetilde{\omega} + i\partial\overline{\partial}v)^{n+1} = 0 & \\ \omega + i\partial_M\overline{\partial}_M v > 0 & \text{on slices} \\ v|_{\mathrm{Re}z=0} = \varphi_0,\ v|_{\mathrm{Re}z=1} = \varphi_1 & \varphi_0, \varphi_1 \in \mathcal{H} \text{ given} \\ v \in C^2(S \times M) & \end{cases} \tag{2.5}$$

The second condition ensures that at each $z \in S$, $v(z,\cdot)$ is in fact in $\mathcal{H}$. It also ensures that the solution v is ω-plurisubharmonic, which agrees with the statement of the homogeneous Monge-Ampère in several complex variables.

In particular, as in C^n, the condition $\omega + i\partial_M\overline{\partial}_M v \geq 0$ yields a variant on a Maximum Principle, which ensures that solutions of (2.5) are unique. In $\mathbb{C}^n$ this is due to Bedford and Taylor [2]; Donaldson was the first to state it in the Kähler setting, in [10]. It has become standardized in the literature as the "Comparison Principle". (See e.g. Theorem 21 in [3].)

Proposition 2.1.2 (Comparison Principle) *For X^n a compact Kähler manifold with boundary, consider $v, \widetilde{v} \in C^2(X, \mathbb{R})$ with $\widetilde{v} \leq v$ on ∂X. If both $\omega + i\partial\overline{\partial}v$ and $\omega + i\partial\overline{\partial}\widetilde{v}$ are semi-positive and $\left(\omega + i\partial\overline{\partial}v\right)^n \geq \left(\omega + i\partial\overline{\partial}\widetilde{v}\right)^n$, then $\widetilde{v} \leq v$ on X.*

Thus if v_1 and v_2 are two solutions to (2.5), applying Proposition (2.1.2) twice, exchanging the roles of v_1 and v_2, yields $v_1 = v_2$ on $S \times M$.

Theorem 2.1.3 ([10] Cor. 7) *If a solution to the system (2.5) exists, then it is unique.*

Note that assuming $v \in C^2$ gives a clear meaning to $i\partial\overline{\partial}v$. But even for $v \in C(X)$, if v is ω-psh the Comparison Principle still holds, using an interpretation of $(i\partial\overline{\partial}v)^n$ discussed in a later section.

2.2 Continuity Method and Results

The system (2.5) is not the first appearance of a Monge-Ampère equation in complex geometry. The Calabi conjecture - for a compact Kähler manifold (M, ω) there exists another Kähler form $\widetilde{\omega}$ cohomologous to ω with prescribed Ricci curvature - is equivalent to the non-homogeneous Monge-Ampère system

$$\begin{cases} (\widetilde{\omega} + i\partial\overline{\partial}v)^n = f\omega^n, \quad f \in C^\infty(M, \mathbb{R}) \\ \omega + i\partial_M\overline{\partial}_M v > 0 \\ v \in C^\infty(M, \mathbb{R}) \end{cases} \tag{MA(f)}$$

for a given $f \in C^\infty(X, \mathbb{R})$ with $f > 0$ and $\int_M f\omega^n = \int_M \omega^n$. The system (MA(f)) was famously solved by Yau in [25].

For a non-linear system of equations such as (2.5), a common method of proof is by the "continuity method": connect the desired f to an f_0 for which the problem is trivial, via a one parameter family $\{f_t = (1-t)f_0 + tf : t \in [0, 1]\}$. Then it suffices to show that the set $T \subset [0, 1]$ for which MA(f_t) has a solution is both open and closed. For the "closedness" portion, it is necessary to have uniform "a priori" estimates on solutions v_t of MA(f_t). Then the Arzelà-Ascoli Theorem ensures that if $t_j \in T$ and $t_j \to t$, some subsequence of v_t converges to a limiting solution v for MA(f).

In [5], X.X. Chen formulated a continuity method approach for the geodesic problem (2.5). His paper established a priori estimates up to order $1+\alpha$, that is, up to an uniform estimate on $|\Delta v_t|$. Later, this result was improved to full $C^{1,1}$ estimates by Chu, Tosatti, and Weinkove [6]. These guarantee the existence of $C^{1,1}$ "weak solutions" for the system (2.5), for any boundary data $\varphi_0, \varphi_1 \in \mathcal{H}$.

For $v \in C(R \times M)$ with $\widetilde{\omega} + i\partial\overline{\partial}v \geq 0$, the quantity $\widetilde{\omega} + i\partial\overline{\partial}v$ makes sense as positive current. (Though for fixed $s \in S$, $v(s, \cdot)$ can no longer be thought of as a potential in $\mathcal{H}$.) Further, in [2], Bedford & Taylor established that $\left(\widetilde{\omega}+i\partial\overline{\partial}v\right)^{n+1}$ is also a well defined current. Thus a $v \in C^{1,1}$ satisfying (2.5) in the sense of currents is referred to as a "weak solution". The Comparison Principle (Proposition 2.1.2) holds for $v, \widetilde{v}$ merely continuous (see e.g. Theorem 21 in [3]), and thus uniqueness follows as well.

It had been hoped that (2.5) would admit at least C^2 (genuine) solutions, for any boundary potentials φ_0 and φ_1. However, in a series of papers, Darvas, Lempert, and Vivas showed that $C^{1,1}$ is the optimal regularity for the general problem (2.5).

2.2.1 Donaldson's formulation

There is a second avenue of approach to the homogeneous Monge-Ampère equation in the Kähler setting. In his second paper on the subject, Donaldson formulated a version of (2.5) aimed at understanding smooth solutions. He replaced the strip S with the unit disk $U \subset \mathbb{C}$ and considered

$$\begin{cases} (\widetilde{\omega}+i\partial\overline{\partial}v)^{n+1}=0 & \\ \omega+i\partial_M\overline{\partial}_M v>0 & \text{on slices } \{z\}\times M \\ v|_{\partial U}=F & F\in C^\infty(\partial U\times M,\mathbb{R}) \text{ given} \\ v\in C^\infty(\overline{U}\times M) & \end{cases} \tag{MA(F)}$$

We are assuming that F is such that $\omega + i\partial_M\overline{\partial}_M F(z,\cdot) > 0$ for all $z \in \partial U$. We will refer to this property by saying that "F is a boundary function".

In [11], Donaldson proved two theorems. First, that the set of boundary functions F for which $\mathrm{MA}(F)$ admits a smooth solution is open in $C^\infty(\partial U \times M)$, and second, that there exist boundary functions F for which no smooth solution is possible. He concluded that it would be interesting to investigate the circumstances under which smooth solutions to $\mathrm{MA}(F)$ exist.

An interesting feature of Donaldson's paper is his proof of his first theorem. He translated solutions of (MA) into families of holomorphic disks, proving that such families are not destroyed by small perturbations of the boundary data. This approach to solving Monge-Ampère type boundary value problems - via an associated foliation by holomorphic maps - is a feature in the history of Monge-Ampère equations in several complex variables. The foliation will be defined in the next section.

2.3 Background on methods

2.3.1 The Monge-Ampère foliation

The distinctive feature of a C^3 solution of the system $\mathrm{MA}(F)$ is that at every point in $\overline{U} \times M$, the $(1,1)$-form $\widetilde{\omega} + i\partial\overline{\partial} v$ must have rank n on an $n+1$ dimensional space. That is, taking $\tilde{u}$ as a local potential for $\widetilde{\omega}$ and local holomorphic coordinates $z_0, \ldots, z_n$ on $\overline{U} \times M$, the matrix $\left[(\tilde{u}+v)_{z_j,\bar{z}_k}\right]_{0 \le j,k \le n}$ has a one dimensional kernel.

This implies a foliation of $\overline{U} \times M$ by graphs of holomorphic functions $U \to M$. Concretely, for each $x \in M$ there is a function $f_x : \overline{U} \to M$, holomorphic on U, with the following properties.

1. The map $\overline{U} \times M \to M$ given by $(z,x) \mapsto f_x(z)$ is C^1.

2. Using local potentials $\tilde{u}$ as above, the map $z \mapsto \big(\partial(\tilde{u}+v)(z, f_x(z)\big) \in T^{1,0}(U \times M)$ is holomorphic on the portion of U where it is defined.

A local version of this is due to Bedford and Kalka in [1].

We refer to this foliation by holomorphic disks as the "Monge-Ampère foliation". In later chapters, we will rely heavily on the fact that when $v \in C^3$ is a solution of $\mathrm{MA}(F)$, the $(1,1)$-form $\widetilde{\omega} + i\partial\overline{\partial} v$ vanishes along the leaves of the corresponding MA foliation. That is, whenever $f_x : \overline{U} \to M$ describes a leaf of the foliation, the pullback $(\mathrm{id} \times f_x)^*\big(\widetilde{\omega} + i\partial\overline{\partial} v\big) = 0$

2.3.2 A Key Theorem

In order to state a 1/2-Hölder estimate for functions valued in a manifold, we employ a generalization of a theorem of Hardy and Littlewood about complex functions, which can be found in [13]. The following generalization has essentially the same proof, but using the distance determined by a metric ω on a Kähler manifold. Denote that distance by d.

Theorem 2.3.1 *Let M be a Kähler manifold with metric ω and distance d. There is a $k \in \mathbb{Z}^+$ such that if $f : U \to M$ is holomorphic with*

$$\left| f_*(z) \frac{\partial}{\partial z} \right| \;\leq\; C(1 - |z|)^{-1/2} \tag{2.6}$$

for any $z \in U$, then for $z, \zeta \in U$,

$$d(f(\zeta), f(z)) \;\leq\; kC|\zeta - z|^{1/2}. \tag{2.7}$$

Proof For $\zeta, z \in \bar{U}$ let $p = |\zeta - z|$. The proof breaks into two parts. First, we consider the case when $1 - |\zeta|, 1 - |z| \;\leq\; 2p$.

Let ζ_1 be the point on the radial line through ζ at distance $2p$ from the boundary. (For now, assume $\zeta \neq 0$. Once the estimate is proved when $\zeta \neq 0$, continuity will take care of $\zeta = 0$.) Define z_1 similarly. Let L_z be the line segment connecting z_1 to z. Since the distance between $f(z_1)$ and $f(z)$ is no larger than the length of the curve $y \in L_z \mapsto f(y)$ in M, we can estimate

$$\mathrm{d}(f(z_1), f(z)) \;\leq\; \int_{z_1}^{z} C(1-|y|)^{-1/2}|dy| \;=\; 2C(1-s)^{1/2}\Big|_{|z|}^{|z_1|} \;\leq\; 2C(1-|z_1|)^{1/2} \;\leq\; 4C\sqrt{p},$$

and similarly for $\mathrm{d}(f(\zeta_1), f(\zeta))$.

Note that $|\zeta_1 - z_1| \leq |\zeta - z| = p$. Also, since discs are convex, the whole line segment between ζ_1 and z_1 lies in $\{1 - |y| \geq p\}$ (because the endpoints do). Thus we can bluntly estimate

$$\mathrm{d}(f(\zeta_1), f(z_1)) \;\leq\; \int_{z_1}^{\zeta_1} C(1 - |y|)^{-1/2}|dy| \;\leq\; Cp^{-1/2}p \;=\; C\sqrt{p}.$$

Therefore,

$$\mathrm{d}(f(\zeta), f(z)) \;\le\; \mathrm{d}(f(\zeta, \zeta_1) + \mathrm{d}(f(\zeta_1), f(z_1)) + \mathrm{d}(f(z_1), f(z)) \;\le\; 9C\sqrt{p}.$$

For the second case, suppose that, say, $1 - |z| > 2p$. By the triangle inequality, it follows that $1-|\zeta| > p$. Thus on the line segment between ζ and z we have $1-|y| > p$, and we estimate as above

$$\mathrm{d}(f(\zeta), f(z)) \;\le\; \int_\zeta^z C(1 - |y|)^{-1/2}|dy| \;\le\; Cp^{-1/2}p \;=\; C\sqrt{p}.$$

Therefore $k = 9$ will do. ■

3. THE 1/2-HOLDER ESTIMATE

Recall our fomulation of the Monge-Ampère problem on the unit disk U, with boundary data F:

$$\begin{cases} (\widetilde{\omega} + i\partial\overline{\partial} v)^{n+1} = 0 & \\ \widetilde{\omega} + i\partial\overline{\partial} v > 0 & \text{on slices } \{z\} \times M \\ v|_{\partial U \times M} = F, & F \in C^\infty(\partial U \times M) \text{ given} \\ v \in C^3(\overline{U} \times M) & \end{cases} \tag{MA(F)}$$

When this system admits a solution which is at least C^3, there is a corresponding Monge-Ampère foliation of $\overline{U} \times M$ given by the graphs of holomorphic maps $\{f_x : \overline{U} \to M\}_{x \in M}$ (see Section 2.3.1). We call the maps f_x "leaf functions" and write $\mathcal{L}(v)$ for the set of leaf functions corresponding to a solution v.

We prove a uniform 1/2-Hölder estimate on leaf functions corresponding to solutions of MA(F) for boundary functions near a fixed F_0 in a C^4 norm. The Banach space neighborhoods are defined as follows.

For X a compact differential manifold, we take a finite cover of X by compact sets inside coordinate charts. Let $\{W_j\}$ be such a cover. For any $k \in \mathbb{N}$, given a C^k function $p : X \to \mathbb{R}$, we let $\|p\|_{C^k(W_j)} := \sup_{W_j} \max_{|\beta| \le k} \left|\partial^\beta p\right|$ (where $\partial^\beta p$ indicates partial derivatives computed in local coordinates). Define a norm

$$\|p\|_{C^k(X)} := \max_{W_j} \|p\|_{C^k(W_j)} .$$

Different choices of covers and coordinates will yield equivalent norms. It is easy to check that under this norm, $C^k(X)$ is a Banach space.

In this manner, we define the Banach space $C^4(\partial U \times M)$ and consider neighborhoods of the form

$$\mathcal{F}_a \;=\; \mathcal{F}_a(F_0) \;=\; \{F \in C^4(\partial U \times M) : \|F - F_0\|_{C^4(\partial U \times M)} < a\} \tag{3.1}$$

for small $a > 0$. We write $\mathcal{L}_a$ as the collection of all leaf functions generated by the family $\mathcal{F}_a$. That is, $\mathcal{L}_a := \{f \in \mathcal{L}(v) : v \text{ a } C^3 \text{ solution of } \mathrm{MA}(F) \text{ for } F \in \mathcal{F}_a\}$.

In order to have a maximum principle for maps into the manifold M, we assume that M is semi-negatively curved. That is, that $T^{1,0}M$ admits a C^2 hermitian metric h_M whose Griffiths curvature is semi-negative (see e.g. Definition VII.6.4 in [9]). We expect that it is possible to weaken or remove this assumption.

Theorem 3.0.1 *Let M be a semi-negatively curved, compact Kähler manifold with ω a Kähler form such that $\{\omega\} \subset H^2_{dR}(M, \mathbb{R})$ is an integral class. Given a C^4 boundary function $F_0 : \partial U \times M \to \mathbb{R}$ there exists a neighborhood $\mathcal{F}_a = \{F : \|F_0 - F\|_{C^4} < a\}$ and $C > 0$ such that if $f : U \to M$ is in $\mathcal{L}_a$, we have*

$$d(f(\zeta), f(z)) \ \leq \ C|\zeta - z|^{1/2}$$

for all $\zeta, z \in \overline{U}$, where d indicates the distance on M determined by ω.

The main result of this chapter is Theorem 3.3.1, which gives a uniform estimate of the form

$$\left| f_*(z) \frac{\partial}{\partial z} \right| \ \leq \ C(1 - |z|)^{-1/2} \text{ for any } z \in U \tag{3.2}$$

for the family $\mathcal{F}_a$. Theorem 3.0.1 will follow from this estimate via the following a generalization of a theorem of Hardy and Littlewood (in [13]). A proof is given in background Section 2.3.2.

Theorem 3.0.2 (Theorem 2.3.1) *Let M be a Kähler manifold with metric ω and distance d. There is a $k \in \mathbb{Z}^+$ such that if $f : U \to M$ is holomorphic with*

$$\left| f_*(z) \frac{\partial}{\partial z} \right| \ \leq \ C(1 - |z|)^{-1/2} \tag{3.3}$$

for any $z \in U$, then for $z, \zeta \in U$,

$$d(f(\zeta), f(z)) \ \leq \ kC|\zeta - z|^{1/2}. \tag{3.4}$$

The proof of Theorem 3.3.1 is built from two component estimates. The first is a uniform estimate on the differential of any holomorphic map of the unit disk into a strongly pseudoconvex domain in a complex manifold (Theorem 3.3.3). This estimate depends on a defining function for the domain, as a way of measuring distance to the boundary. Chapter 4 is devoted to the proof of Theorem 3.3.3, which uses methods from several complex variables.

The second estimate applies only to holomorphic maps with a certain "extremal" property relative to the domain. For such maps, the defining function of the domain can be related directly to distance from the boundary in the unit disk. We will construct strongly pseudoconvex domains depending on the solutions of (MA), in which the leaf functions have this extremal property. Combining this estimate with the first yields Theorem 3.3.1.

The chapter is organized into three sections. The first constructs our setting: extremal maps into strongly pseudoconvex complex manifolds whose boundaries depend on solutions of MA. In the second section we establish two underlying estimates needed for the proof of Theorem 3.3.1. The last section contains the estimate on extremal functions and shows how Theorem 3.3.1 follows, assuming the general estimate of Chapter 4.

3.1 The Line Bundle Construction

We begin by constructing a line bundle over $U \times M$ with a strongly pseudoconvex unit disk bundle. The integrality of the class $\{\omega\} \subset H^2_{dR}(M, \mathbb{R})$ is equivalent to the existence of a holomorphic line bundle $L_M \to M$ with hermitian metric h_M and curvature form $i\Theta(L_M, h_M) = -\omega$ (Theorem 13.9 in [9]). We may pull back this bundle to a line bundle (L, h) over $\overline{U} \times M$ by the projection map $\overline{U} \times M \to M$. The curvature form $\Theta(L, h)$ will be $-\widetilde{\omega}$, for $\widetilde{\omega}$ the pullback of ω to $\overline{U} \times M$.

We are interested in hermitian metrics on L that are determined by solutions to MA(F). Specifically, fixing a boundary function $F : \partial U \times M \to \mathbb{R}$, let $v : \overline{U} \times M \to \mathbb{R}$

be the solution to $\mathrm{MA}(F)$. Define the hermitian metric $h_v := e^v h$ on L. The curvature form corresponding to (L, h_v) is $-\widetilde{\omega} - i\partial\overline{\partial}v$. Thus the vanishing of $\widetilde{\omega} + i\partial\overline{\partial}v$ along the leaves of the MA foliation means that (L, h_v) is flat over each leaf. This flatness is the linchpin of the construction.

For each leaf function f of the MA foliation corresponding to v, we construct a map $\sigma_f : \overline{U} \to L$ such that $\sigma_f(z) \in L_{(z,f(z))}$ for all $z \in \overline{U}$. To do this, consider the pullback bundle $(L_f, h_f) = (\mathrm{id} \times f)^*(L, h_v)$ over $\overline{U}$. This bundle has curvature $(\mathrm{id} \times f)^*(-\widetilde{\omega} - i\partial\overline{\partial}v) = 0$. As a flat line bundle over a simply connected base, L_f has a global holomorphic trivialization which also trivializes the metric (cf. Propositions V.6.7 and V.6.10 in [9]). The trivialization will be continuous up to ∂U since h_v is C^1. Thus there exists a non-vanishing holomorphic section

$$\sigma_f : \overline{U} \to L_f \quad \text{such that} \quad h_f(\sigma_f(z)) = 1. \tag{3.5}$$

The map σ_f defines a section of L over $\mathrm{gr}(f) \subset \overline{U} \times M$, which is holomorphic over $\mathrm{gr}(f\big|_U)$.

Thus for $v \in C^3$ a solution of $\mathrm{MA}(F)$ corresponding to a given boundary function F, we have a family of holomorphic maps $\mathcal{S}(v) := \{\sigma_f : \overline{U} \to L_f \subset L\}$, corresponding to the leaves of the MA foliation. We claim these σ_f's map into a certain strongly pseudoconvex disk bundle of L. The disk bundle will be constructed using another hermitian metric on L depending on F. But this time, we use a strongly $\widetilde{\omega}$-plurisubharmonic extension of F across $\overline{U} \times M$.

Lemma 3.1.1 *Given a C^4 function $F_0 : \partial U \times M \to \mathbb{R}$ with $\omega + i\partial_M\overline{\partial}_M F_0(z, \cdot) > 0$ for all $z \in \partial U$, there exist $a, C_1 > 0$ such that every $F \in \mathcal{F}_a$ has the following C^3 extensions:*

1. *$\psi_F : \overline{U} \times M \to \mathbb{R}$ ω-plurisubharmonic on slices $\{z\} \times M$ for $z \in \overline{U}$ and harmonic along disks $U \times \{x\}$ for $x \in M$*

2. *$\widetilde{F} : \overline{U} \times M \to \mathbb{R}$ strongly $\widetilde{\omega}$-plurisubharmonic, i.e. $\widetilde{\omega} + i\partial\overline{\partial}\widetilde{F} > 0$.*

Further, we have $\|\psi_F\|_{C^3}, \|\widetilde{F}\|_{C^3} \le C_1 \|F\|^2_{C^4}$ for all $F \in \mathcal{F}_a$.

Proof Fixing $F_0 \in C^4(\partial U \times M)$, we initially choose $a_0 > 0$ such that $\omega + i\partial_M \overline{\partial}_M F_0 > 0$ for all $F \in \mathcal{F}_{a_0}$. This choice of family is sufficient to establish (1). Any $F : \partial U \times M \to \mathbb{R}$ has an extension to $\overline{U} \times M$ via Poisson integrals. Define $\psi_F : \overline{U} \times M \to \mathbb{R}$ by

$$\psi_F(z,x) \;=\; \mathbf{P}[F(\cdot,x)](z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} P(z,t)F_0(e^{it},x)\,dt$$

where $P(z,t)$ is the Poisson kernel at $z \in \overline{U}$ and $t \in \mathbb{R}$.

For each $x \in M$, ψ_F is harmonic (and therefore smooth) along disks $U \times \{x\}$ and continuous on $\overline{U} \times \{x\}$. But further, assuming $F \in C^4(\partial U \times M)$ implies that $\psi_F \in C^3(\overline{U} \times M)$ and its derivatives are controlled by those of F. We show this first for $\psi_F(\cdot,x)$ on $\overline{U} \times \{x\}$ for any fixed $x \in M$.

A special case of e.g. Theorem 6.19 in [12] implies that $\mathbf{P}\left[F(\cdot,x)\right] \in C^{k-1}$ whenever $F \in C^k$ for $k \in \mathbb{N}$. Further, as a linear map $C^4(\partial U) \to C^3(\overline{U})$, the Poisson operator $\mathbf{P}$ is closed. Thus the Closed Graph Theorem yields $K > 0$ so that for any $G \in C^4(\partial U, \mathbb{R})$,

$$\big\|\,\mathbf{P}\left[G\right]\big\|_{C^3(\overline{U})} \;\le\; K\,\big\|G\big\|_{C^4(\partial U)}. \tag{3.6}$$

In particular, for any $x \in M$ we have $\big\|\psi_F(\cdot,x)\big\|_{C^3(\overline{U})} \le K\big\|F(\cdot,x)\big\|_{C^4(\partial U)}$.

We can now show that ψ_F is C^3 across $\overline{U} \times M$, using local coordinates on M. Take a multi-index $\beta \in \mathbb{N}^{n+1}$ with $|\beta| \le 3$, and let D^{β_z} and D^{β_x} be the $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial x}$ components of D^β, respectively. Then using (3.6), we have

$$\big|D^\beta \psi_F(z,x)\big| = \big|D^{\beta_z}\mathbf{P}\big[D^{\beta_x}F(\cdot,x)\big](z)\big| \le K\big\|D^{\beta_x}F(\cdot,x)\big\|_{C^{|\beta_z|+1}(\partial U)} \le K\big\|F\big\|_{C^4(\partial U \times M)} \tag{3.7}$$

since $|\beta_z| + |\beta_x| \le 3$.

Thus for any $(z_0,x_0) \in \overline{U} \times M$, we have

$$\begin{aligned}\big|D^\beta\psi_F(z,x) - D^\beta\psi_F(z_0,x_0)\big| \;\le\;& \big|\,D^\beta\mathbf{P}\big[F(\cdot,x) - F(\cdot,x_0)\big](z)\big| \\ &+\; \big|D^\beta\mathbf{P}\big[F(\cdot,x_0)\big](z) - D^\beta\mathbf{P}\big[F(\cdot,x_0)\big](z_0)\big|.\end{aligned}$$

Within a coordinate neighborhood on M, the first term on the right is controlled by $K\left\|D^{\beta_x}\left(F(\cdot,x)-F(\cdot,x_0)\right)\right\|_{C^{\beta_z+1}(\partial U)}$. This goes to 0 as $(z,x)\to(z_0,x_0)$ since $F\in C^4(\partial U\times M)$. For the second term, note that

$$D^{\beta_x}F(\cdot,x_0)\;\in\;C^{4-\beta_x}(\partial U)\;=\;C^{\beta_z+1}(\partial U).$$

It follows that $D^{\beta_z}\mathbf{P}\left[D^{\beta_x}F(\cdot,x_0)\right](z)$ is continuous on $\overline{U}$.

For (1), it remains to show that $\psi_F(z,\cdot)$ is ω-plurisubharmonic for every $z\in\overline{U}$ and $F\in\mathcal{F}_{a_0}$. We show that $\left(\partial_M\overline{\partial}_M\psi_F-i\omega\right)(\xi,\bar{\xi})\geq 0$ for any non-zero $\xi\in T^{1,0}M$. Indeed,

$$\left(\partial_M\overline{\partial}_M\psi_F-i\omega\right)(\xi,\bar{\xi})\;=\frac{1}{2\pi}\int_0^{2\pi}P(z,t)\big(\partial_M\overline{\partial}_M F(e^{it},\cdot)-i\omega\big)(\xi,\bar{\xi})\,dt\;>\;0 \tag{3.8}$$

as an integral of a positive quantity.

For (2), it is necessary to constrain the family $\mathcal{F}_{a_0}$ further. We choose $a>0$ so that for some $\alpha>0$,

$$\left(\partial_M\overline{\partial}_M\psi_F-i\omega\right)(\xi,\bar{\xi})\geq\alpha\left|\xi\right|^2,\quad \xi\in T^{1,0}M \tag{3.9}$$

for every $F\in\mathcal{F}_a$.

We now modify each ψ_F by adding to it a convex function on $\overline{U}$. Take $\chi:\overline{U}\to\mathbb{R}$ be smooth and strictly convex, with $\chi\equiv 0$ on ∂U and $\min_{\overline{U}}|\chi_{z\bar{z}}|=1$. Trivially extend χ to $\widetilde{\chi}:\overline{U}\times M\to\mathbb{R}$. We will define $\widetilde{F}:=\psi_F+m\widetilde{\chi}$ for some sufficiently large $m\in\mathbb{R}^+$.

To find an appropriate m, consider the action of $\widetilde{\omega}+i\partial\overline{\partial}(\psi+m\widetilde{\chi})$ on $\xi\in T^{1,0}(\overline{U}\times M)\simeq\overline{U}\times\mathbb{C}\times T^{1,0}M$. For clarity, write $\xi=(\xi_0,\xi')$ for $\xi_0\in\overline{U}\times\mathbb{C}$ and $\xi'\in T^{1,0}M$. It is also useful to break up $\partial=\partial_M+\partial_z$. Then we have

$$\begin{aligned}-i\big(\widetilde{\omega}+i\partial\overline{\partial}\psi_F\big)(\xi,\bar{\xi})&=-i\big(\omega+\partial\overline{\partial}\psi_F\big)(\xi',\bar{\xi}')+\partial_M\overline{\partial}_z\psi_F(\xi',\bar{\xi}_0)+\partial_z\overline{\partial}_M\psi_F(\xi_0,\bar{\xi}')+0\\&\geq\quad\alpha|\xi'|^2\quad+\quad\partial_M\tfrac{\partial\psi_F}{\partial\bar{z}_0}(\xi')\bar{\xi}_0\quad+\quad\bar{\partial}_M\tfrac{\partial\psi_F}{\partial z_0}(\bar{\xi}')\xi_0\end{aligned} \tag{3.10}$$

since ψ_F is harmonic along $\overline{U}$. The right hand quantity is positive so long as

$$\alpha\left|\xi'\right|^2\;>\;2\left|\partial_M\tfrac{\partial\psi_F}{\partial\bar{z}_0}\right|\left|\xi'\right|\left|\xi_0\right|\quad\text{ i.e. }\quad\left|\xi'\right|\;>\;\frac{2}{\alpha}\max_{\overline{U}\times M}\left|\partial_M\tfrac{\partial\psi_F}{\partial\bar{z}_0}\right|\left|\xi_0\right|=:\lambda\left|\xi_0\right|.$$

Thus for $|\xi'| > \lambda|\xi_0|$, we have $\widetilde{\omega}+i\partial\overline{\partial}(\psi_F+m\widetilde{\chi}) > 0$ regardless of the choice of $m > 0$.

If on the other hand $|\xi'| \leq \lambda\,|\xi_0|$, we see that

$$-i\big(\widetilde{\omega}+i\partial\overline{\partial}\,(\psi_F+m\widetilde{\chi})\big)(\xi,\bar{\xi}) \geq\; 0+\partial_M\overline{\partial}_z\psi_F(\xi',\bar{\xi}_0)+\partial_z\overline{\partial}_M\psi_F(\xi_0,\bar{\xi}')+m\,\partial\overline{\partial}\chi(\xi_0,\bar{\xi}_0)$$

will be positive if $m\,\chi_{z\bar{z}}\,|\xi_0|^2 > 2\left|\partial_M\frac{\partial\psi_F}{\partial\bar{z}_0}\right|\lambda\,|\xi_0|^2$. That is, for $m > \frac{4}{\alpha}\,\|\psi_F\|^2_{C^2(\overline{U}\times M)}$. Recalling that we have $\|\psi_F\|_{C^3} \leq K\big\|F\big\|_{C^4(\partial U\times M)}$ for all $F\in\mathcal{F}_a$ from (3.7), the choice

$$m \;>\; \frac{4}{\alpha}K^2\Big(\,\|F_0\|_{C^4}+a\Big)^2 \;\geq\; \frac{4}{\alpha}K^2\,\|F\|^2_{C^4}$$

will guarantee $\widetilde{\omega}+i\partial\overline{\partial}\widetilde{F}=\widetilde{\omega}+i\partial\overline{\partial}(\psi_F+m\widetilde{\chi}) >\; 0$ for any $F\in\mathcal{F}_a$,

Finally, employing (3.7) again, we see that

$$\big\|\widetilde{F}\big\|_{C^3(\overline{U}\times M)} \;\leq\; K\big\|F\big\|_{C^4(\partial U\times M)} \;+\; \frac{4}{\alpha}K^2\big\|F\big\|^2_{C^4(\partial U\times M)}\big\|\chi\big\|_{C^4(\overline{U})} \tag{3.11}$$

which allows us to choose C_1, since $\big\|F\big\|_{C^4(\partial U\times M)}$ is constrained by the family $\mathcal{F}_a$.

■

Take $\widetilde{F}$ from Lemma 3.1.1 and define $h_{\widetilde{F}} := e^{\widetilde{F}}h$. The corresponding unit disk bundle $\{h_{\widetilde{F}} < 1\}$ is strongly pseudoconvex, since $h_{\widetilde{F}} : L\to\mathbb{R}$ is a strongly plurisubharmonic function. To see this, it will suffice to show that the $(1,1)$-form $i\partial\overline{\partial}\log h_{\widetilde{F}}$ is strictly positive on L minus the zero section, since exponentiation is a strictly convex, strictly increasing function.

Note that the curvature form $\Theta(L,h_{\widetilde{F}}) = -\widetilde{\omega}-i\partial\overline{\partial}\widetilde{F} < 0$. This curvature form is related to $i\partial\overline{\partial}\log h_{\widetilde{F}}$ via the projection map $p : L\to\overline{U}\times M$: $p^*\Theta(L,h_{\widetilde{F}}) = -i\partial\overline{\partial}\log h_{\widetilde{F}}$. For $\xi\in TL$, $p_*(\xi)=0$ only if $\xi\in T(L_{z,x})$ for some fiber of L. Thus as a pullback, $i\partial\overline{\partial}\log h_{\widetilde{F}}$ will be positive except in the fiber direction. But on a fiber $L_{z,x}$, we may write $h_{\widetilde{F}}(\zeta)$ explicitly as $|\zeta|^2e^{(\widetilde{u}+\widetilde{F})(z,x)}$, where $\widetilde{u}$ is a local potential for $\widetilde{\omega}$. Thus $\log h_{\widetilde{F}} = \log|\zeta|^2+\widetilde{u}+\widetilde{F}$. Our choice of $\widetilde{F}$ ensures that this is strongly plurisubharmonic.

Further, we claim that the holomorphic disks $\sigma_f(U)$ lie inside $\{h_{\widetilde{F}} < 1\}$, that is, that $h_{\widetilde{F}}(g(z)) < 1$ for all $z\in U$. To begin, note that $h_{\widetilde{F}} = h_v \equiv 1$ on $\sigma_f(\partial U)$ by construction. Since $h_{\widetilde{F}}$ is strongly psh, $h_{\widetilde{F}} \circ \sigma_f : U \to \mathbb{R}$ cannot attain an internal maximum. Thus $\sigma_f(U) \subset \{h_{\widetilde{F}} < 1\}$.

Consider the collection of all holomorphic maps $g : U \to \{h_{\widetilde{F}} < 1\}$ with the property that $g(z)$ maps into the fiber of L over z. (I.e. $p(g(z)) = (z, \cdot) \in \overline{U} \times M$.) If v solves MA(F), we have $(\widetilde{\omega} + i\partial\overline{\partial}\widetilde{F})^{n+1} \geq (\widetilde{\omega} + i\partial\overline{\partial}v)^{n+1}$. So by the Comparison Principle (Proposition 2.1.2), $\widetilde{F} \leq v$ on $\overline{U} \times M$ and therefore the boundary $\{h_v = 1\}$ must lie inside $\{h_{\widetilde{F}} \leq 1\}$. We say that a map g is "MA-extremal" relative to the solution v if $g(\overline{U}) \subset \{h_v = 1\}$. By definition, our maps σ_f are MA-extremal.

3.2 Two Useful Estimates

The following Lemma establishes the relationship between the family $\mathcal{F}_a$ and the corresponding family of hermitian metrics. For fixed F_0, we call the corresponding C^3, strongly $\widetilde{\omega}$-psh extension $\widetilde{F}_0$ (cf. Lemma 3.1.1). Let $h_{\widetilde{F}_0} : L \to \mathbb{R}$ be the corresponding hermitian metric on L.

Though the line bundle is not compact, we need only the compact set $X_1 := \{\xi \in L : h_{\widetilde{F}_0}(\xi) - 1 \leq 1\}$. (We define X_1 this way because we plan to employ $h_{\widetilde{F}_0}(\xi) - 1$ as the defining function of a domain.) As in 3.1, we have a Banach space $C^3(X_1, \mathbb{R})$ and we consider a neighborhood of the form

$$\mathcal{G}_b \;=\; \mathcal{G}_b(h_{\widetilde{F}_0}) \;=\; \{\tilde{\rho} \in C^3(X_1, \mathbb{R}) : \|\tilde{\rho} - h_{\widetilde{F}_0}\|_{C^3(X_1)} < b\}.$$

Lemma 3.2.1 *For fixed F_0, take $a > 0$ and the extensions $\widetilde{F} : \overline{U} \times M \to \mathbb{R}$ of $F \in \mathcal{F}_a(F_0)$ from Lemma 3.1.1. Given $1 > b > 0$ there is $c > 0$ so that whenever $F \in \mathcal{F}_c$, the corresponding hermitian metric $h_{\widetilde{F}} : L \to R$ lies in $\mathcal{G}_b = \{\tilde{\rho} : \|\tilde{\rho} - h_{\widetilde{F}_0}\|_{C^3(X_1)} < b\}$.*

Proof The proof consists of two stages. First, note that the same argument as (3.7) above, applied to the function $\widetilde{F} - \widetilde{F}_0 = \mathbf{P}\big[F - F_0\big] + (m - m)\widetilde{\chi}$, yields

$$\big\|\widetilde{F} - \widetilde{F}_0\big\|_{C^3(\overline{U} \times M)} \;\leq\; K\big\|F - F_0\big\|_{C^4(\partial U \times M)}. \tag{3.12}$$

Next, we show there is $C_2 > 0$, depending only on the initial hermitian metric h, $\|\widetilde{F}_0\|_{C^3}$, a, and C_1, such that

$$\|h_{\widetilde{F}} - h_{\widetilde{F}_0}\|_{C^3(X_1,\mathbb{R})} \;\leq\; C_2\|\widetilde{F} - \widetilde{F}_0\|_{C^3(\overline{U}\times M)}. \tag{3.13}$$

Recall the definition of $h_{\widetilde{F}}$:

$$h_{\widetilde{F}}(\zeta) \;:=\; h(\zeta)e^{\widetilde{F}(z,x)} \quad \text{for } \zeta \in L_{(z,x)}.$$

Thus we may pull out the quantity $\|h\|_{C^3(X_1,\mathbb{R})}$ and only concern ourselves with $\|e^{\widetilde{F}} - e^{\widetilde{F}_0}\|_{C^3(\overline{U}\times M)}$.

For arbitrary functions $u, u_0 \in C^3(\mathbb{C}^m, \mathbb{R})$, consider the derivative $D^\beta\big(e^u - e^{u_0}\big)$ for $|\beta| \leq 3$. We see that

$$\begin{aligned}\big|D^\beta\big(e^{u_0}\big(e^{(u-u_0)} - 1\big)\big)\big| \;&\leq\; 8\,\|e^{u_0}\|_{C^3}\|e^{(u-u_0)} - 1\|_{C^3} \\ &\leq\; 8\big(5\|u_0\|_{C^3}|e^{u_0}|\big)\big(5\|u - u_0\|_{C^3}|e^{u-u_0}|\big)\end{aligned}$$

by ordinary product and chain rule.

The same estimate will apply to $D^\beta\big(e^{\widetilde{F}} - e^{\widetilde{F}_0}\big)$ in local coordinates:

$$\begin{aligned}\big|D^\beta\big(e^{\widetilde{F}} - e^{\widetilde{F}_0}\big)\big| \;&\leq\; 200\,\|\widetilde{F}_0\|_{C^3}\,|e^{\widetilde{F}_0}|\,|e^{\widetilde{F}-\widetilde{F}_0}|\,\|\widetilde{F} - \widetilde{F}_0\|_{C^3} \\ &\leq\; C_3\,\|\widetilde{F} - \widetilde{F}_0\|_{C^3}\end{aligned}$$

since $\|\widetilde{F}_0\|_{C^3}$ is a constant, $e^{\widetilde{F}_0}$ attains a max on $\overline{U}\times M$, and $e^{\widetilde{F}-\widetilde{F}_0} \leq e^{|F-F_0|_{C^4}}$ using (3.12). Thus we may take $C_2 = \|h\|_{C^3(X_1,\mathbb{R})}C_3$.

Combining the estimates (3.12) and (3.13), it suffices to take $c = \frac{b}{K\,C_2}$.

■

We will also make use of the known uniform $C^{1,1}$ estimate on solutions of MA [5, 6]. In fact, at present we need only a gradient estimate $\|v\|_{C^1(\overline{U}\times M)} < C_{\text{grad}}$, as follows.

Proposition 3.2.2 *Given a boundary function $F_0 \in C^4(\partial U \times M)$, there are positive numbers $a, C_{grad} > 0$ such that if $F \in \mathcal{F}_a$ and $v \in C^2$ solves MA(F), then*

$$|\nabla v| \leq C_{grad}. \tag{3.14}$$

We derive the proposition from the following special case of Theorem 26 in [3].

Theorem 3.2.3 ([3], Theorem 26) *Let (N, Ω) be a compact Kahler manifold of dimension m with nonempty smooth boundary, and assume its bisectional curvatures are bounded below by $B \in \mathbb{R}$. Let $\phi \in C^3(N, \mathbb{R})$ and $p \in (0,1)$. If $\Omega_\phi > 0$ and $\Omega_\phi^m = p\Omega^m$, then*

$$|\nabla \phi| \leq C$$

where C depends only on m, B, and on upper bounds for p, $|\phi|$, and $|(\nabla\phi)_{\partial N}|$.

Proof [Proof of Proposition 3.2.2] We choose $a > 0$ as in Lemma 3.1.1. First we show that there is a $\widetilde{C} > 0$ such that whenever $F \in \mathcal{F}_a$ and $u \in C^3(\overline{U} \times M)$ solves

$$\begin{cases} (\tilde{\omega} + i\partial\bar{\partial}u)^{n+1} = p\,(idz \wedge d\bar{z} + \widetilde{\omega})^{n+1} \\ \tilde{\omega} + i\partial\bar{\partial}u > 0 \\ u|_{\partial U \times M} = F \end{cases} \tag{3.15}$$

for sufficiently small $p \in (0,1)$, then $|\nabla u| \leq \widetilde{C}$.

Let $\Omega = idz \wedge d\bar{z} + \widetilde{\omega}$. Given u solving (3.15), taking $\phi(z,x) = u(z,x) - |z|^2$ satisfies the conditions of Theorem 3.2.3. Thus it will suffice to show that for any $F \in \mathcal{F}_a$ and p sufficiently small, the quantities listed in Theorem 3.2.3 are uniformly bounded with respect to the the Kähler metric $\Omega = \tilde{\omega} + idz \wedge d\bar{z}$ on $\overline{U} \times M$. We need not worry about the dimension and B, which are determined by (M, ω). The remaining quantities we estimate by comparison.

For $F \in \mathcal{F}_a$, take the extensions ψ_F (harmonic across U) and $\widetilde{F}$ (strongly ω-psh) of F from Lemma 3.1.1. If u solves (3.15) for sufficiently small $p > 0$, we have

$$\left(\widetilde{\omega} + i\partial\overline{\partial}u\right)^{n+1} = p\Omega^{n+1} \leq \left(\widetilde{\omega} + i\partial\overline{\partial}\widetilde{F}\right)^{n+1}.$$

Thus by the Comparison Principle, $\widetilde{F} \leq u$ on $\overline{U} \times M$. On the other hand, since u must be subharmonic along $U \times \{x\}$ for any $x \in M$, the Maximum Principle gives $u \leq \psi_F$. Thus

$$\sup_{U \times M} |u| \leq \max\left\{\sup_{U \times M} |\psi_F|\,,\, \sup_{U \times M} \left|\widetilde{F}\right|\right\}$$

and from Lemma 3.1.1, we know that both quantities on the right are controlled by $C_1 \|F\|_{C^4(\overline{U}\times M)}$.

Meanwhile, at boundary points, we have a similar inequality:

$$\nabla \widetilde{F} \;\leq\; \nabla u \;\leq\; \nabla \psi_F$$

which may be seen by considering difference quotients. Derivatives along M at points in $\partial U \times M$ are the same for all three functions. For derivatives along $\{x\} \times U$, the difference quotient for u can be sandwiched between the corresponding quotients for ψ_F and $\widetilde{F}$.

Thus we conclude that

$$|\nabla u| \;\leq\; \max\left(|\nabla \widetilde{F}|, |\nabla \psi_F|\right)$$

which is again controlled by $C_1 \|F\|_{C^4(\overline{U}\times M)}$. Hence for the family $\mathcal{F}_a$, Theorem 3.2.3 yields a constant $\widetilde{C}$ so that $|\nabla u| \leq \widetilde{C}$.

It remains to show that the same estimate holds for solutions of the homogeneous problem MA(F). Suppose v solves MA(F) for some $F \in \mathcal{F}_a$. We take a sequence $F_j \in \mathcal{F}_a \cap C^\infty(\partial U \times M)$ so that $F_j \to F$ in the the C^4 topology. Let $\{p_j\} \subset (0,1)$, $p_j \to 0$.

For each $j = 1, 2, \ldots$, e.g. Theorem 19 in [3] ensures there is a smooth solution $u_j : \overline{U} \times M \to \mathbb{R}$ to the system

$$\begin{cases} (\tilde{\omega} + i\partial\bar{\partial} u_j)^{n+1} = p_j\,(i dz \wedge d\bar{z} + \omega)^{n+1} \\ \tilde{\omega} + i\partial\bar{\partial} u_j > 0 \\ u_j|\partial U \times M = F_j, \end{cases} \tag{3.16}$$

since $\widetilde{F}_j$ gives a smooth subsolution. By our argument above, if $p_j \to 0$ fast enough, $|\nabla u_j| \leq \widetilde{C}$ for all j. Thus the Arzelà-Ascoli theorem guarantees a subsequence $j_k \to \infty$ such that u_{j_k} converges uniformly to some $\widetilde{\omega}$-plurisubharmonic function $u \in C(\overline{U} \times M)$.

According to Corollary III.3.6 in [9] this implies that u is a *weak* solution of the homogeneous system MA(F). However, v is also a solution of MA(F). By uniqueness, $v = u = \lim_j u_{j_k}$, and thus the uniform bound on $|\nabla u_j|$ implies $|\nabla v| \leq \widetilde{C}$. ■

3.3 Proof of Theorems 3.3.1 and Theorem 3.0.1

We are now ready to collect our estimates and prove the following theorem.

Theorem 3.3.1 *Given a C^4 boundary function $F_0 : \partial U \times M \to \mathbb{R}$ there exist a and $C > 0$ such that if $f : U \to M$ is a leaf of a Monge-Ampère foliation corresponding to a solution of MA(F) for $F \in \mathcal{F}_a$, we have*

$$\left| f_*(z) \frac{\partial}{\partial z} \right| \;\leq\; C(1 - |z|)^{-1/2} \text{ for any } z \in U$$

where on the left side, the length $|\cdot|$ is measured in the metric determined by ω.

We first estimate the derivative of general holomorphic maps $g : U \to \{h_{\widetilde{F}} < 1\} \subset L$. This estimate is obtained using the Kobayashi metric, which measures a kind of "holomorphic size" in a complex space.

Recall the definition of the Kobayashi metric,

Definition 3.3.2 *For D a complex manifold, the Kobayashi metric is the function $F_D : T^{1,0}D \to \mathbb{R}^+$ given by*

$$F_D(\xi) = \inf \left\{ \beta > 0 : \exists\, g : U \to D \text{ holomorphic with } g(0) = \pi(\xi) \text{ and } dg_0 \left(\tfrac{\partial}{\partial z}\right) = \tfrac{\xi}{\beta} \right\}.$$

Thus we note that a lower bound on F_D constitutes an upper bound on the differential (at 0) of holomorphic maps $U \to D$.

We want a uniform estimate on holomorphic maps into the unit circle bundle $\{h_{\widetilde{F}} < 1\}$, where the hermitian metric $h_{\widetilde{F}}$ varies within a family $\mathcal{G}_b$ (from Section 3.2). Such an estimate follows from the following Theorem about families of strongly pseudoconvex domains. It is proved in slightly greater generality in Chapter 4.

Note that since defining functions should attain 0 at the boundary, this theorem is stated for the equivalent family $\mathcal{G}_b(h_{\widetilde{F}_0} - 1)$.

Theorem 3.3.3 (Theorem 4.0.2) *There exist δ, b, and $C > 0$ such that for any $\rho \in \mathcal{G}_b(h_{\widetilde{F}_0} - 1)$, if $\zeta \in L$ with $-\delta < \rho(\zeta) < 0$ and $\xi \in T^{1,0}_\zeta L$, we have*

$$F_{\{\rho<0\}}(\xi)^2|\rho(\zeta)| \;\geq\; C|\xi|^2. \tag{3.17}$$

Let us unpack this estimate when $\rho = h_{\widetilde{F}} - 1$. Any holomorphic $g : U \to L$ which lands inside $\{h_{\widetilde{F}} < 1\}$ is a candidate function in the definition of $F_{\{h_{\widetilde{F}}<1\}}(\xi)$, for ξ the unit tangent vector at $g(0)$ in the direction of $dg_0(\frac{\partial}{\partial z})$. In particular, $dg_0\left(\frac{\partial}{\partial z}\right) = \frac{\xi}{\beta}$ for some $\beta \geq F_{\{h_{\widetilde{F}}<1\}}(\xi)$. Thus the estimate (3.17) gives

$$\left|dg_0\left(\tfrac{\partial}{\partial z}\right)\right| \;\leq\; \frac{|\xi|}{F_{\{h_{\widetilde{F}}<1\}}(\xi)} \;\leq\; C^{-1/2}\left|h_{\widetilde{F}}(g(0)) - 1\right|^{1/2}. \tag{3.18}$$

This is a satisfactory estimate on the differential of g at 0. To extend to arbitrary $z \in U$, we compose with automorphisms $\phi_z : U \to U$, $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$. Each ϕ_z has derivative $\phi_z'(0) = -1 + |z|^2$. Thus applying the estimate (3.18) to $g \circ \phi_z$ we obtain

$$\left|dg_z\left(\tfrac{\partial}{\partial z}\right)\right| \;\leq\; C^{-1/2}\left|h_{\widetilde{F}}(g(z)) - 1\right|^{1/2}(1 - |z|^2)^{-1}. \tag{3.19}$$

So far, this estimate applies to arbitrary holomorphic $g : U \to \{h_{\widetilde{F}} < 1\}$. For "MA-extremal" maps, which map $\overline{U}$ into the boundary $\{h_v = 1\}$, we can do still better.

Proposition 3.3.4 *Let $v : \overline{U} \times M \to \mathbb{R}$ be a C^2 solution to MA(F) and $w : \overline{U} \times M \to \mathbb{R}$ any strongly $\widetilde{\omega} -$ psh extension of the boundary function F. If $s : \overline{U} \to \{h_w \leq 1\}$ is MA-extremal with respect to v, then*

$$|1 - h_w(s(z))| \;\leq\; \|h\|_{C^0(\{h_w\leq 1\})}\left(\|e^v\|_{C^1} + \|e^w\|_{C^1}\right)(1 - |z|) \tag{3.20}$$

for any $z \in \overline{U}$.

Proof The proof is quite straightforward, once we recall two facts. The extremal property amounts to $h_v(s(z)) \equiv 1$ on $\overline{U}$, while by assumption $v = w = F$ on ∂U. Thus, if $s(z)$ lies in the fiber over (z, x) and $z^* = \frac{z}{|z|} \in \partial U$, we have

$$\begin{aligned} 1 - h_w(s(z)) &= h(s(z))e^{v(z,x)} - h(s(z))e^{w(z,x)} \\ &\leq h(s(z))\left[\left(e^{v(z,x)} - e^{v(z^*,x)}\right) + \left(e^{w(z^*,x)} - e^{w(z,x)}\right)\right] \\ &\leq \|h\|_{C^0(X_1)}\left(\|e^v\|_{C^1}|z - z^*| + \|e^w\|_{C^1}|z - z^*|\right) \end{aligned}$$

using $\|e^v\|_{C^1} = \|e^v\|_{C^1(\overline{U}\times M)}$ as a Lipschitz constant for e^v on $\overline{U} \times M$. ■

The proof of Theorem 3.3.1 follows from combining Proposition 3.3.4 with the general estimate (3.19). We will use the gradient estimate from Proposition 3.2.2 to achieve a uniform constant on the right hand side of (3.20)

Proof Take $\widetilde{F}_0$ to be the C^3, strongly $\widetilde{\omega}$-psh extension of F_0 and $h_{\widetilde{F}_0} : L \to \mathbb{R}$ the corresponding hermitian metric on L. From Lemma 3.1.1, we have a preliminary $a > 0$ and for every $F \in \mathcal{F}_a$, a C^3 extension $\widetilde{F}$.

Meanwhile, the statement of Theorem 3.3.3 also specifies a choice of family parameter: we take $1 > b > 0$ so that the estimate (3.17) applies whenever $\rho \in \mathcal{G}_b(h_{\widetilde{F}_0} - 1)$. Use Lemma 3.2.1 to choose a fresh $a > 0$ so that if $F \in \mathcal{F}_a$ then $\left\|h_{\widetilde{F}} - h_{\widetilde{F}_0}\right\|_{C^3} < b$. That is, $h_{\widetilde{F}} - 1 \in \mathcal{G}_b(h_{\widetilde{F}_0} - 1)$.

Unpacking the estimate (3.17) from Theorem 3.3.3 as in (3.18, 3.19) above, we have $\delta, C_{\text{holo}} > 0$ such that

$$\left|dg_z\left(\tfrac{\partial}{\partial z}\right)\right| \;\leq\; C_{\text{holo}}\left|h_{\widetilde{F}}(g(z)) - 1\right|^{1/2}(1 - |z|^2)^{-1} \tag{3.19}$$

for any holomorphic $g : U \to \{h_{\widetilde{F}} < 1\}$ with $h_{\widetilde{F}} \in \mathcal{G}_b$, at any $z \in U$ with $-\delta < h_{\widetilde{F}}(g(z)) - 1 < 0$.

Looking at (3.19), we have two good reasons to wish for an estimate which uniformly translates $\left|h_{\widetilde{F}}(g(z)) - 1\right|$ into a condition on the modulus of $z \in U$. First, it would give (3.19) the desired RHS. Further, it would make the condition $-\delta < h_{\widetilde{F}}(g(z)) - 1 < 0$ more tractable. For maps which are MA-extremal, Proposition 3.3.4, together with our uniform control on $\left\|h_{\widetilde{F}}\right\|_{C^3}$ and the gradient estimate of Proposition 3.2.2, gives us precisely such an estimate. In particular, it will apply to the maps σ_f from Section 1.

Recall that for v a C^3 solution of $\mathrm{MA}(F)$, we have a family of leaf functions denoted $\mathcal{L}(v)$. From this set, we constructed a family of sections $\mathcal{S}(v) := \{\sigma_f : \overline{U} \to L : f \in \mathcal{L}(v)\}$ lying over $\mathrm{gr}(f)$ (see (3.5)). By definition, each σ_f is MA-extremal with respect to the solution v. Thus we have, from Proposition 3.3.4,

$$\left|1 - h_{\widetilde{F}}(\sigma_f(z))\right| \;\leq\; \|h\|_{C^0(\{h_{\widetilde{F}}<1\})}\left(\|e^v\|_{C^1} + \|e^{\widetilde{F}}\|_{C^1}\right)(1 - |z|) \tag{3.20}$$

for every $z \in \overline{U}$ and any section $\sigma_f \in \mathcal{S}(v)$.

We claim there is $C_{\mathrm{Ext}} > 0$ such that whenever $F \in \mathcal{F}_a$, the corresponding solution v and strongly psh extension $\widetilde{F}$ satisfy

$$\|h\|_{C^0(\{h_{\widetilde{F}}<1\})}\left(\|e^v\|_{C^1} + \|e^{\widetilde{F}}\|_{C^1}\right) \;\leq\; C_{\mathrm{Ext}}.$$

This claim is the heart of the proof, but it is not very difficult. The requirement $b < 1$ guarantees that $\{h_{\widetilde{F}} \leq 1\} \subset X_1$ for all extensions $\widetilde{F}$ of $F \in \mathcal{F}_a$. So we have $\|h\|_{C^0(\{h_{\widetilde{F}}\leq 1\})} \leq \|h\|_{C^0(X_1)}$, which is a constant. Further, the quantities

$$\big\|e^{\widetilde{F}}\big\|_{C^1} \;\leq\; e^{\|\widetilde{F}\|_{C^0}}\big\|\widetilde{F}\big\|_{C^1} \qquad \text{and} \qquad \big\|e^v\big\|_{C^1} \;\leq\; e^{\|v\|_{C^0}}\big\|v\big\|_{C^1}$$

are controlled by $\big\|\widetilde{F}\big\|_{C^3} \leq C_1\big\|F\big\|_{C^4}$ (Lemma 3.1.1) and $\|v\|_{C^1} < C_{\mathrm{grad}}$ (Proposition 3.2.2), respectively. In turn, $\|F\|_{C^4}$ is controlled by the choice of family $\mathcal{F}_a$.

Thus we obtain the extremely useful estimate

$$\big|1 - h_{\widetilde{F}}(\sigma_f(z))\big| \;\leq\; C_{\mathrm{Ext}}\,(1-|z|) \tag{3.21}$$

for any σ_f corresponding to $F \in \mathcal{F}_a$ and $z \in \overline{U}$. We apply this estimate in two ways. First, note that the condition $-\delta < h_{\widetilde{F}}(\sigma_f(z)) - 1 < 0$ is now satisfied whenever $0 < 1 - |z| < \delta/C_{\mathrm{Ext}}$. Thus our estimate (3.19) holds on the annulus $T_\delta := \{1 - \delta/2C_{\mathrm{Ext}} < |z| < 1\}$, for all σ_f corresponding to $F \in \mathcal{F}_a$. Applying (3.21) a second time, the estimate (3.19) becomes

$$\big|d(\sigma_f)_z\left(\tfrac{\partial}{\partial z}\right)\big| \;\leq\; C_{\mathrm{holo}}\big[C_{\mathrm{Ext}}\,(1-|z|)\,\big]^{1/2}(1-|z|^2)^{-1} \;\leq\; C(1-|z|)^{-1/2}$$

for any σ_f and $z \in T_\delta$.

It remains to unwrap an estimate on the leaf function f from this and extend to the whole unit disk. Using the projection $p : L \to \overline{U} \times M$, we obtain

$$C(1-|z|)^{-1/2} \;\geq\; \big|d(p\circ\sigma_f)_z\left(\tfrac{\partial}{\partial z}\right)\big| \;=\; \big|(\mathrm{id}_z \times df_z)\left(\tfrac{\partial}{\partial z}\right)\big| \;\geq\; \big|df_z\left(\tfrac{\partial}{\partial z}\right)\big|$$

since a projection cannot have larger norm than the original function.

Finally, note that $df_z\frac{\partial}{\partial z} \in T^{1,0}_z M$ is a section of the pull back bundle $f^*T^{1,0}_z M$. Write $\phi(z) = df_z\frac{\partial}{\partial z}$. Since we assumed the Griffiths curvature of h_M is semi-negative, the pullback function $h_M \circ \phi$ is subharmonic. Thus the maximum principle for subharmonic functions allows us to extend the estimate from T_δ to U. This completes the proof of Theorem 3.3.1. ■

Finally, note that using the generalization of Hardy and Littlewood's theorem (Theorem 2.3.1), we immediately obtain a new $C > 0$ with

$$\mathrm{d}(f(\zeta), f(z)) \ \leq \ C|\zeta - z|^{1/2}$$

for any leaf function $f \in \mathcal{L}_a$ and any $\zeta, z \in \overline{U}$.

4. ESTIMATING THE KOBAYASHI METRIC

Let X be a complex manifold, $\pi : T^{1,0}X \to X$ its holomorphic tangent bundle, and $\rho_0 \in C^\infty(X)$ an exhaustion function on X, strongly plurisubharmonic outside of $\{\rho_0 < -2\}$. For a family of functions ρ near ρ_0, we estimate the Kobayashi metric near the boundary of the strongly pseudoconvex, relatively compact domains $\{\rho < 0\}$. Recall the definition:

Definition 4.0.1 *For $D \subset X$, the Kobayashi metric is the function $F_D : T^{1,0}D \to \mathbb{R}^+$ given by*

$$F_D(\xi) = \inf\{\, \beta > 0 : \exists\, f : U \to D \text{ holomorphic with } f(0) = \pi(\xi) \text{ and } df_0\left(\tfrac{\partial}{\partial z}\right) = \tfrac{\xi}{\beta} \,\}$$

where $U \subset \mathbb{C}$ is the unit disk.

We will prove a uniform estimate over a family of domains, given by a family of defining functions ρ. As in Chapter 3, define the Banach space $C^3(X_1)$, for the compact set $X_1 = \{w \in X : \rho_0(w) \leq 1\}$. We consider families in $C^3(X_1)$ of the form

$$\mathcal{G}_b \;=\; \{\rho \in C^3(X_1) : \|\rho - \rho_0\|_{C^3(X_1)} < b\} \tag{4.1}$$

for $0 < b < 1$. Fixing a hermitian metric $|\cdot|^2$ on the holomorphic tangent bundle $T^{1,0}X$, choose b_0 small so that there is an $\alpha > 0$ with $\alpha|\xi|^2 \leq \partial\overline{\partial}\rho(\xi, \overline{\xi})$ for all $\xi \in T^{1,0}X_1$ for all ρ in the closure of $\mathcal{G}_{b_0}$.

Theorem 4.0.2 *Given ρ_0 on X as above, there exist δ, b, and $C > 0$ such that for any $\rho \in \mathcal{G}_b$ and $\xi \in T^{1,0}_w X$ with $-\delta < \rho(w) < 0$, we have $F_{\{\rho<0\}}(\xi)^2|\rho(w)| \geq C|\xi|^2$.*

In [14], Graham proved a similar estimate for a single strongly pseudoconvex bounded domain in $\mathbb{C}^m$, with some parts of the argument extending to Stein manifolds. The idea is to approximate the boundary of each domain $\{\rho < 0\}$ locally with ellipsoids, for which the Kobayashi metric is known explicitly, taking care to preserve uniformity across the family of domains.

For ellipsoids, the Kobayashi metric has a natural lower bound of this form:

Proposition 4.0.3 *For A a positive definite hermitian matrix, let $E \subset \mathbb{C}^m$ be the ellipsoid given by $\{z : \psi(z) = z_1 - \overline{z}_1 + z^* A z < 0\}$. Then $F_E(z,\eta)^2 \, |\psi(z)| \geq \eta^* A \eta$ for all $z \in E$ and $\eta \in \mathbb{C}^m$.*

Proof At any $(z, \eta) \in E \times \mathbb{C}^m \simeq T^{1,0}E$, the Kobayashi metric for E is given by

$$F_E(z,\eta) = \sqrt{\frac{\eta^* A \eta}{|\psi(z)|} + \left| \frac{\eta^* A z - \eta_1}{\psi(z)} \right|^2}$$

(see e.g. Proposition 2.2 in [14]). Thus

$$F_E(z,\eta)^2 |\psi(z)| \; = \; \eta^* A \eta + \frac{|\eta^* A z - \eta_1|^2}{|\psi(z)|} \; \geq \; \eta^* A \eta \, .$$

■

The proof of Theorem 4.0.2 has three stages. The first section estimates $F_{\{\rho<0\}}(\xi)$ near the boundary with $F_{\{\rho<0\}\cap P}(\xi)$, where P is a neighborhood of a boundary point. In the second, we map the domains $\{\rho < 0\} \cap P$ into domains in $\mathbb{C}^m$ which are comparable to ellipsoids. The key is to preserve uniformity across a family of defining functions in both steps. The third section contains the proof of Theorem 4.0.2.

4.1 Reduction to a local neighborhood

Proposition 4.1.1 *Consider a family $\mathcal{G}_b \subset C^3(X_1)$ with $0 < b \leq b_0$. Fix ζ with $\rho_0(\zeta) = 0$ and neighborhoods $\zeta \in P_0 \subset\subset P \subset \{-1 < \rho_0 < 1\}$. Then there exists $\delta > 0$ such that for all $\rho \in \mathcal{G}_b$ we have $F_{\{\rho<0\}\cap P}(\xi) < 2F_{\{\rho<0\}}(\xi)$ for any $\xi \in T^{1,0}_w X$, whenever $w \in P_0 \cap \{-\delta < \rho(w) < 0\}$.*

Note that δ will depend on the neighborhoods $P_0 \subset\subset P$. The proof relies on the following Lemma, which is a restatement of Lemma 4 in [14] (originally due to Royden).

Lemma 4.1.2 *Let D be any relatively compact domain in a complex manifold X and P a neighborhood of a boundary point. For $w \in D \cap P$, define $\mathcal{N}(w) = \mathcal{N}_{D,P}(w)$ by*

$$\mathcal{N}_{D,P}(w) = \inf \{r \in [0,1) : \exists\, f : U \to D \text{ holomorphic with } f(0){=}w \text{ and } f(r) \in D \backslash P\}.$$

Then $\mathcal{N}(w)\, F_{D\cap P}(\xi) \leq F_D(\xi)$ for all $\xi \in T^{1,0}_w X$.

Proof Let $w \in D \cap P$. Choose R such that $0 < R < \mathcal{N}(w)$. For any $f : U \to D$ holomorphic with $f(0) = w$, we must have $f(z) \in D \cap P$ whenever $|z| \leq R$. That is, letting $U_R = \{|z| < R\}$, $f(U_R)$ must land inside P. This gives a map $g : U \to D \cap P$ by letting $g(z) = f(Rz)$.

Thus

$$\begin{aligned} F_{D\cap P}(\xi) &= \inf \left\{\beta : \exists\, g : U \to D \cap P \text{ holo., } g(0) = w,\ g_* \tfrac{\partial}{\partial z} = \tfrac{\xi}{\beta}\right\} \\ &\leq \inf \left\{\beta : g(z) = f(Rz) \text{ for } f : U \to D \text{ holo., } f(0) = w,\ f_* \tfrac{\partial}{\partial z}\right\} \\ &= \tfrac{1}{R} \inf \left\{\alpha = R\,\beta : \exists\, f : U \to D \text{ holo., } f(0) = w,\ f_* \tfrac{\partial}{\partial z} = \tfrac{\xi}{\alpha}\right\} \end{aligned}$$

which is $\frac{1}{R} F_D(\xi)$. Letting $R \to \mathcal{N}(w)$, we have $F_{D\cap P}(\xi) \leq \frac{1}{\mathcal{N}(w)} F_D(\xi)$ for any $\xi \in T^{1,0}_w X$. (If the set by which $\mathcal{N}(w)$ is defined in empty, then we have $F_{D\cap P}(\xi) = 0 = F_D(\xi)$.) ■

The proof of the Proposition now follows from showing that $\mathcal{N}_{\{\rho<0\},P}(w) \to 1$ uniformly as $\rho(w) \to 0$ inside P_0. Note that $\mathcal{N}(w)$ is independent of the vector ξ.

Proof [Proof of Proposition 4.1.1]

To simplify notation, we write D_ρ for the domain $\{\rho < 0\}$ and $\mathcal{N}_\rho$ for $\mathcal{N}_{D_\rho,P}(w)$. By way of contradiction, suppose there is an $\epsilon > 0$ such that for all $\delta > 0$ we have a $\rho \in \mathcal{G}_b$ and a point $w \in D_\rho \cap P_0$, with $-\delta < \rho(w) < 0$ and $\mathcal{N}_\rho(w) \leq 1 - \epsilon$. Choosing a sequence $\{\delta_k\} \to 0$ generates a sequence of defining functions $\{\rho_k\}$ and points $w_k \in D_{\rho_k} \cap P_0$ with $-\delta_k < \rho_k(w_k) < 0$, such that for each k there is a holomorphic function $\{\widetilde{f}_k : U \to D_{\rho_k}\}$ with $\widetilde{f}_k(0) = w_k$ and $\widetilde{f}_k(r_k) \in D_{\rho_k} \setminus P$ for some $r_k \leq 1-\epsilon$. In fact, by shrinking each $\widetilde{f}_k$ slightly as needed, we may assume that $r_k = 1 - \epsilon$ and

$\widetilde{f}_k(r_k) \in \partial P$ for all k. The contradiction will follow from a limiting map f mapping into the boundary of a limiting domain D_∞.

If our functions $\{\widetilde{f}_k\}$ mapped into $\mathbb{C}^m$ for some m, they would form a normal family with a convergent subsequence. Using Grauert's Theorem characterizing exceptional sets from [15], we blow down a compact sub-variety in X, mapping X to a Stein space in some $\mathbb{C}^m$. More precisely, there is a compact subset $Y \subset X$ and a holomorphic map $\Phi : X \to S$, biholomorphic on $X \setminus Y$, taking Y to a finite set of points in a Stein space $S \subset \mathbb{C}^m$ for some m. [15]

Further, we can see that the exceptional set Y lies inside $\{\rho_0 < -2\}$. For if there were a neighborhood $V \subset Y$ on which ρ_0 is strongly plurisubharmonic, ρ_0 would attain a maximum at some point $x \in V$. If x is a smooth point of Y, this contradicts $i\partial\overline{\partial}\rho_0|_V > 0$. Otherwise, Hironaka's resolution of singularities yields a manifold Y' and a holomorphic map $p : Y' \to Y$, which is invertible away from $p^{-1}(\text{sing})$. Then $\rho_0 \circ p$ is a plurisubharmonic function on Y', attaining a max at points in $p^{-1}(\{x\})$, so it must be constant on a neighborhood. This neighborhood will include points outside of $p^{-1}(\text{sing})$. Thus ρ_0 attains the max at smooth points as well. Contradiction. Thus Φ is biholomorphic on $\{-2 < \rho_0\}$.

Take the compositions $\Phi \circ \widetilde{f}_k$ to obtain a new sequence of holomorphic functions $\{f_k\} : U \to S$. This sequence is uniformly bounded since each f_k maps into $\Phi(D_k) \subset \Phi(\{\rho_0 \leq 2\})$. Thus it has a subsequence converging uniformly on compact sets to some holomorphic $f : U \to S$. Note that f cannot be constant, since $f(1-\epsilon) = \lim_{j\to\infty} \Phi\circ\widetilde{f}_j(1-\epsilon) \in \Phi(\partial P)$, while $f(0) \in \Phi(P_0)$. And since Φ is biholomorphic on an open set containing $\overline{P}$, we have $\Phi(P_0) \subset\subset \Phi(P)$.

We show there is a limiting strongly pseudoconvex domain $D_\infty \subset S$ with $f(U)$ in the closure of D_∞ and $f(0) \in \partial D_\infty$. Recall that by the choice of $\mathcal{G}_a$, the defining functions $\{\rho_k\}$ are uniformly bounded in $C^3(X_1)$, with $i\partial\overline{\partial}\rho_k$ bounded away from zero. Thus by the Arzelà-Ascoli theorem, a subsequence converges uniformly up to two derivatives, yielding a C^2 strongly plurisubharmonic limit function on X_1, which we call $\widetilde{\rho}_\infty$.

Let $D_\infty = \Phi(\{\widetilde{\rho}_\infty < 0\}) \subset \mathbb{C}^m$. The composition $\rho_\infty := \widetilde{\rho}_\infty \circ \Phi^{-1}$ gives a strongly plurisubharmonic defining function for D_∞. It is easy to check that $f(U)$ lands in the closure of D_∞, using the uniform convergence of the subsequence $\{\rho_{k_j}\}$. Further, $\rho_\infty(f(0)) = 0$ as the limit of the composition of two uniformly convergent sequences.

Thus f is a holomorphic map into the closure of a strongly pseudoconvex domain, taking 0 to a boundary point, i.e. $\rho_\infty \circ f \leq \rho_\infty \circ f(0)$. But since f is not constant, f_* is injective on some neighborhood of 0. On that neighborhood, $i\partial\overline{\partial}(\rho_\infty \circ f) = i\partial\overline{\partial}\rho_\infty \circ f_* > 0$, and $\rho_\infty \circ f$ cannot attain a max at 0. Contradiction.

■

4.2 Approximating with Ellipsoids

In this section, about each $\zeta \in \{\rho_0{=}0\}$ we choose a neighborhood P such that each $\{\rho < 0\} \cap P$ can be mapped to a domain in $\mathbb{C}^m$ with defining function of the form $\varphi(z) = -2\mathrm{Re}\, z_1 + z^* A z + o(|z|^2)$, for A a positive definite hermitian matrix. For small z, it is reasonable to approximate such neighborhoods with ellipsoids given by $\{\psi(z) = -2\mathrm{Re}\, z_1 + z^* A z - \mu|z|^2 < 0\}$ for small μ.

Much of the work is to choose a cover $\mathcal{P} = \{P_\zeta\}$ of $\{\rho_0 = 0\}$ such that the approximation by ellipsoids can be made uniformly over all $P \in \mathcal{P}$. In turn, the choice of $\mathcal{P}$ constrains the size of the family $\mathcal{G}_b$, since the boundaries $\{\rho{=}0\}$ must be covered by $\mathcal{P}$ for every $\rho \in \mathcal{G}_b$.

Definition 4.2.1 *Let ζ be a boundary point of $\{\rho < 0\} \subset X$. We call a coordinate chart $\theta : V \to \mathbb{C}^m$ about ζ a **normal chart** if*

1. $\theta(\zeta) = 0$

2. *the local defining function $\varphi := \rho \circ \theta^{-1}$ has the form $\varphi(z) = -z_1 - \overline{z_1} + \bar{z} A z + o(|z|^2)$ as $z \to 0$, where A is the complex Hessian of φ at 0.*

Lemma 4.2.2 *Every $\zeta_0 \in \{\rho_0 = 0\}$ has a neighborhood V together with $C, \mu, a > 0$ and a C^1 map $\Theta : \mathcal{G}_a \times V \times V \to \mathbb{C}^m$ such that $\Theta(\rho, \zeta, \cdot) = \theta_{\rho,\zeta}$ is a normal chart on V for all $(\rho, \zeta) \in \mathcal{G}_a \times V$. Further, for $\varphi_{\rho,\zeta} := \rho \circ \theta^{-1}_{\rho,\zeta}$ we have the uniform estimates*

1. $\|\varphi_{\rho,\zeta}\|_{C^3(\theta_{\rho,\zeta}(V))} < C$
2. $\partial\overline{\partial}\varphi_{\rho,\zeta}(z)(\eta, \bar{\eta}) \geq 2\mu|\eta|^2$ *for all* $(z, \eta) \in \theta_{\rho,\zeta}(V) \times \mathbb{C}^m \simeq T^{1,0}\theta_{\rho,\zeta}(V)$

For the proof, we use a particular statement of the Implicit Function Theorem for Banach spaces.

Theorem 4.2.3 *Let B be an open subset of a Banach space, $W \subset \mathbb{C}^m$, and $F : B \times \mathbb{C}^m \times W \to \mathbb{C}^m$ a continuously Fréchet differentiable function with k continuous derivatives with respect to $z \in \mathbb{C}^m$ and $w \in W$. Suppose there is a point $(\beta_0, z_0, w_0) \in B \times \mathbb{C}^m \times W$ such that $F(\beta_0, z_0, w_0) = 0$ and the Jacobian $Jac_w F(\beta_0, z_0, w_0) : \mathbb{C}^m \to \mathbb{C}^m$ is invertible. Then there exist neighborhoods $(\beta_0, z_0) \in B_1 \times U \subset B \times \mathbb{C}^m$ and $w_0 \in V \subset W$ and a C^1 map $G : B_1 \times U \to V$ such that $F(\beta, z, w) = 0$ iff and only if $w = G(\beta, z)$ for all $(\beta, z, w) \in B_1 \times U \times V$. Further, derivatives up to order k of $G(\beta, z)$ with respect to z are continuous.*

Proof The standard Implicit Function Theorem on Banach spaces gives neighborhoods β_0, U, and V and the map $G : B_1 \times U \to V$ such that $F(\beta, z, w) = 0$ iff and only if $w = G(\beta, z)$ for all $(\beta, z, w) \in B_1 \times U \times V$. Then we may differentiate $F(\beta, z, G(\beta, z)) = 0$ with respect to z to obtain

$$Jac_z F + Jac_w F \cdot Jac_z G = 0$$

Since $Jac_w F$ is invertible near (β_0, z_0, w_0), we have

$$Jac_z G(\beta, z) = -Jac_z F \, (Jac_w F)^{-1}$$

on some neighborhood of (β_0, z_0). Since matrix inversion is a smooth function, we are done. ∎

Proof [Proof of Lemma 4.2.2] Take $\zeta_0 \in \{\rho_0{=}0\}$. Let W_0 be a compact neighborhood of ζ_0 inside a coordinate neighborhood W. We may assume $W \subset \mathbb{C}^m$ with $\zeta_0 = 0$. Further, assume the $\overline{\zeta}$ derivative of ρ_0 at 0, which we write as $\nabla\rho_0(0)$, is the vector $(-1, 0, \ldots, 0)$ in $\mathbb{C}^m$. Initially, take $\mathcal{G}_{b_0}$ as in (4.1).

For each pair $(\rho, \zeta) \in \mathcal{G}_{b_0} \times W_0$, let $R_{\rho,\zeta}$ be the matrix obtained by replacing the first row of $I = I_{m\times m}$ with $-\nabla\rho(\zeta)$. Since the map $(\rho, \zeta) \to \nabla\rho(\zeta)$ is continuous and $\nabla\rho_0(\zeta_0) = (-1, 0, \ldots, 0)$, we may take a neighborhood $\mathcal{B} \subset \mathcal{G}_{b_0} \times W_0$ about (ρ_0, ζ_0) on which the matrices $R_{\rho,\zeta}$ are invertible.

For each $(\rho, \zeta) \in \mathcal{B}$, define a coordinate change $\tau_{\rho,\zeta}$ by

$$\tau_{\rho,\zeta}(Z) = (Z_1 - \sum_{i,j=1}^{m} c_{ij}(\rho, \zeta) Z_i Z_j, Z_2, \ldots, Z_m) = (z_1, \ldots, z_m) \tag{4.2}$$

where $c_{ij}(\rho, \zeta)$ is the $Z_i Z_j$ derivative of $\rho(R_{\rho,\zeta}^{-1} Z + \zeta)$ at $Z = 0$.

Make a preliminary definition of Θ as

$$\Theta : \mathcal{B} \times W \to \mathbb{C}^m, \quad \Theta(\rho, \zeta, w) = \theta_{\rho,\zeta}(w) = \tau_{\rho,\zeta}(R_{\rho,\zeta}(w - \zeta)). \tag{4.3}$$

We will show Θ is C^1 and use the Inverse Function Theorem to find uniformly bounded inverses for the maps $\theta_{\rho,\zeta}$. In the course of the argument, it will be necessary to restrict the domain of Θ.

To show Θ is C^1, it suffices to find continuous Fréchet derivatives for $(\rho, \zeta) \to R_{\rho,\zeta}$ and $(\rho, \zeta, Z) \to \tau_{\rho,\zeta}(Z)$ on $\mathcal{B}$ and $\mathcal{B} \times W$, respectively. For the former, consider the map $(\rho, \zeta) \to \nabla\rho(\zeta)$. Its Fréchet derivative, at a point $(\rho, \zeta) \in \mathcal{B}$, is the linear map $T_{\rho,\zeta} : C^3(\overline{X}_1) \times \mathbb{C}^m \to \mathbb{C}^m$ given by $T_{\rho,\zeta}(\sigma, \eta) = \nabla\sigma(\zeta) + D_\eta \nabla\rho(\zeta)$, where D_η denotes the derivative in the direction η. Indeed,

$$\begin{aligned}\nabla(\rho{+}\sigma)(\zeta{+}\eta) - \nabla\rho(\zeta) - \big(\nabla\sigma(\zeta){+}D_\eta\nabla\rho(\zeta)\big) &= \int_0^1 \frac{d}{dt}\big(\nabla(\rho{+}\sigma)(\zeta{+}t\eta)\big)dt - D_\eta\nabla\rho(\zeta) \\ &= \int_0^1 (D_\eta\nabla\sigma)(\zeta + t\eta) + (D_\eta\nabla\rho)(\zeta + t\eta) - (D_\eta\nabla\rho)(\zeta)\, dt.\end{aligned}$$

In the standard norm on $\mathbb{C}^m$, this is bounded by

$$|\eta|\left(m^2\|\sigma\|_{C^3} + \sup_{t\in[0,1]} |\nabla^2\rho(\zeta + t\eta) - \nabla^2\rho(\zeta)|\right) = o\left(\|(\sigma, \eta)\|_{C^3(W_0)\times|\cdot|}\right)$$

since the second derivatives of ρ are uniformly continuous on W_0.

Further, the operator valued map $\mathbf{T} : \mathcal{B} \to L(C^3(\overline{X}_1) \times \mathbb{C}^m, \mathbb{C}^m)$ taking $(\rho, \zeta) \to T_{\rho,\zeta}$, is continuous on $\mathcal{B}$. For taking the sup over $\|(\sigma, \eta)\|_{C^3 \times |\cdot|} = 1$,

$$\begin{aligned}\sup_{(\sigma,\eta)} \left|(T(\rho,\zeta) - T(\rho_1,\zeta_1))(\sigma,\eta)\right| &= \sup \left|\nabla\sigma(\zeta) - \nabla\sigma(\zeta_1) + D_\eta\nabla\rho(\zeta) - D_\eta\nabla\rho_1(\zeta_1)\right| \\ &\leq \sup\left(|\nabla\sigma(\zeta) - \nabla\sigma(\zeta_1)| + |D_\eta\nabla(\rho-\rho_1)(\zeta)| + |D_\eta\nabla\rho_1(\zeta) - D_\eta\nabla\rho_1(\zeta_1)|\right) \\ &\leq m\,|\zeta - \zeta_1| + m^2\|\rho - \rho_1\|_{C^3} + \sup|D_\eta\nabla\rho_1(\zeta) - D_\eta\nabla\rho_1(\zeta_1)|\end{aligned}$$

since each second partial derivative of σ is bounded by 1 and $|\eta| \leq 1$. Thus the continuity of the second derivatives of ρ_1 ensures that $T(\rho,\zeta) - T(\rho_1,\zeta_1) \to 0$ in operator norm as $\|(\rho,\zeta) - (\rho,\zeta_1)\|_{C^3\times|\cdot|} \to 0$. This, in turn, shows that the map $(\rho,\zeta) \to R_{\rho,\zeta}$ is continuously differentiable.

To see that $(\rho,\zeta,Z) \to \tau_{\rho,\zeta}(Z)$ is continuously differentiable, it suffices to consider the terms $c_{ij}(\rho,\zeta) = \frac{\partial^2}{\partial Z_i \partial Z_j}\rho(R^{-1}_{\rho,\zeta}Z + \zeta)|_{Z=0}$. We have already shown $R^{-1}_{\rho,\zeta}$ is C^1, since matrix inversion is a smooth function. The map $(\rho,\zeta) \mapsto \frac{\partial^2}{\partial Z_i \partial Z_j}\rho(\zeta)$ has Fréchet derivative $\Sigma : \mathcal{B} \to L(C^3(\overline{X}_1)\times\mathbb{C}^m, \mathbb{C}^m)$ given by $\Sigma_{\rho,\zeta}(\sigma,\eta) = \frac{\partial^2\sigma}{\partial Z_i \partial Z_j}(\zeta) + D_\eta(\rho_{z_i z_j})(\zeta)$. By a similar argument as for $\mathbf{T}$ above, Σ is continuous as a function on $\mathcal{B}$. At this stage the full three derivatives in the C^3 norm are needed.

We apply the Implicit Function Theorem stated above to the C^1 map $F : \mathcal{B} \times \mathbb{C}^m \times W \to \mathbb{C}^m$ given by $F(\rho,\zeta,z,w) = \Theta(\rho,\zeta,w) - z$. Since taking derivatives of Θ with respect to w is simply differentiating the smooth coordinate change $\tau_{\rho,\zeta}$ and multipling by $R_{\rho,\zeta}$, we may take arbitrarily many continuous derivatives of F with respect to z and w. At the point $(\rho_0,\zeta_0,0,\zeta_0)$, we have $F(\rho_0,\zeta_0,0,\zeta_0) = 0$. Differentiating with respect to w,

$$\operatorname{Jac}_w F\big|_{(\rho_0,\zeta_0,0,\zeta_0)} = \operatorname{Jac}_w\left(\tau_{\rho,\zeta}(R_{\rho,\zeta}(w-\zeta)) - z\right)\big|_{(\rho_0,\zeta_0,0,\zeta_0)} = \operatorname{Jac}_Z \tau_{\rho_0,\zeta_0}(0)\, R_{\rho_0,\zeta_0} I = I$$

since from (4.2) we see that $\operatorname{Jac}_Z \tau_{\rho_0,\zeta_0}(0) = I$.

Thus Theorem 4.2.3 gives neighborhoods $\mathcal{B}_1 \times U_1 \subset \mathcal{B} \times \mathbb{C}^m$ about $((\rho_0,\zeta_0),0)$ and $W_1 \subset W$ about ζ_0, with a C^1 map $G : \mathcal{B}_1 \times U_1 \to W_1$ such that $G(\rho,\zeta,z) = w$ if and only if $\Theta(\rho,\zeta,w) = z$ for all points $(\rho,\zeta,z,w) \in \mathcal{B}_1 \times U_1 \times W_1$. That is,

$g_{\rho,\zeta}(z) := G(\rho,\zeta,z)$ is a right inverse for $\theta_{\rho,\zeta}$ on U_1. Further, the theorem guarantees that the z derivatives of $G(\rho,\zeta,z)$ are continuous on $\mathcal{B}_1 \times U_1$.

We can now establish statement (1), the uniform bound on $\|\varphi_{\rho,\zeta}\|_{C^3}$. Using the continuity of the z derivatives of G on $\mathcal{B}_1 \times U_1$, choose a neighborhood $\mathcal{B}_2 \times U$ about $((\rho_0,\zeta_0),0)$ on which $\|g_{\rho,\zeta}\|_{C^3(U)}$ is uniformly bounded for $(\rho,\zeta) \in \mathcal{B}_2$. With a little more care, we may choose $\mathcal{B}_2$ and a neighborhood $W_2 \subset W_1$ such that $\|\theta_{\rho,\zeta}\|_{C^1(W_2)}$ is also uniformly bounded. Meanwhile, by definition we had $\|\rho\|_{C^3(W_1)}$ uniformly bounded for $\rho \in \mathcal{F}_{b_0}$.

Thus to establish the uniform bound on $\|\varphi_{\rho,\zeta}\|_{C^3} = \|\rho \circ \theta^{-1}_{\rho,\zeta}\|_{C^3}$, it remains ensure that each $g_{\rho,\zeta} : \theta_{\rho,\zeta}(V) \to V$ is a genuine, two sided inverse for $\theta_{\rho,\zeta}$. It will suffice to choose $V \subset W_2$ so that $\theta_{\rho,\zeta}(V) \subset U$ for all pairs (ρ,ζ). For then, we see that $g_{\rho,\zeta}$ is also a left inverse: given any $(\rho,\zeta,w) \in \mathcal{B}_2 \times V$ we have $g_{\rho,\zeta}(\theta_{\rho,\zeta}(w)) = g_{\rho,\zeta}(\theta_{\rho,\zeta}(g_{\rho,\zeta}(z)) = w$, for some $z \in U$.

Finally, using the continuity of Θ and the fact that $\Theta(\rho_0,\zeta_0,\zeta_0) = 0$, choose a product neighborhood $\mathcal{G}_a \times V \times V \subset \mathcal{B}_2 \times W_2$ about (ρ_0,ζ_0,ζ_0) such that $\Theta(\mathcal{G}_a, V, V) \subset U$. Then restricting Θ to $\mathcal{G}_a \times V \times V$, we have established statement (1)).

The existence of μ for statement (2) will follow from the uniform bound on $\|\theta_{\rho,\zeta}\|_{C^1(V)}$. Take any $(\rho,\zeta) \in \mathcal{G}_a \times V$. Given a tangent vector $(z,\eta) \in \theta_{\rho,\zeta}(V) \times \mathbb{C}^m \simeq T^{1,0}\theta_{\rho,\zeta}(V)$, we let $\xi = (\theta^{-1}_{\rho,\zeta}(z))_*\eta \in T^{1,0}_w X$. From the choice of $\mathcal{G}_a$ we had an $\alpha > 0$ such that $\alpha|\xi|^2 < \partial\overline{\partial}\rho(\xi,\overline{\xi})$ for all $\rho \in \mathcal{G}_b$ and $\xi \in T^{1,0}X_1$. So we have

$$\partial\overline{\partial}\varphi_{\rho,\zeta}(z)(\eta,\overline{\eta}) = \left((\theta^{-1}_{\rho,\zeta})^*\partial\overline{\partial}\rho\right)(z)(\eta,\overline{\eta}) = \partial\overline{\partial}\rho(w)(\xi,\overline{\xi}) \geq \alpha|\xi|^2 .$$

Meanwhile, $\mu|\eta|^2 = \mu|(\theta_{\rho,\zeta})_*\xi|^2 \leq \mu C|\xi|^2$, which is less than $\alpha|\xi|^2$ for an appropriate choice of μ.

Finally, note that $\theta_{\rho,\zeta}$ is a normal chart on V for all pairs $(\rho,\zeta) \in \mathcal{G}_a \times V$. For we have

$$\nabla\left(\rho\big(R^{-1}_{\rho,\zeta}(Z)) + \zeta\big)\right)\big|_{Z=0} = \nabla\rho(\zeta) \cdot R^{-1}_{\rho,\zeta} = (-1,0,\ldots,0)$$

for all (ρ,ζ). Therefore, using $Z = \tau_{\rho,\zeta}^{-1}(z)$ as in the definition (4.2), the Taylor series for $\varphi_{\rho,\zeta}(z) = \big(\rho \circ \theta_{\rho,\zeta}^{-1}\big)(z)$ is

$$\begin{aligned}\rho\big(R_{\rho,\zeta}^{-1}Z+\zeta\big) &= -Z_1 - \overline{Z}_1 + \sum_{i,j=1}^{m} c_{ij}Z_iZ_j + \sum_{i,j=1}^{m} \overline{c}_{ij}\overline{Z}_i\overline{Z}_j + \sum_{i,j=1}^{m} a_{ij}Z_i\overline{Z}_j + o(|Z|^2)\\ &= -z_1 - \bar{z}_1 + \sum_{i,j=1}^{m} a_{ij}z_i\bar{z}_j + o(|z|^2)\,.\end{aligned}$$

■

We now have a cover of $\{\rho_0 = 0\}$ by neighborhoods V, each supporting a well controlled family of coordinate maps into C^m. For each V, consider the image domains $\theta_{\rho,\zeta}(\{\rho < 0\} \cap V) \subset \mathbb{C}^m$ for every pair $(\rho,\zeta) \in \mathcal{G}_a \times (\{\rho{=}0\} \cap V)$. Since the $\theta_{\rho,\zeta}$ are normal charts, each image domain has a local defining function $\varphi_{\rho,\zeta}(z) = -2\mathrm{Re}\, z_1 + z^*A_{\rho,\zeta}z + r_{\rho,\zeta}(z)$. Since the $\theta_{\rho,\zeta}$ are biholomorphic, the functions $\varphi_{\rho,\zeta}$ will be strongly plurisubharmonic and each $A_{\rho,\zeta}$ is a positive definite hermitian matrix.

Taking μ from Lemma 4.2.2, we define a family of ellipsoids $E_{\rho,\zeta} \subset \mathbb{C}^m$ given by

$$E_{\rho,\zeta} := \{z \in \mathbb{C}^m : \psi_{\rho,\zeta}(z) = -2\mathrm{Re}\, z_1 + z^*A(\rho,\zeta)z - \mu|z|^2 < 0\}.$$

Each $E_{\rho,\zeta}$ approximates the boundary of $\theta_{\rho,\zeta}(\{\rho < 0\} \cap V)$ from the outside near $0 \in \mathbb{C}^m$. This will be a good comparison, in the sense that the boundary $\{\varphi_{\rho,\zeta} = 0\}$ stays inside $\overline{E_{\rho,\zeta}}$ and the defining functions are comparable, for sufficiently small $z \in \mathbb{C}^m$. The following Lemma ensures that we may shrink the neighborhoods V as needed.

Lemma 4.2.4 *Let $\zeta_0 \in \{\rho_0{=}0\}$ and take the neighborhood V and the map $\Theta : \mathcal{G}_a \times V \times V \to \mathbb{C}^m$ from Lemma 4.2.2. Then there exist $c > 0$ and a neighborhood $\zeta_0 \in P \subset V$ such that for each pair $(\rho,\zeta) \in \mathcal{G}_c \times (\{\rho = 0\} \cap P)$,*

1. *$\theta_{\rho,\zeta}(\{\rho < 0\} \cap P) \subset E_{\rho,\zeta}$ and*
2. *$|\psi_{\rho,\zeta}(z)| < 2|\varphi_{\rho,\zeta}(z)|$ for all $z \in \theta_{\rho,\zeta}(\{\rho < 0\} \cap P)$ lying on the positive $\mathrm{Re}\, z_1$ axis .*

Proof Take $\zeta_0 \in \{\rho_0{=}0\} \cap V$. The first condition is satisfied if $\theta_{\rho,\zeta}(P)$ lands inside a neighborhood of $0 \in \mathbb{C}^m$ on which $\varphi_{\rho,\zeta} \geq \psi_{\rho,\zeta}$. Thus to establish both conditions, it will suffice to find an $\epsilon > 0$ such that for all $z \in D_\epsilon(0)$ and $(\rho, \zeta) \in \mathcal{G}_a \times V$ we have $\psi_{\rho,\zeta}(z) < \varphi_{\rho,\zeta}(z)$, and $|\psi_{\rho,\zeta}(z)| < 2|\varphi_{\rho,\zeta}(z)|$ whenever z lies on the positive Rez_1 axis. Then the continuity of Θ will guarantee a neighborhood $\mathcal{G}_c \times P \times P$ about $(\rho_0, \zeta_0, \zeta_0) \in \mathcal{G}_a \times V \times V$ such that $\Theta(\mathcal{G}_c, P, P) \subset D_\epsilon(0)$.

The inequality $\psi_{\rho,\zeta}(z) < \varphi_{\rho,\zeta}(z)$ is equivalent to $-r_{\rho,\zeta}(z) < \mu|z|^2$. Since $r_{\rho,\zeta}(z)$ is the second order remainder term in the Taylor series for $\varphi_{\rho,\zeta}$, we have for some $t \in (0, 1)$,

$$|r_{\rho,\zeta}(z)| \;=\; \left| \frac{1}{\alpha!} \sum_{|\alpha|=3} \frac{\partial^{|\alpha|} \varphi_{\rho,\zeta}(z)}{\partial z^\alpha}(tz)\, z^\alpha \right| \;\leq\; \frac{1}{6}\, m^3 C\, |z|^3 \tag{4.4}$$

using the uniform bound on $\|\varphi_{\rho,\zeta}\|_{C^3}$ from Lemma 4.2.2. So for condition (1), it suffices to choose $\epsilon < \mu/m^3 C$.

This is very nearly enough for (2) as well. Indeed, for (2) we need, for small z lying on the positive Re z_1 axis,

$$\big|\psi_{\rho,\zeta}(z)\big| - 2\big|\varphi_{\rho,\zeta}(z))\big| = -2\mathrm{Re}\, z_1 + z^* A(\rho, \zeta) z + 2\, r_{\rho,\zeta}(z) + \mu |z|^2 < 0\,.$$

If $|z| < \epsilon$, we have $|r_{\rho,\zeta}(z)| < \mu|z|^2$ (c.f. (4.4)). By the choice of μ we have $\mu|z|^2 < i\partial\overline{\partial}\varphi_{\rho,\zeta}(0)(z, \bar{z}) = z^* A(\rho, \zeta) z$. Thus writing $z = (x, 0, \ldots, 0)$ for some $x \in \mathbb{R}^+$, it is enough to ensure $-2x + 4\,Cx^2 < 0$ or $x < 1/2C$. So we take $\epsilon = \min\{\mu/m^3C\,,\, 1/2C\}$.

■

The comparison of defining functions holds only along the positive real z_1 axis, which we denote Re$^+ z_1$. Thus the estimate in Thm 3.3.1 will be made on a cover of $\{\rho_0 = 0\} \subset X$ by neighborhoods swept out by inverse images of Re$^+ z_1$.

Lemma 4.2.5 *There exists a $d > 0$ and a finite collection $\mathcal{P}$ of neighborhoods $P \subset V$ from Lemma 4.2.4 such that for each $P \in \mathcal{P}$ there is $P_0 \subset\subset P$ such that*

1. *the collection* $\mathcal{P}_0 := \{P_0 : P \in \mathcal{P}\}$ *covers the boundaries* $\{\rho = 0\}$ *for all* $\rho \in \mathcal{G}_d$ *and*

2. *for each pair* $(\rho, w) \in \mathcal{G}_d \times P_0$*, there is a* $\zeta \in \{\rho = 0\} \cap P$ *such that* $\theta_{\rho,\zeta}(w)$ *lands in* $Re^+ z_1$*.*

Proof From the preceding two Lemmas, for $\zeta_0 \in \{\rho_0{=}0\}$ we have a family $\mathcal{G}_c$, a neighborhood P, and a C^1 map $\Theta : \mathcal{G}_c \times P \times P \to \mathbb{C}^m$. We will use the Implicit Function Theorem to find a neighborhood $\mathcal{G}_d \times P_0 \subset \mathcal{G}_c \times P$ of (ρ_0, ζ_0) and a map $g : \mathcal{G}_d \times P_0 \to P$ such that whenever $(\rho, w) \in \mathcal{G}_d \times P_0$, we have $\rho\big(g(\rho, w)\big) = 0$ and $\theta_{\rho,g(\rho,w)}(w) \in \mathrm{Re}^+ z_1$. Taking a finite sub-cover of $\{\rho_0{=}0\}$ by these neighborhoods P_0 will finish the proof.

Define $F : \mathcal{G}_c \times P \times P \to \mathbb{C}^m$ by $F(\rho, \zeta, w) = (\rho(\zeta),\, \pi_{\mathbb{R}^{2n-1}}\theta_{\rho,\zeta}(w))$. We see that $F(\rho_0, \zeta_0, \zeta_0) = (\rho_0(\zeta_0),\, \pi_{\mathbb{R}^{2n-1}}(0)) = 0$. We have already shown F is continuously differentiable in Lemma 4.2.2. It remains to show that the derivative of F with respect to ζ is invertible. We have

$$\mathrm{Jac}_\zeta\big(\rho(\zeta),\, \pi_{\mathbb{R}^{2n-1}}\theta_{\rho,\zeta}(w)\big)\Big|_{(\rho_0,\zeta_0,\zeta_0)} = \Big(\nabla\rho_0(\zeta_0),\, \pi_{\mathbb{R}^{2n-1}}\Big(\mathrm{Jac}_\zeta\,\Theta(\rho,\zeta,w)\big|_{(\rho_0,\zeta_0,\zeta_0)}\Big)\Big)\,.$$

Since $\nabla\rho_0(\zeta_0) = (-1, 0, \ldots, 0)$, it suffices to consider the other $2n-1$ vectors. Using the definition of Θ in 4.3 we have

$$\begin{aligned}\mathrm{Jac}_\zeta\,\tau(\rho,\zeta,R_{\rho,\zeta}(w-\zeta))\Big|_{(\rho_0,\zeta_0,\zeta_0)} &= \Big[D_\zeta\tau \,+\, D_Z\tau\big[(D_\zeta R_{\rho,\zeta})(w-\zeta) + R_{\rho,\zeta}(-I)\big]\Big]\Big|_{(\rho_0,\zeta_0,\zeta_0)}\\ &= 0 \,+\, D_\zeta\tau(\rho_0,\zeta_0,\zeta_0) \,+\, D_Z\tau(\rho_0,\zeta_0,\zeta_0)\,[\,0 - R_{\rho_0,\zeta_0}]\,.\end{aligned}$$

Now referring to (4.2),

$$D_\zeta\tau(\rho_0,\zeta_0,\zeta_0) = \left(Z_1 - \sum_{i,j=1}^{m} D_\zeta c_{ij}(\rho,\zeta)Z_iZ_j,\; Z_2,\ldots,Z_m\right)\Bigg|_{(\rho_0,\zeta_0,\zeta_0)} = 0$$

since $Z\big|_{(\rho_0,\zeta_0,\zeta_0)} = R_{\rho,\zeta}(w-\zeta)\big|_{(\rho_0,\zeta_0,\zeta_0)} = 0$. Similarly, $D_Z\tau(\rho_0,\zeta_0,\zeta_0) = I$. So in the end, we see that $\mathrm{Jac}_\zeta\,\Theta(\rho,\zeta,w)\big|_{(\rho_0,\zeta_0,\zeta_0)} = -I$.

Thus the Implicit Function Theorem gives the neighborhoods and map $g : \mathcal{G}_d \times P_0 \to P_1$ as above. Since we are interested in existence, but not uniqueness of g, we may consider it as a map into the larger set P.

We have now established the existence of our desired neighborhoods $P_0 \subset\subset P$ about each $\zeta_0 \in \{\rho_0{=}0\}$. It remains to choose a finite sub-collection $\mathcal{P}_0$ of the neighborhoods P_0, which covers $\{\rho_0{=}0\}$. Shrinking the family parameter d as needed, we obtain a family $\mathcal{G}_d$ such that $\bigcup_{\mathcal{G}_d}\{\rho = 0\} \subset \bigcup_{\mathcal{P}_0} P_0$.

■

4.3 Proof of Theorem 4.0.2

Theorem 4.3.1 (Theorem 4.0.2) *Given ρ_0 on X as above, there exist b, δ, and $C > 0$ such that for any $\rho \in \mathcal{G}_b$, for all $\xi \in T^{1,0}_w X$ with $-\delta < \rho(w) < 0$ we have*

$$F_{\{\rho<0\}}(\xi)^2\,|\rho(w)| \;\geq\; C|\xi|^2\,. \tag{4.5}$$

Proof From Lemmas 4.2.2, 4.2.4, and 4.2.5 we have a nested pair of finite covers $\mathcal{P}_0$ and $\mathcal{P}$ of $\{\rho_0 = 0\}$ and a $d > 0$ such that for every $\rho \in \mathcal{G}_d$, the boundary $\{\rho = 0\}$ is also covered by $\mathcal{P}_0$.

For each pair of neighborhoods $P_0 \subset\subset P$, we have $\delta_P > 0$ from Proposition 4.1.1 such that whenever $w \in P_0$ with $-\delta_P < \rho(w) < 0$, we have $F_{\{\rho<0\}\cap P}(\xi) < 2F_{\{\rho<0\}}(\xi)$ for all $\xi \in T^{1,0}_w X$. Choose $\delta_0 > 0$ smaller than the minimum of δ_P over the finite cover $\mathcal{P}$. Finally, choose $0 < \delta < \delta_0$ and $0 < b < d$ such that the "δ-collars" $\{-\delta < \rho \leq 0\}$ for $\rho \in \mathcal{G}_b$ are covered by $\mathcal{P}_0$. This defines the family $\mathcal{G}_b$.

Since $\mathcal{P}_0$ is a finite cover, it now suffices to demonstrate (4.5) for w in a single neighborhood $P_0 \in \mathcal{P}_0$. Given $w \in P_0 \subset P$, take any $\rho \in \mathcal{G}_b$ such that $-\delta < \rho(w) < 0$. To simplify notation we let $D := \{\rho < 0\}$.

The map g from Lemma 4.2.5 picks out a boundary point $\zeta = g(\rho, w) \in \{\rho = 0\}$ such that $\theta_{\rho,\zeta}(w) \in \mathrm{Re}^+ z_1$. We will estimate using the domain $\theta_{\rho,\zeta}(D \cap P) \subset \mathbb{C}^m$. This domain has local defining function $\varphi_{\rho,\zeta}(z) = -2\mathrm{Re}\, z_1 + z^* A(\rho,\zeta) z + o(|z|^2)$ and is approximated by the ellipsoid $E_{\rho,\zeta}$ with defining function $\psi_{\rho,\zeta}(z) = -2\mathrm{Re}\, z_1 + z^* A(\rho,\zeta) z - \mu |z|^2$.

Take any $\xi \in T^{1,0}_w X$ and let $(z,\eta) := (\theta_{\rho,\zeta})_*(\xi) \in T^{1,0}\mathbb{C}^m$. The choice of the neighborhood P in Lemma 4.2.4 ensures that $\theta_{\rho,\zeta}(D \cap P) \subset E_{\rho,\zeta}$. So by the monotonicity of the Kobayashi metric and its invariance under biholomorphisms we have

$$F_{E_{\rho,\zeta}}(z,\eta) \;\leq\; F_{\theta_{\rho,\zeta}(D\cap P)}(z,\eta) \;=\; F_{D\cap P}(\xi)\,.$$

Since $-\delta < \rho(w) < 0$, the comparison to a local neighborhood from Lemma 4.1.1 gives $F_{D\cap P}(\xi) < 2\,F_D(\xi)$. It follows that

$$F_{E_{\rho,\zeta}}(z,\eta)^2\,|\psi_{\rho,\zeta}(z)| < 4\,F_D(\xi)^2\,|\psi_{\rho,\zeta}(z)|\,.$$

Since $z = \theta_{\rho,\zeta}(w) \in \mathrm{Re}^+ z_1$, the comparison of defining functions from Lemma 4.2.4 (2) gives

$$|\psi_{\rho,\zeta}(z)| \;<\; 2\,|\varphi_{\rho,\zeta}(z)| \;=\; 2\,|\rho(w)|\,.$$

From Proposition 4.0.3, we have the ellipsoid estimate:

$$F_{E_{\rho,\zeta}}(z,\eta)^2\,|\psi_{\rho,\zeta}(z)| \;>\; \eta^* A_{\rho,\zeta}\eta - \mu|\eta|^2 \;>\; \frac{i}{2}\partial\overline{\partial}\varphi_{\rho,\zeta}(\eta,\bar{\eta}) \;= \frac{i}{2}\partial\overline{\partial}\rho(\xi,\bar{\xi}) \;>\; \frac{1}{2}\alpha|\xi|^2\,.$$

We conclude that

$$F_D(\xi)^2\,|\rho(w)| \;>\; \frac{1}{16}\alpha|\xi|^2\,.$$

■

5. ARBITRARY C^k ESTIMATES

An advantage of working with the MA foliation is that the leaves are holomorphic objects. Thus we may employ the tools of complex analysis to conclude greater regularity from the initial $C^{1/2}$ estimate. Specifically, we employ a variation on the Reflection Principle to extend the leaf functions f across the boundary of the unit disk. Then their higher derivatives near ∂U may be estimated by taking Cauchy integrals inside neighborhoods of controlled size.

This chapter proves higher order estimates on the leaves of MA foliations from the $C^{1/2}$ estimate of Chapter 4 and the known $C^{1,\alpha}$ estimate [5], in the case when the boundary data F and the base metric ω are assumed to be analytic.

Choose a smooth embedding $\Theta : M \to \mathbb{R}^q$ for some q. We define the *k-size* $||f||_k$ of a C^k map $f : U \to M$ as the norm of $\Theta \circ f$ in the Banach space $C^k(U, \mathbb{R}^q)$. A different embedding will give a different norm, but the chain rule shows that the new norm will be comparable to the first one.

For a fixed analytic boundary function $F : \partial U \times M \to \mathbb{R}$, we consider the problem MA(tF) for $t \in [0, 1]$.

Theorem 5.0.1 *For (M, ω) as in Theorem 3.3.1, suppose F is a real analytic boundary function, ω is analytic, and $k \in \mathbb{N}$. Then there is a positive number $C_k > 0$ such that if $f : \overline{U} \to M$ is a leaf function of the Monge-Ampère foliation corresponding to a solution of MA(tF) for any $t \in [0, 1]$, then $||f||_k < C_k$.*

The Arzelà-Ascoli theorem yields the following corollary.

Corollary 5.0.2 *For F is as in Theorem 5.0.1 and $x \in M$, suppose for certain values $t_j \subset [0, 1]$ the equation MA(t_jF) has a C^3 solution v_j, and $f_j : \overline{U} \to M$ is the leaf function of the corresponding MA foliation with $f(0) = x$. Then a subsequence of f_j converges in $C^\infty(\overline{U}, M)$ to an $f \in C^\infty(\overline{U})$, holomorphic on U.*

The chapter has four sections. In the first, we construct holomorphic lifts of the leaf functions into $T^*M^{0,1}$, which take ∂U into a totally real, real analytic submanifold Λ. This construction is based on Semmes in [23] and Donaldson in [11]. In the second section, we establish the existence of an anti-biholomorphic involution ν on a neighborhood of Λ, which restricts to the identity on Λ. The third section collects our existing estimates, in preparation for the proof of Theorem 5.0.1 and Corollary 5.0.2 in Section 4.

5.1 Set Up

5.1.1 Lifted maps

Let F be an analytic boundary function $\partial U \times M \to \mathbb{R}$. Denote by $\mathcal{T}_F$ the set of $t \in [0,1]$ for which $\mathrm{MA}(tF)$ admits a solution $v_t \in C^3(\overline{U} \times M)$. For $t \in \mathcal{T}_F$, we have a MA foliation associated to the solution v_t; that is, a family of holomorphic leaf functions $\mathcal{L}(v_t) = \{f_{t,x} : \overline{U} \to M : \ f_{t,x}(0) = x\}$. We denote by $\mathcal{L}(F)$ the total collection of leaf functions in the families $\mathcal{L}(v_t)$, for $t \in \mathcal{T}_F$. When we work with general leaf functions within the family, for arbitrary $t \in [0,1]$, the subscripts may be omitted.

Abbreviate $(T^*)^{0,1}M$ as T^*M. For each leaf function $f = f_{t,x} \in \mathcal{L}(F)$, define a map

$$g : \overline{U} \to \overline{U} \times T^*M^{1,0}, \quad g(z) = (z, \partial_M v(z, f(z))). \tag{5.1}$$

Note that g describes a smooth section of the pullback bundle $\pi^*(T^*M) \cong \overline{U} \times T^*M$, lying over $\mathrm{gr}(f)$. Following Semmes [23] in spirit and Lempert [19] in notation, we construct a complex structure on the manifold T^*M with respect to which the maps g are holomorphic.

Our new complex structure on T^*M will be determined by the metric ω. It can be defined for any Kähler manifold. Suppose ω admits a global potential: $\omega = i\partial\overline{\partial}u$ on M. Let $\phi : T^*M \to T^*M$ be the diffeomorphism $\phi(\eta) = \eta + \partial u(\pi(\eta))$. We define a new complex manifold $X(\omega)$ on the underlying real manifold of T^*M, by taking the complex structure to be $J(\omega) := \phi^* J$. Thus ϕ becomes a biholomorphism $X(\omega) \to T^*X$.

Note that this construction does not depend on the choice of potential u, so we are justified in calling our new manifold $X(\omega)$. Indeed, if $\omega = i\partial\overline{\partial}u = i\partial\overline{\partial}u_1$, then $\phi - \phi_1 = \partial(u - u_1)$, which is a holomorphic $(1,0)$-form. This implies $\phi^* J$ and $\phi_1^* J$ are the same complex structure. Similarly, if ω admits only local potentials, define the complex structure $J(\omega)$ locally by the same process and then piece them together. On overlaps, where $\omega = i\partial\overline{\partial}u_1 = i\partial\overline{\partial}u_2$, the two maps ϕ_1 and ϕ_2 will differ by the biholomorphism $\eta \to \eta + \partial(u_1 - u_2)(\pi(\eta))$. Thus the induced complex structures agree.

We can now see that according to the complex structure of $\overline{U} \times X(\omega)$, the maps g in (5.1) are holomorphic. It will suffice to show the section $\partial_M v \in C^\infty(\overline{U} \times M, X(\omega))$ is holomorphic along $\mathrm{gr}(f)$. We use the biholomorphism $\phi : X(\omega) \to T^*X$. The composition $\phi \circ \partial_M v : \overline{U} \times M \to T^*X$ is locally the section $\partial_M(v + u)$. That this map is holomorphic along leaves is a fundamental property of the MA foliation. Thus since ϕ is a biholomorphism, $\partial_M v$ is a holomorphic section.

Now, for each $t \in \mathcal{T}_F$, we have a family of holomorphic maps $\{g_{t,x} = \mathrm{id} \times \partial_M v_t(f_{t,x}) : x \in M\}$ corresponding to the leaf functions in $\mathcal{L}(v_t)$. We will be reflecting a slight variation on these maps: let $G_{t,x} = (t, g_{t,x})$ for each $t \in \mathcal{T}_F$, $x \in M$. We denote the whole collection of maps $G_{t,x}$ by $\mathcal{G}(F)$.

5.1.2 Boundary manifold

We now construct a totally real "boundary" manifold Λ, so that $G(\partial U) \subset \Lambda$ for every $G \in \mathcal{G}(F)$. The variation over t is built into this construction. Let $I \subset \mathbb{R}$ be an open interval containing $[0,1]$, chosen so that for $t \in I$ the boundary form $\omega + i\partial\bar{\partial}tF$ stays positive. Let $S := \{s \in \mathbb{C} : \mathrm{Re}\, s \in I\}$. Consider the embedding

$$\Sigma : I \times \partial U \times M \to S \times \mathbb{C} \times X(\omega) =: Z$$

given by $\Sigma(t,z,x) = (t,z,\partial_M(tF)(z,x))$. We take Λ to be its image, so that Λ is a real analytic submanifold of Z.

Note that for any $G = G_{t,x} \in \mathcal{G}(F)$, the restriction of $G_{t,x}(z) = (t,z,\partial_M v_t(z, f_{t,x}(z)))$ to ∂U will land in Λ. Further, note that $\dim_R \Lambda = \dim_{\mathbb{C}} Z = 2n+2$. Recall that an m real-dimensional submanifold Y of a complex m dimensional manifold Z is *totally real* if and only if $TY \cap JTY = \{0\}$.

Lemma 5.1.1 *Λ is totally real in Z.*

Proof For fixed t, z, the image $\Lambda_{t,z}$ of $\Sigma(t,z,\cdot)$ will be totally real in $\{(t,z)\} \times X(\omega)$. (See Lemma 6.5 in [19], but Semmes also knew this).

Meanwhile, with respect to $t \in [0,1]$ and $z \in \partial U$, Σ is simply the identity map in a single coordinate. So

$$\Sigma_* \tfrac{\partial}{\partial t} \;=\; \tfrac{\partial}{\partial t} \;\in\; TS$$

is independent of $J\frac{\partial}{\partial t}$ and any other tangent vectors to Λ. Writing $z = e^{i\theta} \in \partial U$, the same is true for $\frac{\partial}{\partial \theta}$ and $J\frac{\partial}{\partial \theta}$. Thus the image of Σ is totally real. ∎

5.2 Reflection

Theorem 5.2.1 *Let Z be a complex manifold and $Y \subset Z$ a closed, totally real, real analytic submanifold,* $\dim_{\mathbb{R}} Y = \dim_{\mathbb{C}} Z = m$. *Then there is a neighborhood $N \subset Z$ of Y and an anti-biholomorphism $\nu : N \to N$ that restricts to the identity of Y.*

Lemma 5.2.2 *With Z, Y as in the theorem, any $y \in Y$ has a neighborhood $W \subset Z$ with a biholomorphic map $b : W \to V \subset \mathbb{C}^m$ that maps $W \cap Y$ to $V \cap \mathbb{R}^m$.*

Proof On a neighborhood $A \subset \mathbb{R}^m$ about 0, we may take an analytic immersion $\Phi : A \to Y$ with $\Phi(0) = w$. There is a neighborhood $B \subset \mathbb{C}^m$ about A on which Φ extends as a holomorphic function $\Psi : B \to Z$. The Lemma will follow from showing that Ψ is invertible near 0.

Write $z_j = x + iy_j$ for the usual coordinates on $\mathbb{C}^m$. Since $\Phi : A \to Z$ is an embedding, the vectors $v_1, \dots, v_n := \Psi_* \frac{\partial}{\partial x_1}, \dots, \Psi_* \frac{\partial}{\partial x_n} \in T_w Y$ are linearly independent over $\mathbb{R}$. Meanwhile, since Y is totally real, the collection $Jv_1, \dots, Jv_n = \Psi_* \frac{\partial}{\partial y_1}, \dots, \Psi_* \frac{\partial}{\partial y_n} \in T_w Z$ is independent of $v_1, \dots, v_n$, as well as linearly independent in itself. Thus Ψ_* is invertible at 0 and has an inverse $b : W \to V$ between neighborhoods $w \subset W \subset Z$ and $0 \subset V \subset A$. ■

Proof [Proof of Theorem 5.2.2] From Lemma 5.2.2, for any $y \in Y$ we have neighborhoods $W_y \subset Z$ and $V_y \subset \mathbb{C}^m$ with a biholomorphic map $b_y : W_y \to V_y$ taking $Y \cap W_y$ into $\mathbb{R}^m$. Use these biholomorphisms to transport the anti-holomorphic involution $z \mapsto \bar{z} \in \mathbb{C}^m$ to Z, constructing an anti-biholomorphic map $\nu_y : W_y \to W_y$ which restricts to the identity on $Y \cap W_y$. If ν_y agrees with ν_z on $W_y \cap W_z$ we will have a global anti-biholomorphism ν.

Since each ν_y is the identity on $Y \cap W_y$, the identity theorem ensures that $\nu_y = \nu_z$ on any connected component of $W_y \cap W_z$ that intersects Y. We use the tubular neighborhood theorem to chose neighborhoods $W'_y \subset W_y$ and $W'_z \subset W_z$ whose intersection is a connected open set intersecting Y. That is, since Y is closed and analytic, there is a smooth vector bundle $E \to Y$ and a diffeomorphism $\Theta : \Omega \to E$ on a neighborhood $\Omega \subset Z$ of Y, which takes $y \in Y$ to the corresponding point $(y, 0)$ in the zero section of E (Theorem IV.5.1 in [17]). Choose W'_y such that if $\Theta(W'_y)$ intersects a fiber E_x, the intersection is a convex neighborhood of the point $(x, 0) \in E_x$. Then $\nu_y = \nu_z$ on $W'_y \cap W'_z$.

Thus we can define

$$\nu : V \to W \text{ for } V = \bigcup_y W'_y, \quad W = \bigcup_y W_y,$$

by $\nu = \nu_y$ on W'_y, to obtain a global anti-holomorphism.

It remains to ensure that ν is an anti-biholomorphism. The holomorphic map $\nu^2 : V \to Z$ is the identity map on Y. It follows that it is the identity on some, possibly smaller neighborhood N' of Y. So taking $N = N' \cap \nu^{-1}(N')$ yields an anti-biholomorphic involution $\nu = \nu\big|_N : N \to N$. ■

5.3 Existing Estimates

Proposition 5.3.1 *For F a boundary function as in Theorem 3.3.1, there is a positive number C_1 such that every $G \in \mathcal{G}(F)$ satisfies*

$$d(G(\zeta), G(z)) \leq C_1|\zeta - z|^{1/4}, \zeta, z \in \overline{U} \tag{5.2}$$

where d is the distance induced by ω on $S \times \mathbb{C} \times T^*M$.

Proof Since $\{tF : 0 \leq t \leq 1\} \subset C^4(\partial U \times M)$ is compact, we can use Theorem 3.3.1 to produce a $C > 0$ so that any leaf function f corresponding to a C^3 solution v_t of some $\mathrm{MA}(tF)$ satisfies

$$\mathrm{d}(f(\zeta), f(z)) \leq C|\zeta - z|^{1/2}, \zeta, z \in \overline{U}. \tag{5.3}$$

Thus since $G(z) = (t, z, \partial_M v_t(f(z)))$, inequality (5.2) will follow from a 1/2-Hölder estimate on ∂v_t.

We claim this follows from uniform bounds on $\sup|v_t|$ and Δv_t, which will be proved in Proposition 5.3.2. Since M is compact, it is enough to show that for a coordinate neighborhood $V \subset M$, there is $C_V > 0$ with

$$\|v_t\|_{C^{1,1/2}(\overline{U}\times V)} \ \leq \ C_V \left(\|v_t\|_{C^0(\overline{U}\times M)} + \|\Delta v_t\|_{C^0(\overline{U}\times M)}\right).$$

Using Theorem 9.13 in [12], for any $p \in (1, \infty)$ there is a $C_p > 0$ so that

$$\|v_t\|_{W^{2,p}(U\times V)} \ \leq \ C_p \left(\|v_t\|_p + \|\Delta v_t\|_p\right) \ \leq \ C_p' \Big(\|v_t\|_\infty + \|\Delta v_t\|_\infty \Big) \tag{5.4}$$

for the Sobolev norm $\|\cdot\|_{W^{2,p}}$. ($\overline{U} \times V$ may be viewed as a domain in $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$).

The Sobolev embedding theorem (Theorem 7.26 in [12]) translates this Sobolev norm estimate into a local $C^{1,1/2}$ estimate. Choosing $p \in \mathbb{Z}^+$ so that $1/2 \leq 1 - \frac{2n+2}{p}$, there is a continuous embedding $W^{2,p}(U \times V) \hookrightarrow C^{1,1/2}(\overline{U} \times V)$. Thus the closed graph theorem ensures a $C_{1/2} > 0$ with

$$\|v_t\|_{C^{1,1/2}(\overline{U}\times V)} \ \leq \ C_{1/2} \, \|v_t\|_{W^{2,p}(U\times V)}. \tag{5.5}$$

Putting (5.4) and (5.5) together finishes the claim.

■

Proposition 5.3.2 *Given a C^4 boundary function F, there is a $C_F > 0$ such that whenever v_t is a C^3 solution of MA(tF) for some $t \in [0,1]$ we have*

$$|\Delta v_t| \leq C_F$$

where Δ is the Laplacian on $(\overline{U} \times M, \widetilde{\omega})$.

The proposition follows from the estimates of B. Guan [16] and Chen [5]. We use the statements of these second derivative estimates given in Błocki's lecture notes [3].

Theorem 5.3.3 ([3], Theorem 24) *Let (N, Ω) be a compact Kahler manifold of dimension r with nonempty smooth boundary, $u \in C^4(N, \mathbb{R})$, and $p \in (0,1)$. If $\Omega_u > 0$ and $\Omega_u^m = p\Omega^m$, then*

$$|\Delta u| \ \leq \ C_\Delta$$

where C_δ depends only on r, upper bounds for $|u|$, $\sup_{\partial N} \Delta u$, and the scalar curvature of N, and a lower bound for the bisectional curvature of N.

Theorem 5.3.4 ([3], Theorem 27) *Write D_R^- for $D_R(0) \cap \{w_m \leq 0\} \subset \mathbb{C}^m$. Let $\rho, \tilde{\rho} \in C^3(D_R^-)$ with*

$$\partial\overline{\partial}\rho > 0\,,\ \partial\overline{\partial}\tilde{\rho} \geq \lambda\, \partial\overline{\partial}|w|^2 \text{ for some } \lambda > 0, \text{ and } \det(\rho_{z_j\bar{z}_k}) = p \leq \det(\tilde{\rho}_{z_j\bar{z}_k})$$

for some $p \in (0,1)$. Further, suppose that $\tilde{\rho} \leq \rho$ on D_R^- and $\tilde{\rho} = \rho$ on $\{w_m = 0\}$. Then $|D^2\rho(0)| \ \leq \ C_R$ where C_R depends only on m, upper bounds for $\|\tilde{\rho}\|_{C^3(D_R^-)}$ and $|\nabla\rho|$, and lower bounds for λ and R.

Proof [Proof of Proposition 5.3.2] The logic is very similar to the gradient estimate in Chapter 3 (Proposition 3.2.2). Given a boundary function F, we obtain $C_F > 0$ such that for each $t \in [0,1]$, we have $|\nabla\varphi_t| \leq C_F$ whenever $\varphi_t \in C^3(\overline{U} \times M)$ solves

$$\begin{cases} (\tilde{\omega} + i\partial\bar{\partial}\varphi_t)^{n+1} = p\,(idz \wedge d\bar{z} + \widetilde{\omega})^{n+1} \\ \tilde{\omega} + i\partial\bar{\partial}\varphi_t > 0 \\ \varphi_t|_{\partial U \times M} = tF, \end{cases} \tag{5.6}$$

for $p \in (0,1)$ sufficiently small. Then if v_t is a C^3 solution of $\mathrm{MA}(tF)$ for $t \in [0,1]$, taking a sequence $p_j \to 0$, the same estimate applies to v_t.

Let $N = \overline{U} \times M$ and $\Omega = \widetilde{\omega} + i dz \wedge d\bar{z}$. The dimension and curvature bounds for (N, Ω) are fixed constants. We will show that the other quantities in Theorems 5.3.3 and 5.3.4 that control C_Δ and C_R are in turn dominated by a constant independent of $t \in [0,1]$.

Let $\psi_F = \mathbf{P}[F]$ and $\widetilde{F}$ be the extensions of F from Lemma 3.1.1, ψ_F harmonic along $U \times \{x\}$ for $x \in M$ and $\widetilde{F}$ strongly $\widetilde{\omega}$-plurisubharmonic. Note that $t\widetilde{F}$ is also strongly $\widetilde{\omega}$-psh on $\overline{U} \times M$, since $\widetilde{\omega} + i\partial\overline{\partial}(t\widetilde{F}) = (1-t)\widetilde{\omega} + t(\widetilde{\omega} + i\partial\overline{\partial}\widetilde{F}) > 0$. Thus the comparison argument in Proposition 3.2.2 shows that if φ_t is a C^3 solution of (5.6) for sufficiently small p, then $|\varphi_t|$ is controlled by $\sup|t\widetilde{F}|$ and $\sup|t\psi_F|$. These are in turn bounded by $|\widetilde{F}|$ and $|\psi_F|$, and from Lemma 3.1.1 we know that these are controlled by $\|F\|_{C^4}$.

To control $\sup_{\partial N} \Delta\varphi_t$, we look to Theorem 5.3.4. Since $\overline{U} \times M \subset \mathbb{C} \times M$ has flat boundary, we may cover ∂N by a finite collection of coordinate charts $\theta_j : U_j \to V_j \subset \mathbb{C} \times \mathbb{C}^n$ such that $\theta_j\big(\partial N \cap U_j\big) \subset \mathbb{R} \times \mathbb{C}^n$ for each $j \in J$. We also require that for each $j \in J$, there is a local potential $\tilde{u}_j$ for $\widetilde{\omega}$ in a neighborhood of $\overline{U}_j$. Further, using the Lebesgue number lemma (Lemma 27.5 in [22]), we may choose $r > 0$ so that given $\zeta \in \partial N$, the r neighborhood of ζ is contained in U_j for some $j \in J$. Since the collection $\{\theta_j\}_{j \in J}$ is finite, we may further choose an $R > 0$ so that for every $\zeta \in \partial N$, the image of the r neighborhood of ζ under the biholomorphism $\theta_\zeta := \theta_j - \theta_j(\zeta)$ covers $D_R(0) \subset \mathbb{C}^{n+1}$.

Then for each $t \in [0,1]$ and $\zeta \in \partial N$ we are in the setting of Theorem 5.3.4, with $\rho_{t,\zeta} = (\tilde{u}_j + \varphi_t) \circ \theta_\zeta^{-1}$ defined on D_R^-. Let $\widetilde{\rho}_{t,\zeta} := (\tilde{u}_j + t\psi_F + m\widetilde{\chi}) \circ \theta_\zeta^{-1}$ (where $\widetilde{\chi}$ is strictly convex along $\overline{U}$ with $\min_{\overline{U} \times M} |\widetilde{\chi}_{z\bar{z}}| = 1$, as in Lemma 3.1.1.) Then we have $\widetilde{\rho}_{t,\zeta} = \rho_{t,\zeta}$ on $\theta_\zeta(\partial N \cap U_j) \subset \mathbb{R} \times \mathbb{C}^m$ and $\det(\tilde{\rho}_{z_j \bar{z}_k}) \geq p$ for sufficiently small $p > 0$. There must be some $\lambda > 0$ so that

$$\widetilde{\omega} + m\, i dz \wedge d\bar{z} \ \geq\ \lambda(i\partial\overline{\partial}|\theta_\zeta|^2).$$

Since J is finite and θ_ζ is simply some θ_j plus a constant, we may take λ to be independent of $\zeta \in \partial N$. Thus for any $t \in [0,1]$ and $\zeta \in \partial N$, recalling that $\min_{\overline{U}\times M} |\widetilde{\chi}_{z\bar{z}}| = 1$, we have

$$\partial\overline{\partial}\,\widetilde{\rho}_{t,\zeta}(w) \;\geq\; \partial\overline{\partial}\left((\tilde{u}_j + m\widetilde{\chi}) \circ \theta_\zeta^{-1}\right) \;\geq\; (\theta_\zeta^{-1})^*\left(-i\widetilde{\omega} + m\, dz \wedge d\bar{z}\right) \;\geq\; \lambda\partial\overline{\partial}\,|w|^2 .$$

Thus Theorem 5.3.4 will give a $C_R > 0$ such that

$$|D^2\rho_{t,\zeta}(0)| \;\leq\; C_R$$

for all t, ζ, if we demonstrate that the quantities on which C_R depends are independent of $t \in [0,1]$ and $\zeta \in \partial U \times M$.

It will suffice to uniformly control $\left\|\widetilde{\rho}_{t,j}\right\|_{C^3}$ and $|\nabla\rho_{t,j}|$, since n and R are fixed. Further, since $\tilde{u}_j$, θ_ζ^{-1}, and $\widetilde{\chi}$ are smooth on compact sets and J is finite, we need only worry about $\left\|t\psi_F\right\|_{C^3}$ and $|\nabla\varphi_t|$. We know from Lemma 3.1.1 that $\left\|t\psi_F\right\|_{C^3} \leq C_1\left\|F\right\|_{C^4}$ for all $t \in [0,1]$. Meanwhile, as it did for $|\varphi_t|$ above, the comparison argument of Proposition 3.2.2 shows that $|\nabla\varphi_t|$ is controlled by $|\nabla\widetilde{F}|$ and $|\nabla\psi_F|$ for all $t \in [0,1]$, which are in turn controlled by $\|F\|_{C^4}$.

Thus for any $\zeta \in \partial U \times M$, we have $|(D^2\rho_{t,j})(\theta_\zeta(\zeta))| \leq C_R$. Since $\tilde{u}_j$ and θ_ζ are smooth on $\overline{U}_j$ and J is finite, this will yield a uniform bound on $\Delta\varphi_t(\zeta)$ for $\zeta \in \partial U \times M$ and $t \in [0,1]$. So, finally, we have $C_F > 0$ such that $|\nabla\varphi_t| \leq C_F$ for all $t \in [0,1]$.

This estimate transfers to solutions of the homogeneous problem $\mathrm{MA}(tF)$ for $t \in [0,1]$, by a very similar argument to that in Proposition 3.2.2.

■

5.4 Proof of Theorem 5.0.1

Proof Take F as in the statement of the theorem and $k \in \mathbb{Z}^+$. We have a collection $\mathcal{L}(F)$ of leaf functions and a corresponding collection $\mathcal{G}(F)$ of lifts $G : \overline{U} \to Z := S \times \mathbb{C} \times X(\omega)$. The distance estimate from Proposition 5.3.1 implies that there is some compact set $K \subset Z$ such that $G(\overline{U}) \subset K$ for all $G \in \mathcal{G}(F)$. If necessary, expand K to contain the compact set $\Lambda' := \Sigma([0,1] \times \partial U \times M) \subset \Lambda$.

From Theorem 5.2.1, since Λ is a totally real, real analytic submanifold of $Z = S \times \mathbb{C} \times M$, there is an open set $N \subset Z$ about Λ and an anti-biholomorphic involution $\nu : N \to N$ which restricts to the identity on Λ.

We choose a cover of $\Lambda' \subset K$ by relatively compact open sets inside $N \subset Z$, which we require to be invariant under ν. Adding additional open sets disjoint from Λ' yields a cover of K; let $\mathcal{V}$ be a finite sub-cover. We may assume that for each $V \in \mathcal{V}$, there are holomorphic coordinates $y_V = (y_V^1, \ldots, y_V^m)$ for Z on a neighborhood of $\overline{V}$. By compactness, we can take a constant

$$C_2 \;=\; \max\{|y_V(w)| : V \in \mathcal{V}, w \in V\}. \tag{5.7}$$

The Lebesgue number lemma gives an $R > 0$ such that for any $\zeta \in K$, the R-neighborhood of ζ is contained in some $V \in \mathcal{V}$. Using Proposition 5.3.1 again, we may choose $r > 0$ so that for every $z \in \overline{U}$, $G(\overline{U} \cap D_{2r}(z))$ lands inside the R-neighborhood of $G(z)$, whenever $G \in \mathcal{G}(F)$. (E.g. take $r = R^4/(2C_1)^4$.)

We can now estimate the derivatives of $G \in \mathcal{G}(F)$ up to order k at $z \in \overline{U}$ by considering two cases. In the first case, the curve $\partial D_r(z)$ stays inside $\overline{U}$ (i.e. $|z| \leq 1 - r$). Choose $V \in \mathcal{V}$ so that $G(D_r(z)) \subset V$. Then the Cauchy estimate for the holomorphic function $y_V \circ G$ at $\zeta \in D_r(z)$ gives

$$\left|D^j(y_V \circ G(\zeta))\right| \;\leq\; j!\, r^{-j} \max_{w \in \overline{V}} |y_V(w)| \;\leq\; k!\, r^{-k} C_2 \tag{5.8}$$

for any $j \leq k$, using (5.7).

On the other hand, suppose $|z| > 1 - r$. In that case, $\partial D_r(z)$ crosses ∂U. It follows that the R-neighborhood of $g(z)$ lies inside a $V \in \mathcal{V}$ which intersects Λ'. By construction, $V \subset N$ is invariant under ν. Thus for each $G \in \mathcal{G}(F)$ we may define a holomorphic extension

$$\widetilde{G}(\zeta) = \begin{cases} y_V \circ G(\zeta) & \text{if } \zeta \in \overline{U} \cap D_{2r}(z) \\ y_V \circ \nu \circ G(1/\bar{\zeta}) & \text{if } \zeta \in D_{2r}(z) \setminus U \end{cases}$$

of $y_V \circ G$ to $D_{2r}(z)$. $\widetilde{G}$ is continuous on $\partial U \cap D_{2r}(z)$ since its two sub-functions agree there (recall ν is the identity on Λ). Further, it is holomorphic on $D_{2r}(z) \setminus \partial U$, and thus holomorphic on $D_{2r}(z)$ (using Morera's Theorem).

Since $\nu(V) \subset V$, the function $\widetilde{G}$ is also bounded by C_2. Thus the Cauchy estimate again gives

$$\left| D^j(y_V \circ G(\zeta)) \right| \;\leq\; k!\, r^{-k} C_2 \tag{5.9}$$

for any $j \leq k$ and $\zeta \in D_r(z)$.

It remains to recover an estimate on the derivatives up to order k of the leaf functions themselves. For this, we use the inverse of the local coordinates on V (which we call x_V) and the projection $p : Z = S \times \mathbb{C} \times X(\omega) \to M$. Applying $\Theta \circ p \circ x_V$ to $y_V \circ G$ yields $\Theta \circ f$. Since Θ is smooth on the compact manifold M, we obtain a $C_k > 0$ independent of f and $z \in \overline{U}$ so that

$$\left| D^j(\Theta \circ f(z)) \right| \leq C_k$$

for $j = 1, \ldots, k$, as desired. ∎

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