CONNECTIVE BIEBERBACH GROUPS

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Teacher, Grandmother, and Unapologetic Math Lover

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ABSTRACT

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Connectivity is a lifting property of C^*-algebras originally isolated by Dadarlat and Pennig in 2017. In their remarkable paper, they showed that connectivity completely characterizes the separable nuclear C^*-algebras whose E-theory may be unsuspended; that is, A is connective if and only if

$$E(A,B) := [[\mathcal{S}A, \mathcal{S}B \otimes \mathcal{K}]] \cong [[A, B \otimes \mathcal{K}]]$$

for all separable C^*-algebras B. Connectivity offers a wealth of permanence properties including passing to C^*-subalgebras and split extensions — properties not obvious when viewed from a purely E-theoretical perspective. By the contributions of Dadarlat and Pennig and Gabe, connectivity of a C^*-algebra A was also shown to be equivalent to its primitive spectrum containing no non-empty compact open subsets. Although investigating the primitive spectrum is still a challenge, this does provide a testable criterion for connectivity.

Bieberbach groups, of independent interest in physics and chemistry, are exactly the fundamental groups of flat compact Riemannian manifolds. Abstractly, these are torsion free groups fitting into an exact sequence of the form

$$1 \to \mathbb{Z}^n \to G \to D \to 1$$

where $|D| < \infty$. The unitary dual (equivalently the primitive spectrum of the associated group C^*-algebra) of Bieberbach groups benefit from Mackey's machine which allows us to build the unitary dual from "smaller" representations of subgroups. This means that, by careful investigation, we may determine under what conditions the topology of the unitary dual contains non-empty compact open subsets.

The primary result of this document is Theorem 6.1 which shows that if a Bieberbach group has finite abelianization, then its unitary dual contains a compact open subset and thus is not connective. The proof of this theorem is a generalization of a result in Dadarlat and Pennig's paper [1] which shows that the Hantzsche-Wendt group is not connective. Combining this result with other work on Bieberbach groups, we determine that a Bieberbach group G is connective if and only if no non-trivial subgroup of G has finite abelianization.

CHAPTER 1. INTRODUCTION

The E-theory of Connes and Higson is a powerful C^*-homological theory equivalent to KK-theory in the separable nuclear case. E-theory replaces Kasparov-Fredholm C^*-modules with asymptotic morphisms (families of norm-continuous functions indexed by $[0, \infty)$ that approximate $*$-homomorphisms asymptotically) which are arguably more intuitive and practical objects. If we let $[[C, D]]$ represent the asymptotic morphisms from C to D up to a suitable notion of homotopy, then we define the E-theory from A to B by

$$E(A, B) = [[\mathcal{S}A, \mathcal{S}B \otimes \mathcal{K}]]$$

where $\mathcal{S}A$, $\mathcal{S}B$ are the suspensions of A, B and $\mathcal{K}$ is the compact operators on some infinite dimensional separable Hilbert space. As with any homological theory, we want to assign C^*-algebras to abelian groups. In general, $[[A, B \otimes \mathcal{K}]]$ is not a group and thus the suspensions become necessary. But this makes recovering information more directly related to A or B difficult as the suspension will, naturally, alter many of their properties. This raises the motivating question of this document: when may we *unsuspend* the E-theory of a C^*-algebra A?

This question was addressed in 1994 by Dadarlat and Loring when they showed that unsuspension was equivalent to the notion of *homotopy symmetry*, which is exactly the property that $[[\mathrm{id}_A]]$ has an inverse in $[[A, A \otimes \mathcal{K}]]$ [2]. Although elegant, this is a difficult property to check in practice and so, in 2017, Dadarlat and Pennig demonstrated another equivalence: connectivity [3]. Connectivity is a lifting property where we say that a separable C^*-algebra is *connective* if there is an injective $*$-homomorphism such that

$$\phi : A \to \frac{\prod_{n=1}^{\infty} \mathcal{C}B(\mathcal{H})}{\bigoplus_{n=1}^{\infty} \mathcal{C}B(\mathcal{H})}$$

lifts to a completely positive contraction $\Phi : A \to \prod_{n=1}^{\infty} \mathcal{C}B(\mathcal{H})$ where $\mathcal{C}B = C_0(0,1] \otimes B$ is the *cone of* B. Amazingly, the connectivity of a C^*-algebra A guarantees that for any separable C^*-algebra B,

$$E(A,B) \cong [[A, B \otimes \mathcal{K}]]$$

is a group.

Although the connection to unsuspension is not intuitively clear, this lifting property does provide insight into our motivating question. A first observation from the definition of connectivity is that a connective C^*-algebra contains no non-zero projections. This property *almost* characterizes connectivity as lacking non-zero projections is necessary but not quite sufficient. However, observe that by Pasnicu and Rørdam $A \otimes \mathcal{O}_\infty$ is projectionless if and only if the primitive spectrum of A contains no non-empty compact open subsets [4]. The work of Dadarlat and Pennig [1] and Gabe [5] confirms that this topological property of the primitive spectrum is equivalent to connectivity. Thus, we need not only A to lack non-zero projections but $A \otimes \mathcal{O}_\infty$ as well. As an example of a C^*-algebra A which is projectionless but whose spectrum is compact open (and hence $A \otimes \mathcal{O}_\infty$ contains a non-zero projection), consider

$$A = \left\{ f \in C([0,1], M_2(\mathbb{C})) \mid f(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \lambda \in \mathbb{C} \right\}.$$

In the case of groups, this topological characterization is especially helpful. We say a group is connective if the kernel of its trivial representation (called the augmentation ideal) is connective. Using the deep connection between a group's unitary dual and the representation space of its C^*-algebra, we then have a criterion for detecting connectivity when there exist methods for carefully studying a group's irreducible representations. Virtually abelian groups, of which Bieberbach groups are an especially well-studied subclass, are an example of groups with better understood unitary duals. Using Mackey's machine, we investigate the unitary duals of Bieberbach groups and determine under what circumstances do the primitive spectra of their augmentation ideals contain non-empty compact open subsets. This leads to yet another characterization and one of the main results of this document: a Bieberbach group is connective if and only if none of its nontrivial subgroups have finite abelianization.

To add context to the results of this document, there are several chapters of background.

Chapter 2 contains the very basics of Hilbert spaces and C^*-algebras while Chapter 3 addresses the unitary duals of groups and their relationship to the spectra of group C^*-algebras.

Switching topics in Chapter 4, we discuss E-theory and connectivity for separable nuclear C^*-algebras. While the results pertaining to connectivity are of the most importance to this document, E-theory needs some context to be appreciated. For this reason, there is brief discussion of related homological theories — specifically K-theory, K-homology, and KK-theory.

Bieberbach groups are covered in detail in Chapter 5 with particular importance placed on their unitary duals.

Finally, Chapter 6 provides a criteria to check which Bieberbach groups are connective. The main result (Theorem 6.1) is shown in the first section. The second section uses this and other results to provide a characterization of connective Bieberbach groups. The last section investigates under what conditions the point group of a Bieberbach group determines connectivity.

CHAPTER 2. A BASIC INTRODUCTION TO C^*-ALGEBRAS

To begin, we introduce some functional analysis and Hilbert space theory and then state without proof important definitions and theorems for C^*-algebras. Although not every concept or result provided in this chapter will be explicitly used, the included material aims to provide a more complete view of the mathematical foundation of operator algebras. This is still, of course, only scratching the surface. For a more in-depth treatment of all of the topics covered and more related ideas, consider Conways' *A Course in Functional Analysis* [6], Murphy's *C^*-algebras and Operator Theory* [7], Dixmier's *C^*-algebras* [8], or Blackadar's *Operator Algebras: Theory of C^*-algebras and von Neumann Algebras* [9].

For the purposes of this chapter, all vector spaces, Hilbert spaces, and algebras will be over the complex numbers, $\mathbb{C}$.

2.1 Hilbert Spaces

In this section we will focus on the basics of Hilbert spaces. We begin with a defining aspect of these spaces: inner products.

Definition 2.1. For a vector space V, an **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $\lambda \in \mathbb{C}$, $u, v, w \in V$,

(a) $\langle u + \lambda v, w \rangle = \langle u, w \rangle + \lambda \langle v, w \rangle$

(b) $\langle u, v + \lambda w \rangle = \langle u, v \rangle + \overline{\lambda} \langle u, w \rangle$

(c) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

(d) $\langle u, u \rangle \geq 0$

(e) $\langle u, u \rangle = 0$ if and only if $u = 0$

Dot products are an elementary example of inner products and many of the properties we expect from dot products also hold true for inner products.

Theorem 2.2 (Cauchy-Schwarz Inequality (See [6, I.1.4]))**.** For $\langle \cdot, \cdot \rangle$ an inner product on a vector space V, we have for all $u, v \in V$

$$|\langle u, v\rangle| \leq \langle u, u\rangle \langle v, v\rangle.$$

An inner product comes with an associated norm, $\|\cdot\|$, that makes the given vector space into a normed space satisfying the following.

Corollary 2.3 ([6, I.1.4])**.** Define $\|u\| = \langle u, u\rangle^{1/2}$ for all $u \in V$. Then for all $u, v \in V$ and $\lambda \in \mathbb{C}$,

(a) $\|u + v\| \leq \|u\| + \|v\|$

(b) $\|\lambda u\| = |\lambda| \, \|u\|$

(c) $\|u\| = 0$ if and only if $u = 0$

We use $\|\cdot\|$ to induce a metric on V: for all $u, v \in V$,

$$d(u, v) = \|u - v\|.$$

We can now define a Hilbert space.

Definition 2.4. A **Hilbert space** is a vector space $\mathcal{H}$ over $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathcal{H}$ is a complete metric space relative to the metric $d(u, v) = \|u - v\|$ induced by the norm $\|u\| = \langle u, u\rangle^{1/2}$.

Example 2.5 (Examples of Hilbert Spaces)**.**

(a) $\mathbb{C}^n$ equipped with the usual dot product.

(b) Let (X, Ω, μ) be a measure space for X a set, Ω a σ-algebra of subsets of X, and μ a countable additive measure defined on Ω taking values in $\mathbb{R} \cup \{\infty\}$. Then

$L^2(\mu) := L^2(X, \Omega, \mu)$ is the set of square integrable functions in (X, Ω, μ); that is, $L^2(\mu)$ is the collection of measurable functions f such that

$$\int_X |f|^2 \, d\mu < \infty.$$

We can define an inner product on $L^2(\mu)$ by

$$\langle f, g \rangle = \int_X f \, \overline{g} \, d\mu$$

with associated norm

$$\|f\|_2 = \left(\int_X |f|^2 \, d\mu \right)^{1/2}.$$

Using measure theoretic results, it can be shown that $L^2(\mu)$ is a complete metric space with the induced metric and thus a Hilbert space.

(c) For any set S, denote by $\ell^2(S)$ the the collection of functions $f : S \to \mathbb{C}$ such that for all but a countable number of $s \in S$, $f(s) = 0$ and $\sum_{s \in S} |f(s)|^2 < \infty$. We define an inner product on $\ell^2(S)$ by

$$\langle f, g \rangle = \sum_{s \in S} f(s) \, \overline{g(s)}.$$

Showing that $\ell^2(S)$ is a Hilbert space is a routine exercise.

Now, one of the most useful features of Hilbert spaces is the notion of orthogonality, which extends the Euclidean notion of perpendicular vectors. Orthogonality is the idea that, with respect to the given inner product, two elements or sets do not overlap.

Definition 2.6. Let $\mathcal{H}$ be a Hilbert space. For $u, v \in \mathcal{H}$, we say that u, v are **orthogonal**, denoted $u \perp v$, if $\langle u, v \rangle = 0$. If $A, B \subseteq \mathcal{H}$, then we say $A \perp B$ if $f \perp g$ for all $f \in A$, $g \in B$.

We use orthogonality to define a basis for our Hilbert spaces and, consequently, a notion of dimension analogous to the vector space case.

Definition 2.7. E is an **orthonormal** subset of a Hilbert space $\mathcal{H}$ if for all $e \in E$, $\|e\| = 1$ and for all $e_1, e_2 \in E \subseteq \mathcal{H}$ with $e_1 \neq e_2$, $e_1 \perp e_2$. E is a **basis** for $\mathcal{H}$ if it is a maximal orthonomal set. The **dimension** of a Hilbert space $\mathcal{H}$, denoted $\dim \mathcal{H}$, is the cardinality of any basis of $\mathcal{H}$.

Of course, just as with vector spaces, we want dimension to be well-defined. Consider the following proposition.

Proposition 2.8 ([6, Prop I.4.14])**.** Any two bases of a Hilbert space $\mathcal{H}$ have the same cardinality.

When the basis of $\mathcal{H}$ is countable, we call $\mathcal{H}$ **separable** [6, Prop I.4.16]. Many results are often specifically stated for separable Hilbert spaces because countable objects typically provide fewer set-theoretic considerations. In addition, separable infinite dimensional Hilbert spaces are all equivalent (see Theorem 2.10) and thus can be addressed simply as "the Hilbert space". There are examples of non-separable Hilbert spaces but they will not be discussed here. Instead, we will now define what it means for two Hilbert spaces to be equivalent.

Definition 2.9. We say two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are **isomorphic** if there exists a surjective linear $U : \mathcal{H} \to \mathcal{K}$ such that for all $v, w \in \mathcal{H}$,

$$\langle Uv, Uw \rangle = \langle v, w \rangle.$$

Such a map U is said to be an **isomorphism** of Hilbert spaces.

In a result again analogous to the vector space case, we have the following theorem.

Theorem 2.10 ([6, Thm I.5.4])**.** Two Hilbert spaces are isomorphic if and only if they have the same dimension.

We can also form direct sums of Hilbert spaces which we include here for completeness.

Definition 2.11. Let $\mathcal{H}_1, \mathcal{H}_2, ...$ be Hilbert spaces. We define the **direct sum** of the $\mathcal{H}_i$ to be the set

$$\mathcal{H} = \left\{ (h_i)_{i=1}^{\infty} \;\middle|\; h_n \in \mathcal{H}_i \ \forall i \ \text{ and } \ \sum_{i=1}^{\infty} \|h_i\|^2 < \infty \right\}.$$

For $h = (h_i)_i$ and $g = (g_i)_i$ in $\mathcal{H}$, we define the inner product on $\mathcal{H}$ by

$$\langle h, g \rangle = \sum_{i=1}^{\infty} \langle h_i, g_i \rangle_i$$

where $\langle \cdot, \cdot \rangle_i$ is the inner product on $\mathcal{H}_i$. We denote the direct sum of the $\mathcal{H}_i$ by

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots .$$

Proposition 2.12 ([6, Prop I.6.2])**.** If $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, then $\mathcal{H}$ is a Hilbert space.

C^*-algebras are intimately connected with the space of continuous linear maps between Hilbert spaces. The specific connection will be addressed in the next section but we include some results here.

A **linear transformation** is a linear map $L : \mathcal{H} \to \mathcal{K}$ for $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces. If $\mathcal{K} = \mathbb{C}$, we call L a **linear functional** or a **linear form**. We call $L : \mathcal{H} \to \mathcal{H}$ a **linear operator**.

Definition 2.13. We say a linear transformation L on a Hilbert space $\mathcal{H}$ is **bounded** if there exists a constant $c > 0$ such that for all $u \in \mathcal{H}$, $|L(u)| \leq c\,\|u\|$. If L is bounded, then we can define the **norm** of L by

$$\|L\| = \sup\{\, |L(u)| : \|u\| \leq 1\}.$$

In a classic result, we see that the "size" of a linear transformation completely determines its continuity.

Proposition 2.14 ([6, Prop II.1.1])**.** For $\mathcal{H}$, $\mathcal{K}$ Hilbert spaces and $L : \mathcal{H} \to \mathcal{K}$ a linear map, the following are equivalent:

(a) L is continuous

(b) L is continuous at 0

(c) L is continuous at some point $\xi \in \mathcal{H}$

(d) L is bounded

Definition 2.15. $B(\mathcal{H}, \mathcal{K})$ is the set of all bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$. We call $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$ the set of **bounded linear operators**.

The importance of $B(\mathcal{H})$ to the theory of C^*-algebras cannot be overstated, as will be shown in the next section.

Definition 2.16. For $L \in B(\mathcal{H}, \mathcal{K})$, we define $L^* \in B(\mathcal{K}, \mathcal{H})$ by

$$\langle Lu, v\rangle = \langle u, L^* v\rangle$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{K}$. We call L^* the **adjoint** of L.

The adjoint of L is unique [6, Theorem II.2.2]. We call a subset A of $B(\mathcal{H})$ **self-adjoint** if for any $L \in A$, we also have $L^* \in A$.

Proposition 2.17 ([6, Prop II.2.6, Prop II.2.7])**.** For all $L_1, L_2 \in B(\mathcal{H})$ and $\lambda \in \mathbb{C}$,

(a) $(\lambda L_1 + L_2)^* = \overline{\lambda}\, L_1^* + L_2^*$

(b) $(L_1 L_2)^* = L_2^* L_1^*$

(c) $(L_1^*)^* = L_1^{**} = L_1$

(d) $\|L_1\| = \|L_1^*\| = \|L_1^* L_1\|^{1/2}$

This completes our explicit treatment of Hilbert spaces and we move on to C^*-algebras.

2.2 C^*-Algebras

In this section we present some basic properties of C^*-algebras.

Definition 2.18. An **involution** on an algebra A is a map $* : A \to A$ such that for all $x, y \in A$, $\lambda \in \mathbb{C}$,

(i) $(x^*)^* = x$

(ii) $(x + y)^* = x^* + y^*$

(iii) $(\lambda\, x)^* = \overline{\lambda}\, x^*$

(iv) $(x\, y)^* = y^*\, x^*$

We call x^* the **adjoint** of x.

Involution should be thought of as an analogue to complex conjugation.

Definition 2.19. If A is an algebra over $\mathbb{C}$ endowed with an involution, we say that A is a $*$**-algebra**.

Observe that adjoints (Definition 2.16) act as the involution on $B(\mathcal{H})$ making it a $*$-algebra under composition.

Definition 2.20. We say a norm $\|\cdot\|$ is **sub-multiplicative** if, for all $a, b \in A$,

$$\|a \cdot b\| \leq \|a\|\,\|b\|.$$

Sub-multiplicativity ensures that the norm and multiplication in A are compatible. We can now define our primary object of interest object: a C^*-algebra.

Definition 2.21. A C^***-algebra** A is an algebra which is complete under a sub-multiplicative norm $\|\cdot\|$ and endowed with an involution $*$ such that for all $x \in A$, $\|x^*\| = \|x\|$ and

$$\|x\|^2 = \|x^* x\|. \tag{2.1}$$

Equation (2.1) ensures any norm which makes an involutive algebra into a C^*-algebra is unique [7, Cor 2.1.1]. A closed $*$-subalgebra of a C^*-algebra is again a C^*-algebra, which we call a **C^*-subalgebra**. We say that A is **unital** if there exists an **identity** $1_A \in A$ such that for all $a \in A$,

$$a \cdot 1_A = a = 1_A \cdot a.$$

Examples 2.22 (Examples of C^*-algebras)**.**

(a) $\mathbb{C}$ is a unital C^*-algebra with complex conjugation as involution, $\|u\| = (u\,\overline{u})^{1/2} = |u|$ as the norm, and $1 \in \mathbb{C}$ as the identity.

(b) For $n \geq 1$, $M_n(\mathbb{C})$ is a unital C^*-algebra for conjugate transpose as involution, the usual matrix norm, and the identity matrix I_n as the identity.

(c) Suppose X is a Hausdorff space. Define

$$C_0(X) = \{f : X \to \mathbb{C} \,|\, f \text{ is continuous, vanishes at } \infty\}.$$

By "vanishes at ∞", we mean that for all $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq X$ such that $|f(K_\varepsilon)| < \varepsilon$.

If X is compact in addition to Hausdorff, then $C_0(X)$ reduces to the set of continuous functions on X:

$$C_0(X) = C(X) = \{f : X \to \mathbb{C} \,|\, f \text{ is continuous}\}.$$

These are both C^*-algebras with complex conjugation as involution and the usual norm. Observe that $C(X)$ is unital but $C_0(X)$ is not (when X is not compact).

(d) $B(\mathcal{H})$ is a C^*-algebra as are any of its self-adjoint subalgebras. Here, involution is the adjoint and the norm is the usual operator norm. Although $B(\mathcal{H})$ is unital, an arbitrary self-adjoint subalgebra of $B(\mathcal{H})$ need not be.

An especially important self-adjoint subalgebra of $B(\mathcal{H})$ is the collection of *compact operators* of $\mathcal{H}$ which we will denote $\mathcal{K}$. We say $T \in B(\mathcal{H})$ is **compact** if $\overline{T(\mathcal{H})}$ is compact in $\mathcal{H}$. Every compact operator is the norm limit of a sequence of operators whose range is finite dimensional [6, Thm 4.4].

(e) $L^2(\mu)$ is a unital C^*-algebra with involution given by complex conjugation and the norm arising from the inner product. The identity is the constant function $f \equiv 1$.

Although a C^*-algebra A may not be unital, we can uniquely extend A to a unital C^*-algebra which we call the **unitization** of A, denoted $\widetilde{A}$. Consider the set $\widetilde{A} = A \oplus \mathbb{C}$, viewed as a vector space. We make $\widetilde{A}$ an algebra with unit $1_{\widetilde{A}} = (0, 1)$ by multiplication defined via

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

Moreover, we may define involution by $(a, \lambda)^* = (a^*, \overline{\lambda})$ and norm by

$$\|(a, \lambda)\| = \sup\{\, \|ab + \lambda b\| \,:\, \|b\| = 1\}.$$

The map $A \to \widetilde{A}$ given by $a \mapsto (a, 0)$ is an injective homomorphism which identifies A as an ideal of $\widetilde{A}$. In a sense, $\widetilde{A}$ is the one point compactification of A.

Definition 2.23. If A, B are C^*-algebras, a $*$**-homomorphism** $\phi : A \to B$ is a linear multiplicative map such that for all $a \in A$, $\phi(a^*) = \phi(a)^*$. If ϕ is a bijection, then we say A and B are **isomorphic**, which we denote $A \cong B$. If, for all $a \in A$, $\|a\| = \|\phi(a)\|$, we say that ϕ is **isometric**.

C^*-algebras may be viewed as a non-commutative analogue to a topological Hausdorff space, an idea formulated more precisely by the following theorem.

Theorem 2.24 ([7, Thm 2.1.10])**.** If A is a commutative C^*-algebra, then there exists a Hausdorff space X such that A is isometrically isomorphic to $C_0(X)$.

Every C^*-algebra as defined in this section can be viewed concretely as a self-adjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Theorem 2.25 ([6, Thm VIII.5.17])**.** If A is a C^*-algebra, there exists an isometric $*$-homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Thus, there is no real difference between abstract C^*-algebras as per Definition 2.21 and self-adjoint subalgebras of $B(\mathcal{H})$ (which we might term *concrete* C^*-algebras). We next address some classes and properties of C^*-algebras.

Definition 2.26. We say a C^*-algebra A is **separable** if its underlying topological space contains a countable dense subset.

Note 2.27 ([6, Thm VIII.5.17]). By Theorem 2.25, we can represent a C^*-algebra A as a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If A is separable, we may choose $\mathcal{H}$ to also be separable.

Although many (but not all) results may be formulated for both the separable and nonseparable case, separable C^*-algebras by definition have better behaved topological properties. As such, separability is preferred even if it is not necessary. Fortunately, assuming separability is not particularly restrictive and so we consider only the separable case without serious consequences.

Definition 2.28. We say an element $a \in A$ a C^*-algebra is **positive** if there exists $x \in A$ such that $a = x^*x$ and we denote the set of positive elements of A by A_+.

As the name implies, positive elements mimic the elementary notion of positive real numbers. Positivity also extends to functions.

Definition 2.29.

(a) For A and B both C^*-algebras, we say a linear function $\phi : A \to B$ is **positive** if $\phi(A_+) \subseteq B_+$.

(b) Let $M_n(A), M_n(B)$ be $n \times n$ matrices with entries in A, B respectively for some $n \geq 1$. Define $\phi^{(n)} : M_n(A) \to M_n(B)$ by applying ϕ to entry-wise. If $\phi^{(n)}$ is positive, we say that ϕ is n**-positive**.

(c) Any ϕ which is positive for all $n \geq 1$ is called **completely positive**.

(d) A positive linear functional of norm 1 is called a **state**.

Any $*$-homomorphism between C^*-algebras $\phi : A \to B$ is a contraction, that is, $\|\phi(x)\| \leq \|x\|$ for all $x \in A$ [8, Prop 1.3.7]. This brings us to an important class of functions which will be seen again in Section 4.4.

Definition 2.30. For C^*-algebras A and B, $\phi : A \to B$ is a **completely positive contraction (cpc)** if ϕ is completely positive and a contraction. If ϕ is completely positive and $\phi(1_A) = 1_B$, we call ϕ **unital completely positive (ucp)**.

We now introduce *nuclearity*, which defines a particularly well-behaved class of C^*-algebras. As a testament to its importance, nuclearity has many equivalent formulations. Since the specific choice of definition will have no effect on the goals of this document, I will provide the definition which I view to be most elementary. A more thorough treatment of nuclearity may be found in [9].

Definition 2.31. For C^*-algebras A and B, a cpc map $\phi : A \to B$ is **nuclear** if for any $x_1, x_2, ..., x_k \in A$ and $\varepsilon > 0$, there exists n and cpc maps

$$\alpha : A \to M_n(\mathbb{C}), \quad \beta : M_n(\mathbb{C}) \to A$$

such that $\|x_j - \beta \circ \alpha(x_j)\| < \varepsilon$ for $1 \leq j \leq k$. We say that A is **nuclear** if $\mathrm{id}_A : A \to A$ is nuclear.

In essence, the identity map of a nuclear $C*$-algebra approximately factors through matrices.

One important class of examples of nuclear C^*-algebras are the *Cuntz algebras*, $\mathcal{O}_n$. These are the C^*-algebras generated by the set $\{V_1, ..., V_n\}$ subject to the relations

$$V_i^* V_i = 1$$
$$\sum_{i=1}^{n} V_i V_i^* = 1$$

If we replace $\{V_1, ..., V_n\}$ by a sequence $\{V_i\}_i$ subject to an appropriately altered summation, we define $\mathcal{O}_\infty$. The Cuntz algebras appear in many contexts and, in ours, has implications for the primitive spectrum of our C^*-algebras.

As a final comment on nuclearity, I wanted to draw attention to an important application.

Theorem 2.32 (Choi-Effros, [10, Thm 3.10])**.** Suppose A is a separable nuclear C^*-algebra and let B be a C^*-algebra containing a closed two-sided ideal K. If $\phi : A \to B/K$ is cpc, then there exists a cpc $\Phi : A \to B$ such that $\Phi(a) = \phi(a)$ for all $a \in A$.

Put more simply, every cpc map from a separable nuclear C^*-algebra A into a quotient B/K lifts to a cpc mapping from A into B.

The final subject of this section will be definitions that will be needed in Section 4.3.

Definition 2.33. We say a C^*-algebra A is **σ-unital** if there exists a positive element $h \in A_+$ such that hA is dense in A.

Definition 2.34. A **graded C^*-algebra** A is a C^*-algebra equipped with a $*$-automorphism β_A such that $\beta_A^2 = \mathrm{id}_A$. We call β_A the **grading automorphism** of A and say that A is **graded** by β_A.

Remark 2.35 ([11, p. 25]). If A is graded by β_A, we can decompose $A = A_1 \oplus A_2$ into a Banach space decomposition where

$$A_0 = \{\, a \in A \mid \beta_A(a) = a \,\}$$

and

$$A_1 = \{\, a \in A \mid \beta_A(a) = -a \,\}.$$

We say $a \in A_0$ are **degree 0**, denoted $\deg(a) = 0$, and $a \in A_1$ are **degree 1**, denoted $\deg(a) = 1$.

We may always add the trivial grading to a C^*-algebra A by taking $\beta_A = \mathrm{id}_A$, $A_0 = A$, and $A_1 = 0$.

Definition 2.36. Let A and B be graded C^*-algebras. A **graded homomorphism** $\phi : A \to B$ is a $*$-homomorphism such that $\phi \circ \beta_A = \beta_B \circ \phi$.

For a Hilbert A-module E_A (to be seen in Section 4.3) we define a **grading operator** $S_{E_A} : E_A \to E_A$ to be a linear bijection satisfying

$$S_{E_A}(e\, a) = S_{E_A}(e)\beta_A(a), \quad \langle S_{E_A}e, S_{E_A}f\rangle = \beta_A(\langle e, f\rangle), \quad S_{E_A}^2 = \mathrm{id}_{E_A}$$

for all $e, f \in E_A$, $a \in A$.

CHAPTER 3. REPRESENTATIONS OF GROUPS AND C^*-ALGEBRAS

3.1 The Unitary Dual of a Group

This section comes from Dixmier's C^*-*algebras* [8] and Kaniuth and Taylor's *Induced Representations of Locally Compact Groups* [12].

Definition 3.1. A **topological group** is a group endowed with a topology such that the group operations $\cdot : G \times G \to G$ and ${}^{-1} : G \to G$ given by

$$x \cdot y \mapsto xy \quad \text{and} \quad x \mapsto x^{-1}$$

are continuous.

A **discrete group** is a topological group endowed with the discrete topology. Although many results apply more generally to locally compact groups (groups with a locally compact Hausdorff topology), our primary interest will be the discrete case.

Let $U(\mathcal{H})$ be the **unitary group** of some Hilbert space $\mathcal{H}$ defined by

$$U(\mathcal{H}) = \{\, T \in B(\mathcal{H}) \mid T^*T = I = TT^* \,\}.$$

Definition 3.2. A **unitary representation** of a group G is a group homomorphism $\pi : G \to U(\mathcal{H}_\pi)$ such that for all $\xi \in \mathcal{H}_\pi$, $s \mapsto \pi(s)\xi$ is norm continuous. We say the **dimension** of π is the Hilbert space dimension of $\mathcal{H}_\pi$.

When G is discrete, continuity is obviously automatic.

Definition 3.3. Let V be a subspace of $\mathcal{H}_\pi$. We say V is π**-invariant** or an **invariant subspace** if for all $s \in G$ and $\xi \in V$, we have $\pi(s)\xi \in V$.

Note that the orthogonal complement, $V^\perp$, of any invariant subspace V is itself an invariant subspace.

If K is a closed invariant subspace of $\mathcal{H}_\pi$, we define a **subrepresentation** of π by $\pi_K(s) = (\pi|_K)(s)$ for all $s \in G$. π_K is a representation of G on $K = \mathcal{H}_{\pi_K}$.

Definition 3.4. Let $\{\pi_\alpha\}_{\alpha\in I}$ be a collection of representations of G on Hilbert spaces $\{\mathcal{H}_{\pi_\alpha}\}_{\alpha\in I}$. If $\mathcal{H} = \bigoplus_{\alpha\in I}\mathcal{H}_{\pi_\alpha}$ is the Hilbert space direct sum, then we define the **direct sum** of π_α for $\alpha \in I$ by

$$\left(\left(\bigoplus_{\alpha\in I}\pi_\alpha\right)(s)\right)(\xi_\alpha)_{\alpha\in I} = \left(\pi_\alpha(s)(\xi_\alpha)\right)_{\alpha\in I}$$

for all $s \in G$ and $(\xi_\alpha)_\alpha \in \bigoplus_{\alpha\in I}\mathcal{H}_{\pi_\alpha}$.

If $\pi_\alpha = \sigma$ for all $\alpha \in I$, then for ease of notation we write

$$\bigoplus_{\alpha\in I}\pi_\alpha = m\cdot\sigma$$

where $m = \operatorname{card}(I)$.

Definition 3.5. A representation π is **irreducible** if the only closed invariant subspaces of $\mathcal{H}_\pi$ are $\{0\}$ and $\mathcal{H}_\pi$.

The importance of irreducible representations is that they are the representations which cannot be decomposed any further as a direct sum of subrepresentations. In this sense, irreducible representations are the "smallest" a representation can be.

Definition 3.6. If χ is a representation of G of dimension one, we call χ a **character**.

Note 3.7 ([13, p. 184]). When G is abelian, all representations of G are one dimensional.

Because we are representing groups on Hilbert spaces, we don't want to count two representations as different if they only differ by a change of basis. So we have the following definition.

Definition 3.8. Let π and σ be representations of G. We say π and σ are **unitarily equivalent** if there exists a unitary $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that $U\pi(s) = \sigma(s)U$ for all $s \in G$.

Thus, we view representations who differ up to an orthonormal change of basis as equivalent.

Proposition 3.9 ([12, Prop 1.33])**.** Let π be a representation of G such that $\{K_\alpha \,|\, \alpha \in I\}$ is a family of closed invariant subspaces of $\mathcal{H}_\pi$ where the K_α are pairwise orthogonal and $\bigcup_{\alpha \in I} K_\alpha$ has a dense linear span in $\mathcal{H}_\pi$. Then π is equivalent to $\bigoplus_{\alpha \in I} \pi_{K_\alpha}$.

Hence, every representation of G can be decomposed into a direct sum of irreducible representations. We also have the following helpful characterization of irreducibility.

Proposition 3.10 ([12, Prop 1.35])**.** When G is locally compact, a representation π of G is irreducible if and only if the only $T \in B(\mathcal{H}_\pi)$ satisfying $T\pi(s) = \pi(s)T$ for all $s \in G$ is of the form $T = \lambda I$ for some $\lambda \in \mathbb{C}$ and I the identity operator on $\mathcal{H}_\pi$.

Note there are "enough" irreducible representations to fully describe a locally compact group G.

Theorem 3.11 ([12, Thm 1.56, Gelfand-Raikov])**.** For G a locally compact group with $s, t \in G$ such that $s \neq t$, there exists an irreducible representation π of G such that $\pi(s) \neq \pi(t)$.

We now define the *unitary dual* of G.

Definition 3.12. Let $\widehat{G}$ denote the set of irreducible representations of G up to unitary equivalence. This is called the **unitary dual** or **dual space** of G.

In practice, it is an arduous task to compute $\widehat{G}$ for arbitrary groups. However, as we will see in Section 5.2, the special structure of Bieberbach groups allows a set-theoretic description of $\widehat{G}$. Although $\widehat{G}$ does not have an intrinsic topology, we may add one by noting it is canonically isomorphic to a space that does come equipped with a topology, as we will see in Section 3.3.

3.2 The Spectrum of a C^*-algebra

In this section, we address the spectrum of a C^*-algebra which is the C^*-algebraic version of the unitary dual of a group. The source for this section is Dixmier's monograph, C^*-*algebras* [8], which is an essential read for anyone interested in a serious study of C^*-algebras (although it may perhaps not be the best introductory material).

Definition 3.13. Let A be a $*$-algebra. A **representation of** A is a map $\pi : A \to B(\mathcal{H}_\pi)$ such that for all $a, b \in A$, $\lambda \in \mathbb{C}$,

$$\pi(ab) = \pi(a)\pi(b)$$
$$\pi(a + \lambda b) = \pi(a) + \lambda\pi(b)$$
$$\pi(a^*) = \pi(a)^*$$

The **dimension** of π is the dimension of $\mathcal{H}_\pi$.

In a manner analogous to the group case, we say two representations π and π' are **equivalent**, denoted $\pi \simeq \pi'$, if there exists an isomorphism $U : H_\pi \to H_{\pi'}$ such that for all $a \in A$, $U\pi(a) = \pi'(a)U$. Invariant subspaces, subrepresentations, and Definition 3.4 all extend from groups in exactly the manner one would expect.

If ρ and ρ' are representations of A such that ρ' is equivalent to a subrepresentation of ρ, then we say that ρ' is **contained** in ρ or that ρ **contains** ρ'. We denote this $\rho' \leq \rho$ or $\rho \geq \rho'$.

Although representations of groups and $*$-algebras share many similarities, we do need to consider an additional situation called *degeneracy* when discussing representations of $*$-algebras.

Definition 3.14. Let K be the closed subspace of $\mathcal{H}_\pi$ generated by $\pi(a)\xi$ for all $a \in A$ and $\xi \in \mathcal{H}_\pi$. We call K the **essential subspace** of π.

Proposition 3.15 ([8, Prop 2.2.6])**.** If $K \subseteq \mathcal{H}_\pi$ is the essential subspace of π, then the closed subspace K' which consists of all $\xi \in \mathcal{H}_\pi$ such that $\pi(a)\xi = 0$ for $a \in A$ is orthogonal to K and $\mathcal{H}_\pi = K \oplus K'$.

The essential subspace is then the part of $\mathcal{H}_\pi$ on which π meaningfully acts. We say that π is **non-degenerate** if $K = \mathcal{H}_\pi$ and **degenerate** otherwise.

We now address what it means for a representation of a $*$-algebra to be irreducible.

Proposition 3.16 ([8, Prop 2.3.1])**.** Let A be a $*$-algebra and π a representation of A on $\mathcal{H}_\pi$. The following are equivalent:

(i) The only closed invariant subspaces of $\mathcal{H}_\pi$ are $\{0\}$ and $\mathcal{H}_\pi$.

(ii) Let $T \in B(\mathcal{H}_\pi)$. If $T\pi(a) = \pi(a)T$ for all $a \in A$, then $T = \lambda I$ where $\lambda \in \mathbb{C}$ and I is the identity operator on $\mathcal{H}_\pi$.

Definition 3.17. If either condition above is met and $\mathcal{H}_\pi \neq \{0\}$, we say that π is **irreducible**.

Observe that if π is irreducible, then π is automatically non-degenerate.

Note 3.18. There is a stronger notion of irreducibility: we say that π is **algebraically irreducible** if the only invariant subspaces (not necessarily closed) of π are $\{0\}$ and $\mathcal{H}_\pi$. When $\dim \mathcal{H}_\pi = +\infty$, this is more stringent than our notion of irreducibility. However, these are the *same* when A is a C^*-algebra [8, Cor 2.8.4].

C^*-algebras have enough irreducible representations to separate points and preserve the norm.

Theorem 3.19 ([8, Thm 2.7.3])**.** Let A be a C^*-algebra. Then there is a family $(\pi_\alpha)_{\alpha \in I}$ of irreducible representation of A such that for all $a \in A$,

$$\|a\| = \sup_\alpha \|\pi_\alpha(a)\|.$$

Before we can define the spectrum of A, we must discuss the collection of primitive ideals of A. Note that a **primitive** ideal is the kernel of a non-zero *algebraically* irreducible representations of A.

Definition 3.20. $\widehat{A}$ is the set of equivalence classes under $\simeq$ of non-trivial irreducible representations of A. Prim(A) is the set of all two-sided primitive ideals of A.

Theorem 3.21 ([8, Thm 2.9.7])**.** Let A be a C^*-algebra.

(i) The primitive two-sided ideals of A are the kernels of the non-zero irreducible representations of A.

(ii) Every closed two-sided ideal of A is the intersection of the primitive two-sided ideals containing it.

From this, we see there is a canonical surjective mapping [8, p. 58]:

$$\widehat{A} \xrightarrow{\ker \pi} \mathrm{Prim}\,(A).$$

We begin by defining a topology on $\mathrm{Prim}\,A$. For each subset $T \subseteq \mathrm{Prim}\,(A)$, we define $\mathtt{I}(T) = \bigcap_{\ker \pi \in T} \ker \pi$. We define $\overline{T}$ to be the collection of all primitive ideals of A which contain $\mathtt{I}(T)$, that is,

$$\overline{T} = \{\, \ker \pi \mid \mathtt{I}(T) \subseteq \ker \pi,\, \pi \in \widehat{A}\,\}.$$

Definition 3.22. The topology on $\mathrm{Prim}\,A$ defined above is the **Jacobson topology**.

We call the set $\mathrm{Prim}\,(A)$ endowed with the Jacobson topology the **primitive spectrum** of A. This topology is so poorly behaved that we cannot generally expect it to be Hausdorff. We do, however, have the following.

Proposition 3.23 ([8, Prop 3.1.3])**.** $\mathrm{Prim}\,(A)$ is a T_0-space, i.e., for any two $I_1, I_2 \in \mathrm{Prim}\,(A)$, there exists an open set $S = \mathrm{Prim}\,(A) \setminus \overline{T}$ for some $T \subseteq \mathrm{Prim}\,(A)$ where either

$$I_1 \in S \text{ and } I_2 \notin S \quad \text{ or } \quad I_1 \notin S \text{ and } I_2 \in S.$$

Amazingly, even through $\mathrm{Prim}\,(A)$ is generally not Hausdorff, it may be second countable!

Corollary 3.24 ([14, Cor II.6.5.7])**.** If A is separable, then $\mathrm{Prim}\,(A)$ is second countable.

Using the topology on the primitive spectrum, we may define a topology on $\widehat{A}$.

Definition 3.25. The **spectrum** of A is the set $\widehat{A}$ endowed with the pullback of the Jacobson topology under the canonical map $\widehat{A} \xrightarrow{\ker \pi} \operatorname{Prim}(A)$.

As pathological as the topology of $\operatorname{Prim}(A)$ is, $\widehat{A}$ is worse. In general, $\widehat{A}$ is **not** a T_0-space. If, out of stubbornness, we demand that $\widehat{A}$ be T_0, then $\widehat{A}$ and $\operatorname{Prim}(A)$ coincide.

Proposition 3.26 ([8, Prop 3.1.6])**.** $\widehat{A}$ is a T_0-space if and only if the canonical map $\widehat{A} \xrightarrow{\ker \pi} \operatorname{Prim}(A)$ is a homeomorphism.

Thus, in the case that the spectrum of A is better behaved, it suffices to only consider $\operatorname{Prim}(A)$. As unruly as the topology of $\widehat{A}$ is, we do have the following results related to convergence.

Proposition 3.27 ([8, Sec. 3.5])**.** Let π and $(\pi_i)_i$ be irreducible representations of a C^*-algebra A such that $\mathcal{H} = \mathcal{H}_\pi = \mathcal{H}_{\pi_i}$ for all i. Suppose for all $\xi \in \mathcal{H}$ and $a \in A$,

$$\|\pi_i(a)\xi - \pi(a)\xi\| \to 0.$$

Then $(\pi_i)_i$ converges to π in $\widehat{A}$.

Proposition 3.28 ([13, Prop VII.3.5, p. 193])**.** For $\sigma \in \widehat{A}$, the map $\sigma \mapsto \|\sigma(a)\|$ is lower semi-continuous for each $a \in A$. For $B \subseteq A$ a dense subset, define

$$O_B = \{\, \sigma \in \widehat{A} \ : \ \|\sigma(a)\| > 1 \text{ for all } a \in A \,\}.$$

The collection of O_B forms a base for the pullback of the Jacobson topology on $\widehat{A}$. A sequence σ_n has a limit point $\rho \in \widehat{A}$ if and only if

$$\liminf_n \|\sigma_n(a)\| \geq \|\rho(a)\| \text{ for all } a \in A.$$

We are also able to slice up $\widehat{A}$ based on the dimensions of the irreducible representations in a way which respects the topology.

Proposition 3.29 ([8, Prop 3.6.3])**.** Let $\widehat{A}_n$ be the set of $\pi \in \widehat{A}$ such that $\dim \pi \leq n$ and $\widehat{A}^n$ be the set of $\pi \in \widehat{A}$ such that $\dim \pi = n$. Then $\widehat{A}_n$ is closed in $\widehat{A}$ and $\widehat{A}^n$ is open in $\widehat{A}_n$.

In this next section, we introduce group C^*-algebras and discuss spectra.

3.3 The Representation Space of Group C^*-algebras

This section primarily comes from Chapter 7 of Davidson's *C^*-algebras by Example* as it addresses the ideas of group C^*-algebras most relevant for our purposes.

Because we will be addressing measures, recall the following:

Definition 3.30. A **Borel set** in a topological space X is any subset of X which can be formed from open sets of X through countable union, countable intersection, or relative complement. A **Borel measure** is any measure defined on Borel sets.

Definition 3.31. Let X be a topological space with measure μ. We say that μ is **regular** if for all measurable sets A we have

$$\mu(A) = \sup\{\, \mu(K) \,|\, K \subseteq A,\ K \text{ measurable, compact} \,\}$$

and

$$\mu(A) = \inf\{\, \mu(O) \,|\, A \subseteq O,\ O \text{ measurable, open} \,\}.$$

Let G be a locally compact group. The **left Haar measure**, μ_G, is a regular Borel measure which is invariant under left translation; that is, for any Borel subset E of G and $s \in G$, $\mu_G(s \cdot E) = \mu_G(E)$ where

$$s \cdot E = \{\, s \cdot g \,|\, g \in E \,\}.$$

When G is compact, this measure will be finite and we may normalize by $\mu_G(G) = 1$. If G is infinite and discrete, then we require $\mu_G(\{e\}) = 1$ for e the identity element of G.

In general, left Haar measures are not right translation invariant and we introduce the **modular function** $\Delta : G \to \mathbb{R}_+$; this is a continuous homomorphism which measures how far μ_G is from being right translation invariant:

$$\mu_G(E \cdot s) = \Delta(s)\, \mu_G(E)$$

where $E \cdot s = \{\, g \cdot s : g \in E \,\}$. We call a group **unimodular** if $\Delta(s) = 1$ for all $s \in G$, which is to say that μ_G is both left and right translation invariant. Discrete groups are unimodular.

Now, define

$$L^1(G) = \left\{ f : G \to \mathbb{C} \mid \int_G |f(s)|\, d\mu_G(s) < \infty \right\}.$$

This is a $*$-algebra with respect to convolution and involution:

$$f * g(t) = \int_G f(s)\, g(s^{-1}t)\, d\mu_G(s)$$
$$f^*(t) = \Delta(t)^{-1}\, \overline{f(t^{-1})}$$

By [12, Prop 1.2], these two operations make $L^1(G)$ a Banach $*$-algebra but **not** a C^*-algebra as it fails the C^*-condition (Equation (2.1)).

We note that when G is discrete, $L^1(G)$ is unital and μ_G becomes the counting measure. In this case, we may write $\ell^1(g)$ instead of $L^1(G)$. Let $\mathbb{C}[G]$ be the set of all finite sums $\sum_{s\in G} \lambda_s \delta_s$ where $\lambda_s \in \mathbb{C}$ and $\delta_s : G \to \{0, 1\}$ is the characteristic function. We call $\mathbb{C}[G]$ the *group algebra* and note that it is a dense subset of $\ell^1(G)$.

Now, for π a unitary representation of G, we can canonically induce a representation of $L^1(G)$ by

$$\widetilde{\pi}(f) = \int_G f(t)\, \pi(t)\, d\mu_G(t).$$

Conversely, if $\widetilde{\pi}$ is a non-degenerate representation of $L^1(G)$, then it determines a unique unitary representation π of G. For details, see [13, p. 183-184]. Because of this 1-1 correspondence, we suppress the $\sim$ and simply refer to both the representation of G and $L^1(G)$ by π.

Let π_u be the direct sum of all irreducible representations of G up to unitary equivalence.

Definition 3.32. The **group C^*-algebra** of G is

$$C^*(G) = \overline{\pi_u(L^1(G))}^{\|\cdot\|}.$$

As might be expected, this can be an enormous algebra since the universal representation potentially contains a huge number of representations. However, there are situations in which the group C^*-algebra (also called the *universal group C^*-algebra* or *full group C^*-algebra*) is isomorphic to a seemingly smaller algebra.

Definition 3.33. $\lambda : G \to L^2(G)$ defined by

$$\lambda(s)\, g(t) = g(s^{-1}t)$$

is called the **left regular representation**. After extending λ to $L^1(G)$, we define the **reduced group C^*-algebra** of G by

$$C_r^*(G) = \overline{\lambda(L^1(G))}^{\|\cdot\|_2}.$$

Note 3.34. $L^2(G)$ is a C^*-algebra under the usual norm

$$\|f\|_2 = \left(\int_G |f(t)|^2 \, d\mu_G(t) \right)^{1/2}.$$

Since $C_r^*(G)$ arises from just one representation, it is a priori smaller than $C^*(G)$. Yet, in special cases we actually have $C_r^*(G) \cong C^*(G)$.

Definition 3.35. A **mean** is a positive linear functional on $L^\infty(G)$ (the space of essentially bounded functions on G) of norm 1. We say that a group G is **amenable** if there is a left translation invariant mean on G.

Example 3.36. Examples of amenable groups:

(a) compact groups [13, p.185]

(b) abelian groups [13, Cor VII.2.2]

(c) discrete *virtually abelian* groups [13, Prop VII.2.3]

Theorem 3.37 ([13, Thm VII.2.8])**.** When G is a discrete amenable group, $C_r^*(G) \cong C^*(G)$.

Our groups of choice are Bieberbach groups, which are discrete and *virtually abelian* (see p. 27). Thus, we will not distinguish between $C^*(G)$ and $C_r^*(G)$.

Now, when G is locally compact, there is a bijective correspondence between representations of G and non-degenerate representations of $C^*(G)$, which is to say there is a canonical bijection of $\widehat{C^*(G)} \to \widehat{G}$ [8, 13.9.3]. Then, we may add a topology to the unitary dual of G from the topology of the spectrum of $C^*(G)$. Therefore, studying $\widehat{G}$ is the same as studying $\widehat{C^*(G)}$.

Note 3.38.

(a) $\widehat{\mathbb{Z}}^n \cong \mathbb{T}^n$ [12, Ex 1.75]

(b) If G is discrete, then $\widehat{G}$ is compact [12, Prop 1.70]

Before we move on to the next chapter, we quickly comment on the class of **virtually abelian groups**, which are groups formed by extending an abelian group by a finite group. Virtually abelian groups are called *Type I* groups and their C^*-algebras have some nice properties. The technical definition of Type I groups involve von Neumann algebras which is a topic more involved than necessary for our purposes. Instead, we take Thoma's result as a definition.

Definition 3.39 (Thoma, [15])**.** A countable discrete group G is **Type I** if G is virtually abelian.

If G is Type I, we will say that $C^*(G)$ is **Type I** as well. Type I C^*-algebras have the property that their spectrum is homeomorphic to their primitive spectrum; that is, $\widehat{A} \cong \mathrm{Prim}\,(A)$ when A is Type I [8].

Moreover, Moore showed Type I groups have uniformly bounded representations.

Proposition 3.40 (Moore, [16, Prop 2.1])**.** Suppose G is locally compact and that H is an open subgroup of finite index. All the irreducible representations of G are uniformly bounded by some integer i if and only if all irreducible representations of H are uniformly bounded by some integer j.

In fact, the proof of this proposition shows more. If $k = [G : H]$, $\dim \pi \leq i$ for all $\pi \in \widehat{G}$, and $\dim \sigma \leq j$ for all $\sigma \in \widehat{H}$, then

$$i \leq k \cdot j.$$

This leads to the following useful corollary:

Corollary 3.41. Suppose G is a discrete group with a subgroup $N \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$ such that $[G : N] = k$. Then $\dim \pi \leq k$ for all $\pi \in \widehat{G}$.

Proof. Because G is discrete, N is open in G. By Note 3.7, $\dim \sigma = 1$ for all $\sigma \in \widehat{N}$. Hence, by the proof of the previous proposition, we see that for all $\pi \in \widehat{G}$,

$$\dim \pi \leq k \cdot 1 = k.$$

$\square$

CHAPTER 4. CONNECTIVITY OF GROUPS

C^*-algebras have a rich homology theory which captures topological and geometric invariants. Using these invariants, we distinguish C^*-algebras and, in special cases, classify them (see, for example, Elliott's paper [17]). The purpose of this chapter is to introduce connectivity, which is an E-theoretic property. However, for completeness, we very briefly address the related topics of K-theory, K-homology, and KK-theory in an effort to provide sufficient context.

4.1 Categories and Functors

The goal of a homology theory is to assign a mathematical object to an abelian group. To address this in a more technical way, we will need to introduce categories and functors.

A **category** $\mathbf{C}$ is a class of *objects*, $\mathcal{O}(\mathbf{C})$, such that for each pair of objects $A, B \in \mathcal{O}(\mathbf{C})$, there is a set of *morphisms* from A to B, denoted $\mathrm{Mor}\,(A, B)$, which respect the associative rule of composition in the following sense:

$$\begin{aligned} \mathrm{Mor}\,(A,B) \times \mathrm{Mor}\,(B,C) &\to \mathrm{Mor}\,(A,C) \\ (\phi, \psi) &\mapsto \psi \circ \phi \end{aligned}$$

Moreover, we require that for each $A \in \mathcal{O}(\mathbf{C})$, there exists $\mathrm{id}_A \in \mathrm{Mor}\,(A, A)$ such that for every $B \in \mathcal{O}(\mathbf{C})$ and every $\phi \in \mathrm{Mor}\,(A, B)$,

$$\mathrm{id}_B \circ \phi = \phi = \phi \circ \mathrm{id}_A.$$

We call $N \in \mathcal{O}(\mathbf{C})$ a **zero object** if for all $A \in \mathcal{O}(\mathbf{C})$, $\mathrm{Mor}\,(A, N)$ and $\mathrm{Mor}\,(N, A)$ each contain exactly one element.

Between categories $\mathbf{C}$ and $\mathbf{D}$, we define a **covariant functor** $\mathcal{F}$ as a map $A \mapsto \mathcal{F}(A)$ from $\mathcal{O}(\mathbf{C})$ to $\mathcal{O}(\mathbf{D})$ and, for each pair $A, B \in \mathcal{O}(\mathbf{C})$, a collection of maps $\phi \mapsto \mathcal{F}(\phi)$ from $\mathrm{Mor}\,(A, B)$ to $\mathrm{Mor}\,(\mathcal{F}(A), \mathcal{F}(B))$ satisfying the following:

(i) $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ for all $A \in \mathcal{O}(\mathbf{C})$

(ii) $\mathcal{F}(\phi \circ \psi) = \mathcal{F}(\psi) \circ \mathcal{F}(\phi)$ for all $A, B, C \in \mathbf{C}$, $\phi \in \mathrm{Mor}\,(A, B)$, $\psi \in \mathrm{Mor}\,(B, C)$

A **contravariant functor** $\mathcal{G}$ satisfies mostly the same conditions as a covariant functor except that it reverses the direction of the morphisms. We will focus on covariant functors and refer to them just as functors.

The categories that will interest us are the category of C^*-algebras, **C*-alg**, and abelian groups, **Ab**. The objects in $\mathcal{O}(\mathbf{C^*\text{-}alg})$ are, of course, C^*-algebras with morphisms given by $*$-homomorphisms which need not preserve units. The objects in $\mathcal{O}(\mathbf{Ab})$ are abelian groups with morphisms given by group homomorphisms. We observe that $\mathcal{O}(\mathbf{C^*\text{-}alg})$ has a zero object, namely the $\{0\}$ algebra.

4.2 A Brief Description of K-theory and K-homology

4.2.1 K-theory for C^*-algebras

K-theory is a homological theory which extracts invariants from vector bundles over topological spaces (see Atiyah [18]). This theory is especially powerful when applied to C^*-algebras which we may think of as a non-commutative analogue to this original construction. For the sake of space, we will focus specifically on the theory as applied to C^*-algebras. A wonderful resource for an initiation into this material is Rørdam, Larsen, and Laustsen's aptly titled *An Introduction to K-theory for C^*-algebras* [19].

Let A be a C^*-algebra and let $M_n(A)$ be the set of $n \times n$ matrices with entries in A. For two matrices $p \in M_n(A)$, $q \in M_m(A)$, we define $p \oplus q$ to be the matrix in $M_{n+m}(A)$ of the form

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

where 0 represents the appropriately sized zero matrix. We add $\oplus$ as a binary operation to the set $M_\infty(A) := \bigcup_{n\in\mathbb{N}} M_n(A)$. We use $\mathcal{P}_n(A), \mathcal{P}_\infty(A)$ to represent the set of all projections of $M_n(A), M_\infty(A)$ respectively.

Defining $K_0(A)$

$K_0(A)$ is an abelian group which arises from equivalence classes on $\mathcal{P}_\infty(A)$. Consider the following.

Definition 4.1. If p and q are projections of A, we say they are **Murray-von Neumann equivalent**, denoted $p \sim q$, if there exists $v \in A$ such that $p = v^*v$ and $q = vv^*$.

For $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say p and q are **Murray-von Neumann** equivalent, denoted $p \sim_0 q$, if there exists $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. Hence, we extend $\sim_0$ to all of $\mathcal{P}_\infty(A)$.

Remark 4.2 ([19, Prop 2.3.2 (i),(iii)]). This definition does indeed define an equivalence relation. We note that $p \oplus 0_n \sim_0 p$ where 0_n is the zero matrix in $M_n(A)$ and $p \oplus q \sim_0 q \oplus p$.

Now, define

$$\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim_0 .$$

This forms an abelian semi-group under $+$ where, if $[x]_D \in \mathcal{D}(A)$ represents the equivalence class of $x \in \mathcal{P}_\infty(A)$, then $[p]_D + [q]_D = [p \oplus q]_D$ [19, Prop 2.3.2]. Observe that by the Remark 4.2, $[0_n]_D$ acts as the identity element.

$\mathcal{D}(A)$ fails to be a group because a generic element does not have an inverse. We then apply the Grothendieck construction, a functor that assigns an abelian group to an abelian semi-group [19, Sec 3.1].

Proposition 4.3 (Grothendieck Construction, [19, Sec 3.1])**.** Let $(S, +)$ be an abelian semi-group. Define a binary relation $\sim_G$ on $S \times S$ by $(x_1, y_1) \sim_G (x_2, y_2)$ if and only if there is $z \in S$ such that

$$x_1 + y_2 + z = x_2 + y_1 + z.$$

$\sim_G$ is an equivalence relation.

Let $\mathcal{G}(S) := S \times S/\sim_G$. Then $\mathcal{G}(S)$ is an abelian group under $+$ given by

$$[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$$

where $[x, y]$ is the equivalence class of (x, y). The inverse and zero are defined by

$$-[x, y] = [y, x] \quad \text{and} \quad [x, x] = 0.$$

Suppose for the moment that A is a unital C^*-algebra. Then

$$K_0(A) := \mathcal{G}(\mathcal{D}(A)).$$

If $p \in \mathcal{P}_\infty(A)$, we let $[p]_0$ be its image in $K_0(A)$.

When A is not unital, this definition is no longer suitable. We want K_0 to be half-exact, that is, if we have the following exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0,$$

then we want

$$K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B)$$

to also be exact for $K_0(\eta) : K_0(\bullet) \to K_0(\bullet)$ where $K_0(\eta([p]_0)) = [\eta(p)]_0$. The definition given above does not produce a half-exact functor when A is non-unital, so we must do a little more work.

Recall that $\widetilde{A}$ is the unitization of A (a process which can be applied to unital or non-unital C^*-algebra, see Section 2.2). Then $\widetilde{A} \cong A \oplus \mathbb{C}$ as algebras (although not as C^*-algebras since $\widetilde{A}$ is unital and $A \oplus \mathbb{C}$ is not). If A is non-unital, consider the split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \widetilde{A} \arrow[r, bend left, "\pi"] & \mathbb{C} \arrow[l, bend left, "\lambda"] \arrow[r] & 0 \end{tikzcd}$$

where $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$. Let $K_0(\pi) : \widetilde{A} \to \mathbb{C}$ be the map induced by π and consider the exact sequence

$$K_0(A) \longrightarrow K_0(\widetilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}).$$

When A is not unital, we define

$$K_0(A) := \ker\left(K_0(\pi)\right).$$

This definition of K_0 for non-unital A provides the desired functoriality.

Let $s : \widetilde{A} \to \widetilde{A}$ defined by $s = \lambda \circ \pi$ be the **scalar map**. We may extend s naturally to $M_n(\widetilde{A})$ by applying s entrywise; call this extended function s_n. Then, consider the restriction of s_n to $\mathcal{P}_n(\widetilde{A})$. When the size of the matrix is not important, we abuse notation and simply refer to this extended function as $s : \mathcal{P}_\infty(\widetilde{A}) \to \mathcal{P}_\infty(\widetilde{A})$. We may then define the **standard picture** of $K_0(A)$ [19, Prop 4.2.2]:

$$K_0(A) = \left\{ [p]_0 - [s(p)]_0 \mid p \in \mathcal{P}_\infty(\widetilde{A}) \right\}.$$

Before we move on to defining K_1, I would like to mention two important properties of K_0.

(a) When A and B are homotopy equivalent C^*-algebras, $K_0(A) \cong K_0(B)$ [19, Prop 4.1.4(ii)].

(b) If $\mathcal{K}$ is the compact operators on some separable Hilbert space, then $K_0(A) \cong K_0(A \otimes \mathcal{K})$. This is to say that tensoring with the compacts (or even finite dimensional matrices) does not impact the K-theory of a C^*-algebra [19, Prop 6.4.1].

Defining $K_1(A)$

Defining the K_1 group is a less arduous task than defining K_0. Because K_1 replaces projections by unitaries, the C^*-algebra A must be unital from the start. In the case A is not unital, we consider $\widetilde{A}$ instead. Interestingly, there is no confusion in this construction if we only consider $\widetilde{A}$, regardless of whether A is unital or not. But for the moment assume that A is unital.

Let $\mathcal{U}_n(A), \mathcal{U}_\infty(A)$ represent the set of all unitaries of $M_n(A)$, $M_\infty(A)$ respectively.

Definition 4.4. For X some topological space, we say that $f, g \in X$ are **homotopic** if there exists a continuous $\Phi : [0,1] \times X \to X$ such that $\Phi(0) = f$ and $\Phi(1) = g$. We denote this by $f \sim_h g$.

For $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we say $u \sim_1 v$ if and only if there exists $k \geq \max\{m, n\}$ such that

$$\begin{pmatrix} u & 0 \\ 0 & I_{k-m} \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & I_{k-n} \end{pmatrix}$$

in $\mathcal{U}_k(A)$ where 0 is the appropriately sized zero matrix and I_ℓ is the identity matrix of $\mathcal{U}_\ell(A)$.

For a C^*-algebra A (not necessarily unital), we define

$$K_1(A) := \mathcal{U}_\infty(\widetilde{A})/\!\sim_1$$

where $[u]_1$ denotes the equivalence class of $u \in \mathcal{U}_\infty(\widetilde{A})$. $K_1(A)$ is an abelian group under addition: $[u]_1 + [v]_1 = [u \oplus v]_1$ [19, Prop 8.1.4].

Note 4.5. If A is unital, $K_1(A) \cong K_1(\widetilde{A})$. See chapter 8 in [19] for more details.

As with $K_0(A)$, we have a standard picture of $K_1(A)$ [19, Prop 8.1.4]:

$$K_1(A) = \left\{ [u]_1 \mid u \in \mathcal{U}_\infty(\widetilde{A}) \right\}.$$

Moreover, just as with $K_0(A)$,

(a) $K_1(A)$ is half-exact

(b) $K_1(A) \cong K_1(M_n(A)) \cong K_1(A \otimes \mathcal{K})$ for any $n \in \mathbb{N}$

(c) if A, B are homotopy equivalent, then $K_1(A) \cong K_1(B)$

Higher K-groups

We define the **cone** of A by

$$\mathcal{C}A := \{\, f : [0,1] \to A \text{ continuous} \mid f(0) = 0 \,\}$$

and the **suspension** of A by

$$\mathcal{S}A := \{\, f : [0,1] \to A \text{ continuous} \mid f(0) = 0 = f(1) \,\}.$$

Observe that $\mathcal{C}A \cong C_0(0,1] \otimes A$ and $\mathcal{S}A \cong C_0(0,1) \otimes A$.

Suspension is a functor: if $\phi : A \to B$ is a $*$-homomorphism, then $\mathcal{S}\phi : \mathcal{S}A \to \mathcal{S}B$ defined by $(\mathcal{S}\phi)(f(t)) = \phi(f(t))$ for all $f(t) \in \mathcal{S}A$. In fact, by Proposition 10.1.2 in [19], this functor is exact — which is to say given an exact sequence of C^*-algebras,

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0,$$

then the following sequence is also exact

$$0 \longrightarrow \mathcal{S}I \xrightarrow{\mathcal{S}\phi} \mathcal{S}A \xrightarrow{\mathcal{S}\psi} \mathcal{S}B \longrightarrow 0.$$

We define higher K-groups as follows: if we let $\mathcal{S}^n\bullet = \mathcal{S}(\mathcal{S}^{n-1}\bullet)$, then

$$K_n(A) = K_1(\mathcal{S}^{n-1}A) \tag{4.1}$$

for all $n \geq 2$. By Theorem 10.1.3 in [19], $K_1(A) \cong K_0(\mathcal{S}A)$. Combining this with Equation (4.1),

$$K_n(A) = K_1(\mathcal{S}^{n-1}A) \cong K_0(\mathcal{S}^n A).$$

Although it is not clear in the C^*-algebra context why this is the correct definition for higher K-groups, it does arise more naturally in the vector bundle context (see Atiyah [18]).

Theorem 4.6 (Bott Periodicity, [19, Thm 11.1.2])**.** For every C^*-algebra A,

$$K_0(A) \cong K_1(\mathcal{S}A).$$

We then have an obvious corollary which explicitly demonstrates this periodicity.

Corollary 4.7 ([19, Cor 11.3.1])**.** For every C^*-algebra A and $n \geq 0$,

$$K_{n+2}(A) \cong K_n(A).$$

Bott's remarkable result leads to perhaps the most useful property of K-theory: the six-term exact sequence.

Theorem 4.8 (The six-term exact sequence, [19, Thm 12.1.2])**.** For every short exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0,$$

the associated six-term exact sequence

$$\begin{tikzcd}
K_0(I) \arrow[r, "K_0(\phi)"] & K_0(A) \arrow[r, "K_0(\psi)"] & K_0(B) \arrow[d, "\delta_0"] \\
K_1(B) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "K_1(\psi)"] & K_1(I) \arrow[l, "K_1(\phi)"]
\end{tikzcd}$$

is exact.

The maps δ_0, δ_1 (the exponential and index maps, respectively) are of vital importance to the theory but I will not address them here. Instead, I direct the curious reader to [19].

4.2.2 K-Homology for C^*-algebras

The most helpful description of the connection between K-theory, K-homology, and KK-theory that I could find was Higson and Roe's *Analytic K-theory* [20], which is the basis of this subsection.

Let A be a separable unital C^*-algebra (we will address the non-unital case later) and let $\pi : A \to B(\mathcal{H})$ be a representation of A on a separable Hilbert space $\mathcal{H}$.

Definition 4.9. We say that π is **ample** if (1) $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$, (2) $\pi(a) = 0$ implies $a = 0$, and (3) $\pi(A) \cap \mathcal{K} = \{0\}$.

Definition 4.10. The **dual algebra** of A is the C^*-subalgebra of $B(\mathcal{H})$ defined by

$$\mathsf{D}_\pi = \{\, T \in B(\mathcal{H}) \mid T\pi(a) - \pi(a)T \in \mathcal{K} \text{ for all } a \in A \,\}$$

where π is a representation of A onto a separable Hilbert space.

By Section 5.2 in [20], any two ample representations of a C^*-algebra A produce isomorphic dual algebras of A. Thus, the dual algebra does not depend on the choice of ample representation and the following is well-defined:

Definition 4.11. The **reduced analytic K-homology groups** of a separable unital C^*-algebra A are

$$\widetilde{K}^1(A) = K_0(\mathsf{D}(A)) \quad \text{and} \quad \widetilde{K}^0(A) = K_1(\mathsf{D}(A))$$

where $\mathsf{D}(A)$ is a dual algebra of some ample representation of A.

We note that the indexing is done intentionally to reverse the order since this is the dual to K-theory. Note that K-theoretic equivalence of ample representations coincides with Murray-von Neumann equivalence [20, Prop 5.1.4].

Definition 4.12. The **unreduced analytic K-homology groups** of a separable C^*-algebra A are

$$K^1(A) = K_0(\mathsf{D}(\widetilde{A})) \quad \text{and} \quad K^0(A) = K_1(\mathsf{D}(\widetilde{A}))$$

where $\mathsf{D}(A)$ is a dual algebra of some ample representation of A.

Although this second definition clearly lends itself to C^*-algebras unital or not, if A already has a unit, then there is a difference between reduced and unreduced K-homology.

Corollary 4.13 ([20, Cor 5.2.11])**.** When A is separable, unital, and commutative, then

$$K^0(A) \cong \widetilde{K}^0(A) \oplus \mathbb{Z} \quad \text{and} \quad K^1(A) \cong \widetilde{K}^1(A).$$

We see that K-homology fits with our notion of K-theory. However, choosing an ample representation presents some set-theoretic difficulties. To overcome this, other methods for defining K-homology have been devised but we will not address them here. See [20] for a treatment of some alternate methods.

4.3 A Briefer Description of KK-theory for C^*-algebras

KK-theory combines K-theory and K-homology into a single bivariant functor. For this section, we rely on Jensen and Thomsen's *Elements of KK-theory* [11].

Let B be a C^*-algebra with norm $\|\cdot\|_B$ and let E_B denote a right B-module.

Definition 4.14. We say that E_B is a **pre-Hilbert B-module** if it is a complex vector space equipped with a map $\langle \cdot, \cdot \rangle : E_B \times E_B \to B$ such that the following is true for all $b \in B,\ e, f, g \in E_B,\ \lambda \in \mathbb{C}$:

(i) $\langle e, f + \lambda g \rangle = \langle e, f \rangle + \lambda \langle e, g \rangle$

(ii) $\langle e, f\, b \rangle = \langle e, f \rangle\, b$

(iii) $\langle e, f \rangle^* = \langle f, g \rangle$

(iv) $\langle e, e \rangle \geq 0$

(v) $e \neq 0$ implies $\langle e, e \rangle \neq 0$

We add a norm to E_B by defining $\|e\|_{E_B} = \|\langle e, e \rangle\|_B^{1/2}$ for $e \in E_B$.

Proposition 4.15 ([11, Lemma 1.1.2])**.** Suppose E_B is a pre-Hilbert B-module with $\|e\|_{E_B}$ as defined above. Then for all $e, f \in E_B$ and $b \in B$,

(1) $\|e\, b\|_{E_B} \leq \|e\|_{E_B}\, \|b\|_B$

(2) $\|\langle e, f\rangle\|_B \leq \|e\|_{E_B}\, \|f\|_{E_B}$

This proposition tells us that this norm is well-behaved and is compatible with the norm on B.

Definition 4.16. A **Hilbert** B**-module** is a pre-Hilbert B-module E_B which is complete in the norm $\|\cdot\|_{E_B}$.

Now, suppose we have two Hilbert B-modules, E_1 and E_2. Let $\mathcal{L}_B(E_1, E_2)$ be the set of all $T : E_1 \to E_2$ such that there exists $T^* : E_2 \to E_1$ satisfying

$$\langle Te_1, e_2\rangle = \langle e_1, T^* e_2\rangle$$

for all $e_1 \in E_1$ and $e_2 \in E_2$. Essentially, this means that $\mathcal{L}_B(E_1, E_2)$ is the set of all operators with adjoints. The existence of an adjoint implies T is linear and a Hilbert B-module map [11, Sec 1.1].

Further, we define $\Theta_{e_2,e_1} : E_1 \to E_2$ by

$$\Theta_{e_2,e_1}(f_1) = e_2\, \langle e_1, f_1\rangle$$

for all $e_1, f_1 \in E_1$ and $e_2 \in E_2$. If we let $\Theta^*_{e_2,e_1} = \Theta_{e_1,e_2}$, then we see that $\Theta_{e_2,e_1} \in \mathcal{L}_B(E_1, E_2)$. Define

$$\mathcal{K}_B(E_1, E_2) = \overline{\text{Span}\ \{\, \Theta_{e_2,e_1} \mid e_1 \in E_1, e_2 \in E_2 \,\}}.$$

Proposition 4.17 ([11, Lemma 1.1.7, Lemma 1.1.9])**.**

(a) $\mathcal{L}_B(E_B) = \mathcal{L}_B(E_B, E_B)$ is a C^*-algebra.

(b) $\mathcal{K}_B(E_B) = \mathcal{K}_B(E_B, E_B)$ is a closed two-sided ideal in $\mathcal{L}_B(E_B)$.

This implies that $\mathcal{K}_B(E_B)$ is also a C^*-algebra. We note that when $E_{\mathbb{C}} = \mathcal{H}$ is a Hilbert space, $\mathcal{K} = \mathcal{K}_{\mathbb{C}}(\mathcal{H})$ is just the compact operators.

Remark 4.18. $\mathcal{L}_B(E_B)$ is the **multiplier algebra** of B, typically denoted $\mathcal{M}(B)$. This algebra has several different formulations (see [7] for another setup involving essential ideals).

Now, suppose A and B are σ-unital C^*-algebras such that $\mathcal{K}_B(E_B)$ is also σ-unital (see Section 2.2).

Definition 4.19. A **Kasparov** $A-B$**-module** is a triple $\mathcal{E} = (E_B, \phi, F)$ where $\phi : A \to \mathcal{L}_B(E_B)$ is a graded homomorphism and $F \in \mathcal{L}_B(E_B)$ with $\deg(F) = 1$ such that for all $a \in A$,

(i) $F\phi(a) - \phi(a)F \in \mathcal{K}_B(E_B)$

(ii) $(F^2 - 1)\phi(a) \in \mathcal{K}_B(E_B)$

(iii) $(F^* - F)\phi(a) \in \mathcal{K}_B(E_B)$

See Section 2.2 for the definition of *graded homomorphism*.

Let $\mathbb{E}(A, B)$ be the set of all Kasparov $A-B$-modules. To get from $\mathbb{E}(A, B)$ to $KK(A, B)$, we need to define what it means for two of these triples to be *isomorphic* and *homotopic*.

Definition 4.20. $\mathcal{E}_1 = (E_1, \phi_1, F_1)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2)$ are **isomorphic** if there is an isomorphism $\psi : E_1 \to E_2$ of Hilbert B-modules satisfying

(i) $S_{E_2} \circ \psi = \psi \circ S_{E_1}$

(ii) $F_2 \circ \psi = \psi \circ F_1$

(iii) $\phi_2(b) \circ \psi = \psi \circ \phi_1(b)$ for all $b \in B$

When E_1 and E_2 are isomorphic, we write $E_1 \simeq E_2$.

For $t \in [0, 1]$, let $\mathrm{ev}_t : IB = B \otimes C[0, 1] \to A$ be the surjection which evaluates at t. Then ev_t is a graded homomorphism on IB for each t with grading automorphism $\beta_{IB} \otimes \mathrm{id}_{C[0,1]}$. Let $\mathcal{G}_{\pi_t} = (E_{IB}, \pi_t, F)$ where $F \in \mathcal{L}_{IB}(E_{IB})$ and $\deg(F) = 1$.

Definition 4.21. Two Kasparov $A-B$-modules $\mathcal{E}, \mathcal{F} \in \mathbb{E}(A, B)$ are **homotopic** if there is $\mathcal{G} \in \mathbb{E}(A, IB)$ such that $\mathcal{G}_{\mathrm{ev}_0} \simeq \mathcal{E}$ and $\mathcal{G}_{\mathrm{ev}_1} \simeq \mathcal{F}$. If there is a finite set $\{\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n\} \subseteq \mathbb{E}(A, B)$ where $\mathcal{E}_i$ is homotopic to $\mathcal{E}_{i+1}$ for $i = 1, 2, ..., n-1$ such that $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_n = \mathcal{F}$, then we write $\mathcal{E} \sim \mathcal{F}$.

This is an equivalence relation [11, Lemma 2.1.12] and we can finally define

$$KK(A, B) := \mathbb{E}(A, B)/\sim .$$

We have the following properties:

Proposition 4.22. Let A, B, C be C^*-algebras.

(a) $KK(A, B) \cong KK(A \otimes M_n(\mathbb{C}), B \otimes M_m(\mathbb{C}))$ for $n, m \in \mathbb{N}$ [14, Cor 17.8.8]

(b) $KK(A, B) \cong KK(A \otimes \mathcal{K}, B) \cong KK(A, B \otimes \mathcal{K}) \cong KK(A \otimes \mathcal{K}, B \otimes \mathcal{K})$ [14, Cor 17.8.8]

(c) $KK(A, B) \cong KK(\mathcal{S}A, \mathcal{S}B)$ [14, Cor 17.8.9]

(d) If A and B are homotopy equivalent, then by [14, Prop 17.9.1]

$$KK(A, C) \cong KK(B, C) \quad \text{and} \quad KK(C, A) \cong KK(C, B)$$

This construction recovers K-theory and K-homology for C^*-algebras:

$$KK(\mathbb{C}, A) \cong K_0(A) \quad \text{and} \quad KK(\mathcal{S}\mathbb{C}, A) \cong K_1(A)$$

and

$$KK(A, \mathbb{C}) \cong K^0(A) \quad \text{and} \quad KK^1(A, \mathcal{S}\mathbb{C}) \cong K^1(A),$$

see Chapter VIII in [14].

4.4 E-theory and Connectivity

Finally, we address the primary homology theory of this document. In their seminal 1990 paper [21], Connes and Higson introduced E-theory, which simplifies and extends KK-theory. This is done by trading Kasparov $A-B$-modules for *asymptotic morphisms.*

Let A and B be separable C^*-algebras.

Definition 4.23. A family of functions $\{\phi_t : A \to B\}_{t\in I}$ satisfying

(a) for all $a \in A$, $t \mapsto \phi_t(a)$ is norm-continuous and bounded; and

(b) for all $a, b \in A$ and $\lambda \in \mathbb{C}$,

$$\lim_{t\to\infty} \|\phi_t(a+\lambda b) - (\phi_t(a) + \lambda\, \phi_t(b))\| = 0$$
$$\lim_{t\to\infty} \|\phi_t(ab) - \phi_t(a)\, \phi_t(b)\| = 0$$
$$\lim_{t\to\infty} \|\phi_t(a^*) - \phi_t(a)^*\| = 0,$$

is called a **discrete asymptotic morphism** if $I = \mathbb{N}$ and an **asymptotic morphism** if $I = [0, \infty)$.

We say that a (discrete) asymptotic morphism $(\phi_t)_{t\in I}$ is a **completely positive contraction (cpc)** if ϕ_t is cpc for each $t \in I$.

We may view asymptotic morphisms as families of functions which asymptotically behave like $*$-homomorphisms. Note that the constant asymptotic morphism $\phi_t = \phi$ for all t is a $*$-homomorphism. We want this theory to be homotopy invariant but we need to explain what we mean by a homotopy of asymptotic morphisms.

Definition 4.24. Let $(\phi_t)_{t\in I}$ and $(\psi_t)_{t\in I}$ be (discrete) asymptotic morphisms from A to B. A **homotopy** between $(\phi_t)_{t\in I}$ and $(\psi_t)_{t\in I}$ is a (discrete) asymptotic morphism $\{\Phi_t : A \to C[0,1] \otimes B\}_{t\in I}$ such that $\mathrm{ev}_0 \circ \Phi_t = \phi_t$ and $\mathrm{ev}_1 \circ \Phi_t = \psi_t$.

We have the following notation depending on the type of asymptotic morphism we are referencing:

- $[[A, B]]$ denotes the homotopy classes of asymptotic morphisms
- $[[A, B]]_{\mathbb{N}}$ denotes the homotopy classes of discrete asmptotic morphisms

- $[[A,B]]^{cp}$ denotes the homotopy classes of cpc asymptotic morphisms

This leads us to our definition of E-theory.

Definition 4.25. $E(A,B) \cong [[\mathcal{S}A, \mathcal{S}B \otimes \mathcal{K}]]$

When A is nuclear, Choi-Effros (Theorem 2.32) implies $[[A,B]] \cong [[A,B]]^{cp}$. Moreover, [22] shows $KK(A,B) \cong [[SA, SB \otimes \mathcal{K}]]^{cp}$. Putting this together

$$E(A,B) \cong KK(A,B)$$

in the separable nuclear case.

Although this may seem strange, $[[A, B \otimes \mathcal{K}]]$ is not generally a group (in fact, it fails when $A = B = \mathbb{C}$) and so for $E(A,B)$ to be an abelian group, we need the suspensions [21, Thm 7(2)]. But this raises the natural question: when do we have $E(A,B) \cong [[A, B \otimes \mathcal{K}]]$? That is, under what conditions can we *unsuspend* the E-theory? Introduced originally as *property (QH)* in [3], connectivity is a lifting property with many useful equivalencies and which completely characterizes those C^*-algebras for which we can unsuspend E-theory [1].

Definition 4.26 ([3, Def 2.6])**.** Let A be a C^*-algebra. We say A is **connective** if there is an injective $*$-homomorphism

$$\Phi : A \longrightarrow \prod_{n=1}^{\infty} \mathcal{C}B(\mathcal{H}) / \bigoplus_{n=1}^{\infty} \mathcal{C}B(\mathcal{H})$$

which is liftable to a cpc map $\phi : A \to \mathcal{C}B(\mathcal{H})$ where $\mathcal{C}B(\mathcal{H}) = C(0,1] \otimes B(\mathcal{H})$.

Theorem 4.27 ([3, Thm 3.1])**.** Let A be a separable nuclear C^*-algebra. A is connective if and only if

$$[[A, B \otimes \mathcal{K}]] \cong E(A,B)$$

for any separable C^*-algebra B.

That connectivity would completely characterize unsuspension is not obvious nor intuitive and the reader is encouraged to read Dadarlat and Pennig's proof in [3]. One benefit of phrasing unsuspension in terms of connectivity is that that connectivity has a wealth of permanence properties which are not clear when viewed from the perspective of unsuspension. We list two particularly important properties below.

Theorem 4.28 ([3, Thm 3.3])**.**

(a) If A is a connective separable nuclear C^*-algebra, then any C^*-subalgebra $B \subseteq A$ is also connective.

(b) If

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A \arrow[r, bend left] & B \arrow[l, bend left] \arrow[r] & 0 \end{tikzcd}$$

is a split short exact sequence of C^*-algebras of which two are connective, then so is the third.

By Theorem 2.4 in [1], we can extend (b) to include non-split extensions when I and B are connective.

Now, a key property of connective C^*-algebras is the absence of non-zero projections (else the $*$-monomorphism described in the definition would not exist) which of course implies connective C^*-algebras are non-unital [3, Rmk 2.8]. Although a lack of non-zero projections is *not* sufficient to conclude that a C^*-algebra is connective (see the counter-example in the introduction), it is *almost* enough. We require, instead, $A \otimes \mathcal{O}_2$ posses no non-zero projections (which is strictly stronger than A possessing no non-zero projections). By Pasnicu and Rørdam's paper [4], $A \otimes \mathcal{O}_2$ contains no non-zero projections if and only if the primitive spectrum of A contains no nonempty compact open subsets. Connectivity is then, in addition to being a geometric lifting property, a topological property of the primitive spectrum.

Theorem 4.29 ([1, Prop 2.7], [5, Cor E])**.** Let A be a separable nuclear C^*-algebra. A is connective if and only if $\mathrm{Prim}(A)$ contains no non-empty compact open subsets.

This is a remarkable result which points to a potential methodology for detecting connectivity. Investigating the primitive spectrum is still a difficult task but the following definition and related proposition do suggest a possible strategy.

Definition 4.30. For A a separable C^*-algebra, we say a point $\pi \in \widehat{A}$ is **shielded** if any sequence $(\pi_i)_i \in \widehat{A} \setminus \{\pi\} \neq \varnothing$ which converges to π has a convergent subsequence to another point $\eta \in \widehat{A} \setminus \{\pi\}$.

Proposition 4.31 ([1, Lemma 2.10])**.** Let A be a unital separable C^*-algebra. If $\pi \in \widehat{A}$ is closed and shielded, then $I = \ker \pi$ is not connective.

This is the key idea we use in Section 6.1. We also note that connectivity is preserved under isomorphism.

Proposition 4.32 ([1, Prop 2.8])**.** For A and B separable nuclear C^*-algebras with homeomorphic spectra, A is connective if and only if B is connective.

In the next section, we will define what it means for a *group* to be connective.

4.5 Connective Groups

Let $\iota : C^*(G) \to \mathbb{C}$ be the trivial representation. We call $I(G) = \ker \iota$ the **augmentation ideal** which is a C^*-subalgebra of $C^*(G)$.

Definition 4.33. For a discrete group G, we say that G is **connective** if $I(G)$ is connective.

Because connective C^*-algebras do not contain any non-zero projections, G must be torisonfree (else we introduce projections into $C^*(G)$ which will find their way into $I(G)$). Moreover, because connectivity passes to C^*-subalgebras, connectivity then necessarily passes to subgroups. However, as may be expected, simply possessing a non-trivial connective subgroup is not sufficient to conclude that the group itself is connective. Yet, we do have the following results.

Proposition 4.34 ([1, Lemma 3.5])**.** Let $m > 1$ and Γ, G countable discrete groups which fit into the following short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1.$$

Suppose that $\phi : G \to \mathbb{Z}$ and $q : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ are group homomorphisms such that $\pi = q \circ \phi$. Then, if Γ is connective, so is G.

This implies that if a connective subgroup sits correctly in the group with torsion cyclic quotient, then the group must necessarily be connective as well. When $\Gamma \cong \mathbb{Z}^n$, we may weaken this requirement.

Theorem 4.35 ([1, Thm 3.8])**.** Let $m, n \in \mathbb{N}$. If G is a countable torsion free discrete group which fits into the following exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow G \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0,$$

then G is connective.

This is a interesting result which immediately provides an infinite number of concrete examples of connective groups. Of course, there are many more groups which do not fit into this sort of exact sequence and so we will need to develop a new strategy to address a larger variety of groups. Consider the following application of shielding:

Theorem 4.36 ([1, Cor 2.11])**.** Let G be a countable discrete group. If $\iota \in \widehat{G}$ is shielded, then $I(G)$ is not connective.

This was the method used by Dadarlat and Pennig in [1, Sec. 3] to prove that the Hantzsche-Wendt group is **not** connective. In Section 6.1, we will generalize this strategy to show that Bieberbach groups with finite abelianization are not connective by proving that shielding occurs at the trivial representation. Before we can show this, however, we must first discuss Bieberbach groups.

CHAPTER 5. BIEBERBACH GROUPS AND THEIR REPRESENTATIONS

5.1 An Introduction to Bieberbach Groups

The primary reference for this section is *Bieberbach Groups and Flat Manifolds* by Charlap [23] which introduces and synthesizes the topic in an effective manner. Although Bieberbach groups are exactly the fundamental groups of compact flat Riemannian manifolds [23, Cor 5.1], this deep connection will not be explored here. There are many treatments of Bieberbach groups which I recommend, including Hiller's article [24] for a light introduction and Szczepanski's book [25] for a more in depth approach. For those interested in a discussion with on emphasis on the connection to flat manifolds, consider chapter 3 in Wolf's book [26].

We begin by introducing concrete Bieberbach groups, which are special subgroups of the isometries of $\mathbb{R}^n$. Then we will discuss abstract Bieberbach groups which will be our preferred formulation.

Let $M_n(\mathbb{R})$ represent the $n \times n$ matrices with entries in $\mathbb{R}$. Two subgroups of $M_n(\mathbb{R})$ of special importance to us will be O_n, the $n \times n$ **orthogonal group**, and GL_n, $n \times n$ **real general linear group**, which are, more explicitly

$$O_n = \{\, A \in M_n(\mathbb{R}) \mid \det A = \pm 1 \,\}$$

and

$$GL_n = \{\, B \in M_n(\mathbb{R}) \mid \det B \neq 0 \,\}.$$

Note 5.1. We may also think of O_n as the collection of $A \in M_n(\mathbb{R})$ such that $AA^T = I$, A^T the transpose of A.

Definition 5.2. A **rigid motion** is an ordered pair $(A, \mathbf{s})$ for $A \in O_n$ and $\mathbf{s} \in \mathbb{R}^n$ which acts on $\mathbb{R}^n$ via

$$(A, \mathbf{s}) \cdot \mathbf{x} = A\mathbf{x} + \mathbf{s}$$

for all $\mathbf{x} \in \mathbb{R}^n$. We denote the collection of rigid motions by $\mathcal{R}_n$.

Definition 5.3. An **isometry** $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which preserves distance, i.e., for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ in the Euclidean metric.

For each $A \in O_n$, there exists a linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^n$ which realizes A for some basis of $\mathbb{R}^n$. Then, if we let $l_{\mathbf{s}} : \mathbb{R}^n \to \mathbb{R}^n$ be the function defined by

$$l_{\mathbf{s}}(\mathbf{x}) = \mathbf{x} + \mathbf{s},$$

every isometry f is of the form

$$f(\mathbf{x}) = l_{\mathbf{s}} \circ L_A(\mathbf{x})$$

for an appropriate L_A and $\mathbf{s}$ [24, Thm (1.0)]. This is to say that isometries and rigid motions of $\mathbb{R}^n$ are, in fact, one and the same. We will use the rigid motion notation in our discussion of concrete Bieberbach groups.

We make $\mathcal{R}_n$ a group under multiplication defined by

$$(A, \mathbf{s}) \cdot (B, \mathbf{t}) = (AB, A\mathbf{t} + \mathbf{s})$$

for $A, B \in O_n$, $\mathbf{s}, \mathbf{t} \in \mathbb{R}^n$. Under this operation, $\mathcal{R}_n \cong O_n \ltimes \mathbb{R}^n$.

Let $(A, \mathbf{s}) \in \mathcal{R}_n$. We define $r : \mathcal{R}_n \to O_n$ by $r(A, \mathbf{s}) = A$. Observe that r is a homomorphism and we will call $r(A, \mathbf{s}) = A$ the **rotational part of** $(A, \mathbf{s})$. We say $(A, \mathbf{s})$ is a **pure rotation** if $\mathbf{s} = \mathbf{0}$.

Next, we define $t : \mathcal{R}_n \to \mathbb{R}^n$ by $t(A, \mathbf{s}) = \mathbf{s}$. Note that t is not a homomorphism. We will call $t(A, \mathbf{s}) = \mathbf{s}$ the **translation part of** $(A, \mathbf{s})$. $(A, \mathbf{s})$ will be called a **pure translation** if $A = I$ and, by abuse of notation, for any G a subgroup of $\mathcal{R}_n$, we will denote the subgroup of pure translations of G by $t(G) = \{(A, \mathbf{s}) \in G \mid A = I\}$.

For $G \leq \mathcal{R}_n$, let $G(\mathbf{x}) = \{(A, \mathbf{s}) \cdot \mathbf{x} \mid (A, \mathbf{s}) \in G\} \subseteq \mathbb{R}^n$ denote the **orbit of x**. We say the set $G(\mathbf{x})$ is **discrete** if each singleton in $G(\mathbf{x})$ is open in the subspace topology on $\mathbb{R}^n$.

We define an equivalence relation on $\mathbb{R}^n$ by $\mathbf{x} \sim \mathbf{y}$ if and only if $G(\mathbf{x}) = G(\mathbf{y})$. The **orbit space** of $\mathbb{R}^n$ over $\sim$ is given by

$$\mathbb{R}^n/G := \mathbb{R}^n/\sim$$

in the quotient topology.

Definition 5.4. We say $G \leq \mathcal{R}_n$ is a **crystallography** group if G is discrete and $\mathbb{R}^n/G$ is compact. Further, we say that G is **Bieberbach** if it is a torsion free crystallography group.

We now discuss Bieberbach's Three Theorems. For the original statements in his native German, see [27], [28].

Theorem 5.5 (Bieberbach's First Theorem, [23, Thm 3.1])**.** If G is a crystallography group, then

(i) $r(G)$ is finite and

(ii) $t(G)$ is a finitely generated free abelian group which spans $\mathbb{R}^n$.

This first part of the theorem tells us that the pure rotations associated to any crystallography group form a finite group and the second part informs us that the pure translations are a lattice of $\mathbb{R}^n$. Because $t(G)$ is a finitely generated free abelian group and spans $\mathbb{R}^n$, this implies that $t(G) \cong \mathbb{Z}^n$ (see [23, p.21]). In fact, any crystallography group $G \leq \mathcal{R}_n$ fits into an exact sequence of the form

$$0 \to t(G) \to G \to r(G) \to 1$$

where $t(G) \cong \mathbb{Z}^n$ and $|r(G)| < \infty$ [23, p.18, 21]. Moreover, $t(G)$ is the unique normal, maximal abelian subgroup of G [23, Prop 4.1]. This exact sequence will allow us to define an abstract Bieberbach group, but we continue with the concrete case for now.

Definition 5.6. An **affine motion** is an ordered pair $(B, \mathbf{t})$ for $B \in GL_n$, $\mathbf{t} \in \mathbb{R}^n$ which acts on $\mathbb{R}^n$ via

$$(B, \mathbf{t}) \cdot \mathbf{x} = B\mathbf{x} + \mathbf{t}$$

for all $\mathbf{x} \in \mathbb{R}^n$. We denote the collection of affine motions by $\mathcal{A}_n$.

Theorem 5.7 (Bieberbach's Second Theorem, see [23, Thm 4.1])**.** Suppose G and G' are crystallographic groups in $\mathcal{R}_n$ and $\phi : G \to G'$ is an isomorphism. Then there exists $(B, \mathbf{t}) \in \mathcal{A}_n$ such that for all $(A, \mathbf{s}) \in G$,

$$\phi((A, \mathbf{s})) = (B, \mathbf{t})(A, \mathbf{s})(B, \mathbf{t})^{-1}.$$

This theorem shows that Bieberbach groups are unique up to an affine change of basis.

Theorem 5.8 (Bieberbach's Third Theorem, [23, Thm 7.1])**.** Up to an affine change in coordinates, there are only finitely many crystallography groups of $\mathcal{R}_n$.

This final theorem means that for each n, there are at most finitely many crystallography groups up to isomorphism. We note that $\mathbb{Z}^n$ is a crystallography group and so there is at least one crystallography group for each n.

Now that we have discussed Bieberbach groups in the original way they were studied, we define abstract Bieberbach groups.

Definition 5.9. We say G is a **crystallography group** of dimension n if G fits into an exact sequence of groups of the form

$$1 \to N \to G \to D \to 1 \tag{5.1}$$

where $N \cong \mathbb{Z}^n$ is maximal abelian and $|D| = k < \infty$. We call N the **lattice** of G and D the **point group** or **holonomy group**. If G is also torsion free, we say G is a **Bieberbach group**.

We will adopt this notation for the remainder of this document.

Note 5.10. N is automatically normal in G.

At first glance, it seems that Definition 5.9 describes a larger class of groups than Definition 5.4. However, as the following theorem shows, these definitions are equivalent.

Theorem 5.11 (Auslander-Kuranishi, [29, Thm 2])**.** If G is a group fitting into an exact sequence of the form in Equation (5.1) above, then there exists an injective homomophism $\Phi : G \to \mathcal{R}_n$ such that $\Phi(G)$ is a crystallography subgroup of $\mathcal{R}_n$.

This allows us to view crystallography groups (and hence Bieberbach groups) as group extensions of $\mathbb{Z}^n$ by a finite group D (making them virtually abelian). Auslander and Kuranshi also showed there are no restrictions on our choice of D.

Theorem 5.12 ([29, Thm 3])**.** Any finite group D is the point group of a Bieberbach group for some dimension n.

We finish this section with a useful Bieberbach group decomposition. In 1957, Calabi proposed a method for decomposing a dimension 4 Bieberbach group with infinite abelianization as an iterated semi-direct product [30] which was later extended to higher dimensions, see Wolf [26] or Szczepanski [25].

Theorem 5.13 (Calabi Decomposition, [25, Prop 3.1])**.** Suppose G is a Bieberbach group of dimension $n \geq 1$ such that there exists a surjective homomorphism $f : G \to \mathbb{Z}$. Then $\ker f = G'$ is a Bieberbach group of dimension $n - 1$.

We observe that the existence of this surjective homomorphism is equivalent to $H_1(G, \mathbb{Z}) = G/[G, G]$ (that is, the abelianization of G) being an infinite group. This means, given a Bieberbach group of dimension n with $H_1(G, \mathbb{Z})$ infinite, we may decompose G as

$$G \cong ((H \rtimes \mathbb{Z}) \rtimes \cdots) \rtimes \mathbb{Z}$$

where we have $m \leq n$ copies of $\mathbb{Z}$ and either

(a) H is trivial or

(b) H is a Bieberbach group with $H_1(H, \mathbb{Z})$ finite.

5.2 Representations of Bieberbach Groups

In general, computing representations for non-abelian groups is difficult. However, in the case of virtually abelian groups (that is, a group which is an extension of an abelian group by a finite group), there is more structure. Because Bieberbach groups are virtually abelian and discrete, the so called Mackey machine is particularly powerful — even if the picture is still frustratingly incomplete. The basic reference for this section is the superb *Induced Representations of Locally Compact Groups* by Kaniuth and Taylor [12].

For G a Bieberbach group, we have an exact sequence of groups

$$1 \to N \to G \xrightarrow{q} D \to 1$$

such that $N \cong \mathbb{Z}^n$ is maximal abelian and $|D| = k < \infty$. Let $q(g) = \dot{g}$. For the sake of convenience, fix a section

$$D = \{\dot{d}_1, \dot{d}_2, ..., \dot{d}_k\}$$

such that $d_1 = e$. We define an action $G \curvearrowright N$ by

$$g \cdot h = ghg^{-1}$$

which descends to an action $D \curvearrowright N$ by

$$\dot{d}_i \cdot h = d_i h d_i^{-1}.$$

We show this is well-defined. For some $\dot{d} \in D$, choose $c \in G$ such that $\dot{c} = \dot{d}$. Then, because $D \cong G/N$, there exists $h_c \in N$ such that

$$ch_c = d \quad \Rightarrow \quad h_c^{-1} c^{-1} = d^{-1}.$$

Thus,

$$\dot{d} \cdot h = dhd^{-1} = \underbrace{(ch_c)}_{d}\, h\, \underbrace{(h_c^{-1}c^{-1})}_{d^{-1}} = c\,(h_c h h_c^{-1})\,c^{-1} = c\,(h_c h_c^{-1} h)\,c^{-1} = chc^{-1} = \dot{c} \cdot h$$

since N abelian.

Using $G \curvearrowright N$, we may also define an action $G \curvearrowright \widehat{N}$ (which will also descend to a well-defined action $D \curvearrowright \widehat{N}$ using reasoning similar to above) by

$$g \cdot \chi(h) = \chi(g^{-1} \cdot h) = \chi(g^{-1}hg).$$

To each $\chi \in \widehat{N}$, we associate its **stabilizer** group

$$G_\chi = \{\, g \in G \mid g \cdot \chi = \chi \,\}$$

and its **orbit**

$$O_\chi = \{\, g \cdot \chi \in \widehat{N} \mid g \in G \,\}.$$

We observe that because N is abelian, $N \leq G_\chi$ for all $\chi \in \widehat{N}$. Moreover, note that the map $G/G_\chi \to O_\chi$ given by $[g] \mapsto g \cdot \chi$ is a bijection. Hence,

$$|G/G_\chi| = |O_\chi|. \tag{5.2}$$

By [12, Prop 4.2], if we are given $\pi \in \widehat{G}$, then there exists $\chi \in \widehat{N}$ such that

$$\pi|_N \sim m_\pi \bigoplus_{g \in G/G_\chi} g \cdot \chi$$

where $\sim$ is unitary equivalence. In fact, we can describe all of $\widehat{G}$ as a set by inducing from irreducible representations of G_χ for carefully chosen χ.

Let H a subgroup in G of finite index and let σ be an irreducible representation of H on the Hilbert space $\mathcal{H}_\sigma$. The **induced representation** $\pi = \mathrm{ind}_H^G \sigma$ is a representation of G on

$$\mathcal{H}_\pi = \{\, \xi : G \to H_\sigma \mid \xi(gh) = \sigma(h^{-1})\xi(g),\, g \in G,\, h \in H \,\} \tag{5.3}$$

defined by the action

$$\pi(x)\xi(y) = \xi(x^{-1}y) \tag{5.4}$$

for all $x, y \in G$. If we denote $G/H = \{\dot{a}_1, \dot{a}_2, ..., \dot{a}_\ell\}$, then we may write

$$\mathcal{H}_\pi \ni \xi \longmapsto (\xi(a_1), \xi(a_2), ..., \xi(a_\ell)) \in \underbrace{\mathcal{H}_\sigma \oplus \mathcal{H}_\sigma \oplus \cdots \oplus \mathcal{H}_\sigma}_{\ell} = \mathcal{H}_\sigma^{\oplus \ell}.$$

In fact, this map a bijective isometry of $\mathcal{H}_\pi$ and $\mathcal{H}_\sigma^{\oplus\ell}$ as Hilbert spaces [12, Sec 2.1].

Let $\Omega \subseteq \widehat{N}$ be a subset which intersects each orbit of $\widehat{N}$ under the action of G exactly once. For each $\chi \in \widehat{N}$, let $\widehat{G}_\chi$ be the unitary dual of G_χ and $\widehat{G}_\chi^{(\chi)}$ denote the subset of $\sigma \in \widehat{G}_\chi$ such that $\sigma|_N \sim m_\sigma \cdot \chi$. We can finally state the primary tool of our main result:

Theorem 5.14 ([12, Thm 4.28])**.** For G a Bieberbach group,

$$\widehat{G} = \{ \operatorname{ind}_{G_\chi}^G \sigma \mid \sigma \in \widehat{G}_\chi^{(\chi)},\, \chi \in \Omega \}.$$

We observe that because we are taking one χ from each orbit in $\widehat{N}$, each $\pi \in \widehat{G}$ induced from $\sigma \in \widehat{G}_\chi^{(\chi)}$ may be sorted into an orbit class based on the order of O_χ. If we let R_t represent the collection of $\pi \in \widehat{G}$ which lives over a character of orbit length t, then we observe that

$$\widehat{G} = \bigsqcup_{t \mid |D|} R_t.$$

Let $N = \langle h_1, h_2, ..., h_n \rangle$, where $\{h_i\}$ is a set of free abelian generators. For any $h \in N$, there exists a vector $[h] = [m_1, m_2, ..., m_n] \in \mathbb{Z}^n$ such that

$$h = h_1^{m_1} h_2^{m_2} \cdots h_n^{m_n}.$$

Clearly, there is a 1-1 correspondence between $h \in N$ and $[h] \in \mathbb{Z}^n$ which is not unsurprising as $N \cong \mathbb{Z}^n$. Using this identification, define the **holonomy representation** $\eta_G : D \to GL_n(\mathbb{Z})$ by

$$\eta_G(\dot{d}_j)(h_i) = [\dot{d}_j \cdot h_i] = [d_j h_i d_j^{-1}].$$

Thus, we can think of D as a subgroup of $GL_n(\mathbb{Z})$.

Note 5.15. If $\dot{d}_s, \dot{d}_t \in D$ such that $d_s h d_s^{-1} = d_t h d_t^{-1}$ for all $h \in N$, then $d_t^{-1} d_s$ commutes with every element of N. Thus $d_t^{-1} d_s$ must be an element of N since N is maximal abelian in G. Since $\dot{d}_s$ and $\dot{d}_t$ are elements of $D \cong G/N$, this means that $\dot{d}_t^{-1} \dot{d}_s = e$ and hence $s = t$.

Now that we have the necessary information about how we build our representations, we proceed to showing some lemmas which will be used in Chapter 6.

5.3 Some Needed Lemmas

Lemma 5.16. Suppose G is a Bieberbach group with point group D and lattice N. If $\pi \in \widehat{G}$, then $\dim \pi \leq |D|$.

Proof. We observe that G is discrete, $N \cong \mathbb{Z}^n$, and $[G : N] = |D|$. By Corollary 3.41, we see $\dim \pi \leq |D|$ for all $\pi \in \widehat{G}$. $\square$

Lemma 5.17. If G is a Bieberbach group which has a finite number of characters such that each character of G has a finite image, then there exists $K \leq N \leq G$ such that $[G : K] < \infty$ and every character of G is trivial on K.

Proof. Let $\widehat{G}_1$ be the set of characters of G which, by assumption, is a finite collection. Define

$$K = \left(\bigcap_{\sigma \in \widehat{G}_1} \ker \sigma \right) \cap N.$$

Clearly, $K \leq N$ and every character of G is trivial on K. All that is left to show is that K is of finite index in G.

Let $\sigma \in \widehat{G}_1$. Then there exists $\ell_\sigma \in \mathbb{N}$ such that the image $\sigma(G) \subseteq \mathbb{T}$ is contained in the ℓ_σ-roots of unity. Because there are only a finite number of characters, there exists a finite $\ell \in \mathbb{N}$ such that the image of every $\sigma \in \widehat{G}_1$ is contained in the ℓ-roots of unity. Hence, if $H = \bigcap_{\sigma \in \widehat{G}_1} \ker \sigma$, then

$$G/H \hookrightarrow \mathbb{Z}/\ell\mathbb{Z}$$

and we conclude that $[G : H] \leq \ell < \infty$. Observing by the second isomorphism theorem that $N/K = N/H \cap N \cong NH/H$,

$$\begin{aligned}
[G : K] &= [G : H \cap N] \\
&= [G : N]\,[N : H \cap N] \\
&= [G : N]\,[NH : H] \\
&\leq [G : N]\,[G : H] \\
&\leq |D| \cdot \ell < \infty
\end{aligned}$$

□

Lemma 5.18. If $G = \langle g_1, ..., g_a | r_1 = \cdots = r_b = e\rangle$ is finitely presented and $(\tau_n)_n$ is a sequence of irreducible representations of G on a shared finite dimensional Hilbert space H_τ, then there is a unitary representation $\tau : G \to U(H_\tau)$ such that a subsequence $(\tau_{n_i})_{n_i}$ converges to τ in the point norm topology.

Proof. Since H_τ is finite dimensional, $U(H_\tau)$ is compact in the point norm topology. Let $(T^i_n)_n = (\tau_n(g_i))_n$ and consider the following process.

Because $U(H_\tau)$ is compact, the sequence $(T^1_n)_n$ has a convergent subsequence to some $T^1 \in U(H_\tau)$. Label this subsequence $n(1)$. Then $(T^2_{n(1)})_{n(1)}$ has a convergent subsequence to some $T^2 \in U(H_\tau)$. Let this subsequence of $n(1)$ be labeled $n(2)$. For each $2 \leq i \leq a$, let T^i be the limit of some subsequence of $(T^i_{n(i-1)})_{n(i-1)}$ and label the subsequence $n(a-1) = m$.

Let $r_i = p_i(g_1, ..., g_a)$ be the appropriate noncommutative polynomial in the g_j describing the i^{th} relation in the given presentation of G. For $2 \leq \ell \leq b$, define

$$(R^\ell_{m(\ell-1)})_{m(\ell-1)} = (p_i(T^1_{m(\ell-1)}, ..., T^a_{m(\ell-1)}))_{m(\ell-1)}.$$

At each stage, choose $m(\ell)$ to be a convergent subsequence of $m(\ell-1)$ and set $o = m(b-1)$.

The map $\tau : G \to U(H_\tau)$ defined by $\tau(g_i) = T^i$ is the limit of the subsequence $(\tau_o)_o$. Moreover, because this subsequence respects the generators and relations of G by construction, τ is a representation of G. □

Lemma 5.19. If α is an inner automorphism of a group G, then $G \rtimes_\alpha \mathbb{Z} \cong G \times \mathbb{Z}$.

Proof. Let $\alpha : \mathbb{Z} \to \mathrm{Aut}\,(G)$ be such that

$$\alpha_z(g) = zgz^{-1} \quad \text{for all } z \in \mathbb{Z}, g \in G$$

and let $f : \mathbb{Z} \to G$ be any homomorphism.

Consider

$$\begin{aligned}(g,e)(f(z),z^{-1})(g,e)^{-1} &= (g,e)(f(z),z^{-1})(g^{-1},e)\\ &= (gf(z),z^{-1})(g^{-1},e)\\ &= (gf(z)\alpha_{f(z)}(g^{-1}),z^{-1})\\ &= (gf(z)(f(z)^{-1}g^{-1}f(z)),z^{-1})\\ &= (f(z),z^{-1})\end{aligned}$$

If we let $Z_f = \{(f(z),z^{-1}) \,|\, z \in \mathbb{Z}\}$, then $\beta : \mathbb{Z} \to Z_f$ given by $\beta(z) = (f(z),z^{-1})$ is an isomorphism and

$$G \rtimes_\alpha \mathbb{Z} \cong G \rtimes_{\alpha\circ\beta^{-1}} Z_f.$$

We observe that Z_f commutes with G. Hence, $G \rtimes_{\alpha\circ\beta^{-1}} Z_f \cong G \times Z_f$. Since $Z_f \cong \mathbb{Z}$, we conclude

$$G \rtimes_\alpha \mathbb{Z} \cong G \times \mathbb{Z}.$$

$\square$

CHAPTER 6. CONNECTIVE BIEBERBACH GROUPS

This chapter is sourced from an article by Dadarlat and myself [31]. I have rearranged the presentation and made some small changes to the proofs but its content is the same. The first section details the main result, the second addresses some unexpected characterizations of connectivity for Bieberbach groups, and the third investigates in which cases connectivity may be detected from the point group.

6.1 Main Result

Theorem 6.1 ([31, Thm 2.4])**.** Suppose G is a Bieberbach group with trivial center. If ω is a character of G, then $\widehat{G} \setminus \{\omega\}$ is a compact open subset of $\widehat{G}$.

We use this theorem to prove:

Theorem 6.2 ([31, Thm 1.1])**.** If G is a Bieberbach group with $H_1(G, \mathbb{Z})$ finite, then G is not connective.

Recall that $H_1(G, \mathbb{Z}) = G/[G, G]$ where $[G, G]$ is the commutator subgroup of G. Thus, the first homology group of G is equivalent to the abelianiziation of G.

The proof of these theorems generalizes a result of Dadarlat and Pennig in [1, Cor 3.2] where they show that the Hantzsche-Wendt group (the only Bieberbach group of dimension 3 with finite first homology) is not connective. Before we begin the proof of Theorem 6.1, we will need two lemmas pertaining to the action of the point group on the lattice of G. Recall that if G is a Bieberbach group, it is a torsion free group fitting into an exact sequence of the form

$$1 \to N \to G \to D \to 1$$

where $N \cong \mathbb{Z}^n$ and $|D| = k < \infty$. In terms of notation, if B acts on A, we let the set of fixed points under this action be denoted

$$A^B = \{a \in A \,|\, b \cdot a = a\}.$$

Definition 6.3. If H is an abelian group, the **rank** of H is

$$\operatorname{rank}(H) = \dim(H \otimes_{\mathbb{Z}} \mathbb{R}).$$

We have the following proposition.

Proposition 6.4 ([31, Prop 2.1])**.** For G a Bieberbach group,

$$\operatorname{rank}(H_1(G, \mathbb{Z})) = \operatorname{rank}(Z(G)) = \operatorname{rank}(N^G).$$

Proof. We begin by showing $Z(G) = N^G$. Indeed, let $z \in Z(G)$. Since z commutes with every element of G, we see that it also commutes with every element of N. Because N is maximal abelian (see Definition 5.9), $z \in N$. Further, for all $g \in G$, $gzg^{-1} = z$ and thus $z \in N^G$. We conclude $Z(G) \subseteq N^G$. To see that $N^G \subseteq Z(G)$, let $h \in N^G$. Then for all $g \in G$, $ghg^{-1} = h$ which means h commutes with every element of G and so $h \in Z(G)$. In particular, $\operatorname{rank}(Z(G)) = \operatorname{rank}(N^G)$.

Next, by [32, Cor 6.4], the short exact sequence of groups

$$1 \to N \to G \to D \to 1$$

induces an exact sequence of groups

$$H_2(D, \mathbb{Z}) \to H_1(N, \mathbb{Z})^D \to H_1(G, \mathbb{Z}) \to H_1(D, \mathbb{Z}) \to 0.$$

Observe that since D is finite, $H_1(D, \mathbb{Z})$ and $H_2(D, \mathbb{Z})$ are also finite groups. This implies by exactness that

$$\operatorname{rank}(H_1(N, \mathbb{Z})^D) = \operatorname{rank}(H_1(G, \mathbb{Z})).$$

Further, $H_1(N, \mathbb{Z}) = N/[N, N] = N$ since N is abelian. Because the action of G on N descends to an action of D on N (see Section 5.2), we see that $N^G = N^D$ and thus $H_1(N, \mathbb{Z})^D = N^G$. Putting this all together, we conclude

$$\operatorname{rank}(N^G) = \operatorname{rank}(H_1(G, \mathbb{Z}))$$

as desired. $\square$

This next proposition requires some knowledge of compact Lie groups and their associated Lie algebras. A helpful resource on this subject is Sepanski's *Compact Lie Groups* [33].

Observe by Note 3.38, $\widehat{\mathbb{Z}}^n \cong \mathbb{T}^n$. And thus, if G acts on $N \cong \mathbb{Z}^n$, G also acts on $\widehat{N} \cong \mathbb{T}^n$ by $g \cdot \chi(h) = \chi(g^{-1}hg)$ for all $h \in N$. By commutativity, the action of G on $\mathbb{T}^n$ descends to an action of D. Thus, we can consider the fixed points of these actions: $(\mathbb{Z}^n)^D$ and $(\mathbb{T}^n)^D$. We note that $(\mathbb{Z}^n)^D$ is a subgroup of $\mathbb{Z}^n$ and $(\mathbb{T}^n)^D$ is a closed subgroup of the Lie group $\mathbb{T}^n$. By [33, Thm 4.14], $(\mathbb{T}^n)^D$ is then in fact a Lie subgroup of $\mathbb{T}^n$.

In a way analogous to the discrete case, we say the **rank** of a Lie group K, denoted $\operatorname{rank}(K)$, is the dimension of one of its Cartan subgroups. Recall that a Cartan subgroup of a compact connected Lie group is a maximal connected abelian subgroup. Cartan subgroups all have the same dimension, so rank is well-defined. Because Cartan subgroups are maximal abelian, if K is itself abelian, then $\operatorname{rank}(K)$ is the same as the dimension of K (see Chapter 5 in [33] for more details).

Proposition 6.5 ([31, Prop 2.2])**.** Suppose G is a Bieberbach group with point group D viewed as a subgroup of $GL_n(\mathbb{Z})$ (see p. 54), then (a) $\operatorname{rank}((\mathbb{Z}^n)^D) = \operatorname{rank}((\mathbb{T}^n)^D)$ and (b) $\operatorname{rank}(N^G) = \operatorname{rank}(\widehat{N}^G)$.

Proof. (b) follows from (a) by Proposition 6.4, so we show (a).

Since D acts on N by automorphisms, we have the following representation $\theta : D \to GL_n(\mathbb{Z}) = \operatorname{Aut}(\mathbb{Z}^n)$ corresponding to this action. By Note 5.15, θ is injective. By choosing a basis, we may write $\theta(s)$ as a $n \times n$ integer matrix, $A(s)$. For $\mathbf{v} \in \mathbb{Z}^n$, the action of D on $\mathbb{Z}^n$ is given by $\mathbf{v} \mapsto A(s)\mathbf{v}$. We can also formulate the dual action for $\chi : \mathbb{Z}^n \to \mathbb{T}$ by $\chi \mapsto \chi(A(s^{-1})\,\cdot)$.

We aim to show that $\operatorname{rank}((\mathbb{Z}^n)^D) = \operatorname{rank}((\mathbb{T}^n)^D)$. Consider the following spaces:

$$\begin{aligned}\mathbf{H} &:= (\mathbb{Z}^n)^D = \{\, \mathbf{v} \in \mathbb{Z}^n \mid A(s)\mathbf{v} = \mathbf{v}, s \in D \,\}\\ \mathbf{K} &:= (\mathbb{T}^n)^D = \{\, \chi \in \widehat{\mathbb{Z}}^n \mid \chi(A(s^{-1}\mathbf{v})) = \chi(\mathbf{v}), s \in D, \mathbf{v} \in \mathbb{Z}^n \,\}\\ \mathbf{W} &:= \{\, \mathbf{a} \in \mathbb{R}^n \mid A(s)^T\mathbf{a} = \mathbf{a}, s \in D \,\}\end{aligned}$$

where B^T is the matrix transpose of B. Observe that

$$\{\, \mathbf{v} \in \mathbb{R}^n \mid A(s)\mathbf{v} = \mathbf{v}, s \in D \,\}$$

is the set of points in $\mathbb{R}^n$ fixed by the action of D. Because D is acting by integer-valued matrices,

$$\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R} \cong \{\, \mathbf{v} \in \mathbb{R}^n \mid A(s)\mathbf{v} = \mathbf{v}, s \in D \,\}.$$

We will first demonstrate that

$$\operatorname{rank}(\mathbf{H}) = \dim(\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}) \tag{6.1}$$

and

$$\operatorname{rank}(\mathbf{K}) = \dim(\mathbf{W}). \tag{6.2}$$

Second, we will show that

$$\dim(\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}) = \dim(\mathbf{W}). \tag{6.3}$$

We begin with Equation (6.1). $\mathbf{H}$ is a subgroup of $\mathbb{Z}^n$ and thus there must be $0 \leq p \leq n$ such that $\mathbf{H} \cong \mathbb{Z}^p$, which is to say that $\operatorname{rank}(\mathbf{H}) = p$. Now, $\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space over $\mathbb{R}$ and is, in fact, the solution space of $A(s)\mathbf{v} - \mathbf{v} = \mathbf{0}$. Using Gaussian elimination, it should be clear that this solution space is

$$\mathbb{R}^p \cong \mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R} \cong \{\, \mathbf{v} \in \mathbb{R}^n \mid A(s)\mathbf{v} - \mathbf{v} = \mathbf{0}, s \in D \,\}.$$

Therefore, $\dim(\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}) = p = \operatorname{rank}(\mathbf{H})$ and we have Equation (6.1).

Next, we show Equation (6.2). $\mathbf{K}$, as mentioned above, is a compact abelian Lie subgroup of $\mathbb{T}^n$. By Theorem 5.2(a) in [33], there exists a connected closed subgroup

$\mathbf{T}$ of $\mathbb{T}^n$ isomorphic to $\mathbb{T}^q$ for some $0 \le q \le n$ and a finite group $F \subseteq \mathbb{T}^n$ such that $\mathbf{K} = F \times \mathbf{T}$ chosen so $F \cap \mathbf{T} = \{1\}$. By Theorem 5.4 and the main theorem of Section 7.22 in [33], $\dim(\mathbf{T}) = q = \operatorname{rank}(\mathbf{K})$.

Now, consider the set $\mathbf{W}' = \{\mathbf{a} \in \mathbb{R}^n \mid e^{2\pi i \mathbf{a}} \in \mathbf{T}, t \in \mathbb{R}\}$. This is the Lie algebra of $\mathbf{T}$ and thus $\operatorname{rank}(\mathbf{W}') = \dim(\mathbf{T})$ by definition. We will show that $\mathbf{W}' = \mathbf{W}$.

For $\mathbf{a} \in \mathbf{W}'$ and $t \in \mathbb{R}$, let χ be the character of $\mathbb{Z}^n$ corresponding to $e^{2\pi i t \mathbf{a}}$. The condition $\chi(A(s^{-1})\mathbf{v}) = \chi(\mathbf{v})$ is equivalent to

$$e^{2\pi i \langle t\mathbf{a}, \mathbf{v}\rangle} = e^{2\pi i \langle t\mathbf{a}, A(s^{-1})\mathbf{v}\rangle} = e^{2\pi i \langle A(s^{-1})^T t\mathbf{a}, \mathbf{v}\rangle} \quad \text{for } \mathbf{v} \in \mathbb{Z}^n. \tag{6.4}$$

We know for all $s \in D$,

$$1 = e^{2\pi i \langle t\mathbf{a}, \mathbf{v}\rangle} e^{-2\pi i \langle A(s^{-1})^T t\mathbf{a}, \mathbf{v}\rangle} = e^{2\pi i \langle t\mathbf{a} - A(s^{-1})^T t\mathbf{a}, \mathbf{v}\rangle},$$

which implies $\langle t\mathbf{a} - A(s^{-1})^T t\mathbf{a}, \mathbf{v}\rangle$ is an integer. Since $\mathbf{v} \in \mathbb{Z}^n$, we conclude that $t\mathbf{a} - A(s^{-1})^T t\mathbf{a} \in \mathbb{Z}^n$ for all $s \in D$ as well. But $\mathbb{Z}^n$ is a discrete space and so for all $\mathbf{a} \in \mathbf{W}'$, $s \in D$, $t \in \mathbb{R}$,

$$t\mathbf{a} - A(s^{-1})^T t\mathbf{a} = \mathbf{a} - A(e)^T \mathbf{a} = \mathbf{0}. \tag{6.5}$$

Hence, $\mathbf{W}' \subseteq \mathbf{W}$. If $\mathbf{a} \in \mathbf{W}$, then clearly Equation (6.5) satisfies Equation (6.4) and so $\mathbf{W} \subseteq \mathbf{W}'$.

We have then shown $\dim(\mathbf{T}) = q = \operatorname{rank}(\mathbf{K})$, $\operatorname{rank}(\mathbf{W}') = \dim(\mathbf{T})$, and $\mathbf{W} = \mathbf{W}'$, which establishes Equation (6.2).

Finally, we show $\mathbf{W}$ and $\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}$ have the same dimension as vector spaces (Equation (6.3)) where we observe that

$$\mathbf{W} = \{\, \mathbf{a} \in \mathbb{R}^n \mid A(s^{-1})^T \mathbf{a} = \mathbf{a}, s \in D \,\} = \{\, \mathbf{a} \in \mathbb{R}^n \mid A(s)^T \mathbf{a} = \mathbf{a}, s \in D \,\}.$$

Let V be a real vector space and $\theta : D \to GL_n(\mathbb{R})$ a finite group representation. We define

$$V^D = \{\, \mathbf{v} \in V \mid \theta(s)\mathbf{v} = \mathbf{v}, s \in D \,\}.$$

Our goal is to show that $\dim V^D = \dim(V^*)^D$, which we will then apply to $\mathbf{W}$ and $\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}$.

Recall from elementary linear algebra that for a linear map $T : V \to V$, we may define its dual map $T^* : V^* \to V^*$ with the second dual $T^{**} : V^{**} \to V^{**}$ canonically identified with T, and thus we may identify V^D with $(V^{**})^D$. Because $\dim V^D = \dim(V^{**})^D$, if we show that

$$\dim V^D \leq \dim(V^*)^D \quad \text{and hence} \quad \dim(V^*)^D \leq \dim(V^{**})^D,$$

we have shown that $\dim V^D = \dim(V^*)^D$.

Let $E : V \to V$ be a linear map such that $E^2 = E$ and $E(V) = V^D$. From E, we define an idempotent $P : V \to V$ by

$$P = \frac{1}{|D|} \sum_{t \in D} \theta(t^{-1}) E \theta(t).$$

Claim:

(i) P commutes with $\theta(s)$ for all $s \in D$

(ii) P is a projection with $P(V) = E(V) = V^D$

(iii) $P\theta(s) = P$ for all $s \in D$

To show (i), observe that for $s \in D$,

$$\begin{aligned} \theta(s)P &= \theta(s)\frac{1}{|D|} \sum_{t \in D} \theta(t^{-1}) E \theta(t) \\ &= \frac{1}{|D|} \sum_{t \in D} \theta(s)\theta(t^{-1}) E \theta(t) \\ &= \frac{1}{|D|} \sum_{t \in D} \theta(st^{-1}) E \theta(t) \end{aligned}$$

If we let $r^{-1} = st^{-1}$, then $t = rs$. So, we may write

$$\begin{aligned} \frac{1}{|D|} \sum_{t \in D} \theta(st^{-1}) E \theta(t) &= \frac{1}{|D|} \sum_{r \in D} \theta(r^{-1}) E \theta(rs) \\ &= \frac{1}{|D|} \sum_{r \in D} \theta(r^{-1}) E \theta(r)\theta(s) \\ &= P\theta(s) \end{aligned}$$

To show (ii), we check that P is a projection onto V^D. Let $w \in V^D$ and so $\theta(s)w = w$ for all $s \in D$. Consider

$$\begin{aligned}
Pw &= \frac{1}{|D|}\sum_{t\in D}\theta(t^{-1})E\underbrace{\theta(t)w}_{=w}\\
&= \frac{1}{|D|}\sum_{t\in D}\theta(t^{-1})\underbrace{Ew}_{=w}\\
&= \frac{1}{|D|}\sum_{t\in D}\underbrace{\theta(t^{-1})w}_{=w}\\
&= \frac{1}{|D|}\sum_{t\in D}w = w
\end{aligned}$$

Suppose $v \in V$ with $\theta(s)v = v_s$. Then

$$\begin{aligned}
Pv &= \frac{1}{|D|}\sum_{t\in D}\theta(t^{-1})E\theta(t)v\\
&= \frac{1}{|D|}\sum_{t\in D}\theta(t^{-1})\underbrace{Ev_t}_{=w\in V^D}\\
&= \frac{1}{|D|}\sum_{t\in D}\underbrace{\theta(t^{-1})w}_{=w}\\
&= \frac{1}{|D|}\sum_{t\in D}w = w \in V^D
\end{aligned}$$

Thus, we see that $P(V) = V^D$ and $P^2 = P$.

Now, to check (iii), note that $Pv \in V^D$ for all $v \in V$ and for any $w \in V^D$, $\theta(s)w = w$. This finishes the claim.

We return to showing $\dim V^D = \dim(V^*)^D$. By the claim, we have

$$\theta(s)P = P\theta(s) = P = P^2 \quad \text{for all } s \in D.$$

Passing to the dual,

$$P^*\theta(s)^* = \theta(s)^*P^* = P^* = (P^*)^2 \quad \text{for all } s \in D.$$

But this implies that $P^*(V^*) \subseteq (V^*)^D$ because $\theta(s)^*v^* = v^*$ for $v^* \in (V^*)^D$. Since $\operatorname{rank}(P) = \operatorname{rank}(P^*)$, we see that $\dim(V^D) \le \dim(V^*)^D$. In light of our previous comments, we may conclude $\dim(V^D) = \dim(V^*)^D$.

Observe that

$$\mathbf{W} = \{\, \mathbf{a} \in \mathbb{R}^n \mid A(s)^T \mathbf{v} = \mathbf{v} \,\} \quad \text{and} \quad \mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R} = \{\, \mathbf{a} \in \mathbb{R}^n \mid A(s)\mathbf{v} = \mathbf{v} \,\}$$

are dual to each other. We conclude that $\dim(\mathbf{W}) = \dim(\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R})$, which shows Equation (6.3). Therefore, $p = q$ and so $\operatorname{rank}((\mathbb{Z}^n)^D) = \operatorname{rank}((\mathbb{T}^n)^D)$. □

Proposition 6.6 ([34, Prop 1.4], [31, Prop 2.2])**.** For G a Bieberbach group, the following are equivalent:

(i) $H_1(G, \mathbb{Z})$ is finite

(ii) G has trivial center, that is, $Z(G) = \{e\}$

(iii) The action of G on N has exactly one fixed point, that is, $N^G = \{e\}$

(iv) The action of G on $\widehat{N}$ has finitely many fixed points, that is, $\widehat{N}^G$ is a finite group

Items (i)-(iii) were shown to be equivalent in [34] and all were shown to be equivalent in [31].

Proof.

((i)⇒(ii)): Suppose $H_1(G, \mathbb{Z}) = G/[G, G]$ is finite. By Proposition 6.4, we know that $\operatorname{rank}(H_1(G, \mathbb{Z})) = \operatorname{rank}(Z(G))$ and so $Z(G)$ is also finite. However, since $Z(G) \leq G$ for G torsion free, we conclude that $Z(G)$ is trivial.

((ii)⇒(iii)): Suppose $Z(G) = \{e\}$. By the proof of Proposition 6.4, we know that $Z(G) = N^G$, hence $N^G = \{e\}$.

((iii)⇒(iv)): Suppose $N^G = \{e\}$. By Proposition 6.5, $\operatorname{rank}(N^G) = \operatorname{rank}(\widehat{N}^G)$ and so $\widehat{N}^G$ is finite.

((iv)⇒(iii)): If $\widehat{N}^G$ is finite and it has the same rank as N^G, then N^G is also finite. But, $N^G \leq G$ for G torsion free. Hence, N^G is trivial.

((iii)⇒(i)): Suppose $N^G = \{e\}$. By Proposition 6.5, $\operatorname{rank}(H_1(G, \mathbb{Z})) = \operatorname{rank}(N^G)$. This implies that $H_1(G, \mathbb{Z})$ is finite. □

We can now address the proof of Theorem 6.1. What is provided here is the proof developed by Dadarlat and this author found in [31, Thm 2.4] up to a slight change in wording. By design, this proof generalizes the method used in [1] to show that the Hantzsche-Wendt group is not connective.

Proof of Theorem 6.1. Suppose G is a Bieberbach group with trivial center. By Proposition 6.6, this implies that $H_1(G, \mathbb{Z}) = G/[G, G]$ is finite. Note that the characters of G must factor through $[G, G]$ and so G has only finitely many characters. Let $\omega \in \widehat{G}$ be one of these characters. The kernels of characters are closed in the topology of $\widehat{G}$ since they maximal among primitive ideals [8, Prop 3.1.4] and thus $\widehat{G} \setminus \{\omega\}$ is open. $\widehat{G}$ is also second countable because $C^*(G)$ is separable (Corollary 3.24). This means we may use the sequential definition of compactness. Our goal will then be to show that for any sequence $(\pi_i)_i \in \widehat{G} \setminus \{\omega\}$ converging to ω, $(\pi_i)_i$ contains a subsequence which converges to point $\gamma \in \widehat{G} \setminus \{\omega\}$. This is to say that every character in $\widehat{G}$ is shielded (Theorem 4.36).

Let $(\pi_i)_i \in \widehat{G} \setminus \{\omega\}$ be a sequence which converges to ω. For any irreducible representation π of G, $\dim \pi \leq |D|$ by Lemma 5.16. Thus, $\dim \pi_i \leq |D|$ for all i and, by passing to a subsequence, we may assume without loss of generality that $\dim \pi_i = m$ for some $1 \leq m \leq |D|$. By Mackey's remarkable result (Theorem 5.14), we know that associated to each π_i, there is $\chi_i \in \Omega$ and $\sigma_i \in \widehat{G}_{\chi_i}^{(\chi_i)}$ such that $\pi_i = \mathrm{ind}_{G_{\chi_i}}^G \sigma_i$ up to unitary equivalence.

For each $\chi \in N$, its associated stabilizer group G_χ contains N. Because $D = G/N$ is finite, there can only be a finite number of groups, say L, such that $N \leq L \leq G$. Hence, we conclude there are only a finite number of distinct G_χ for $\chi \in \Omega$. Passing again to a subsequence, we may assume without loss of generality that $L = G_{\chi_i}$ for all χ_i.

We have two cases we must consider: (a) $L = G$ or (b) $L \neq G$.

Case (a): Suppose that $G_{\chi_i} = L = G$ for all i. This is to say, for all i, G fixes χ_i. We will prove that in this case we reach a contradiction wherein $(\pi_i)_i$ does not converge to ω as assumed.

Since each χ_i is fixed by the action of G, $\chi_i \in \widehat{N}^G$ for all i. Because we have assumed $Z(G)$ is trivial, Proposition 6.6 implies that $\widehat{N}^G$ is finite. By again passing to a subsequence in $(\pi_i)_i$, we may assume that $\chi = \chi_i$ for all i. Hence, for all i,

$$\pi_i = \sigma_i \in \widehat{G}^{(\chi)}_{\chi} = \widehat{G}^{(\chi)}.$$

This means that $\pi_i|_N = m \cdot \chi$ for all i where $\dim \pi_i = m$. We argue that this implies $\chi = \omega|_N$.

Suppose by way of contradiction that $\chi \neq \omega|_N$, which means there exists some $h \in N$ such that $\chi(h) - \omega(h) \neq 0$. Consider the element $a = \chi(h)e - h$ in $C^*(G)$. By abuse of notation, view χ as the representation extended to all of $C^*(G)$. Then

$$\chi(a) = \chi(\chi(h)e - h) = \chi(h)\underbrace{\chi(e)}_{1} - \chi(h) = 0.$$

Because $\pi_i|_N = m \cdot \chi$ and $e, h \in N$, we see $\pi_i(a) = 0$ for all i. We may reformulate this as $a \in \bigcap_i \ker \pi_i$. But $a \notin \ker \omega$ as

$$\omega(a) = \omega(\chi(h)e - h) = \chi(h)\underbrace{\omega(e)}_{1} - \omega(h) \neq 0$$

by design. We conclude that $(\pi_i)_i$ does not converge to ω, contradicting our initial assumption.

Hence, we must instead have $\pi_i|_N = m \cdot \chi = m \cdot \omega|_N$. We make the following claim.

Claim: If $\sigma : G \to \mathbb{C}$ is a character of G, then $\sigma(G)$ is a finite group.

Since σ is a homomorphism, $\sigma(G)$ is indeed a subgroup of $\mathbb{C} \setminus \{0\}$. Moreover, for any $g, h \in G$,

$$\sigma(h^{-1}g^{-1}hg) = \sigma(h)^{-1}\sigma(g)^{-1}\sigma(h)\sigma(g) = \sigma(g)^{-1}\sigma(g)\sigma(h)^{-1}\sigma(h) = 1$$

by commutativity of $\mathbb{C}$. Therefore, σ factors through $H_1(G, \mathbb{Z}) = G/[G, G]$, a finite group by assumption. We conclude $\sigma(G)$ must also be finite. This completes the claim.

By the previous comments, there are finitely many characters of G and the image of each character is finite, and so by applying Lemma 5.17, there exists $K \leq N \leq G$ such that $[G : K] < \infty$ and $\sigma(K) = \{1\}$ for all σ characters of G. Hence,

$$\pi_i|_K = m \cdot \omega|_K = m \cdot \iota|_K.$$

Because π_i is trivial on K, π_i factors through $q : G \to G/K$ for all i where we note that G/K is a finite group. Thus, each π_i is associated uniquely to one irreducible representation of G/K. The sequence $(\pi_i)_i$, then, has only finitely many distinct terms and thus must be eventually constant for it to converge to ω as $\{\omega\}$ is closed in $\widehat{G}$. But, this means that π_i is eventually equal to ω contradicting our assumption that $\pi_i \in \widehat{G} \setminus \{\omega\}$. This concludes the case when $G_{\chi_i} = L = G$.

Case (b): Assume that $G_{\chi_i} = L \neq G$ for all i. Assign $r = [G : L] > 1$ and choose elements $e_1, ..., e_r \in G$ such that e_1 is the identity element of G and we may decompose $G = e_1 L \cup \cdots \cup e_r L$. Recalling that $\pi_i = \operatorname{ind}_L^G \sigma_i$ and so if $\dim \pi_i$ is equal for all i, then $\dim \sigma_i$ must also be equal for all i. Thus, regardless of i, we may fix a single Hilbert space, $\mathcal{H}_\sigma = \mathbb{C}^d$, on which all the σ_i are realized. By Equation (5.3), we then have the irreducible representation

$$\pi_i = \operatorname{ind}_L^G \sigma_i : G \to B(\mathcal{H}_{\pi_i})$$

acting on

$$\mathcal{H}_{\pi_i} = \{\, \xi : G \to \mathcal{H}_\sigma \mid \xi(gh) = \sigma_i(h)^{-1}\xi(g), g \in G, h \in L \,\}$$

by $\pi_i(g)\xi(x) = \xi(g^{-1}x)$ for all $g, x \in G$. Since $\xi(e_j h) = \sigma_i(h)^{-1}\xi(e_j)$ for $h \in L$, $\xi \mapsto \xi(e_j)$ is well-defined.

Let $\mathcal{H}_\pi = \mathcal{H}_\sigma^{\oplus r}$. For each i, define $V_i : \mathcal{H}_{\pi_i} \to \mathcal{H}_\pi$ by

$$V_i(\xi) = (\xi(e_1), ..., \xi(e_r)).$$

The adjoint of V_i, $V_i^* : \mathcal{H}_\pi \to \mathcal{H}_{\pi_i}$, maps $(\xi_1, ..., \xi_r) \in \mathcal{H}_\pi$ to a function $\xi : G \to \mathcal{H}_\sigma$ such that $\xi(e_j h) = \sigma_i(h)^{-1}\xi_j$ for all $h \in L$. Observe that V_i is a unitary operator.

Consider the unitary representation $\rho_i = V_i \pi_i(\cdot) V_i^*$. Let $\mathbf{v} = (\xi_1, 0, ..., 0)$ for some $\xi_1 \in \mathcal{H}_\sigma$. Then

$$\begin{aligned} \rho(e_r)\mathbf{v} &= V_i \pi_i(e_r) V_i^*(\xi_1, 0, ..., 0) \\ &= V_i \pi_i(e_r)\xi \text{ such that } \xi(h) = \sigma_i(h)^{-1}\xi_1 \text{ for } h \in L \\ &= V_i \xi(e_r^{-1}\cdot) \end{aligned}$$

Because ξ is supported entirely on $e_1 L$, we see that $\xi(e_r^{-1}\cdot)$ is supported entirely on $e_r L$. This give us

$$\begin{aligned} V_i \xi(e_r^{-1}\cdot) &= (\xi(e_r^{-1}e_1), \xi(e_r^{-1}e_2), ..., \xi(e_r^{-1}e_r)) \\ &= (0, ..., 0, \xi(e_1)) \\ &= (0, ..., 0, \xi_1) \end{aligned}$$

Let $E : H_\sigma^{\oplus r} \to \mathcal{H}_\sigma^{\oplus r}$ be the projection defined by $E(\xi_1, ..., \xi_r) = (\xi_1, 0, ..., 0)$. By our work above, $E\rho_i E = 0$ for all i.

Because H_π is finite dimensional and Bieberbach groups are finitely presented, we may apply Lemma 5.18 to extract a subsequence $(\rho_{i_j})_{i_j}$ of $(\rho_i)_i$ which converges in the point norm topology of $U(\mathcal{H}_\pi)$ to some unitary representation ρ of G, which is to say $\|\rho_{i_j}(g) - \rho(g)\| \to 0$ for all $g \in G$. Thus, $E\rho E = 0$.

Now, suppose $\rho = m \cdot \omega$. Then $\rho(e_r) = m \cdot \omega(e_r)$ which, as a scalar multiple of the identity, would commute with E. Note that $E\rho(e_r) = E(m \cdot \omega(e_r)) = \omega(e_r)$ because E projects onto the first summand. This implies

$$0 = \|E\rho(e_r)E\| = \|\omega(e_r)E\| = \|\omega(e_r)\| = 1$$

since ω is a character and hence of norm 1. Clearly this is a contradiction and we conclude that ρ is **not** a multiple of ω.

Decomposing ρ into irreducible representations of G, we conclude there must be at least one irreducible γ which is distinct from ω. Thus, there exists a subsequence of $(\rho_i)_i$ which converges to γ. But each ρ_i is π_i via a change of basis. Therefore, by passing to a subsequence, we conclude that $(\pi_i)_i$ converges to $\gamma \in \widehat{G} \setminus \{\omega\}$. This completes the proof. $\square$

We can now show Theorem 6.2.

Proof of Theorem 6.2. Let G be a Bieberbach group with $|H_1(G,\mathbb{Z})| < \infty$. By Theorem 6.1, we see that $\widehat{G} \setminus \{\iota\}$ is a compact open subset of $\widehat{G}$ where ι is the trivial character of G. This means that ι is shielded. Applying Theorem 4.36, we conclude that G is not connective. $\square$

This is a compelling result but it is not quite an equivalence. There are, in fact, Bieberbach groups with infinite first homology but which contain *nonconnective* subgroups. Fortunately, a sufficient modification of this statement will produce a complete characterization of connectivity for Bieberbach groups. There are, in addition, some interesting group theoretic properties that are also (unexpectedly) equivalent to connectivity.

6.2 Characterization of Connective Bieberbach Groups

Although formulated as a geometric property of C^*-algebras, connectivity for Bieberbach groups also has a purely group theoretic realization. This section will address how to characterize connectivity for Bieberbach groups and is again a slight rewriting of [31].

Definition 6.7. Let G be a group. We say that G is **poly-$\mathbb{Z}$** if G contains a finite increasing series of subgroups

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_\ell = G$$

where $G_i \trianglelefteq G_{i+1}$ and $G_{i+1}/G_i \cong \mathbb{Z}$ for all $1 \leq i \leq \ell - 1$.

Observe that for each $1 \leq i \leq \ell - 1$, we have the following short exact sequence

$$1 \longrightarrow G_i \xrightarrow{i} G_{i+1} \xrightarrow{q} \mathbb{Z} \longrightarrow 1.$$

This sequence splits, which is to say there exists $s : \mathbb{Z} \to G_{i+1}$ such that $q \circ s = \mathrm{id}_{G_{i+1}}$. Thus, $G_{i+1} \cong G_i \rtimes_\alpha \mathbb{Z}$ for some $\alpha \in \mathrm{Aut}\,(G_i)$ by the splitting lemma (see, for example, [32, Prop (2.1)]).

Recall that an inner automorphism of a group H is a group homomorphism $\phi_h : H \to H$ defined by $\phi_h(x) = hxh^{-1}$ for all $x \in H$. We denote the collection of inner automorphisms by $\mathrm{Inn}\,(H)$.

Corollary 6.8 ([31, Cor 3.2])**.** Let G be a countable discrete connective group. Suppose $\alpha \in \mathrm{Aut}\,(G)$ is such that $\alpha^m \in \mathrm{Inn}\,(G)$ for some $m \geq 1$. Then $G \rtimes_\alpha \mathbb{Z}$ is connective.

Proof. Let $\beta := \alpha^m$. Because $\beta \in \mathrm{Inn}\,(G)$ by assumption, we know $G \rtimes_\beta \mathbb{Z} \cong G \times \mathbb{Z}$ by Lemma 5.19. By Cor 3.3 of [35], $G \times \mathbb{Z}$ is connective and thus so is $G \rtimes_\beta \mathbb{Z}$. Define $\phi : G \rtimes_\beta \mathbb{Z} \to G \rtimes_\alpha \mathbb{Z}$ by $\phi(x, k) = (x, mk)$ for $x \in G$, $m, k \in \mathbb{Z}$. This is an injective map and induces an exact sequence of groups

$$1 \longrightarrow G \rtimes_\beta \mathbb{Z} \xrightarrow{\phi} G \rtimes_\alpha \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1$$

where $\pi = q_2 \circ q_1$ is the composition of quotient maps $q_1 : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ and $q_2 : G \rtimes_\alpha \mathbb{Z} \to \mathbb{Z}$. By applying Proposition 4.34, we conclude that $G \rtimes_\alpha \mathbb{Z}$ is connective. $\square$

We need two more results before we can state our characterization.

Proposition 6.9 ([25, Thm 3.2])**.** Let G be a Bieberbach group with $\alpha \in \mathrm{Aut}\,(G)$. $G \rtimes_\alpha \mathbb{Z}$ is Bieberbach if and only if $\alpha^m \in \mathrm{Inn}\,(G)$ for some $m \geq 1$.

Theorem 6.10 ([36, Thm 23])**.** Suppose G is a Bieberbach group. G is poly-$\mathbb{Z}$ if and only if every nontrivial subgroup of G has nontrivial center.

We can now state and prove the following theorem:

Theorem 6.11 ([31, Thm 1.2])**.** Let G be a Bieberbach group. The following are equivalent:

(i) G is connective.

(ii) G is poly-$\mathbb{Z}$.

(iii) Every nontrivial subgroup of G has a nontrivial center.

(iv) $\widehat{G} \setminus \{\iota\}$ has no non-empty compact open subsets.

Proof.

((i)⇒(ii)): Suppose G is connective. We recall that connectivity passes to subgroups and thus each subgroup of G is also connective. Because G is assumed to be Bieberbach, we know that it has a nontrivial center by Theorem 6.2 and Proposition 6.6. As we saw before on p. 51, G with infinite abelianization may be written as an iterated semidirect product

$$G \cong ((H \rtimes \mathbb{Z}) \rtimes \cdots) \rtimes \mathbb{Z}$$

where either

(a) $H = \{e\}$ or

(b) H is a Bieberbach group with finite abelianization.

Case (b) is not viable by Theorem 6.2 combined with the comments above. Hence, we find ourselves in case (a) and conclude

$$G \cong ((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \cdots) \rtimes \mathbb{Z},$$

which is to say that G is poly-$\mathbb{Z}$.

((ii)⇒(i)): Suppose G is poly-$\mathbb{Z}$ and Bieberbach of dimension $n \geq 1$. Let

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_\ell = G$$

be the finite increasing series of subgroups as in Definition 6.7. Because $G_{i+1}/G_i \cong \mathbb{Z}$ for all $1 \leq i \leq \ell - 1$, we must have $G_1 = \mathbb{Z}$ (which we observe to be connective). Further, $G_{i+1} = G_i \rtimes_{\alpha_i} \mathbb{Z}$ for some $\alpha_i \in \mathrm{Aut}\,(G_i)$, $1 \leq i \leq \ell - 1$. Since G is of dimension n, this construction combined with $G_1 = \mathbb{Z}$, $G_\ell = G$ implies that $n = \ell$.

By Definition 6.7, we also have $G_i \trianglelefteq G_{i+1}$ for all $1 \leq i \leq n - 1$. By lemma on p. 576 in [36], if $K \trianglelefteq G$ for G Bieberbach, then K is also Bieberbach. This implies

G_i is Bieberbach for each $1 \leq i \leq n-1$. Therefore, by Proposition 6.9, there exists m_i such that $\alpha_i^{m_i} \in \mathrm{Inn}\,(G_i)$ for all $1 \leq i \leq n-1$.

Because $G_1 = \mathbb{Z}$ is connective, $G_2 = G_1 \rtimes_{\alpha_1} \mathbb{Z}$ is also connective by Corollary 6.8. We apply this process iteratively to see $G_{i+1} = G_i \rtimes_{\alpha_i} \mathbb{Z}$ is connective for all $1 \leq i \leq n-1$. We conclude that $G_n = G$ is connective as desired.

((ii) $\Longleftrightarrow$ (iii)): Apply Theorem 6.10.

((i) $\Longleftrightarrow$ (iv)): By Thoma's theorem [15], G virtually abelian implies $C^*(G)$ is Type I and as such by p. 27 we have

$$\mathrm{Prim}\,(I(G)) = \widehat{I(G)} = \widehat{G} \setminus \{\iota\}.$$

We apply Theorem 4.29 to complete the proof. □

6.3 Detecting Connectivity from the Point Group

Due to the complexity of Bieberbach groups of dimension 4 or higher, concisely written group presentations are typically not readily available. However, one piece of datum always included in the description of a Bieberbach group is the point group. Thus, a method which allows us to determine the connectivity of Bieberbach groups from just the point group would be very useful. Although the connectivity of a Bieberbach group is not totally determined by its point group, we can say something definite in certain cases. As with the previous two sections, this is taken from [31].

We begin with a concept first introduced by Bowditch in [37].

Definition 6.12. Let G be a torsion free group. We say that G is **diffuse** if for any non-empty finite subset A of G, there is an element $a \in A$ such that for all $g \in G$, either ga or $g^{-1}a$ is not in A.

Diffuseness relates to Kaplansky's unit conjecture as diffuse groups have the unique product property. Moreover, diffuseness is equivalent to *local indicability* when G is discrete and amenable [38].

Definition 6.13. A group G is **locally indicable** if every nontrivial finitely generated subgroup of G has an infinite cyclic quotient.

G is then locally indicable if, for every nontrivial finitely generated $L \leq G$, $H_1(L, \mathbb{Z})$ is infinite.

Theorem 6.14 ([38, Thm 6.4])**.** An amenable discrete group is diffuse if and only if it is locally indicable.

We observe that Bieberbach groups are finitely generated. Any subgroup of a Bieberbach group is a finitely generated free abelian group extended by a finite group. Thus, every subgroup of a Bieberbach group is finitely generated. We see by Theorem 6.11 that Bieberbach groups are connective if and only if they are locally indicable if and only if they are diffuse. We can use these ideas to formulate a corollary to Theorem 6.11.

Corollary 6.15 ([31, Cor 1.3])**.** Let D be a finite group.

(a) If D is not solvable, then any Bieberbach group with point group D is not connective.

(b) If all the Sylow p-subgroups of D are cyclic (in which case D is automatically solvable), then any Bieberbach group with point group D is connective.

(c) If D is solvable and has a non-cyclic Sylow subgroup, then there are Bieberbach groups G_1 connective and G_2 not connective both with point group D.

Before we prove this corollary, we briefly address Sylow p-subgroups and *primitive* groups.

Definition 6.16. Let G be a group of order $p^{\alpha}m$ ($\alpha \geq 0$) where p is prime and does not divide m. Then any subgroup of G of order p^{α} is a **Sylow p-subgroup**.

Sylow's theorem tells us that such a subgroup always exists and any two Sylow p-subgroups in G are conjugate (see [39, Thm 18]).

One application of Sylow p-subgroups and diffuseness to Bieberbach groups is the following result by Kionke and Raimbault:

Theorem 6.17 ([40, Thm 3.5(iii)])**.** Suppose D is a finite group. If D is solvable with a non-cyclic Sylow subgroup, then D is the point group for Bieberbach groups G_1 and G_2 such that G_1 is diffuse and G_2 is not.

Moreover, consider the following.

Definition 6.18. Suppose $H \leq G$. We say that H has a **normal complement** if there exists $N \trianglelefteq G$ such that $G = HN$ and $H \cap N = \{e\}$.

Definition 6.19. Suppose $H \leq G$. We say that H has a **normal p-complement** if there exists $N \trianglelefteq G$ such that $G = HN$, $H \cap N = \{e\}$, $|N|$ is relatively prime to p, and $[G : N] = p^\alpha$ for some $\alpha \geq 1$.

Now, Hiller and Sah called a finite group **primitive** if it was the point group of a Bieberbach group G with finite $H_1(G, \mathbb{Z})$. They also showed the following:

Theorem 6.20 ([34, Thm p.178])**.** A finite group D is primitive if and only if no Sylow p-subgroup of D has a normal complement.

Observe that the trivial group and cyclic groups $\mathbb{Z}/m\mathbb{Z}$ are **not** primitive since we can always take $\{e\}$ as the normal complement. However, $(\mathbb{Z}/p)^m$ is primitive for p prime and $m \geq 2$ since it has no *cyclic* Sylow p-subgroups and the condition holds vacuously.

Recall that Burnside's normal p-complement theorem [41, Sec. 4] states that if D is a finite group containing P as a Sylow p-subgroup such that P commutes with every element of $N_D(P) = \{\, d \in D \mid dP = Pd \,\}$, then there exists $K \trianglelefteq D$ which is a normal p-complement to P. As a consequence of this theorem, if p is the smallest prime dividing $|D|$ and the Sylow p-subgroups of D are cyclic, then G has a normal p-complement. Therefore, if all the Sylow p-subgroups of D are cyclic, then D is **not** primitive. This leads to the following application.

Proposition 6.21. Let $\mathcal{G}$ be the class of finite groups whose Sylow p-subgroups are all cyclic. If G is a Bieberbach group with point group $D \in \mathcal{G}$, then G is poly-$\mathbb{Z}$.

Proof. Let D be a finite group with all cyclic Sylow p-subgroups. Say P is one such Sylow p-subgroup. If $K \trianglelefteq D$, then $K \cap P$ is a Sylow p-subgroup of K. Moreover, all Sylow p-subgroups of K arise in this way. Thus, any normal subgroup of D must also have all cyclic Sylow p-subgroups.

Next, consider the quotient D/K for $K \trianglelefteq D$. All Sylow p-subgroups Q of D/K are of the form $PK/K \cong P/P \cap K$ for P a Sylow p-subgroup of D. Because $P/P \cap K$ is a cyclic group modulo a cyclic group, it is also cyclic. Thus, we conclude that each quotient of D has all cyclic Sylow p-subgroups. Therefore, $\mathcal{G}$ is closed under normal subgroups and quotients.

We will now apply induction on the dimension, n, of a Bieberbach group G with point group D to show that if $D \in \mathcal{G}$, then G is poly-$\mathbb{Z}$.

Base Case: $n = 1$

When $n = 1$, there is exactly one Bieberbach group $G \cong \mathbb{Z}$ which is poly-$\mathbb{Z}$.

Induction Case: $n > 1$

By the comments before this proof, we know that $D \in \mathcal{G}$ is not primitive and so by definition $H_1(G, \mathbb{Z})$ is infinite. Applying Calabi's construction (p. 51), we see that

$$G \cong ((H \rtimes \mathbb{Z}) \rtimes \cdots) \rtimes \mathbb{Z}$$

for H either trivial or H a Bieberbach group with $H_1(H, \mathbb{Z})$ finite. Let G' be the group such that $G \cong G' \rtimes \mathbb{Z}$. Then $G' \trianglelefteq G$ and, by [36, Lemma p. 576], we see that G' is also Bieberbach. Because we assumed G is of dimension n and we have "peeled off" a copy of $\mathbb{Z}$ to find G', the dimension of G' is $n - 1$.

Let N be the lattice of G and D_0 the quotient $G'/N \cap G'$. Consider the following diagram with exact rows and columns.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \cap G' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & D_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N/N \cap G' \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & D/D_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

We note that $N \cap G' \cong \mathbb{Z}^{n-1}$. From this diagram, we see that D_0 is isomorphic to a normal subgroup of D though we cannot conclude D_0 is the point group of G' as $N \cap G'$ may not be maximal abelian. By the proof of Theorem 3.1 in [42], the maximal abelian normal subgroup of G' is the centralizer of $N \cap G'$ in G'. Call this group N'. Of course, $N' \cong \mathbb{Z}^{n-1}$ and the point group of G' is $D' \cong G'/N'$. Since $N \cap G' \leq N'$, D' is isomorphic to a quotient of D_0. By the comments at the beginning of the proof, because D is in $\mathcal{G}$ and D_0 is isomorphic to a normal subgroup of D, $D_0 \in \mathcal{G}$. Since D' is a quotient of D_0, we conclude that $D' \in \mathcal{G}$.

By the induction hypothesis, G' is poly-$\mathbb{Z}$ because it is Bieberbach of dimension $n-1$ with $D' \in \mathcal{G}$. Therefore, $G \cong G' \rtimes \mathbb{Z}$ is also poly-$\mathbb{Z}$. $\square$

We can now proceed with the proof of Corollary 6.15.

Proof of Corollary 6.15.

(a) By Theorem 6.11, we know that if G is a connective Bieberbach group, it must by poly-$\mathbb{Z}$. Quotients of solvable groups are solvable and, as poly-$\mathbb{Z}$ groups are solvable, we conclude that D must be solvable.

(b) Suppose all the Sylow p-subgroups of D are cyclic. By Proposition 6.21, G is poly-$\mathbb{Z}$. Applying Theorem 6.11, G is connective.

(c) The result follows by Theorem 6.17 and the equivalence of connectivity and diffuseness for Bieberbach groups.

□

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