QUANTUM TOROIDAL SUPERALGEBRAS

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Luan Pereira Bezerra

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THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. Evgeny Mukhin, Chair

> Department of Mathematical Sciences

Dr. Daniel Ramras

> Department of Mathematical Sciences

Dr. Roland Roeder

> Department of Mathematical Sciences

Dr. Vitaly Tarasov

> Department of Mathematical Sciences

Approved by:

> Dr. Evgeny Mukhin
>
> > Head of the Graduate Program

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ABSTRACT

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We introduce the quantum toroidal superalgebra $\mathcal{E}_{m|n}$ associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$ and initiate its study. For each choice of parity $\mathbf{s}$ of $\mathfrak{gl}_{m|n}$, a corresponding quantum toroidal superalgebra $\mathcal{E}_{\mathbf{s}}$ is defined.

To show that all such superalgebras are isomorphic, an action of the toroidal braid group $\widehat{\mathcal{B}}_{m+n}$ on the direct sum $\oplus_{\mathbf{s}}\mathcal{E}_{\mathbf{s}}$ is constructed.

The superalgebra $\mathcal{E}_{\mathbf{s}}$ contains two distinguished subalgebras, both isomorphic to the quantum affine superalgebra $U_q\widehat{\mathfrak{sl}}_{m|n}$ with parity $\mathbf{s}$, called vertical and horizontal subalgebras. We show the existence of Miki automorphism of $\mathcal{E}_{\mathbf{s}}$, which exchanges the vertical and horizontal subalgebras.

For $m \neq n$ and standard parity, we give a construction of level 1 $\mathcal{E}_{m|n}$-modules through vertex operators. We also construct an evaluation map from $\mathcal{E}_{m|n}(q_1, q_2, q_3)$ to the quantum affine algebra $U_q\widehat{\mathfrak{gl}}_{m|n}$ at level $c = q_3^{(m-n)/2}$.

1. INTRODUCTION

Quantum toroidal algebras are affinizations of quantum affine algebras. They were first introduced 25 years ago in [19], motivated by the study of Hecke operators in algebraic surfaces.

Since that time, several applications were discovered in geometry, algebra, and mathematical physics. The quantum toroidal algebras appear as Hall algebras of elliptic curves, [8], [38], they also act on equivariant K-groups of Hilbert schemes and Laumon moduli spaces, [18], [39], [40]. The quantum toroidal algebras are natural dual objects to double affine Hecke algebras, [45]. The quantum toroidal algebras provide integrable systems of XXY–type, among them is a deformation of quantum KdV flows, [16]. Characters of representations of quantum toroidal algebras appear in topological field theory, [13], AGT conjecture, [2]. The full list is much longer.

However, the supersymmetric version of quantum toroidal algebras remained unexplored. Our goal is to introduce the *quantum toroidal superalgebras* $\mathcal{E}_{m|n}(q_1, q_2, q_3)$ related to the superalgebras $\mathfrak{gl}_{m|n}$, with $m \neq n$, and initiate their study. We expect these algebras to have many properties similar to the quantum toroidal algebras $\mathcal{E}_{m|0}(q_1, q_2, q_3)$ associated with $\mathfrak{gl}_m$ which can be used in similar way, but with various new features occurring due to the supersymmetry. In particular, the Cartan matrix of $\mathfrak{sl}_{m|n}$ is not unique, and it depends on the choice of parity for $\mathfrak{sl}_{m|n}$. The parity choices of $\mathfrak{sl}_{m|n}$ are parameterized by sequences $\mathbf{s} = (s_1, \ldots, s_{m+n})$, where $s_i = \pm 1$, and 1 occurs m times, -1 occurs n times. Let $\mathcal{S}_{m|n}$ be the set of all such sequences. We often replace the index $m|n$ by $\mathbf{s} \in \mathcal{S}_{m|n}$, for example, we denote the algebra $\mathfrak{sl}_{m|n}$ given in parity $\mathbf{s}$ by $\mathfrak{sl}_{\mathbf{s}}$.

Many insights on quantum toroidal algebras have a geometric origin. For the quantum toroidal superalgebras, the geometric point of view is not broadly available, since very few results on the geometry of supersymmetric spaces are known. Our

approach is purely algebraic. We use known results on quantum affine superalgebras and on quantum toroidal algebras to obtain their generalizations to quantum toroidal superalgebras.

This text is organized as follows.

In Chapter 1, we recall definitions and well–known facts on Lie superalgebras. Both *Drinfeld–Jimbo* and *new Drinfeld* realizations of the quantum affine superalgebras $U_q\widehat{\mathfrak{sl}}_{m|n}$, for any choice of parity, are recalled in Section 1.2. In Section 1.3, we collect results of [46], where an action of the affine braid group of $\mathfrak{sl}_{m|n}$ was used to show that these two presentations are equivalent, and that $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ are isomorphic for all $\mathbf{s} \in \mathcal{S}_{m|n}$. The *Drinfeld–Jimbo* and *new Drinfeld* presentations of $U_q\widehat{\mathfrak{sl}}_{m|n}$ are a key ingredient in the definition of the quantum toroidal superalgebra $\mathcal{E}_{m|n}$, and their equivalence motivates the *Miki automorphism*.

In Chapter 2, we define the quantum toroidal algebra associated with $\mathfrak{gl}_{m|n}$, for any choice of parity, and give a few properties. The quantum toroidal superalgebras $\mathcal{E}_{m|n}$ were first introduced in [6] with standard parity, and in [7] for any choice of parity. As in the even case, they depend on complex parameters q_1, q_2, q_3 such that $q_1q_2q_3 = 1$. We require that the superalgebra $\mathcal{E}_{\mathbf{s}}$ has a "vertical" quantum affine subalgebra $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ given in new Drinfeld realization, and a "horizontal" quantum affine subalgebra $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ given in Drinfeld–Jimbo realization. We always have $q_2 = q^2$. In addition, we want our construction to be invariant under rotations $\widehat{\tau}$ of the Dynkin diagram which connects $\mathcal{E}_{\mathbf{s}}$ with $\mathcal{E}_{\tau\mathbf{s}}$, where $\tau\mathbf{s} = (s_2, \ldots, s_{m+n}, s_1)$. This leads us to the generators and relations presentation of $\mathcal{E}_{\mathbf{s}}$, see Definition 2.1.1. Naturally, the algebra $\mathcal{E}_{\mathbf{s}}$ is generated by currents $E_i(z), F_i(z)$, and half currents $K_i^{\pm}(z)$, $i = 0, \ldots, m + n - 1$, labeled by nodes of the affine Dynkin diagram of type $\widehat{\mathfrak{sl}}_{\mathbf{s}}$, and the relations are written in terms of the corresponding affine Cartan matrix. Similar to the even case, the quantum toroidal superalgebra $\mathcal{E}_{\mathbf{s}}$ has a two–dimensional center.

It is natural to expect that all algebras $\mathcal{E}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, should be isomorphic. In Chapter 3, we use the toroidal braid group $\widehat{\mathcal{B}}_{m+n}$ to prove that this is indeed so, see Corollary 3.1.2. As a byproduct, we also obtain the *Miki automorphism*, see Theorem

3.1.9, which is central to the study of quantum toroidal algebras in the even case, see [28], [14]. The *Miki automorphism* is the highly non–explicit automorphism which maps vertical and horizontal subalgebras to each other. Note that the isomorphism from $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ in new Drinfeld realization to $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ in Drinfeld–Jimbo realization is already not explicit. The Miki automorphism originates in the well known Fourier transform Φ for toroidal braid group, see Lemma 3.1.6, which maps commutative generators $\widehat{\mathcal{Y}}_i \in \widehat{\mathcal{B}}_{m+n}$ to Knizhnik-Zamolodchikov elements.

Recently, a promising application of quantum toroidal algebras and superalgebras on toric Calabi–Yau manifolds has been noted on [48] and [25]. In particular, the choice of parity of $\mathcal{E}_{m|n}$ is identified with the choice of the toric diagram resolution, and the isomorphism of the superalgebras $\mathcal{E}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, corresponds to the statement that different quiver gauge theories describe the same geometry.

In Chapter 4, we focus on $\mathcal{E}_{m|n}$ in standard parity. We use bosonization techniques to construct level one representations of $\mathcal{E}_{m|n}$, see Theorem 4.1.3. Our formulas are built on work [21] and generalize the known result in the even case [37]. We expect that the irreducible level one modules stay irreducible when restricted to the vertical $U_q\,\widehat{\mathfrak{gl}}_{m|n}$ subalgebra. However, unlike the even case, the precise structure of irreducible level one modules for the quantum affine $\mathfrak{gl}_{m|n}$ is not fully understood, see [21], [24], and Conjecture 4.1.5.

Finally, we proceed to the evaluation map. The evaluation map is a surjective algebra homomorphism $\mathcal{E}_{m|n}(q_1, q_2, q_3) \to \widetilde{U}_q\,\widehat{\mathfrak{gl}}_{m|n}$ to the quantum affine algebra at level c completed with respect to the homogeneous grading, where $q_3^{m-n} = c^2$, see Theorem 4.2.2. The evaluation map has the property that its restriction to the vertical subalgebra is the identity map. In the even case, the evaluation map was found in [27], see also [15].

1.1 Superalgebras

In this section, we give a brief review on superalgebras. We follow [9].

A *vector superspace* V is a vector space with a $\mathbb{Z}_2$-gradation, i.e., V decomposes as a direct sum of vector subspaces $V = V_0 \oplus V_1$. The *parity* of an element $v \in V_i$ is given by $|v| = i$, $i \in \mathbb{Z}_2$. The elements of V_0 are called *even* and the elements of V_1 are called *odd*. Throughout this text, the notation $|v|$ will always assume that v is a homogeneous element.

Let $m, n \in \mathbb{Z}_{\geq 0}$. The vector superspace with $\dim V_0 = m$ and $\dim V_1 = n$ is denoted by $\mathbb{C}^{m|n}$. If $n = 0$, we often write $\mathbb{C}^m$ instead of $\mathbb{C}^{m|0}$. We use this convention for other algebraic objects throughout this text.

Example 1.1.1. Let V and W be two vector superspaces. The space of linear transformations from V to W is a superspace. In particular, the space of endomorphisms $\mathrm{End}(V)$ of V is a vector superspace.

A *superalgebra* is a vector superspace $A = A_0 \oplus A_1$ with a bilinear multiplication satisfying $A_i A_j \subseteq A_{i+j}$, $i, j \in \mathbb{Z}_2$.

A *Lie superalgebra* is a superalgebra whose product $[\cdot, \cdot]$, called supercommutator or superbracket, satisfies

- $[a, b] = -(-1)^{|a||b|}[b, a]$ (super skew symmetry);

- $(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||c|}[c, [a, b]] + (-1)^{|a||b|}[b, [c, a]] = 0$ (super Jacobi identity).

Example 1.1.2. Any associative superalgebra A can be given a Lie superalgebra structure with superbracket defined on homogeneous elements by

$$[a, b] = ab - (-1)^{|a||b|}ba, \tag{1.1.1}$$

and extended to all elements by linearity. In particular, the superalgebra $\mathrm{End}(\mathbb{C}^{m|n})$, equipped with the superbracket above, is a Lie superalgebra called the *general linear Lie superalgebra* and denoted by $\mathfrak{gl}_{m|n}$.

The Lie superalgebra $\mathfrak{gl}_{m|n}$ can be realized as follows.

For notation convenience, let $N := m + n$.

Fix a basis $v_1, \ldots, v_m, v_{m+1}, \ldots, v_N$ of $\mathbb{C}^{m|n}$ such that $|v_i| = 0$, if $i = 1, \ldots, m$, and $|v_i| = 1$, if $i = m+1, \ldots, N$. A basis with this property is called *standard*.

Then, the Lie superalgebra $\mathfrak{gl}_{m|n}$ is isomorphic to the superalgebra of $N \times N$ matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.1.2}$$

with the superbracket (1.1.1), where $A, B, C,$ and D are $m \times m$, $m \times n$, $n \times m$, and $n \times n$ matrices, respectively.

The even subalgebra $(\mathfrak{gl}_{m|n})_0$ is isomorphic to the superalgebra of $N \times N$ matrices of the form (1.1.2), with $B = 0$ and $C = 0$. Note that $(\mathfrak{gl}_{m|n})_0 \cong \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ as Lie algebras.

The subalgebra of $N \times N$ diagonal matrices is the *Cartan subalgebra of $\mathfrak{gl}_{m|n}$.*

The *supertrace* of a matrix X of the form (1.1.2) is defined as

$$\mathrm{str}(X) := \mathrm{tr}(A) - \mathrm{tr}(D),$$

where tr denotes the usual trace of square matrices.

It follows that str is linear and satisfies

$$\mathrm{str}([X, Y]) = 0 \qquad X, Y \in \mathfrak{gl}_{m|n}.$$

Hence, $\{X \in \mathfrak{gl}_{m|n} \mid \mathrm{str}(X) = 0\}$ is a subalgebra of $\mathfrak{gl}_{m|n}$, called the *special linear Lie superalgebra* and denoted by $\mathfrak{sl}_{m|n}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_{m|n}$, i.e, the subalgebra of $N \times N$ diagonal matrices and supertrace zero.

If $m \neq n$, the Lie superalgebra $\mathfrak{sl}_{m|n}$ is simple. However, if $m = n$, the identity matrix $\mathrm{Id}_{N \times N}$ generates the center of $\mathfrak{sl}_{m|m}$. The subalgebra $\mathfrak{sl}_{m|m}/\mathbb{C}\,\mathrm{Id}_{N \times N}$ is called the *projective special linear Lie superalgebra* and denoted by $\mathfrak{psl}_{m|m}$.

The Lie superalgebras $\mathfrak{sl}_{m|n}$, $m \neq n$, and $\mathfrak{psl}_{m|m}$ are called the *classical simple Lie superalgebras of type A*. We don't discuss classical Lie superalgebras of other types in this text.

1.1.1   Root systems and Cartan matrices

Fix $m \neq n$. Let $N := m + n$ and $I = \{1, 2, \ldots, N - 1\}$.

Let $\mathfrak{g} = \mathfrak{sl}_{m|n}$, and let $\mathfrak{h}$ be its Cartan subalgebra.

For $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \; \forall\, h \in \mathfrak{h}\}.$$

The root system of $\mathfrak{g}$ is defined as

$$\Phi := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}.$$

A root α is *even* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq 0$, and *odd* if $\mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq 0$. Let Φ_0 be the set of even roots, and Φ_1 be the set of odd roots. The roots of $\mathfrak{g}$ generate $\mathfrak{h}^*$. The set of *simple roots* of $\mathfrak{g}$ is defined by fixing a basis of $\mathfrak{h}^*$ composed of roots.

The supertrace str defines a non–degenerate supersymmetric bilinear form on $\mathfrak{g}$ by $\langle x|y \rangle := \mathrm{str}(xy)$. The restriction of this bilinear form to the Cartan subalgebra $\mathfrak{h}$ is non–degenerate and symmetric. We use the same notation for the induced bilinear form on $\mathfrak{h}^*$.

A root α is called *isotropic* if $\langle \alpha|\alpha \rangle = 0$. Isotropic roots are always odd.

The Weyl group of $\mathfrak{g}$ is defined as the Weyl group of $\mathfrak{g}_0$. It is generated by the (even) reflections

$$r_\alpha(x) := x - 2\frac{\langle x|\alpha \rangle}{\langle \alpha|\alpha \rangle}\alpha \qquad \alpha \in \Phi_0, \; x \in \mathfrak{h}^*. \tag{1.1.3}$$

It is isomorphic to $\mathfrak{S}_m \times \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters.

Explicitly, the root system of $\mathfrak{g}$ can be described as follows.

Let E_{ii}, $i = 1, \ldots, N$, denote the $N \times N$ diagonal matrix with only non-zero entry at position (i, i).

Define $\epsilon_i \in \mathrm{End}(\mathbb{C}^{m|n})^*$ by

$$\epsilon_i := \mathrm{str}(E_{ii}) \langle E_{ii}|\cdot \rangle = \begin{cases} \langle E_{ii}|\cdot \rangle & i = 1, \ldots, m; \\ -\langle E_{ii}|\cdot \rangle & i = m + 1, \ldots, N. \end{cases}$$

Under this identification, we have

$$\Phi_0 := \{\pm(\epsilon_i - \epsilon_j) \mid 1 \le i < j \le m, \text{ or } m < i < j \le N\},$$

$$\Phi_1 := \{\pm(\epsilon_i - \epsilon_j) \mid 1 \le i \le m, \text{ and } m < j \le N\}.$$

The *standard simple roots* of $\mathfrak{g}$ are defined by $\alpha_i := \epsilon_i - \epsilon_{i+1}$, $i \in I$, and the *standard Cartan matrix* A of $\mathfrak{g}$ is given by $A_{i,j} := \langle \alpha_i | \alpha_j \rangle$, $i, j \in I$. We have

$$A = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & \ddots & \ddots & & & & & \\ & \ddots & 2 & -1 & & & & \\ & & -1 & 0 & 1 & & & \\ & & & 1 & -2 & \ddots & & \\ & & & & \ddots & \ddots & 1 & \\ & & & & & 1 & -2 \end{pmatrix} \leftarrow\!-\!-\, m$$

$$\uparrow$$
$$m$$

Note that α_m is the only isotropic simple root.

The Dynkin diagram associated with the Cartan matrix A is constructed under the usual conventions for simple Lie algebras, but simple isotropic roots are represented by $\otimes$ and non-isotropic simple roots by $\bigcirc$. For Lie superalgebras of other types, non-isotropic odd roots can occur, but we don't discuss them here.

The *standard Dynkin diagram* of $\mathfrak{g}$ is

Fig. 1.1. Standard Dynkin diagram of $\mathfrak{sl}_{m|n}$.

Given a choice of simple roots Δ, the set Φ^+ of *positive roots* is defined as ($\mathbb{Z}_{>0}$-span of Δ) $\cap\, \Phi$. Define

$$\mathfrak{n}^{\pm} := \bigoplus_{\alpha \in \pm\Phi^+} \mathfrak{g}_\alpha.$$

We have a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The *Borel subalgebra* corresponding to this choice is defined by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

In the Lie algebra ($n = 0$) setting, it is well known that the Cartan matrix is unique, all choices of simple roots are equivalent under the Weyl group action, and all Borel subalgebras are naturally isomorphic.

In the supersymmetric case, this is no longer true. Recall that the Weyl group of $\mathfrak{g}$ is defined as the Weyl group of its even subalgebra $\mathfrak{g}_0$ and it is isomorphic to $\mathfrak{S}_m \times \mathfrak{S}_n$. The number of different choices of simple roots of $\mathfrak{sl}_{m|n}$ and $\mathfrak{sl}_N$ should be the same, but the Weyl group of $\mathfrak{sl}_N$ is isomorphic to $\mathfrak{S}_N$.

In order to classify the $\binom{N}{m}$ choices of simple roots that are not equivalent under the Weyl group action it is useful to introduce parity sequences.

A *parity sequence* is a N-tuple of ± 1 with exactly m positive coordinates. Set

$$\mathcal{S}_{m|n} = \{(s_1, \ldots, s_N)| \ s_i \in \{-1, 1\}, \ \#\{i \,|\, s_i = 1\} = m\}.$$

The parity sequence of the form $\mathbf{s} = (1, \ldots, 1, -1, \ldots, -1)$ is called the *standard parity sequence.*

Given a parity sequence $\mathbf{s} \in \mathcal{S}_{m|n}$, we have the Cartan matrix $A^{\mathbf{s}} = (A^{\mathbf{s}}_{i,j})_{i,j \in I}$, where

$$A^{\mathbf{s}}_{i,j} = (s_i + s_{i+1})\delta_{i,j} - s_i\delta_{i,j+1} - s_j\delta_{i+1,j} \qquad (i, j \in I). \qquad (1.1.4)$$

The symmetric group $\mathfrak{S}_N$ acts naturally on $\mathcal{S}_{m|n}$ by permuting indices, $\sigma\mathbf{s} := (s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(N)})$ for all $\sigma \in \mathfrak{S}_N$, $\mathbf{s} \in \mathcal{S}_{m|n}$.

Let $\mathbf{s}$ be a parity sequence. The Lie superalgebra $\mathfrak{gl}_{m|n}$ with parity $\mathbf{s}$, denoted by $\mathfrak{gl}_{\mathbf{s}}$, can be identified as the superalgebra of $N \times N$ matrices by choosing a basis $v_1, \ldots, v_N$ of $\mathbb{C}^{m|n}$ such that $|v_i| = (1 - s_i)/2$. It follows that the Lie superalgebras $\mathfrak{gl}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, are all isomorphic as Lie superalgebras. However, their Borel subalgebras are not isomorphic, each $\mathbf{s} \in \mathcal{S}_{m|n}$ yields a different Cartan matrix, and, consequently, a different Dynkin diagram.

The supertrace of a $N \times N$ matrix X is defined by

$$\text{str}(X) = \sum_{i=1}^{N} s_i M_{ii},$$

and all definitions discussed above for $\mathfrak{gl}_{m|n}$ with standard parity are extended to $\mathfrak{gl}_{\mathbf{s}}$ using this definition of the supertrace.

We will often use the parity sequence $\mathbf{s}$ as index instead of the dimensions $m|n$ to emphasize the dependence on $\mathbf{s}$. Any superspace $X_{\mathbf{s}}$ should be understood as $X_{m|n}$ with parity choice given by $\mathbf{s} \in \mathcal{S}_{m|n}$.

The parity of a simple root α_i of $\mathfrak{sl}_{\mathbf{s}}$ is given by $|\alpha_i| = (1 - s_i s_{i+1})/2$. For notation convenience, we write $|i| = |\alpha_i|$, $i \in I$.

1.1.2 Odd reflections

Let $\Delta_{\mathbf{s}} = \{\alpha_i \,|\, i \in I\}$ be the set of simple roots of $\mathfrak{sl}_{\mathbf{s}}$, for some $\mathbf{s} \in \mathcal{S}_{m|n}$, and let $Q_{\mathbf{s}}$ be the root lattice.

Let $\alpha_j \in \Delta_{\mathbf{s}}$ be an even simple root. Then, the definition of the reflection r_{α_j}, see (1.1.3), on the simple roots of $\mathfrak{sl}_{\mathbf{s}}$ reads

$$r_{\alpha_j}(\alpha_i) := \begin{cases} \alpha_i & i \neq j, j \pm 1; \\ -\alpha_j & i = j; \\ \alpha_i + \alpha_j & i = j \pm 1. \end{cases} \tag{1.1.5}$$

If $\alpha_j \in \Delta_{\mathbf{s}}$ is an odd simple root, we define the *odd reflection* r_{α_j} on the simple roots of $\mathfrak{sl}_{\mathbf{s}}$ by (1.1.5), and extended the definition to all roots by linearity. However, the action of an odd reflection does not preserve the parity. The action on the parity is as follows.

If we forget the parity of the roots, i.e., if we assume that all roots are even, the action of all reflections coincide with the action of the Weyl group of $\mathfrak{sl}_N$, which is isomorphic to $\mathfrak{S}_N$. The isomorphism between the Weyl group of $\mathfrak{sl}_N$ and $\mathfrak{S}_N$ is obtained by identifying the reflection r_{α_j}, for a simple root α_j, with the transposition $(j, j+1) \in \mathfrak{S}_N$.

Let $\alpha_j \in \Delta_\mathbf{s}$ be a simple root. Then, r_{α_j} maps the roots of $\mathfrak{sl}_{m|n}$ with parity $\mathbf{s}$ to the roots of $\mathfrak{sl}_{m|n}$ with parity $r_{\alpha_j}\mathbf{s}$. Note that, if α_j is even, then $r_{\alpha_j}\mathbf{s} = \mathbf{s}$.

Example 1.1.3. We illustrate all Dynkin diagrams of $\mathfrak{sl}_{3|1}$ and how the odd reflections affect the parity.
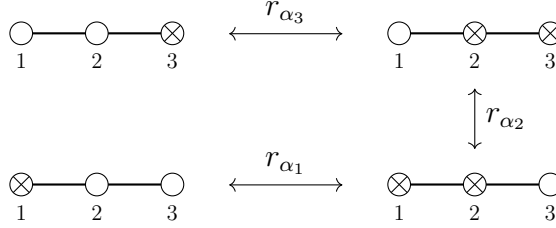

Fig. 1.2. Dynkin diagrams of $\mathfrak{sl}_{3|1}$.

1.1.3 Chevalley generators of $\mathfrak{sl}_{m|n}$

The Lie superalgebra $\mathfrak{sl}_{m|n}$ can be also defined by generators and relations. The presentation of $\mathfrak{sl}_{m|n}$ in terms of Chevalley generators will be a key ingredient in the following chapters.

Fix a parity sequence $\mathbf{s} \in \mathcal{S}_{m|n}$.

The Lie superalgebra $\mathfrak{sl}_\mathbf{s}$ is isomorphic to the Lie superalgebra generated by elements e_i, f_i, h_i, $i \in I$ subject to the relations

$$[e_i, f_j] = \delta_{i,j} h_i, \qquad [h_i, h_j] = 0,$$
$$[h_i, e_j] = A^\mathbf{s}_{i,j} e_j, \qquad [h_i, f_j] = -A^\mathbf{s}_{i,j} f_j,$$
$$[e_i, e_j] = [f_i, f_j] = 0 \qquad\qquad (A^\mathbf{s}_{i,j} = 0),$$

for all $i, j \in I$, and the following Serre relations.

If $|i| = 0$ and $A^\mathbf{s}_{i,j} = \pm 1$, then

$$[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] = 0.$$

If $|i| = 1$ and $i \neq 1, N-1$, i.e., i is not an extremal node of the Dynkin diagram, then

$$[e_i, [e_{i-1}, [e_i, e_{i+1}]]] = [f_i, [f_{i-1}, [f_i, f_{i+1}]]] = 0.$$

The correspondence with the matrix realization is given by identifying $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$, and $h_i = s_i E_{ii} - s_{i+1} E_{i+1,i+1}$, for all $i \in I$. Also, if α_i is a simple root, $(\mathfrak{sl_s})_{\alpha_i} = \mathbb{C} e_i$ and $(\mathfrak{sl_s})_{-\alpha_i} = \mathbb{C} f_i$.

Note that this more abstract presentation of $\mathfrak{sl_s}$ heavily depends on the parity sequence $\mathbf{s} \in \mathcal{S}_{m|n}$, and it doesn't follow automatically that the superalgebras $\mathfrak{sl_s}$ are all isomorphic. However, the presentation in terms of Chevalley generators is convenient when dealing with deformations and generalizations of these algebras.

1.1.4 The quantum superalgebras $U_q \mathfrak{sl}_{m|n}$

Fix $q \in \mathbb{C}^\times$ not a root of unity and let $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$, $k \in \mathbb{Z}$.

We also use the notation $[X, Y]_a = XY - (-1)^{|X||Y|} a Y X$. For simplicity, we write $[X, Y]_1 = [X, Y]$. The bracket $[X, Y]_a$ satisfy the following Jacobi identity

$$[[X, Y]_a, Z]_b = [X, [Y, Z]_c]_{abc^{-1}} + (-1)^{|Y||Z|} c [[X, Z]_{bc^{-1}}, Y]_{ac^{-1}}. \tag{1.1.6}$$

The quantum superalgebra $U_q \mathfrak{sl}_{m|n}$ is a deformation of the universal enveloping algebra of $\mathfrak{sl}_{m|n}$ with deformation parameter q.

Let $\mathbf{s} \in \mathcal{S}_{m|n}$.

The quantum superalgebra $U_q \mathfrak{sl_s}$ is the unital associative superalgebra generated by *Chevalley generators* e_i, f_i, $t_i^{\pm 1}$, $i \in I$. The parity of the generators is given by $|e_i| = |f_i| = |i| = (1 - s_i s_{i+1})/2$, and $|t_i^{\pm 1}| = 0$.

The defining relations are as follows.

$$t_i t_j = t_j t_i, \quad t_i e_j t_i^{-1} = q^{A_{i,j}^{\mathbf{s}}} e_j, \quad t_i f_j t_i^{-1} = q^{-A_{i,j}^{\mathbf{s}}} f_j,$$

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}},$$

$$[e_i, e_j] = [f_i, f_j] = 0 \qquad\qquad (A_{i,j}^{\mathbf{s}} = 0),$$

for all $i, j \in I$, and the following Serre relations.

If $|i| = 0$ and $A^{\mathbf{s}}_{i,j} = \pm 1$, then

$$[\![e_i, [\![e_i, e_j]\!]]\!] = [\![f_i, [\![f_i, f_j]\!]]\!] = 0,$$

where $[\![X, Y]\!] = [X, Y]_{q^{-\langle \beta | \gamma \rangle}}$ if X, Y have weights $\beta, \gamma \in Q_{\mathbf{s}}$, i.e., if $t_i X t_i^{-1} = q^{\langle \alpha_i | \beta \rangle} X$ and $t_i Y t_i^{-1} = q^{\langle \alpha_i | \gamma \rangle} Y$ for $i \in I$.

If $|i| = 1$ and $i \neq 1, N - 1$, i.e., i is not an extremal node of the Dynkin diagram, then

$$[\![e_i, [\![e_{i+1}, [\![e_i, e_{i-1}]\!]]\!]]\!] = [\![f_i, [\![f_{i+1}, [\![f_i, f_{i-1}]\!]]\!]]\!] = 0.$$

1.2 The quantum affine superalgebras $U_q \widehat{\mathfrak{sl}}_{m|n}$

In this section, we review definitions of the quantum affine superalgebra $U_q \widehat{\mathfrak{sl}}_{m|n}$.

We consider two presentations of the quantum affine superalgebra $U_q \widehat{\mathfrak{sl}}_{m|n}$, the *Drinfeld-Jimbo* realization and the *new Drinfeld* realization.

Roughly speaking, the Drinfeld-Jimbo presentation of $U_q \widehat{\mathfrak{sl}}_{m|n}$ should be thought as the Lie superalgebra with Chevalley type generators, and relations given in terms of the *affine Cartan matrix*.

The affine Cartan matrix $\hat{A}$ is obtained by including an even *null root* δ to the root system of $\mathfrak{sl}_{m|n}$, such that $\langle \delta | \delta \rangle = \langle \delta | \alpha_i \rangle = 0$, for all $i \in I$. Then, $\alpha_0 := \delta - \sum_{i \in I} \alpha_i$ is regarded as a new simple root with parity $|\alpha_0| = \sum_{i \in I} |\alpha_i|$, and the affine Dynkin diagram has an extra 0 node. Let $\hat{I} := \{0, 1, \ldots, N - 1\}$ be the set of Dynkin nodes. We will consider the indices in $\hat{I}$ module N.

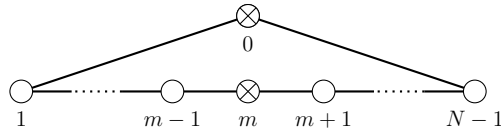In standard parity, the affine Dynkin diagram of $\widehat{\mathfrak{sl}}_{m|n}$ has the following shape.

Fig. 1.3. Dynkin diagram of $\widehat{\mathfrak{sl}}_{m|n}$ in standard parity.

On the other hand, the new Drinfeld realization of $U_q\widehat{\mathfrak{sl}}_{m|n}$ should be thought as the affinization of $U_q\mathfrak{sl}_{m|n}$, i.e., $\widehat{\mathfrak{sl}}_{m|n}$ is a central extension of $\mathfrak{sl}_{m|n} \otimes \mathbb{C}[t^{\pm}]$. It is generated by an invertible central element c and the coefficients of the currents $x_i^{\pm}(z) := \sum_{r\in\mathbb{Z}} x_{i,r}^{\pm} z^{-r}$, $k_i^{\pm}(z) = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{r>0} h_{i,\pm r} z^{\mp r}\right)$, $i \in I$. The relations are given in terms of the Cartan matrix of $\mathfrak{sl}_{m|n}$. Note that the indices of the currents are in I.

The presentations of the superalgebra $U_q\widehat{\mathfrak{sl}}_{m|n}$ in Drinfeld-Jimbo and new Drinfeld forms were given in [46]. We recall them here for an arbitrary choice of parity $\mathbf{s} \in \mathcal{S}_{m|n}$.

Drinfeld–Jimbo

Let $\mathbf{s}$ be a parity sequence.

In the *Drinfeld-Jimbo realization*, the superalgebra $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is generated by *Chevalley generators* $e_i, f_i, t_i^{\pm 1}$, $i \in \hat{I}$. The parity of generators is given by $|e_i| = |f_i| = |i| = (1 - s_i s_{i+1})/2$, and $|t_i^{\pm 1}| = 0$.

The defining relations are as follows.

$$t_i t_j = t_j t_i, \quad t_i e_j t_i^{-1} = q^{A_{i,j}^{\mathbf{s}}} e_j, \quad t_i f_j t_i^{-1} = q^{-A_{i,j}^{\mathbf{s}}} f_j,$$

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}},$$

$$[e_i, e_j] = [f_i, f_j] = 0 \qquad\qquad\qquad\qquad (A_{i,j}^{\mathbf{s}} = 0),$$

$$[\![e_i, [\![e_i, e_{i\pm 1}]\!]]\!] = [\![f_i, [\![f_i, f_{i\pm 1}]\!]]\!] = 0 \qquad\qquad (A_{i,i}^{\mathbf{s}} \neq 0),$$

$$[\![e_i, [\![e_{i+1}, [\![e_i, e_{i-1}]\!]]\!]]\!] = [\![f_i, [\![f_{i+1}, [\![f_i, f_{i-1}]\!]]\!]]\!] = 0 \qquad (mn \neq 2, A_{i,i}^{\mathbf{s}} = 0),$$

$$[\![e_{i+1}, [\![e_{i-1}, [\![e_{i+1}, [\![e_{i-1}, e_i]\!]]\!]]\!]]\!] = [\![e_{i-1}, [\![e_{i+1}, [\![e_{i-1}, [\![e_{i+1}, e_i]\!]]\!]]\!]]\!] \quad (mn = 2, A_{i,i}^{\mathbf{s}} \neq 0),$$

$$[\![f_{i+1}, [\![f_{i-1}, [\![f_{i+1}, [\![f_{i-1}, f_i]\!]]\!]]\!]]\!] = [\![f_{i-1}, [\![f_{i+1}, [\![f_{i-1}, [\![f_{i+1}, f_i]\!]]\!]]\!]]\!] \quad (mn = 2, A_{i,i}^{\mathbf{s}} \neq 0).$$

The element $t_0 t_1 \ldots t_{N-1}$ is central.

The subalgebra of $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ generated by e_i, f_i, t_i, $i \in I$, is isomorphic to $U_q\mathfrak{sl}_{\mathbf{s}}$.

The superalgebra $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ in the Drinfeld-Jimbo realization has a $\mathbb{Z}^N$-grading given by

$$\deg^h(x) = \left(\deg_0^h(x), \deg_1^h(x), \ldots, \deg_{N-1}^h(x)\right), \qquad (1.2.1)$$

where

$$\deg_i^h(e_j) = \delta_{i,j}, \quad \deg_i^h(f_j) = -\delta_{i,j}, \quad \deg_i^h(t_j) = 0 \qquad (i, j \in \hat{I}).$$

New Drinfeld Realization

In the *new Drinfeld realization*, the superalgebra $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is generated by *current generators* $x_{i,r}^{\pm}, h_{i,r}, k_i^{\pm 1}, c^{\pm 1}, i \in I, r \in \mathbb{Z}'$. Here and below, we use the following convention: $r \in \mathbb{Z}'$ means $r \in \mathbb{Z}$ if r is an index of a non-Cartan current generator $x_{i,r}^{\pm}$, and $r \in \mathbb{Z}'$ means $r \in \mathbb{Z} \setminus \{0\}$ if r is an index of a Cartan current generator $h_{i,r}$.

The parity of generators is given by $|x_{i,r}^{\pm}| = |i| = (1 - s_i s_{i+1})/2$, and all remaining generators have parity 0.

The defining relations are as follows.

c is central, $\quad k_i k_j = k_j k_i, \quad k_i x_j^{\pm}(z) k_i^{-1} = q^{\pm A_{i,j}^{\mathbf{s}}} x_j^{\pm}(z),$

$$[h_{i,r}, h_{j,s}] = \delta_{r+s,0} \frac{[r A_{i,j}^{\mathbf{s}}]}{r} \frac{c^r - c^{-r}}{q - q^{-1}},$$

$$[h_{i,r}, x_j^{\pm}(z)] = \pm \frac{[r A_{i,j}^{\mathbf{s}}]}{r} c^{-(r \pm |r|)/2} z^r x_j^{\pm}(z),$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \left(\delta\left(c\frac{w}{z}\right) k_i^+(w) - \delta\left(c\frac{z}{w}\right) k_i^-(z) \right),$$

$$(z - q^{\pm A_{i,j}^{\mathbf{s}}} w) x_i^{\pm}(z) x_j^{\pm}(w) + (-1)^{|i||j|}(w - q^{\pm A_{i,j}^{\mathbf{s}}} z) x_j^{\pm}(w) x_i^{\pm}(z) = 0 \quad (A_{i,j}^{\mathbf{s}} \neq 0),$$

$$[x_i^{\pm}(z), x_j^{\pm}(w)] = 0 \qquad\qquad\qquad\qquad (A_{i,j}^{\mathbf{s}} = 0),$$

$$\mathrm{Sym}_{z_1,z_2} [x_i^{\pm}(z_1), [x_i^{\pm}(z_2), x_{i\pm 1}^{\pm}(w)]] = 0 \qquad (A_{i,i}^{\mathbf{s}} \neq 0,\ i \pm 1 \in I),$$

$$\mathrm{Sym}_{z_1,z_2} [x_i^{\pm}(z_1), [x_{i+1}^{\pm}(w_1), [x_i^{\pm}(z_2), x_{i-1}^{\pm}(w_2)]]] = 0 \qquad (A_{i,i}^{\mathbf{s}} = 0,\ i \pm 1 \in I),$$

where $x_i^{\pm}(z) = \sum_{k \in \mathbb{Z}} x_{i,k}^{\pm} z^{-k}$, $k_i^{\pm}(z) = k_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{r>0} h_{i,\pm r} z^{\mp r}\right)$.

The subalgebra of $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ generated by $x_{i,0}^{\pm}, k_i, i \in I$, is isomorphic to $U_q\mathfrak{sl}_{\mathbf{s}}$.

The superalgebra $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ in the new Drinfeld realization has a $\mathbb{Z}^N$-grading given by

$$\deg^v(x) = \left(\deg_1^v(x), \ldots, \deg_{N-1}^v(x); \deg_\delta(x)\right), \qquad (1.2.2)$$

where

$$\deg_i^v(x_{j,r}^\pm) = \pm\delta_{i,j}, \quad \deg_i^v(k_j) = \deg_i^v(h_{j,r}) = \deg_i^v(c) = 0 \qquad (i,j \in I,\ r \in \mathbb{Z}'),$$

$$\deg_\delta(x_{i,r}^\pm) = \deg_\delta(h_{i,r}) = r, \quad \deg_\delta(k_j) = \deg_\delta(c) = 0 \qquad (i \in I,\ r \in \mathbb{Z}').$$

The isomorphism between Drinfeld-Jimbo and new Drinfeld realizations is described in Proposition 1.3.3.

For $J \subset I$, we call the subalgebra of $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ generated by $a_{j,r}$, $j \in J$, $r \in \mathbb{Z}'$ and $a = x^+, x^-, h$, the *diagram subalgebra associated with J* and denote it by $U_q^J\widehat{\mathfrak{sl}}_{\mathbf{s}}$. Any diagram subalgebra is isomorphic to a tensor product of $U_q\widehat{\mathfrak{sl}}_{k|l}$ algebras.

The quantum affine superalgebra $U_q\,\widehat{\mathfrak{gl}}_{\mathbf{s}}$ is obtained from $U_q\,\widehat{\mathfrak{sl}}_{\mathbf{s}}$ in the new Drinfeld realization by including the currents $k_0^\pm(z)$ subject to the same relations.

1.3 Affine braid group

It is well known that the role of the Weyl group in the simple Lie algebras is played by an appropriate braid group in the quantum setting, see [4], [26]. In this section, we recall the action of extended affine braid group of type A in $U_q\widehat{\mathfrak{sl}}_{\bullet} = \bigoplus_{\mathbf{s}\in\mathcal{S}_{m|n}} U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$. We follow [46].

In this section, we always assume $N \geq 4$.

1.3.1 Extended affine braid group of type A

We recall the extended affine braid group of type A.

Let $\mathcal{B}_N$ be the group generated by elements τ, T_i, $i \in \hat{I}$, with defining relations

$$T_iT_j = T_jT_i \qquad\qquad (j \neq i, i \pm 1), \qquad (1.3.1)$$

$$T_jT_iT_j = T_iT_jT_i \qquad\qquad (j = i \pm 1), \qquad (1.3.2)$$

$$\tau T_{i-1}\tau^{-1} = T_i \qquad\qquad (i \in \hat{I}). \qquad (1.3.3)$$

The group $\mathcal{B}_N$ is called the *extended affine braid group* of type A.

Alternatively, $\mathcal{B}_N$ can be described as the group generated by elements $\mathcal{X}_i$, T_i, $i \in I$, with defining relations

$$T_i T_j = T_j T_i \qquad (j \neq i, i \pm 1), \qquad (1.3.4)$$

$$T_j T_i T_j = T_i T_j T_i \qquad (j = i \pm 1), \qquad (1.3.5)$$

$$\mathcal{X}_i \mathcal{X}_j = \mathcal{X}_j \mathcal{X}_i \qquad (i, j \in I), \qquad (1.3.6)$$

$$T_i \mathcal{X}_j = \mathcal{X}_j T_i \qquad (i \neq j), \qquad (1.3.7)$$

$$T_1^{-1} \mathcal{X}_1 T_1^{-1} = \mathcal{X}_2 \mathcal{X}_1^{-1}, \qquad (1.3.8)$$

$$T_{N-1}^{-1} \mathcal{X}_{N-1} T_{N-1}^{-1} = \mathcal{X}_{N-2} \mathcal{X}_{N-1}^{-1}, \qquad (1.3.9)$$

$$T_i^{-1} \mathcal{X}_i T_i^{-1} = \mathcal{X}_{i-1} \mathcal{X}_{i+1} \mathcal{X}_i^{-1} \qquad (2 \leq i \leq N-2). \qquad (1.3.10)$$

An isomorphism γ between the two realizations is given by

$$\gamma : \mathcal{X}_1 \mapsto \tau T_{N-1} \cdots T_1, \quad T_i \mapsto T_i \qquad (i \in I). \qquad (1.3.11)$$

We have a surjective group homomorphism

$$\pi : \mathcal{B}_N \to \mathfrak{S}_N, \quad \tau \mapsto \tau, \quad T_i \mapsto \sigma_i \qquad (i \in \hat{I}), \qquad (1.3.12)$$

where we denoted $\sigma_i = (i, i+1)$, $i \in I$, $\sigma_0 = (1, N)$, and, by an abuse of notation, $\tau = (1, 2, \ldots, N)$.

1.3.2   Action of $\mathcal{B}_N$ on Drinfeld-Jimbo realization of $U_q \widehat{\mathfrak{sl}}_\bullet$

The symmetric group $\mathfrak{S}_N$ acts naturally on $\mathcal{S}_{m|n}$ by permuting indices, $\sigma \mathbf{s} := (s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(N)})$ for all $\sigma \in \mathfrak{S}$, $\mathbf{s} \in \mathcal{S}_{m|n}$.

The extended affine braid group also acts on $\mathcal{S}_{m|n}$ by $T\mathbf{s} = \pi(T)\mathbf{s}$, for $T \in \mathcal{B}_N$, $\mathbf{s} \in \mathcal{S}_{m|n}$, see (1.3.12).

The next proposition describes a family of isomorphisms of quantum affine superalgebras.

Proposition 1.3.1 ([46]). *We have the following.*

(i) *For $i \in \hat{I}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, there exists an isomorphism of superalgebras $T_{i,\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \to U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$ given on Chevalley generators by*

$$T_{i,\mathbf{s}}(t_i) = t_i^{-1}, \qquad\qquad\qquad T_{i,\mathbf{s}}(t_{i\pm 1}) = t_i t_{i\pm 1},$$

$$T_{i,\mathbf{s}}(e_i) = -s_i f_i t_i, \qquad\qquad\qquad T_{i,\mathbf{s}}(f_i) = -s_{i+1} t_i^{-1} e_i,$$

$$T_{i,\mathbf{s}}(e_{i-1}) = s_{i+1} q^{-s_{i+1}} [\![e_{i-1}, e_i]\!], \qquad T_{i,\mathbf{s}}(f_{i-1}) = -(-1)^{|f_i||f_{i-1}|} [\![f_{i-1}, f_i]\!],$$

$$T_{i,\mathbf{s}}(e_{i+1}) = s_i q^{-s_i} (-1)^{|e_i||e_{i+1}|} [\![e_{i+1}, e_i]\!], \quad T_{i,\mathbf{s}}(f_{i+1}) = -[\![f_{i+1}, f_i]\!],$$

$$T_{i,\mathbf{s}}(e_j) = e_j, \qquad T_{i,\mathbf{s}}(f_j) = f_j, \qquad T_{i,\mathbf{s}}(t_j) = t_j \qquad (j \neq i, i \pm 1).$$

The parities on the r.h.s. correspond to the generators of target algebra $U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}}$.

(ii) *The left-inverse of $T_{i,\mathbf{s}}$, $(T_{i,\mathbf{s}})^{-1} : U_q \widehat{\mathfrak{sl}}_{\sigma_i \mathbf{s}} \to U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$, is given by*

$$(T_{i,\mathbf{s}})^{-1}(t_i) = t_i^{-1}, \qquad\qquad\qquad (T_{i,\mathbf{s}})^{-1}(t_{i\pm 1}) = t_i t_{i\pm 1},$$

$$(T_{i,\mathbf{s}})^{-1}(e_i) = -s_{i+1} t_i^{-1} f_i, \qquad\qquad (T_{i,\mathbf{s}})^{-1}(f_i) = -s_i e_i t_i,$$

$$(T_{i,\mathbf{s}})^{-1}(e_{i-1}) = s_i q^{-s_i} (-1)^{|e_i||e_{i-1}|} [\![e_i, e_{i-1}]\!], \qquad (T_{i,\mathbf{s}})^{-1}(f_{i-1}) = -[\![f_i, f_{i-1}]\!],$$

$$(T_{i,\mathbf{s}})^{-1}(e_{i+1}) = s_{i+1} q^{-s_{i+1}} [\![e_i, e_{i+1}]\!],$$

$$(T_{i,\mathbf{s}})^{-1}(f_{i+1}) = -(-1)^{|f_i||f_{i+1}|} [\![f_i, f_{i+1}]\!],$$

$$(T_{i,\mathbf{s}})^{-1}(e_j) = e_j, \qquad (T_{i,\mathbf{s}})^{-1}(f_j) = f_j, \qquad (T_{i,\mathbf{s}})^{-1}(t_j) = t_j \qquad (j \neq i, i \pm 1).$$

The parities on the r.h.s. correspond to the generators of target algebra $U_q \widehat{\mathfrak{sl}}_{\mathbf{s}}$.

(iii) *For $\mathbf{s} \in \mathcal{S}_{m|n}$, there exist an isomorphism of superalgebras $\tau_{\mathbf{s}} : U_q \widehat{\mathfrak{sl}}_{\mathbf{s}} \to U_q \widehat{\mathfrak{sl}}_{\tau \mathbf{s}}$ given on Chevalley generators by*

$$\tau_{\mathbf{s}}(x_i) = x_{i+1} \qquad\qquad (x = e, f, t).$$

$\square$

We note the following useful formula

$$(T_i \, T_{i\pm 1})_{\mathbf{s}}(x_i) = x_{i\pm 1} \qquad\qquad (x = e, f, t). \qquad\qquad (1.3.13)$$

The isomorphisms $T_{i,\mathbf{s}}$ and $\tau_{\mathbf{s}}$ change the grading in the Drinfeld-Jimbo realization as follows.

If $\deg^h(x) = (d_0, d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_{N-1})$, then

$$\deg^h(T_{i,\mathbf{s}}(x)) = (d_0, d_1, \ldots, d_{i-1}, d_{i-1} + d_{i+1} - d_i, d_{i+1}, \ldots, d_{N-1}) \qquad (i \in \hat{I}),$$

$$\deg^h(\tau_{\mathbf{s}}(x)) = (d_{N-1}, d_0, d_1, \ldots, d_{N-2}).$$

(1.3.14)

The isomorphisms generate a groupoid if one considers the category whose objects are the superalgebras $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, and whose morphisms are $\tau_{\mathbf{s}}$, $T_{i,\mathbf{s}}$, $i \in \hat{I}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, their compositions and inverses.

In our situation, the groupoid structure is equivalent to the group action as follows.

Define the following automorphisms of $U_q\widehat{\mathfrak{sl}}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$

$$\tau = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \tau_{\mathbf{s}}, \qquad\qquad T_i = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} T_{i,\mathbf{s}} \qquad\qquad (i \in \hat{I}). \qquad (1.3.15)$$

Note that, by abuse of notation, we denote by τ both the automorphism above and the element of $\mathfrak{S}_N$.

Proposition 1.3.2 ([46]). *The automorphisms τ, T_i, $i \in \hat{I}$, define an action of the extended affine braid group $\mathcal{B}_N$ on $U_q\widehat{\mathfrak{sl}}_{\bullet}$, i.e., they satisfy the relations* (1.3.1)–(1.3.3). $\qquad\qquad\square$

We adopt the following convention. For $T \in \mathcal{B}_N$, we denote $T_{\mathbf{s}}$ the restriction of T to the $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ summand in $U_q\widehat{\mathfrak{sl}}_{\bullet}$. Note that the image of $T_{\mathbf{s}}$ is also a particular summand in $U_q\widehat{\mathfrak{sl}}_{\bullet}$, namely $U_q\widehat{\mathfrak{sl}}_{T\mathbf{s}}$. For example, $(\tau T_i T_j T_k)_{\mathbf{s}} = \tau_{\sigma_i \sigma_j \sigma_k \mathbf{s}} T_{i, \sigma_j \sigma_k \mathbf{s}} T_{j, \sigma_k \mathbf{s}} T_{k, \mathbf{s}}$ is mapping $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ to $U_q\widehat{\mathfrak{sl}}_{\tau \sigma_i \sigma_j \sigma_k \mathbf{s}}$. We use a similar convention with other maps, see, for example, Theorem 3.1.1 below.

Note that the action of $\mathcal{B}_N$ on $\mathcal{S}_{m|n}$ is transitive. In particular, Proposition 1.3.1 implies that all superalgebras $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, are isomorphic.

1.3.3 Action of $\mathcal{B}_N$ on new Drinfeld realization of $U_q\widehat{\mathfrak{sl}}_\bullet$

We have an action of extended affine braid group $\mathcal{B}_N$ on $U_q\widehat{\mathfrak{sl}}_\bullet$ given in Chevalley generators, see Section 1.3.2. The group $\mathcal{B}_N$ contains elements $\mathcal{X}_i$, $i \in I$, see Section 1.3.1. The elements $\mathcal{X}_i$ preserve the parity, $\mathcal{X}_i\mathbf{s} = \mathbf{s}$, for all $\mathbf{s} \in \mathcal{S}_{m|n}$, and, therefore, $(\mathcal{X}_i)_\mathbf{s}$ is an automorphism of $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$. These automorphisms are used to obtain an isomorphism between the two different realizations of $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$.

Proposition 1.3.3 ([46]). *There exists an isomorphism $\iota_\mathbf{s}$ between new Drinfeld and Drinfeld-Jimbo realizations of $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$ mapping:*

$$k_i \mapsto t_i, \qquad\qquad c \mapsto t_0 t_1 \cdots t_{N-1},$$
$$x_{i,r}^+ \mapsto (-1)^{ir}\mathcal{X}_{i,\mathbf{s}}^{-r}(e_i), \qquad x_{i,r}^- \mapsto (-1)^{ir}\mathcal{X}_{i,\mathbf{s}}^{r}(f_i) \qquad (r \in \mathbb{Z}, i \in I).$$

$\square$

The identifications $\iota_\mathbf{s}$ allow us to study the action of $\mathcal{B}_N$ on the new Drinfeld realization. One can describe action of the $\mathcal{X}_{i,\mathbf{s}}$ in current generators explicitly.

Proposition 1.3.4 ([46]). *For $i \in I$, $\mathbf{s} \in \mathcal{S}_{m|n}$, the action of $\mathcal{X}_{i,\mathbf{s}}$ in current generators is given by*

$$\mathcal{X}_{i,\mathbf{s}}(x_{j,r}^\pm) = (-1)^{i\delta_{ij}}x_{j,r\mp\delta_{ij}}^\pm, \qquad \mathcal{X}_{i,\mathbf{s}}(k_j) = c^{-\delta_{ij}}k_j,$$
$$\mathcal{X}_{i,\mathbf{s}}(h_{j,r}) = h_{j,r}, \qquad\qquad \mathcal{X}_{i,\mathbf{s}}(c) = c \qquad (r \in \mathbb{Z}', j \in I).$$

$\square$

For the action of $T_{i,\mathbf{s}}$ in current generators we have some partial information.

Lemma 1.3.5. *For $i \in I$, we have*

$$T_{i,\mathbf{s}}(a_{j,r}) = a_{j,r} \qquad (r \in \mathbb{Z}', \ j \in I, \ i \neq j, j \pm 1, \ a = x^+, x^-, h). \qquad (1.3.16)$$

Moreover,

$$T_{i,\mathbf{s}}(x^+_{i+1,r}) = s_i q^{-s_i}(-1)^{|i||i+1|}[\![x^+_{i+1,r}, x^+_{i,0}]\!] \qquad (r \in \mathbb{Z}), \qquad (1.3.17)$$

$$T_{i,\mathbf{s}}(x^+_{i-1,r}) = s_{i+1} q^{-s_{i+1}}[\![x^+_{i-1,r}, x^+_{i,0}]\!] \qquad (r \in \mathbb{Z}), \qquad (1.3.18)$$

$$T_{i,\mathbf{s}}(x^-_{i+1,r}) = -[\![x^-_{i+1,r}, x^-_{i,0}]\!] \qquad (r \in \mathbb{Z}), \qquad (1.3.19)$$

$$T_{i,\mathbf{s}}(x^-_{i-1,r}) = -(-1)^{|i||i-1|}[\![x^-_{i-1,r}, x^-_{i,0}]\!] \qquad (r \in \mathbb{Z}). \qquad (1.3.20)$$

The parities on the r.h.s. correspond to the generators of target algebra $U_q\widehat{\mathfrak{sl}}_{\sigma_i\mathbf{s}}$.

We also have, $T_{i,\mathbf{s}}U_q^{\{i\}}\widehat{\mathfrak{sl}}_\mathbf{s} \subset U_q^{\{i\}}\widehat{\mathfrak{sl}}_{\sigma_i\mathbf{s}}$ *if $i \neq 1, N-1$.*

Finally, $T_{1,\mathbf{s}}U_q^{\{1\}}\widehat{\mathfrak{sl}}_\mathbf{s} \subset U_q^{\{1,2\}}\widehat{\mathfrak{sl}}_{\sigma_1\mathbf{s}}$ *and* $T_{N-1,\mathbf{s}}U_q^{\{N-1\}}\widehat{\mathfrak{sl}}_\mathbf{s} \subset U_q^{\{N-1,N-2\}}\widehat{\mathfrak{sl}}_{\sigma_{N-1}\mathbf{s}}$.

Proof. Equations (1.3.16)-(1.3.20) follow from relation (1.3.7) and Proposition 1.3.1.

To prove the last part, we note that if $i \neq 1$ then the algebra generated by $U_q^{\{i\}}\widehat{\mathfrak{sl}}_\mathbf{s}$, is a subalgebra of the algebra generated by $x^\pm_{i,0}$ and $h_{i-1,\pm1}$. Therefore $T_{i,\mathbf{s}}U_q^{\{i\}}\widehat{\mathfrak{sl}}_\mathbf{s} \subset U_q^{\{i,i-1\}}\widehat{\mathfrak{sl}}_{\sigma_1\mathbf{s}}$. Similarly, if $i \neq N-1$, we have $T_{i,\mathbf{s}}U_q^{\{i\}}\widehat{\mathfrak{sl}}_\mathbf{s} \subset U_q^{\{i,i+1\}}\widehat{\mathfrak{sl}}_{\sigma_1\mathbf{s}}$. $\qquad\square$

We can also write the inverse of the isomorphism $\iota_\mathbf{s}$.

Lemma 1.3.6. *The isomorphism $\iota_\mathbf{s}^{-1}$ maps*

$$e_i \mapsto x^+_{i,0}, \quad f_i \mapsto x^-_{i,0}, \quad t_i \mapsto k_i \qquad (i \in I), \qquad (1.3.21)$$

$$t_0 \mapsto c(k_1 k_2 \cdots k_{m+n-1})^{-1}, \qquad (1.3.22)$$

$$e_0 \mapsto \left(\mathcal{X}_1 T_{N-1} \cdots T_2 T_1^{-1}\right)_\mathbf{s}(x^+_{1,0}), \qquad (1.3.23)$$

$$f_0 \mapsto \left(\mathcal{X}_1 T_{N-1} \cdots T_2 T_1^{-1}\right)_\mathbf{s}(x^-_{1,0}). \qquad (1.3.24)$$

Proof. It is sufficient to use (1.3.11). $\qquad\square$

Note that in Lemma 1.3.6 we apply T_i only to Chevalley generators, therefore the formulas are explicit.

The correspondence between the $\mathbb{Z}^N$-grading in the two realizations of $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$ is as follows.

If $x \in U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is given in the new Drinfeld realization and the grading is $\deg^v(x) = (d_1^v, \ldots, d_{N-1}^v; d_\delta)$, then the grading in the Drinfeld-Jimbo realization is

$$\deg^h(\iota_{\mathbf{s}}(x)) = (d_\delta, d_1^v + d_\delta, \ldots, d_{N-1}^v + d_\delta). \tag{1.3.25}$$

Similarly, if $x \in U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is given in the Drinfeld-Jimbo realization and $\deg^h(x) = (d_0^h, d_1^h, \ldots, d_{N-1}^h)$, then the grading in the new Drinfeld realization is

$$\deg^v(\iota_{\mathbf{s}}^{-1}(x)) = (d_1^h - d_0^h, \ldots, d_{N-1}^h - d_0^h; -d_0^h). \tag{1.3.26}$$

1.3.4 The anti-automorphisms φ and η

We have two anti-automorphisms of $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ which will be used in Sections 2.1 and 3.1.

Lemma 1.3.7. *For $\mathbf{s} \in \mathcal{S}_{m|n}$, we have a superalgebra anti-automorphism $\varphi_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \to U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ given on Chevalley generators by*

$$\varphi_{\mathbf{s}}(e_i) = e_i, \qquad \varphi_{\mathbf{s}}(f_i) = f_i, \qquad \varphi_{\mathbf{s}}(t_i) = t_i^{-1} \qquad (i \in \hat{I}).$$

Moreover, the anti-automorphisms $\varphi_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, satisfy

$$(\varphi \, T_i \, \varphi)_{\mathbf{s}} = (T_{i,\sigma_i\mathbf{s}})^{-1} \qquad (i \in \hat{I}).$$

Proof. This is checked by a straightforward computation. $\qquad\square$

Note that $\varphi_{\mathbf{s}}$ preserves the grading (1.2.1).

Lemma 1.3.8. *For $\mathbf{s} \in \mathcal{S}_{m|n}$, we have a superalgebra anti-automorphism $\eta_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \to U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ given on current generators by*

$$\eta_{\mathbf{s}}(c) = c, \qquad \eta_{\mathbf{s}}(k_i^\pm(z)) = k_i^\mp(cz^{-1}), \qquad \eta_{\mathbf{s}}(x_i^\pm(z)) = x_i^\pm(z^{-1}) \qquad (i \in I).$$

Moreover, the anti-automorphisms $\eta_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, satisfy

$$(\eta \, T_i \, \eta)_{\mathbf{s}} = (T_{i,\sigma_i\mathbf{s}})^{-1} \qquad (i \in I).$$

Proof. The existence of this anti-automorphism is checked directly.

For the last equality, note that $\eta_{\mathbf{s}}$ coincides with $\varphi_{\mathbf{s}}$ on the subalgebra generated by $x_{i,0}^{\pm}, k_i, c, \ i \in I$. Also, by the isomorphism between the new Drinfeld and Drinfeld–Jimbo realizations of $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$, it is sufficient to check the identity on $x_{1,\mp 1}^{\pm}$.

We verify $(\eta \, T_i \, \eta)_{\mathbf{s}} = (T_{i,\sigma_i \mathbf{s}})^{-1}$ on $x_{1,1}^-$ for $i = 1, 2$. The remaining values of i are trivial. The check for $x_{1,-1}^+$ is analogous.

Using the relation (1.3.8) we have

$$(\eta T_1 \eta)_{\mathbf{s}}(x_{1,1}^-) = -(\eta T_1 \mathcal{X}_1^{-1})_{\mathbf{s}}(x_{1,0}^-) = -(\eta \mathcal{X}_1 \mathcal{X}_2^{-1} T_1^{-1})_{\mathbf{s}}(x_{1,0}^-) = \eta_{\sigma_1 \mathbf{s}}(-s_1 c^{-1} x_{1,-1}^+ k_1)$$
$$= -s_1 c^{-1} k_1^{-1} x_{1,1}^+,$$
$$(T_{1,\sigma_1 \mathbf{s}})^{-1}(x_{1,1}^-) = -(T_1^{-1} \mathcal{X}_1)_{\mathbf{s}}(x_{1,0}^-) = -(\mathcal{X}_2 \mathcal{X}_1^{-1} T_1)_{\mathbf{s}}(x_{1,0}^-) = -s_1 c^{-1} k_1^{-1} x_{1,1}^+.$$

And using the relation (1.3.7) we have

$$(\eta T_2 \eta)_{\mathbf{s}}(x_{1,1}^-) = -(\eta T_2 \mathcal{X}_1^{-1})_{\mathbf{s}}(x_{1,0}^-) = (\eta \mathcal{X}_1^{-1})_{\sigma_2 \mathbf{s}}((-1)^{|1||2|} [\![x_{1,0}^-, x_{2,0}^-]\!]) = -[\![x_{2,0}^-, x_{1,1}^-]\!],$$
$$(T_{2,\sigma_2 \mathbf{s}})^{-1}(x_{1,1}^-) = -(T_2^{-1} \mathcal{X}_1)_{\mathbf{s}}(x_{1,0}^-) = \mathcal{X}_{1,\sigma_2 \mathbf{s}}([\![x_{2,0}^-, x_{1,0}^-]\!]) = -[\![x_{2,0}^-, x_{1,1}^-]\!].$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that, if $x \in U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is given in the new Drinfeld realization and $\deg^v(x) = (d_1, \ldots, d_{N-1}; d_\delta)$, then

$$\deg^v(\eta_{\mathbf{s}}(x)) = (d_1, \ldots, d_{N-1}; -d_\delta). \tag{1.3.27}$$

Both anti-automorphisms $\varphi_{\mathbf{s}}$ and η_s are anti-involutions: $\varphi_s^2 = \eta_{\mathbf{s}}^2 = 1$.

2. THE QUANTUM TOROIDAL SUPERALGEBRAS

2.1 Quantum toroidal superalgebra $\mathcal{E}_\mathbf{s}$

In this section, we introduce the quantum toroidal superalgebra $\mathcal{E}_\mathbf{s}$ associated with $\mathfrak{gl}_{m|n}$ for any choice of parity $\mathbf{s} \in \mathcal{S}_{m|n}$. We give a few properties of these algebras.

2.1.1 Definition of $\mathcal{E}_\mathbf{s}$

Fix $d, q \in \mathbb{C}^\times$ and define

$$q_1 = d\,q^{-1}, \quad q_2 = q^2, \quad q_3 = d^{-1}q^{-1}.$$

Note that $q_1 q_2 q_3 = 1$. In this text, we always assume that q_1, q_2 are generic, meaning that $q_1^{n_1} q_2^{n_2} q_3^{n_3} = 1$, $n_1, n_2, n_3 \in \mathbb{Z}$, iff $n_1 = n_2 = n_3$. Fix also $d^{1/2}, q^{1/2} \in \mathbb{C}^\times$ such that $(d^{1/2})^2 = d$, $(q^{1/2})^2 = q$.

Recall the affine Cartan matrix $\hat{A}^\mathbf{s} = (A^\mathbf{s}_{i,j})_{i,j \in \hat{I}}$, see (1.1.4).

We also define the matrix $M^\mathbf{s} = (M^\mathbf{s}_{i,j})_{i,j \in \hat{I}}$ by $M^\mathbf{s}_{i+1,i} = -M^\mathbf{s}_{i,i+1} = s_{i+1}$, and $M^\mathbf{s}_{i,j} = 0$, $i \neq j \pm 1$.

For example, if $\mathbf{s}$ is the standard parity sequence, we have

$$\hat{A}^\mathbf{s} = \begin{pmatrix} 0 & -1 & & & & & & & 1 \\ -1 & 2 & -1 & & & & & & \\ & -1 & \ddots & \ddots & & & & & \\ & & \ddots & 2 & -1 & & & & \\ & & & -1 & 0 & 1 & & & \\ & & & & 1 & -2 & \ddots & & \\ & & & & & \ddots & \ddots & 1 & \\ 1 & & & & & & 1 & -2 \end{pmatrix}, \quad \hat{M}^\mathbf{s} = \begin{pmatrix} 0 & -1 & & & & & & & -1 \\ 1 & 0 & -1 & & & & & & \\ & 1 & \ddots & \ddots & & & & & \\ & & \ddots & 0 & -1 & & & & \\ & & & 1 & 0 & 1 & & & \\ & & & & -1 & 0 & \ddots & & \\ & & & & & \ddots & \ddots & 1 & \\ 1 & & & & & & -1 & 0 \end{pmatrix} \begin{matrix} \leftarrow\!- & 0 \\ \\ \\ \\ \leftarrow\!- & m \\ \\ \\ \\ \end{matrix}$$

Definition 2.1.1. *The quantum toroidal algebra associated with* $\mathfrak{gl}_{m|n}$ *and parity sequence* **s** *is the unital associative superalgebra* $\mathcal{E}_\mathbf{s} = \mathcal{E}_\mathbf{s}(q_1, q_2, q_3)$ *generated by* $E_{i,r}$, $F_{i,r}$, $H_{i,r}$, *and invertible elements* K_i, C, *where* $i \in \hat{I}$, $r \in \mathbb{Z}'$, *subject to the defining relations* (2.1.1)-(2.1.16) *below. The parity of the generators is given by* $|E_{i,r}| = |F_{i,r}| = |i| = (1 - s_i s_{i+1})/2$, *and all remaining generators have parity 0.*

We use generating series

$$E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k},$$

$$K_i^\pm(z) = K_i^{\pm 1} \exp\Big(\pm(q - q^{-1}) \sum_{r > 0} H_{i,\pm r} z^{\mp r}\Big).$$

Let also $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ be the formal delta function.

Then, the defining relations are as follows.

C, K **relations**

$$C \text{ is central}, \qquad\qquad K_i K_j = K_j K_i, \qquad\qquad (2.1.1)$$

$$K_i E_j(z) K_i^{-1} = q^{A_{i,j}^\mathbf{s}} E_j(z), \qquad K_i F_j(z) K_i^{-1} = q^{-A_{i,j}^\mathbf{s}} F_j(z). \qquad (2.1.2)$$

K-K, K-E **and** K-F **relations**

$$K_i^\pm(z) K_j^\pm(w) = K_j^\pm(w) K_i^\pm(z), \tag{2.1.3}$$

$$\frac{d^{M_{i,j}^\mathbf{s}} C^{-1} z - q^{A_{i,j}^\mathbf{s}} w}{d^{M_{i,j}^\mathbf{s}} C z - q^{A_{i,j}^\mathbf{s}} w} K_i^-(z) K_j^+(w) = \frac{d^{M_{i,j}^\mathbf{s}} q^{A_{i,j}^\mathbf{s}} C^{-1} z - w}{d^{M_{i,j}^\mathbf{s}} q^{A_{i,j}^\mathbf{s}} C z - w} K_j^+(w) K_i^-(z), \tag{2.1.4}$$

$$(d^{M_{i,j}^\mathbf{s}} z - q^{A_{i,j}^\mathbf{s}} w) K_i^\pm(C^{-(1 \pm 1)/2} z) E_j(w) = (d^{M_{i,j}^\mathbf{s}} q^{A_{i,j}^\mathbf{s}} z - w) E_j(w) K_i^\pm(C^{-(1 \pm 1)/2} z),$$

$$\tag{2.1.5}$$

$$(d^{M_{i,j}^\mathbf{s}} z - q^{-A_{i,j}^\mathbf{s}} w) K_i^\pm(C^{-(1 \mp 1)/2} z) F_j(w) = (d^{M_{i,j}^\mathbf{s}} q^{-A_{i,j}^\mathbf{s}} z - w) F_j(w) K_i^\pm(C^{-(1 \mp 1)/2} z).$$

$$\tag{2.1.6}$$

E-F **relations**

$$[E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} (\delta\left(C\frac{w}{z}\right) K_i^+(w) - \delta\left(C\frac{z}{w}\right) K_i^-(z)). \qquad (2.1.7)$$

E-E **and** *F-F* **relations**

$$[E_i(z), E_j(w)] = 0, \quad [F_i(z), F_j(w)] = 0 \qquad (A_{i,j}^{\mathbf{s}} = 0), \qquad (2.1.8)$$

$$(d^{M_{i,j}^{\mathbf{s}}} z - q^{A_{i,j}^{\mathbf{s}}} w) E_i(z) E_j(w) = (-1)^{|i||j|} (d^{M_{i,j}^{\mathbf{s}}} q^{A_{i,j}^{\mathbf{s}}} z - w) E_j(w) E_i(z) \qquad (A_{i,j}^{\mathbf{s}} \neq 0),$$
$$(2.1.9)$$

$$(d^{M_{i,j}^{\mathbf{s}}} z - q^{-A_{i,j}^{\mathbf{s}}} w) F_i(z) F_j(w) = (-1)^{|i||j|} (d^{M_{i,j}^{\mathbf{s}}} q^{-A_{i,j}^{\mathbf{s}}} z - w) F_j(w) F_i(z) \qquad (A_{i,j}^{\mathbf{s}} \neq 0).$$
$$(2.1.10)$$

Serre relations

$$\mathrm{Sym}_{z_1,z_2}[\![E_i(z_1), [\![E_i(z_2), E_{i\pm1}(w)]\!]]\!] = 0 \qquad (A_{i,i}^{\mathbf{s}} \neq 0), \qquad (2.1.11)$$

$$\mathrm{Sym}_{z_1,z_2}[\![F_i(z_1), [\![F_i(z_2), F_{i\pm1}(w)]\!]]\!] = 0 \qquad (A_{i,i}^{\mathbf{s}} \neq 0), \qquad (2.1.12)$$

If $mn \neq 2$,

$$\mathrm{Sym}_{z_1,z_2}[\![E_i(z_1), [\![E_{i+1}(w_1), [\![E_i(z_2), E_{i-1}(w_2)]\!]]\!]]\!] = 0 \qquad (A_{i,i}^{\mathbf{s}} = 0), \qquad (2.1.13)$$

$$\mathrm{Sym}_{z_1,z_2}[\![F_i(z_1), [\![F_{i+1}(w_1), [\![F_i(z_2), F_{i-1}(w_2)]\!]]\!]]\!] = 0 \qquad (A_{i,i}^{\mathbf{s}} = 0). \qquad (2.1.14)$$

If $mn = 2$,

$$\mathrm{Sym}_{z_1,z_2}\mathrm{Sym}_{w_1,w_2}[\![E_{i-1}(z_1), [\![E_{i+1}(w_1), [\![E_{i-1}(z_2), [\![E_{i+1}(w_2), E_i(y)]\!]]\!]]\!]]\!] = \qquad (2.1.15)$$

$$= \mathrm{Sym}_{z_1,z_2}\mathrm{Sym}_{w_1,w_2}[\![E_{i+1}(w_1), [\![E_{i-1}(z_1), [\![E_{i+1}(w_2), [\![E_{i-1}(z_2), E_i(y)]\!]]\!]]\!]]\!] \quad (A_{i,i}^{\mathbf{s}} \neq 0),$$

$$\mathrm{Sym}_{z_1,z_2}\mathrm{Sym}_{w_1,w_2}[\![F_{i-1}(z_1), [\![F_{i+1}(w_1), [\![F_{i-1}(z_2), [\![F_{i+1}(w_2), F_i(y)]\!]]\!]]\!]]\!] = \qquad (2.1.16)$$

$$= \mathrm{Sym}_{z_1,z_2}\mathrm{Sym}_{w_1,w_2}[\![F_{i+1}(w_1), [\![F_{i-1}(z_1), [\![F_{i+1}(w_2), [\![F_{i-1}(z_2), F_i(y)]\!]]\!]]\!]]\!] \quad (A_{i,i}^{\mathbf{s}} \neq 0).$$

The relations (2.1.3)-(2.1.6) are equivalent to

$$[H_{i,r}, E_j(z)] = \frac{[rA_{i,j}]}{r} d^{-rM_{i,j}} C^{-(r+|r|)/2} z^r E_j(z), \tag{2.1.17}$$

$$[H_{i,r}, F_j(z)] = -\frac{[rA_{i,j}]}{r} d^{-rM_{i,j}} C^{-(r-|r|)/2} z^r F_j(z), \tag{2.1.18}$$

$$[H_{i,r}, H_{j,s}] = \delta_{r+s,0} \cdot \frac{[rA_{i,j}]}{r} d^{-rM_{i,j}} \frac{C^r - C^{-r}}{q - q^{-1}}, \tag{2.1.19}$$

for all $r \in \mathbb{Z}'$, $i, j \in \hat{I}$.

The relations (2.1.11) and (2.1.12) are also satisfied if $A_{i,i}^{\mathbf{s}} = 0$, due to the quadratic relations (2.1.8).

The element $K := K_0 K_1 \cdots K_{N-1}$ is central.

In standard parity, the poles of the correlation functions of currents $E_i(z)$ are depicted in Figure 2.1. For example, the correlation function of $E_0(z)E_1(w)$ has a pole at $z = q_1 w$, while the correlation function of $E_1(z)E_0(w)$ has a pole at $z = q_3 w$.
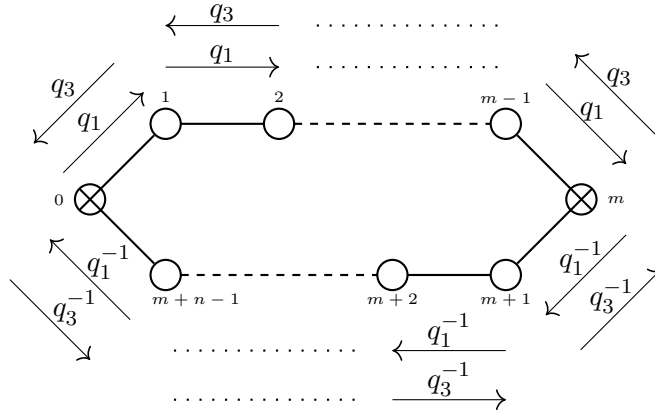
Fig. 2.1. Standard Dynkin diagram of type $\widehat{\mathfrak{sl}}_{m|n}$ with poles of correlation functions for the $E_i(z)$ currents of $\mathcal{E}_{m|n}$.

The poles of the correlation functions of the currents $F_i(z)$ are obtained from the Figure 2.1 replacing q by q^{-1}, i.e., $q_1^{\pm 1}$ is replaced by $q_3^{\mp 1}$, and $q_3^{\pm 1}$ by $q_1^{\mp 1}$.

For other choices of parity, as a general rule, the poles are reversed in every odd node. See, for example, the changes on the nodes 0 and m in the Figure 2.1.

For any $J \subset \hat{I}$, let $\mathcal{E}_\mathbf{s}^J \subset \mathcal{E}_\mathbf{s}$ be the subalgebra generated by $E_i(z), F_i(z), C, i \in J$. We call $\mathcal{E}_\mathbf{s}^J$ the *diagram subalgebra associated with J* of $\mathcal{E}_\mathbf{s}$.

2.1.2   Some properties of $\mathcal{E}_\mathbf{s}$

For each $i \in \hat{I}$, the superalgebra $\mathcal{E}_\mathbf{s}$ has a $\mathbb{Z}$-grading given by

$$\deg_i(E_{j,r}) = \delta_{i,j}, \quad \deg_i(F_{j,r}) = -\delta_{i,j}, \quad \deg_i(H_{j,r}) = \deg_i(K_j) = \deg_i(C) = 0,$$

for all $j \in \hat{I}$, $r \in \mathbb{Z}'$.

There is also the *homogeneous $\mathbb{Z}$-grading* given by

$$\deg_\delta(E_{j,r}) = \deg_\delta(F_{j,r}) = r, \quad \deg_\delta(H_{j,r}) = r, \quad \deg_\delta(K_j) = \deg_\delta(C) = 0,$$

for all $j \in \hat{I}$, $r \in \mathbb{Z}'$.

Thus, the superalgebra $\mathcal{E}_\mathbf{s}$ has a $\mathbb{Z}^{N+1}$-grading given on a homogeneous element $X \in \mathcal{E}_\mathbf{s}$ by

$$\deg(X) = \big(\deg_0(X), \deg_1(X), \ldots, \deg_{N-1}(X); \deg_\delta(X)\big). \tag{2.1.20}$$

The superalgebra $\mathcal{E}_\mathbf{s}$ has a graded topological Hopf superalgebra structure given on generators by

$$\Delta E_i(z) = E_i(z) \otimes 1 + K_i^-(z) \otimes E_i(C_1 z),$$

$$\Delta F_i(z) = F_i(C_2 z) \otimes K_i^+(z) + 1 \otimes F_i(z),$$

$$\Delta K_i^+(z) = K_i^+(C_2 z) \otimes K_i^+(z),$$

$$\Delta K_i^-(z) = K_i^-(z) \otimes K_i^-(C_1 z),$$

$$\Delta C = C \otimes C,$$

$$\varepsilon(E_i(z)) = \varepsilon(F_i(z)) = 0, \quad \varepsilon(K_i^\pm(z)) = \varepsilon(C) = 1,$$

$$S(E_i(z)) = -\big(K_i^-(C^{-1}z)\big)^{-1} E_i(C^{-1}z),$$

$$S(F_i(z)) = -F_i(C^{-1}z)\big(K_i^+(C^{-1}z)\big)^{-1},$$

$$S(K_i^\pm(z)) = \big(K_i^\pm(C^{-1}z)\big)^{-1}, \quad S(C) = C^{-1},$$

where $C_1 = C \otimes 1$, $C_2 = 1 \otimes C$. The maps Δ and ε are extended to algebra homomorphisms, and the map S to a superalgebra anti-homomorphism, $S(xy) = (-1)^{|x||y|}S(y)S(x)$. Note that the tensor product multiplication is defined for homogeneous elements $x_1, x_2, y_1, y_2 \in \mathcal{E}_{\mathbf{s}}$ by $(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{|y_1||x_2|}x_1x_2 \otimes y_1y_2$ and extended to the whole algebra by linearity.

2.1.3 Horizontal and vertical subalgebras

Let $\mathbf{s}$ be a parity sequence. For $i \in I$, define $\mu_{\mathbf{s}}(i) = -\sum_{j=1}^{i} s_j$. Define the *vertical homomorphism* of superalgebras $v_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$ by

$$v_{\mathbf{s}}(x_i^+(z)) = E_i(d^{\mu_{\mathbf{s}}(i)}z), \quad v_{\mathbf{s}}(x_i^-(z)) = F_i(d^{\mu_{\mathbf{s}}(i)}z),$$
$$v_{\mathbf{s}}(k_i^\pm(z)) = K_i^\pm(d^{\mu_{\mathbf{s}}(i)}z), \quad v_{\mathbf{s}}(c) = C \qquad (i \in I).$$

Note that if $x \in U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ and $\deg^v(x) = (d_1, d_2, \dots, d_{N-1}; d_\delta)$, then

$$\deg(v_{\mathbf{s}}(x)) = (0, d_1, d_2, \dots, d_{N-1}; d_\delta). \qquad (2.1.21)$$

Proposition 2.1.2. *The vertical homomorphism $v_{\mathbf{s}}$ is injective for generic values of parameters.*

Proof. In standard parity, the evaluation map constructed in Theorem 4.2.2 produces a left-inverse of $v_{\mathbf{s}}$ for generic values of parameters, see Lemma 4.2.1. Thus, $v_{\mathbf{s}}$ is an embedding with image $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$.

For other parity choices, an evaluation map is constructed in [7]. □

The image of the vertical homomorphism coincides with $\mathcal{E}_{\mathbf{s}}^I$. We denote this subalgebra $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ and call it the *vertical quantum affine* $\mathfrak{sl}_{m|n}$.

The vertical subalgebra $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is a Hopf subalgebra of $\mathcal{E}_{\mathbf{s}}$.

The *vertical quantum affine* $\mathfrak{gl}_{m|n}$, denoted by $U_q^{ver}\widehat{\mathfrak{gl}}_{\mathbf{s}}$, is obtained by including the currents $K_0^\pm(z)$. Note that $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ and $U_q^{ver}\widehat{\mathfrak{gl}}_{\mathbf{s}}$ are given in new Drinfeld realization.

The currents $K_0^\pm(z)$ do not commute with $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$. To obtain a current in $U_q^{ver}\widehat{\mathfrak{gl}}_{\mathbf{s}}$ commuting with $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ we proceed as follows.

For each $r \in \mathbb{Z}^\times$,

$$\det\left([rA_{i,j}^{\mathbf{s}}]d^{-rM_{i,j}^{\mathbf{s}}}\right)_{i,j \in \hat{I}} = [r]^{m+n}\left(d^{r(m-n)} + d^{r(n-m)} - q^{r(m-n)} - q^{r(n-m)}\right) \neq 0.$$

Thus, the system

$$\sum_{i \in \hat{I}} \gamma_{i,r}[rA_{i,j}^{\mathbf{s}}]d^{-rM_{i,j}^{\mathbf{s}}} = 0 \qquad (j \in I), \tag{2.1.22}$$

has a one-dimensional space of solutions. The element $H_r^{ver} = \sum_{i \in \hat{I}} \gamma_{i,r} H_{i,r} \in U_q^{ver}\widehat{\mathfrak{gl}}_{\mathbf{s}}$ commutes with $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}} \subset U_q^{ver}\widehat{\mathfrak{gl}}_{\mathbf{s}}$. Such element is unique up to scalar. We fix a normalization by requiring $\gamma_{0,r} = 1$, $r \in \mathbb{Z}_{<0}$, and

$$[H_r^{ver}, H_s^{ver}] = \delta_{r+s,0}[(n-m)r]\frac{1}{r}\frac{C^r - C^{-r}}{q - q^{-1}}.$$

Set $H^{ver}(z) = \sum_{r \in \mathbb{Z}^\times} H_r^{ver} z^{-r}$. This current will be used in Section 4.2.

Corollary 2.1.3. *Let $J \subset \hat{I}$, $J \neq \hat{I}$. Then the diagram subalgebra $\mathcal{E}_{\mathbf{s}}^J$ is isomorphic to tensor product of quantum affine superalgebras $U_q\widehat{\mathfrak{sl}}_{k|l}$ for generic values of parameters.*

$\square$

We denote $U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ the subalgebra of $\mathcal{E}_{\mathbf{s}}$ generated by $E_{i,0}, F_{i,0}, K_i, i \in \hat{I}$, and we call it the *horizontal quantum affine* $\mathfrak{sl}_{m|n}$.

We have a *horizontal homomorphism* of superalgebras $h_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$ given by

$$e_i \mapsto E_{i,0}, \quad f_i \mapsto F_{i,0}, \quad t_i \mapsto K_i \qquad (i \in \hat{I}),$$

with image $U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$.

We will later prove (for $N > 3$) that, for generic values of parameters, the horizontal homomorphism $h_{\mathbf{s}}$ is injective, see Corollary 3.1.10. Note that it is not a Hopf algebra map.

Note that, if $x \in U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$ and $\deg^h(x) = (d_0, d_1, d_2, \ldots, d_{N-1})$, then

$$\deg(h_{\mathbf{s}}(x)) = (d_0, d_1, d_2, \ldots, d_{N-1}; 0). \tag{2.1.23}$$

Lemma 2.1.4. *The quantum toroidal algebra $\mathcal{E}_{\mathbf{s}}$ is generated by the vertical and horizontal subalgebras $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ and $U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$.*

Proof. The only generators which are not generators of either the vertical or horizontal subalgebras are $E_{0,r}$, $F_{0,r}$, $r \in \mathbb{Z}^\times$. These generators are obtained as commutators of $E_{0,0}$ and $F_{0,0}$ with $H_{1,\pm 1}$, see (2.1.17), (2.1.18). $\qquad\square$

We often use a shortcut notation

$$X_0^+(z) := E_0(z), \qquad X_0^-(z) := F_0(z), \qquad \tilde{K}_0^\pm(z) := K_0^\pm(z),$$
$$X_i^\pm(z) := v_\mathbf{s}(x_i^\pm(z)), \qquad \tilde{K}_i^\pm(z) := v_\mathbf{s}(k_i^\pm(z)) \qquad (i \in I). \tag{2.1.24}$$

2.1.4 Morphisms

We list some symmetries of the superalgebras $\mathcal{E}_\mathbf{s}$.

Given $\mathbf{s} \in \mathcal{S}_{m|n}$, let $\mathbf{s}' = (s_{m-1}, s_{m-2}, \ldots, s_{-n})$. Then, the *diagram isomorphism* $\omega_\mathbf{s} : \mathcal{E}_\mathbf{s}(q_1, q_2, q_3) \to \mathcal{E}_{\mathbf{s}'}(q_3, q_2, q_1)$ defined by

$$\omega_\mathbf{s}(C) = C, \qquad \omega_\mathbf{s}(A_i(z)) = A_{m-i}(z) \qquad (i \in \hat{I},\ A = K^\pm, E, F),$$

changes d to d^{-1}.

Given $\mathbf{s} \in \mathcal{S}_{m|n}$, let $-\mathbf{s} = (-s_1, -s_2, \ldots, -s_N) \in \mathcal{S}_{n|m}$. The *change of parity* isomorphism $\nu_\mathbf{s} : \mathcal{E}_\mathbf{s}(q_1, q_2, q_3) \to \mathcal{E}_{-\mathbf{s}}(q_3^{-1}, q_2^{-1}, q_1^{-1})$ defined by

$$\nu_\mathbf{s}(C) = C, \qquad \nu_\mathbf{s}(K_i^\pm(z)) = -K_{-i}^\pm(z),$$
$$\nu_\mathbf{s}(E_i(z)) = E_{-i}(z), \qquad \nu_\mathbf{s}(F_i(z)) = F_{-i}(z) \qquad (i \in \hat{I}),$$

changes q to q^{-1}.

For $u \in \mathbb{C}^\times$, the *shift of spectral parameter* $\gamma_{u,\mathbf{s}} : \mathcal{E}_\mathbf{s} \to \mathcal{E}_\mathbf{s}$ is an isomorphism defined by

$$\gamma_{u,\mathbf{s}}(C) = C, \qquad \gamma_{u,\mathbf{s}}(A_i(z)) = A_i(uz) \qquad (i \in \hat{I},\ A = K^\pm, E, F).$$

For $\mathbf{s} \in \mathcal{S}_{m|n}$, there exists an isomorphism of superalgebras $\hat{\tau}_\mathbf{s} : \mathcal{E}_\mathbf{s} \to \mathcal{E}_{\tau\mathbf{s}}$ given by

$$\hat{\tau}_\mathbf{s}(C) = C, \qquad \hat{\tau}_\mathbf{s}(A_i(z)) = A_{i+1}(-d^{-s_N} z) \qquad (i \in \hat{I},\ A = K^\pm, E, F). \tag{2.1.25}$$

Recall our notation (2.1.24). In this notation, the map $\hat{\tau}_\mathbf{s}$ takes the form

$$\hat{\tau}_\mathbf{s}(C) = C, \qquad \hat{\tau}_\mathbf{s}(A_i(z)) = A_{i+1}(-d^{(n-m)\delta_{i,N-1}} z) \qquad (i \in \hat{I},\ A = \tilde{K}^\pm, X^+, X^-).$$

Proposition 2.1.5. *The isomorphisms $\widehat{\tau}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, satisfy*

$$(\widehat{\tau} h)_{\mathbf{s}} = (h \tau)_{\mathbf{s}}, \tag{2.1.26}$$

$$(\widehat{\tau} v)_{\mathbf{s}}(a_i(z)) = v_{\tau \mathbf{s}}(a_{i+1}(-z)) \qquad (i \in I \setminus \{N - 1\}, \, a = k^{\pm}, x^{\pm}). \tag{2.1.27}$$

$$(\widehat{\tau} v T_i)_{\mathbf{s}}(a_j(z)) = (v T_{i+1})_{\tau \mathbf{s}}(a_{j+1}(-z)) \quad (i, j \in I \setminus \{N - 1\}, \, a = k^{\pm}, x^{\pm}). \tag{2.1.28}$$

The maps $\widehat{\tau}_{\mathbf{s}}$ preserve the homogeneous grading and $\deg_i(\widehat{\tau}_{\mathbf{s}}(X)) = \deg_{i-1}(X)$, $i \in \hat{I}$.

Proof. A straightforward computation shows that $\widehat{\tau}_{\mathbf{s}}$ preserve the homogeneous grading, satisfy equality (2.1.26), and $\deg_i(\widehat{\tau}_{\mathbf{s}}(X)) = \deg_{i-1}(X)$, $i \in \hat{I}$, if $X \in \mathcal{E}_{\mathbf{s}}$ is homogeneous.

We check $(\widehat{\tau} v)_{\mathbf{s}}(x_i^+(z)) = v_{\tau \mathbf{s}}(x_{i+1}^+(-z))$ for $1 \leq i \leq N - 2$.

By definition, we have

$$(\widehat{\tau} v)_{\mathbf{s}}(x_i^+(z)) = \widehat{\tau}_{\mathbf{s}}(E_i(d^{\mu_{\mathbf{s}}(i)} z)) = E_{i+1}(-d^{\mu_{\mathbf{s}}(i) - s_N} z),$$

$$v_{\tau \mathbf{s}}(x_{i+1}^+(-z)) = E_{i+1}(-d^{\mu_{\tau \mathbf{s}}(i+1)} z).$$

But $\tau \mathbf{s} = (s_N, s_1, \ldots, s_{N-1})$, thus

$$\mu_{\tau \mathbf{s}}(i + 1) = -s_N - s_1 - \cdots - s_i = \mu_{\mathbf{s}}(i) - s_N.$$

The proofs for $x_i^-(z)$ and $k_i^{\pm}(z)$ are analogous.

Equation (2.1.28) for $i \neq j$ follows from Lemma 1.3.5 and equation (2.1.27).

To show (2.1.28) with $i = j$, set $l = i - 1$ if $i \neq 1$, and $l = 2$ if $i = 1$. In particular, $A_{l,i}^{\mathbf{s}} \neq 0$. Then, since $l \neq i$, we have

$$(\widehat{\tau} v T_i)_{\mathbf{s}}(h_{l,\pm 1}) = -(v T_{i+1})_{\tau \mathbf{s}}(h_{l+1,\pm 1}).$$

Also, by a direct computation, we have

$$(\widehat{\tau} v T_i)_{\mathbf{s}}(x_{i,0}^{\pm}) = (v T_{i+1})_{\tau \mathbf{s}}(x_{i+1,0}^{\pm}).$$

Therefore, the constant terms of left hand side and right hand side of (2.1.28) coincide. The equality of other terms follow from the commutator

$$[h_{l,\pm 1}, x_i^+(z)] = [A_{l,i}^{\mathbf{s}}]c^{-(1 \pm 1)/2} z^{\pm 1} x_i^{\pm}(z)$$

and a similar one for $x_i^-(z)$. $\qquad \square$

The homomorphism $v_{\mathbf{s}}, h_{\mathbf{s}}$ and $\widehat{\tau}_{\mathbf{s}}$ previously defined correspond to the algebra $\mathcal{E}_{\mathbf{s}}(q_1, q_2, q_3)$. Let $v'_{\mathbf{s}}, h'_{\mathbf{s}}$ and $\widehat{\tau}'_{\mathbf{s}}$ be the analogous homomorphisms corresponding to the algebra $\mathcal{E}_{\mathbf{s}}(q_3, q_2, q_1)$, i.e., the parameter d is switched to d^{-1}.

The map $\eta_{\mathbf{s}}$ defined on Lemma 1.3.8 has the following toroidal counterpart.

For $\mathbf{s} \in \mathcal{S}_{m|n}$, there exists an anti-isomorphism of superalgebras $\hat{\eta}_{\mathbf{s}} : \mathcal{E}_{\mathbf{s}}(q_1, q_2, q_3) \to \mathcal{E}_{\mathbf{s}}(q_3, q_2, q_1)$ given by

$$\hat{\eta}_{\mathbf{s}}(C) = C, \qquad\qquad \hat{\eta}_{\mathbf{s}}(K_i^{\pm}(z)) = K_i^{\mp}(Cz^{-1}),$$
$$\hat{\eta}_{\mathbf{s}}(E_i(z)) = E_i(z^{-1}), \qquad \hat{\eta}_{\mathbf{s}}(F_i(z)) = F_i(z^{-1}) \qquad\qquad (i \in \hat{I}).$$

The anti-isomorphism $\hat{\eta}_{\mathbf{s}}$ preserves $\deg_i$, $i \in \hat{I}$, and $\deg_\delta(\hat{\eta}_{\mathbf{s}}(X)) = -\deg_\delta(X)$ if $X \in \mathcal{E}_{\mathbf{s}}$ is homogeneous.

Lemma 2.1.6. *The anti-isomorphisms $\hat{\eta}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, satisfy*

$$(\hat{\eta}\, v)_{\mathbf{s}} = (v'\, \eta)_{\mathbf{s}}, \qquad\qquad (\hat{\eta}\, h)_{\mathbf{s}} = (h'\, \varphi)_{\mathbf{s}}, \qquad\qquad (\hat{\eta}\, \widehat{\tau})_{\mathbf{s}} = (\widehat{\tau}'\, \hat{\eta})_{\mathbf{s}}.$$

Proof. The equality $(\hat{\eta}\, h)_{\mathbf{s}} = (h'\, \varphi)_{\mathbf{s}}$ is clear.

We check, for example, $(\hat{\eta}\, v)_{\mathbf{s}} = (v'\, \eta)_{\mathbf{s}}$ on $x_i^+(z)$ and $(\hat{\eta}\, \widehat{\tau})_{\mathbf{s}} = (\widehat{\tau}'\, \hat{\eta})_{\mathbf{s}}$ on $E_{i,r}$. The other cases are analogous. Note that $\hat{\eta}_{\mathbf{s}}$ interchanges the parameters q_1 and q_3, i.e., d and d^{-1} are interchanged. Thus,

$$(\hat{\eta}\, v)_{\mathbf{s}}(x_{i,r}^+) = \hat{\eta}_{\mathbf{s}}(d^{-r\mu_{\mathbf{s}}(i)} E_{i,r}) = d^{-r\mu_{\mathbf{s}}(i)} E_{i,-r} = v'_{\mathbf{s}}(x_{i,-r}^+) = (v'\, \eta)_{\mathbf{s}}(x_{i,r}^+),$$
$$(\hat{\eta}\, \widehat{\tau})_{\mathbf{s}}(E_{i,r}) = \hat{\eta}_{\tau\mathbf{s}}(-d^{r\, s_N} E_{i+1,r}) = -d^{r\, s_N} E_{i+1,-r} = \widehat{\tau}'_{\mathbf{s}}(E_{i,-r}) = (\widehat{\tau}'\, \hat{\eta})_{\mathbf{s}}(E_{i,r}).$$

$\square$

The maps $\mathcal{X}_{i,\mathbf{s}}$ defined on Proposition 1.3.4 also have toroidal analogs as follows.

There exist automorphisms of superalgebras $\widehat{\mathcal{X}}_{i,\mathbf{s}} : \mathcal{E}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$, $i \in \hat{I}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, given by

$$\widehat{\mathcal{X}}_{i,\mathbf{s}}(C) = C, \qquad \widehat{\mathcal{X}}_{i,\mathbf{s}}(K_j^{\pm}(z)) = C^{\mp\delta_{ij}} K_j^{\pm}(z),$$
$$\widehat{\mathcal{X}}_{i,\mathbf{s}}(X_j^{\pm}(z)) = ((-1)^i z^{\mp 1})^{\delta_{ij}} X_j^{\pm}(z), \qquad\qquad (j \in \hat{I}). \tag{2.1.29}$$

The automorphism $\widehat{\mathcal{X}}_{i,\mathbf{s}}$ preserves $\deg_j$, $j \in \hat{I}$, and $\deg_\delta(\widehat{\mathcal{X}}_{i,\mathbf{s}}(X)) = \deg_\delta(X) - \deg_i(X)$ if $X \in \mathcal{E}_{\mathbf{s}}$ is homogeneous. Let also $\widehat{\mathcal{X}}'_{i,\mathbf{s}}$ be the analogous automorphism corresponding to the algebra $\mathcal{E}_{\mathbf{s}}(q_3, q_2, q_1)$.

Let $\zeta_{\mathbf{s}} : \mathcal{E}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, be the rescaling automorphism given by

$$\zeta_{\mathbf{s}}(C) = C, \quad \zeta_{\mathbf{s}}(X_i^\pm(z)) = ((-1)^N d^{n-m})^{\pm\delta_{i,0}} X_i^\pm(z), \quad \zeta_{\mathbf{s}}(K_i^\pm(z)) = K_i^\pm(z) \quad (i \in \hat{I}).$$

Proposition 2.1.7. *The automorphisms* $\widehat{\mathcal{X}}_{i,\mathbf{s}}$, $\zeta_{\mathbf{s}}$ *satisfy*

$$(\hat{\eta}' \, \widehat{\mathcal{X}}'_i \, \hat{\eta})_{\mathbf{s}} = \widehat{\mathcal{X}}_{i,\mathbf{s}}^{-1}, \qquad (\widehat{\mathcal{X}}_j \, v)_{\mathbf{s}} = (v \, \mathcal{X}_j)_{\mathbf{s}} \qquad\qquad (i \in \hat{I}, \; j \in I), \qquad (2.1.30)$$

$$(\hat{\tau} \, \widehat{\mathcal{X}}_{j-1})_{\mathbf{s}} = (\widehat{\mathcal{X}}_j \, \hat{\tau})_{\mathbf{s}}, \qquad (\hat{\tau} \, \widehat{\mathcal{X}}_{N-1})_{\mathbf{s}} = (\zeta \, \widehat{\mathcal{X}}_0 \, \hat{\tau})_{\mathbf{s}} \qquad\qquad (j \in I). \qquad (2.1.31)$$

Proof. Identities (2.1.30) and the first equality of (2.1.31) are clear.

We check $(\hat{\tau} \, \widehat{\mathcal{X}}_{N-1})_{\mathbf{s}} = (\zeta \, \widehat{\mathcal{X}}_0 \, \hat{\tau})_{\mathbf{s}}$ applied to $X_{N-1,r}^\pm$:

$$(\hat{\tau} \, \widehat{\mathcal{X}}_{N-1})_{\mathbf{s}}(X_{N-1,r}^\pm) = \hat{\tau}_{\mathbf{s}}((-1)^{N-1} X_{N-1,r\mp 1}^\pm) = (-1)^{N+r} d^{(r\pm 1)(m-n)} X_{0,r\mp 1}^\pm,$$

$$(\zeta \widehat{\mathcal{X}}_0 \, \hat{\tau})_{\mathbf{s}}(X_{N-1,r}^\pm) = (\zeta \widehat{\mathcal{X}}_0)_{\mathbf{s}}((-1)^r d^{r(m-n)} X_{0,r}^\pm) = \zeta_{\mathbf{s}}((-1)^r d^{r(m-n)} X_{0,r\mp 1}^\pm)$$

$$= (-1)^{N+r} d^{(r\pm 1)(m-n)} X_{0,r\mp 1}^\pm.$$

The check on the remaining generators is similar. $\qquad\square$

3. BRAID GROUP ACTIONS

3.1 Toroidal braid group

We construct an action of the toroidal braid group $\widehat{\mathcal{B}}_N$ associated with $\mathfrak{sl}_{m|n}$ on $\mathcal{E}_\bullet = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \mathcal{E}_\mathbf{s}$. As a consequence, we show that the algebras $\mathcal{E}_\mathbf{s}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, are all isomorphic. It also gives us the *Miki automorphism* of $\mathcal{E}_\mathbf{s}$, which interchanges the horizontal and vertical subalgebras.

In this section, we assume $N \geq 4$.

3.1.1 Action of $\mathcal{B}_N$ on $\mathcal{E}_\bullet$

We start with extending the action of affine braid group $\mathcal{B}_N$ from the vertical subalgebra $U_q^{ver}\widehat{\mathfrak{sl}}_\bullet$ given in Proposition 1.3.1 to the toroidal algebra $\mathcal{E}_\bullet$.

Note that the map $\widehat{\tau}_\mathbf{s}$ was already defined in (2.1.25). We also recall that the prime symbol $'$ indicates the action of the operator with q_3 and q_1 switched, see Section 2.1.4.

Theorem 3.1.1. *Let $N > 3$. For $i \in \hat{I}$, $\mathbf{s} \in \mathcal{S}_{m|n}$, there exists an isomorphism of superalgebras $\widehat{T}_{i,\mathbf{s}} : \mathcal{E}_\mathbf{s} \to \mathcal{E}_{\sigma_i \mathbf{s}}$ satisfying*

$$(\widehat{T}_i \, v)_\mathbf{s} = (v T_i)_\mathbf{s} \qquad (i \in I), \qquad (3.1.1)$$

$$(\widehat{T}_i \, h)_\mathbf{s} = (h \, T_i)_\mathbf{s} \qquad (i \in \hat{I}), \qquad (3.1.2)$$

$$(\widehat{\tau} \widehat{T}_i)_\mathbf{s} = (\widehat{T}_{i+1} \widehat{\tau})_\mathbf{s} \qquad (i \in \hat{I}), \qquad (3.1.3)$$

$$(\hat{\eta}' \, \widehat{T}_i' \, \hat{\eta})_\mathbf{s} = (\widehat{T}_i^{-1})_\mathbf{s} \qquad (i \in \hat{I}). \qquad (3.1.4)$$

Moreover, the isomorphisms $\widehat{T}_{i,\mathbf{s}}$ satisfy the Coxeter relations

$$(\widehat{T}_{i+1} \widehat{T}_i \widehat{T}_{i+1})_\mathbf{s} = (\widehat{T}_i \widehat{T}_{i+1} \widehat{T}_i)_\mathbf{s} \qquad (i \in \hat{I}), \qquad (3.1.5)$$

$$(\widehat{T}_i \widehat{T}_j)_\mathbf{s} = (\widehat{T}_j \widehat{T}_i)_\mathbf{s} \qquad (i \neq j \pm 1). \qquad (3.1.6)$$

Finally, $\widehat{T}_{i,\mathbf{s}}$ are graded with respect to homogeneous grading.

Proof. Throughout this proof we write similar formulas for $X_i^{\pm}(z)$ and $\tilde{K}_i^{\pm}(z)$. To avoid repeating the same formula four times, we use the letters $A_i(z)$ to denote $X_i^{\pm}(z)$ or $\tilde{K}_i^{\pm}(z)$, and $a_i(z)$ to denote $x_i^{\pm}(z)$ or $k_i^{\pm}(z)$. Note that $A_i(z)$ and $a_i(z)$ in the same formula are all of the same kind – e.g. all $X_i^+(z)$ and $x_i^+(z)$.

Define the map $\widehat{T}_{1,\mathbf{s}}$ on generators of $\mathcal{E}_\mathbf{s}$ by

$$\widehat{T}_{1,\mathbf{s}}(A_0(z)) = (\widehat{\tau}^{-1}vT_2)_{\tau\mathbf{s}}(a_1(-z)), \qquad \widehat{T}_{1,\mathbf{s}}(A_i(z)) = (vT_1)_\mathbf{s}(a_i(z)) \qquad (i \in I).$$

Note that $\widehat{T}_{1,\mathbf{s}}(A_i(z)) = A_i(z)$ if $i = 3, \ldots, N-1$. Moreover, the action of $\widehat{T}_{1,\mathbf{s}}$ on $A_0(z), A_2(z), A_{1,0}$ is explicit by Lemma 1.3.5. The map $\widehat{T}_{1,\mathbf{s}}$ respects homogeneous grading because $T_{1,\mathbf{s}}$ and $T_{2,\mathbf{s}}$ do.

We claim that this extends to an isomorphism of superalgebras. In fact, all relations which do not involve node 1 (that is, the ones which do not contain $E_1(z), F_1(z), K_1^{\pm}(z)$) can be checked by a direct computation. To check the relations which do involve node 1, and to reduce the calculations in other cases, we can use the following arguments.

The relations not involving node 0 are satisfied since we can compute in the vertical algebra and $T_{1,\mathbf{s}}$ is a homomorphism.

To check the relations involving node 0, note that by Proposition 2.1.5 we have

$$\widehat{\tau}_\mathbf{s}(A_{N-1}(z)) = A_0(-d^{n-m}z), \qquad \widehat{\tau}_\mathbf{s}(A_i(z)) = A_{i+1}(-z) \qquad\qquad (i \neq N-1),$$
$$\widehat{T}_{1,\mathbf{s}}(A_i(z)) = (\widehat{\tau}^{1-k}vT_k)_{\tau^{k-1}\mathbf{s}}(a_{k+i-1}((-1)^{1-k}z)) \qquad\qquad (k, k+i-1 \in I),$$
$$\widehat{T}_{1,\mathbf{s}}(A_{N-1}(d^{m-n}z)) = (\widehat{\tau}^{-2}vT_3)_{\tau^2\mathbf{s}}(a_1(z)) = (\widehat{\tau}^{-3}vT_4)_{\tau^3\mathbf{s}}(a_2(-z)) \qquad (N > 4),$$
$$\widehat{T}_{1,\mathbf{s}}(A_{N-2}(d^{m-n}z)) = (\widehat{\tau}^{-3}vT_4)_{\tau^3\mathbf{s}}(a_2(-z)) \qquad\qquad (N > 4).$$

Let us first assume that $N > 4$.

We prove the relations involving the nodes $0, 1$, and 2 by moving these nodes to $1, 2$ and 3 using τ and $\widehat{\tau}$. Namely, by (1.3.3) and (2.1.27), we have

$$\widehat{T}_{1,\mathbf{s}}(A_i(z)) = (\widehat{\tau}^{-1}vT_2)_{\tau\mathbf{s}}(a_{i+1}(-z)) \qquad\qquad (i = 0, 1, 2).$$

Observe that the defining relations between generators of $\mathcal{E}_\mathbf{s}$ involving nodes $0, 1, 2$ are the same as the defining relations in $U_q\widehat{\mathfrak{sl}}_{\tau\mathbf{s}}$ involving nodes $1, 2, 3$. Since $(\widehat{\tau}^{-1}vT_2)_{\tau\mathbf{s}}$ is a homomorphism, it maps the the relations of $U_q\widehat{\mathfrak{sl}}_{\tau\mathbf{s}}$ involving the nodes $1, 2, 3$ to zero. Therefore $\widehat{T}_{1,\mathbf{s}}$ maps defining relations of $\mathcal{E}_\mathbf{s}$ involving nodes $0, 1, 2$ to zero.

The relations involving the nodes $N - 1, 0, 1$ are treated similarly: we go to nodes $1, 2, 3$ again by using

$$\widehat{T}_{1,\mathbf{s}}(A_i(z)) = (\widehat{\tau}^{-2}vT_3)_{\tau^2\mathbf{s}}(a_{i+2}(z)) \qquad\qquad (i = 0, 1),$$
$$\widehat{T}_{1,\mathbf{s}}(A_{N-1}(d^{m-n}z)) = (\widehat{\tau}^{-2}vT_3)_{\tau^2\mathbf{s}}(a_1(z)).$$

Note that the defining relations between generators of $\mathcal{E}_\mathbf{s}$ involving nodes $N - 1, 0, 1$ are the same as the defining relations in $U_q\widehat{\mathfrak{sl}}_{\tau^2\mathbf{s}}$ involving nodes $1, 2, 3$ if the shift of spectral parameter in the generating series related to node $N-1$ is taken into account. Therefore, as before, $\widehat{T}_{1,\mathbf{s}}$ maps defining relations of $\mathcal{E}_\mathbf{s}$ involving nodes $N - 1, 0, 1$ to zero.

For the relations involving the nodes $N - 2, N - 1, 0$ we proceed in the same way by using

$$\widehat{T}_{1,\mathbf{s}}(A_{N-i}(d^{m-n}z)) = (\widehat{\tau}^{-3}vT_4)_{\tau^3\mathbf{s}}(a_{3-i}(-z)) \qquad\qquad (i = 1, 2),$$
$$\widehat{T}_{1,\mathbf{s}}(A_0(z)) = (\widehat{\tau}^{-3}vT_4)_{\tau^3\mathbf{s}}(a_3(-z)).$$

and reducing to nodes $1, 2, 3$ once again. We omit further details. Thus for $N > 4$ all relations follow without extra computations.

If $N = 4$, the previous argument applies for the relations involving the nodes $0, 1, 2$, or $-1, 0, 1$, or $0, 1$ or the nodes involving $2, 0$. The additional relations (2.1.13) and (2.1.14) for $i = 3$ in the case $A_{3,3}^\mathbf{s} = 0$ are checked directly. We check (2.1.13) with $i = 3$ as an example.

First, by identity (1.1.6) and relation (2.1.13) we have

$$0 = \mathrm{Sym}_{z_1,z_2}[\![[\![[\![E_3(z_1), [\![E_0(w_1), [\![E_3(z_2), E_2(w_2)]\!]]\!]]\!], E_{1,0}]\!], E_{1,0}]\!]$$
$$= (q + q^{-1})\mathrm{Sym}_{z_1,z_2}[\![[\![E_3(z_1), [\![E_0(w_1), E_{1,0}]\!]]\!], [\![E_3(z_2), [\![E_2(w_2), E_{1,0}]\!]]\!]]\!].$$

Using Lemma 1.3.5 we can compute the action of $\widehat{T}_{1,\mathbf{s}}$ on (2.1.13) explicitly. Note that $A_{3,3}^{\mathbf{s}} = 0$, $N = 4$, $m \neq n$ imply $mn = 3$ and $|1| = 0$. We have

$$
\begin{aligned}
&\widehat{T}_{1,\mathbf{s}}(\mathrm{Sym}_{z_1,z_2}[\![E_3(z_1), [\![E_0(w_1), [\![E_3(z_2), E_2(w_2)]\!]]\!]]\!]) \\
&= \widehat{T}_{1,\mathbf{s}}(\mathrm{Sym}_{z_1,z_2}[\![[\![E_3(z_1), E_0(w_1)]\!], [\![E_3(z_2), E_2(w_2)]\!]]\!]) \\
&= q^{-A_{1,1}^{\mathbf{s}}}\mathrm{Sym}_{z_1,z_2}[\![[\![E_3(z_1), [\![E_0(w_1), E_{1,0}]\!]]\!], [\![E_3(z_2), [\![E_2(w_2), E_{1,0}]\!]]\!]]\!] = 0.
\end{aligned}
$$

This shows that $\widehat{T}_{1,\mathbf{s}}$ is a homomorphism.

For $i \in \hat{I}$ define

$$
\widehat{T}_{i,\mathbf{s}} = (\widehat{\tau}^{\,i-1}\,\widehat{T}_1\,\widehat{\tau}^{\,1-i})_{\mathbf{s}}. \tag{3.1.7}
$$

Since $\widehat{T}_{1,\mathbf{s}}$ and $\widehat{\tau}_{\mathbf{s}}$ are homomorphisms for all $\mathbf{s} \in \mathcal{S}_{m|n}$, the maps $\widehat{T}_{i,\mathbf{s}}$ are well defined homomorphisms for all $i \in \hat{I}$ and $\mathbf{s} \in \mathcal{S}_{m|n}$.

Note that

$$
\widehat{T}_{i,\mathbf{s}}(A_j(z)) = A_j(z) \qquad\qquad (j \neq i, i \pm 1). \tag{3.1.8}
$$

Also, $\widehat{T}_{i,\mathbf{s}}(A_j(z))$ is explicit if $j = i \pm 1$ and so is $\widehat{T}_{i,\mathbf{s}}(A_{i,0})$.

Now, we show that the homomorphisms $\widehat{T}_{i,\mathbf{s}}$ satisfy equations (3.1.1) using induction on i. For $i = 1$ the statement follows from definition of $\widehat{T}_1$. Suppose (3.1.1) is true for $i = j \leq N - 2$. Let us prove it for $i = j + 1$.

If $l \in I$, $l \neq 1$, we have

$$
\begin{aligned}
(\widehat{T}_{j+1}v)_{\mathbf{s}}(a_l(z)) &= (\widehat{\tau}\,\widehat{T}_j\,\widehat{\tau}^{-1}v)_{\mathbf{s}}(a_l(z)) = (\widehat{\tau}\,\widehat{T}_j v)_{\tau^{-1}\mathbf{s}}(a_{l-1}(-z)) \\
&= (\widehat{\tau}\,v\,T_j)_{\tau^{-1}\mathbf{s}}(a_{l-1}(-z)) = (v\,T_{j+1})_{\mathbf{s}}(a_l(z)).
\end{aligned}
$$

Here the second equality is (2.1.27), the third equality is the induction hypothesis, and the last equality is (2.1.28).

If $l = 1$ then

$$
(\widehat{T}_{j+1}\,v)_{\mathbf{s}}(a_1(z)) = (\widehat{\tau}\,\widehat{T}_j\,\widehat{\tau}^{-1})_{\mathbf{s}}(A_1(z)) = (\widehat{\tau}\,\widehat{T}_j)_{\tau^{-1}\mathbf{s}}(A_0(-z)) = (v T_{j+1})_{\mathbf{s}}(a_1(z)).
$$

Here the last equation is a definition if $j = 1$ and a trivial statement if $j > 1$ since in that case $\widehat{T}_{j,\mathbf{s}}(A_0(z)) = A_0(z)$ and $T_{j+1,\mathbf{s}}(a_1(z)) = a_1(z)$.

Thus, $\widehat{T}_{i,\mathbf{s}}$ satisfy equation (3.1.1).

Next we show (3.1.4) for $i = 1$. By (3.1.1), Lemmas 1.3.8 and 2.1.6, we have

$$(\widehat{T}_1 \hat{\eta}' \widehat{T}_1' \hat{\eta})_{\mathbf{s}}(A_i(z)) = (v\, T_1\, \eta\, T_1\, \eta)_{\mathbf{s}}(a_i(z)) = v_{\mathbf{s}}(a_i(z)) = A_i(z) \qquad (i \in I),$$

$$(\widehat{T}_1 \hat{\eta}' \widehat{T}_1' \hat{\eta})_{\mathbf{s}}(A_0(z)) = (\widehat{\tau}^{-1} v\, T_2\, \eta\, T_2\, \eta)_{\tau\mathbf{s}}(a_1(-z)) = (\widehat{\tau}^{-1} v)_{\tau\mathbf{s}}(a_1(-z)) = A_0(z).$$

Thus, equation (3.1.4) holds for $i = 1$. In particular, it implies that $\widehat{T}_{1,\mathbf{s}}$ has an inverse and therefore is an isomorphism.

By Lemma 2.1.6, $\hat{\eta}_{\mathbf{s}}$ commutes with $\widehat{\tau}_{\mathbf{s}}$. It implies (3.1.4) for all $i \in \hat{I}$. In particular, $\widehat{T}_{i,\mathbf{s}}$ is isomorphism for all $i \in \hat{I}$.

By (2.1.26), the isomorphisms $\widehat{T}_{i,\mathbf{s}}$ satisfy equation (3.1.2).

Finally, we show Coxeter relations (3.1.5) and (3.1.6). By equation (3.1.3), it is sufficient to show these relations when $i = 1$ and $j \neq 0$. By Proposition 1.3.2, Coxeter relations are satisfied by the homomorphisms $T_{i,\mathbf{s}}$. Then by (3.1.1), relations (3.1.5) and (3.1.6) are satisfied on the image of $v_{\mathbf{s}}$. By (3.1.2), these relations are also satisfied on the image of $h_{\mathbf{s}}$. Since horizontal and vertical subalgebras generate the whole algebra $\mathcal{E}_{\mathbf{s}}$, we obtain the proof of (3.1.5) and (3.1.6)

$\square$

Corollary 3.1.2. *The superalgebras $\mathcal{E}_{\mathbf{s}}$ are isomorphic for all $\mathbf{s} \in \mathcal{S}_{m|n}$.*

Proof. The corollary follows from Theorem 3.1.1. $\square$

Remark 3.1.3. *The corollary above treats the case $N \geq 4$. For $N = 3$, the isomorphisms between all three algebras $\mathcal{E}_{\mathbf{s}}$ are given by the map $\widehat{\tau}$.*

Define the following automorphisms of $\mathcal{E}_{\bullet} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \mathcal{E}_{\mathbf{s}}$

$$\widehat{T}_i = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \widehat{T}_{i,\mathbf{s}}, \qquad\qquad \widehat{\tau} = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \widehat{\tau}_{\mathbf{s}} \qquad\qquad (i \in I).$$

Corollary 3.1.4. *Let $N > 3$. The automorphisms $\widehat{\tau}$, $\widehat{T}_i$, $i \in \hat{I}$, define an action of the extended affine braid group $\mathcal{B}_N$ on $\mathcal{E}_{\bullet}$, i.e., they satisfy the relations (1.3.1)-(1.3.3).*

3.1.2 Toroidal braid group

We recall the definition of the toroidal braid group of $\mathfrak{sl}_N$, see [28].

Definition 3.1.5. *The toroidal braid group $\widehat{\mathcal{B}}_N$ of $\mathfrak{sl}_N$ is the group generated by elements $\widehat{\tau}$, $\widehat{T}_i$, $\widehat{\mathcal{Y}}_j$, $i \in I, j \in \hat{I}$, satisfying the relations*

$$\widehat{T}_i \widehat{T}_j = \widehat{T}_j \widehat{T}_i \qquad (j \neq i, i \pm 1), \qquad (3.1.9)$$

$$\widehat{T}_j \widehat{T}_i \widehat{T}_j = \widehat{T}_i \widehat{T}_j \widehat{T}_i \qquad (j = i \pm 1), \qquad (3.1.10)$$

$$\widehat{\mathcal{Y}}_i \widehat{\mathcal{Y}}_j = \widehat{\mathcal{Y}}_j \widehat{\mathcal{Y}}_i, \qquad (3.1.11)$$

$$\widehat{T}_i \widehat{\mathcal{Y}}_j = \widehat{\mathcal{Y}}_j \widehat{T}_i \qquad (j \neq i, i + 1), \qquad (3.1.12)$$

$$\widehat{T}_i^{-1} \widehat{\mathcal{Y}}_i \widehat{T}_i^{-1} = \widehat{\mathcal{Y}}_{i+1} \qquad (i \in I), \qquad (3.1.13)$$

$$\widehat{\tau} \widehat{T}_i \widehat{\tau}^{-1} = \widehat{T}_{i+1}, \qquad (1 \leq i \leq N - 2), \qquad (3.1.14)$$

$$\widehat{\tau}^2 \widehat{T}_{N-1} \widehat{\tau}^{-2} = \widehat{T}_1, \qquad (3.1.15)$$

$$\widehat{\tau} \widehat{\mathcal{Y}}_i \widehat{\tau}^{-1} = \widehat{\mathcal{Y}}_{i+1} \qquad (i \in I). \qquad (3.1.16)$$

We remark that the toroidal braid group $\widehat{\mathcal{B}}_N$ quotient by the relation $\widehat{\tau} \widehat{\mathcal{Y}}_0 \widehat{\tau}^{-1} = \widehat{\mathcal{Y}}_1$ is isomorphic to double affine Hecke group with central element set to 1, see Definition 4.1 in [10].

The toroidal braid group has the following Fourier transform given by qKZ elements.

Lemma 3.1.6 ([10],[28]). *There exists an automorphism Φ of $\widehat{\mathcal{B}}_N$ given by*

$$\Phi(\widehat{T}_i) = \widehat{T}_i, \qquad \Phi(\widehat{\mathcal{Y}}_j) = \widehat{T}_{j-1}^{-1} \cdots \widehat{T}_1^{-1} \widehat{\tau} \widehat{T}_{N-1} \cdots \widehat{T}_j, \qquad \Phi(\widehat{\tau}) = \widehat{\mathcal{Y}}_1^{-1} \widehat{T}_1 \cdots \widehat{T}_{N-1}.$$

$\square$

Note that the subgroup $G \subset \widehat{\mathcal{B}}_N$ generated by $\widehat{T}_1$ and $\widehat{\tau}$, and the subgroup $H \subset \widehat{\mathcal{B}}_N$ generated by $\widehat{T}_1$ and $\widehat{\mathcal{Y}}_1$ are both isomorphic to the extended affine braid group $\mathcal{B}_N$. The isomorphism γ between these two presentations of $\mathcal{B}_N$ is described in (1.3.11).

Let i_G be the inclusion $i_G : G \cong \mathcal{B}_N \to \widehat{\mathcal{B}}_N$ given by

$$i_G(T_1) = \widehat{T}_1, \qquad i_G(\tau) = \widehat{\tau},$$

and i_H be the inclusion $i_H : H \cong \mathcal{B}_N \to \widehat{\mathcal{B}}_N$ given by

$$i_H(T_1) = \widehat{T}_1, \qquad i_H(\mathcal{X}_1) = \widehat{\mathcal{Y}}_1.$$

The following lemma is easily checked on generators.

Lemma 3.1.7. *The homomorphism γ, i_G, i_H and Φ satisfy the following commutative diagram*

$$
\begin{array}{ccc}
\widehat{\mathcal{B}}_N & \xrightarrow[\Phi]{\sim} & \widehat{\mathcal{B}}_N \\
{\scriptstyle i_H}\Big\uparrow & & \Big\uparrow{\scriptstyle i_G} \\
\mathcal{B}_N & \xrightarrow[\gamma]{\sim} & \mathcal{B}_N
\end{array}
$$

$\square$

Recall automorphisms $\zeta_{\mathbf{s}}$, $\widehat{\mathcal{X}}_{i,\mathbf{s}}$ of $\mathcal{E}_{\mathbf{s}}$ described in Proposition 2.1.7. Define the following automorphisms of $\mathcal{E}_{\mathbf{s}}$

$$\widehat{\mathcal{Y}}_{0,\mathbf{s}} = (\zeta \widehat{\mathcal{X}}_0 \widehat{\mathcal{X}}_{N-1}^{-1})_{\mathbf{s}}\,, \qquad \widehat{\mathcal{Y}}_{i,\mathbf{s}} = (\widehat{\mathcal{X}}_i \widehat{\mathcal{X}}_{i-1}^{-1})_{\mathbf{s}} \qquad (i \in I). \qquad (3.1.17)$$

Let also

$$\widehat{\mathcal{Y}}_i = \bigoplus_{\mathbf{s} \in \mathcal{S}_{m|n}} \widehat{\mathcal{Y}}_{i,\mathbf{s}} \qquad\qquad (i \in \hat{I}).$$

Proposition 3.1.8. *The automorphisms $\widehat{\tau}, \widehat{T}_i, \widehat{\mathcal{Y}}_j$, $i \in I$, $j \in \hat{I}$, define an action of the toroidal braid group $\widehat{\mathcal{B}}_N$ on $\mathcal{E}_{\bullet}$.*

Proof. The relations for $\widehat{T}_i$ and $\widehat{\tau}$ follow from Theorem 3.1.1. Relation (3.1.16) between $\widehat{\mathcal{Y}}_i$ and $\widehat{\tau}$ follows from (2.1.31). Relations (3.1.12) are clear due to (3.1.8). Equation (3.1.13) between $\widehat{\mathcal{Y}}_i$ and $\widehat{T}_j$ on vertical subalgebra follows from (1.3.10) and (1.3.8) (note that $(\widehat{\mathcal{X}}_0 v)_{\mathbf{s}} = v_{\mathbf{s}}$). To check the relations on $A_0(z)$ we write $A_0(z) = \widehat{\tau}_{\mathbf{s}}^{-1}(A_1(-z))$ and use the already established relations with $\widehat{\tau}$. $\square$

3.1.3 Miki automorphism

Now we are ready to prove the existence of the Miki automorphism of $\mathcal{E}_\mathbf{s}$ which switches horizontal and vertical subalgebras.

Recall the isomorphism $\iota_\mathbf{s}$ identifying the new Drinfeld and Drinfeld-Jimbo realizations of $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$, see Proposition 1.3.3.

Theorem 3.1.9. *Let $N > 3$. There exists a superalgebra automorphism $\psi_\mathbf{s}$ of $\mathcal{E}_\mathbf{s}$ satisfying*

$$(\psi\,v)_\mathbf{s} = (h\,\iota)_\mathbf{s}, \qquad (\psi\,h\,\iota)_\mathbf{s} = (v\,\eta\,\iota^{-1}\,\varphi\,\iota)_\mathbf{s}, \qquad \psi_\mathbf{s}^{-1} = (\hat{\eta}'\,\psi'\,\hat{\eta})_\mathbf{s}, \qquad (3.1.18)$$

where the first two equalities are equalities of maps from the new Drinfeld realization of $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$ to $\mathcal{E}_\mathbf{s}$.

Proof. We often write equation of maps from $U_q\widehat{\mathfrak{sl}}_\mathbf{s}$ to $\mathcal{E}_\mathbf{s}$, similar to (3.1.18). We understand that new Drinfeld realization is identified with the Drinfeld-Jimbo realization via map ι and do not distinguish between them. In particular, we skip ι from our formulas.

Recall notation (2.1.24). For $i \in I$, let $\mathcal{Z}_{i,\mathbf{s}} = (\widehat{\mathcal{Y}}_1 \cdots \widehat{\mathcal{Y}}_i)_\mathbf{s}$.

Using equations (1.3.11), (2.1.26) and (3.1.2) we get

$$\Phi(\widehat{\mathcal{Y}}_{1,\mathbf{s}})h_\mathbf{s} = h_\mathbf{s}(\tau T_{N-1} \cdots T_1)_\mathbf{s} = (h\,\mathcal{X}_1)_\mathbf{s}\,.$$

And, by relations (1.3.8)-(1.3.10), we have

$$\Phi(\widehat{\mathcal{Y}}_{i,\mathbf{s}})h_\mathbf{s} = (h\,\mathcal{X}_i\,\mathcal{X}_{i-1}^{-1})_\mathbf{s} \qquad\qquad (i \in I \setminus \{1\}).$$

Thus,

$$\Phi(\mathcal{Z}_{i,\mathbf{s}})h_\mathbf{s} = h_\mathbf{s}\mathcal{X}_{i,\mathbf{s}} \qquad\qquad (i \in I). \qquad (3.1.19)$$

Define $\psi_\mathbf{s}$ on the $\mathcal{E}_\mathbf{s}$ generators by

$$\psi_\mathbf{s}(X_{i,r}^\pm) = (-1)^{ir}\Phi(\mathcal{Z}_i^{\mp r})_\mathbf{s}(X_{i,0}^\pm), \qquad \psi_\mathbf{s}(K_i) = K_i \qquad (i \in I, r \in \mathbb{Z}),$$

$$\psi_\mathbf{s}(X_{0,r}^\pm) = (-1)^r\Phi(\widehat{\tau}^{-1}\mathcal{Z}_1^{\mp r})_{\tau\mathbf{s}}(X_{1,0}^\pm), \quad \psi_\mathbf{s}(K_0) = \Phi(\widehat{\tau}^{-1})_{\tau\mathbf{s}}(K_1) \quad (r \in \mathbb{Z}).$$

For $i \in I$, $r \in \mathbb{Z}'$, by equation (3.1.19), we have

$$(\psi\, v)_{\mathbf{s}}(x_{i,r}^{\pm}) = (-1)^{ir}\Phi(\mathcal{Z}_i^{\mp r})_{\mathbf{s}}(X_{i,0}^{\pm}) = (-1)^{ir}(\Phi(\mathcal{Z}_i^{\mp r})h)_{\mathbf{s}}(x_{i,0}^{\pm})$$

$$= (-1)^{ir}h_{\mathbf{s}}\mathcal{X}_{i,\mathbf{s}}^{\mp r}(x_{i,0}^{\pm}) = h_{\mathbf{s}}(x_{i,r}^{\pm}).$$

This implies $\psi_{\mathbf{s}} v_{\mathbf{s}} = h_{\mathbf{s}}$. Thus, $\psi_{\mathbf{s}}$ extends to a homomorphism $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$.

We now check that $\psi_{\mathbf{s}}$ satisfy the relations involving the node 0. This is done in a similar way as in Theorem 3.1.1. For $i \in I$, using (1.3.13) and the relations in the group, we obtain

$$\Phi(\widehat{\tau})_{\mathbf{s}}(X_{i,0}^{\pm}) = (\widehat{\mathcal{Y}}_1^{-1}\widehat{T}_1 \cdots \widehat{T}_{N-1})_{\mathbf{s}}(X_{i,0}^{\pm}) = X_{i+1,0}^{\pm} \qquad (i \neq N-1),$$

$$\Phi(\widehat{\tau}^2)_{\mathbf{s}}(X_{N-1,0}^{\pm}) = (\mathcal{Z}_2^{-1}\widehat{T}_2\widehat{T}_1\widehat{T}_3\widehat{T}_2 \cdots \widehat{T}_{N-1}\widehat{T}_{N-2})_{\mathbf{s}}(X_{N-1,0}^{\pm}) = X_{1,0}^{\pm}.$$

Thus, the relations involving the nodes $0, 1$ and 2 follow from the relations involving the nodes $1, 2$ and 3 in $U_q^{ver}\widehat{\mathfrak{sl}}_{\tau\mathbf{s}}$ using the equation

$$\psi_{\mathbf{s}}(X_{i,r}^{\pm}) = (-1)^r\Phi(\widehat{\tau}^{-1})_{\tau\mathbf{s}}\psi_{\tau\mathbf{s}}(X_{i+1,r}^{\pm}) \qquad (i \in \hat{I}). \qquad (3.1.20)$$

For the relations involving the nodes $0, 1$ and $N-1$ we use

$$\psi_{\mathbf{s}}(X_{i,r}^{\pm}) = (-1)^r(d^{m-n})^{\pm r\delta_{i,N-1}}\Phi(\widehat{\tau}^{-2})_{\tau^2\mathbf{s}}\psi_{\tau^2\mathbf{s}}(X_{i+2,r}^{\pm}) \qquad (i = 0, 1, N-1).$$

And for the relations involving the nodes $0, N-1$ and $N-2$ we use

$$\psi_{\mathbf{s}}(X_{i,r}^{\pm}) = (-1)^r(d^{m-n})^{\pm r(1-\delta_{i,0})}\Phi(\widehat{\tau}^{-3})_{\tau^3\mathbf{s}}\psi_{\tau^3\mathbf{s}}(X_{i+3,r}^{\pm}) \qquad (i = 0, N-1, N-2).$$

We check the equation $(\psi\, h)_{\mathbf{s}} = (v\, \eta\, \varphi)_{\mathbf{s}}$ on the Chevalley generators e_i, $i \in \hat{I}$. The proof for f_i, t_i, $i \in \hat{I}$, is analogous. By (1.3.11) and (1.3.23), we have

$$(\psi\, h)_{\mathbf{s}}(e_i) = \psi_{\mathbf{s}}(X_{i,0}^+) = X_{i,0}^+ = v_{\mathbf{s}}(x_{i,0}^+) = (v\, \eta\, \varphi)_{\mathbf{s}}(e_i) \qquad (i \in I),$$

$$(\psi\, h)_{\mathbf{s}}(e_0) = \psi_{\mathbf{s}}(X_{0,0}^+) = \Phi(\widehat{\tau}^{-1})_{\tau\mathbf{s}}(X_{1,0}^{\pm}) = (\widehat{T}_{N-1}^{-1} \cdots \widehat{T}_1^{-1}\widehat{\mathcal{X}}_1\widehat{\mathcal{X}}_2^{-1})_{\tau\mathbf{s}}(X_{1,0}^+)$$

$$= (\widehat{T}_{N-1}^{-1} \cdots \widehat{T}_1^{-1}\widehat{\mathcal{X}}_1)_{\tau\mathbf{s}}(X_{1,0}^+),$$

$$(v\, \eta\, \varphi)_{\mathbf{s}}(e_0) = v_{\mathbf{s}}\eta_{\mathbf{s}}(T_{N-1} \cdots T_1\mathcal{X}_1^{-1})_{\mathbf{s}}(x_{1,0}^+) = v_{\mathbf{s}}(T_{N-1}^{-1} \cdots T_1^{-1}\mathcal{X}_1)_{\tau\mathbf{s}}(x_{1,0}^+)$$

$$= (\widehat{T}_{N-1}^{-1} \cdots \widehat{T}_1^{-1}\widehat{\mathcal{X}}_1)_{\tau\mathbf{s}}(X_{1,0}^+).$$

Finally, we show $\psi_{\mathbf{s}}^{-1} = (\hat{\eta}'\,\psi'\,\hat{\eta})_{\mathbf{s}}$.

By Lemma 2.1.6 and the identities $(\psi\,v)_{\mathbf{s}} = h_{\mathbf{s}}$, $(\psi\,h)_{\mathbf{s}} = (v\,\eta\,\varphi)_{\mathbf{s}}$, we have

$$(\psi\,\hat{\eta}'\,\psi'(\hat{\eta}\,v))_{\mathbf{s}} = (\psi\,\hat{\eta}'(\psi'\,v')\eta)_{\mathbf{s}} = (\psi(\hat{\eta}'\,h')\eta)_{\mathbf{s}} = ((\psi\,h)\varphi\,\eta)_{\mathbf{s}} = (v\,\eta\,\varphi\,\varphi\,\eta)_{\mathbf{s}} = v_{\mathbf{s}}\,,$$

$$(\psi\,\hat{\eta}'\,\psi'(\hat{\eta}h))_{\mathbf{s}} = (\psi\,\hat{\eta}'(\psi'\,h')\varphi)_{\mathbf{s}} = (\psi(\hat{\eta}'\,v')\eta)_{\mathbf{s}} = (\psi\,v)_{\mathbf{s}} = h_{\mathbf{s}}\,.$$

Thus, $(\psi\,\hat{\eta}'\,\psi'\,\hat{\eta})_{\mathbf{s}} = 1_{\mathbf{s}}$ on both $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ and $U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$, but they generate $\mathcal{E}_{\mathbf{s}}$. Therefore $(\psi\,\hat{\eta}'\,\psi'\,\hat{\eta})_{\mathbf{s}} = 1_{\mathbf{s}}$ on $\mathcal{E}_{\mathbf{s}}$.

This completes the proof. $\qquad\square$

Since the Miki automorphism sends the vertical subalgebra $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ to the horizontal subalgebra $U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$, from Proposition 2.1.2 we obtain the following corollary.

Corollary 3.1.10. *Let $N > 3$. For generic values of parameters, horizontal map $h_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$ is injective. In particular, $U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is isomorphic to $U_q\widehat{\mathfrak{sl}}_{\mathbf{s}}$.* $\qquad\square$

The key property used in the proof of Theorem 3.1.9 was the compatibility of ψ with the braid group action which we describe in the next proposition.

Proposition 3.1.11. *The automorphism $\psi_{\mathbf{s}}$ satisfies*

$$(\psi B)_{\mathbf{s}} = (\Phi(B)\psi)_{\mathbf{s}} \qquad\qquad (B \in \widehat{\mathcal{B}}_N,). \qquad (3.1.21)$$

Proof. If $B = \widehat{T}_i$, $i \in I$, we use (3.1.1), (3.1.2) and the first two equalities of (3.1.18) to show (3.1.21) is satisfied on the horizontal and vertical subalgebras. We have

$$(\psi\,\widehat{T}_i\,v)_{\mathbf{s}} = (\psi\,v\,T_i)_{\mathbf{s}} = (h\,T_i)_{\mathbf{s}} = (\widehat{T}_i\,h)_{\mathbf{s}} = (\widehat{T}_i\,\psi\,v)_{\mathbf{s}} = (\Phi(\widehat{T}_i)\,\psi\,v)_{\mathbf{s}},$$

$$(\psi\,\widehat{T}_i\,h)_{\mathbf{s}} = (\psi\,h\,T_i)_{\mathbf{s}} = (v\,\eta\,\varphi\,T_i)_{\mathbf{s}} = (\widehat{T}_i\,v\,\eta\,\varphi)_{\mathbf{s}} = (\widehat{T}_i\,\psi\,h)_{\mathbf{s}} = (\Phi(\widehat{T}_i)\,\psi\,h)_{\mathbf{s}}.$$

Since the horizontal and vertical subalgebras generate $\mathcal{E}_{\mathbf{s}}$, we have $(\psi\widehat{T}_i)_{\mathbf{s}} = (\Phi(\widehat{T}_i)\psi)_{\mathbf{s}}$ on $\mathcal{E}_{\mathbf{s}}$.

The case $B = \widehat{\tau}$ is equation (3.1.20).

Since $\widehat{\mathcal{B}}_N$ is generated by $\widehat{T}_i, \widehat{\mathcal{Y}}_1$ and $\widehat{\tau}$, it it remains to check the case $B = \widehat{\mathcal{Y}}_1$. From the previous cases we have

$$(\psi\,\widehat{\tau})_{\mathbf{s}} = (\Phi(\widehat{\tau})\,\psi)_{\mathbf{s}} = (\widehat{\mathcal{Y}}_1^{-1}\,\widehat{T}_1\cdots\widehat{T}_{N-1}\,\psi)_{\mathbf{s}}.$$

Thus,

$$(\widehat{\tau}^{-1}\,\psi^{-1})_{\mathbf{s}} = (\psi^{-1}\,\widehat{T}_{N-1}^{-1}\cdots\widehat{T}_1^{-1}\,\widehat{\mathcal{Y}}_1)_{\mathbf{s}} = (\widehat{T}_{N-1}^{-1}\cdots\widehat{T}_1^{-1}\,\psi^{-1}\,\widehat{\mathcal{Y}}_1)_{\mathbf{s}},$$

or equivalently

$$(\psi^{-1}\,\widehat{\mathcal{Y}}_1^{-1})_{\mathbf{s}} = (\widehat{\tau}\,\widehat{T}_{N-1}^{-1}\cdots\widehat{T}_1^{-1}\,\psi^{-1})_{\mathbf{s}}.$$

By the first equality of (2.1.30), equation (3.1.4), and the last equalities of (2.1.6) and (3.1.18), we have

$$
\begin{aligned}
(\psi\,\widehat{\mathcal{Y}}_1)_{\mathbf{s}} &= (\hat{\eta}'(\psi')^{-1}(\widehat{\mathcal{Y}}_1')^{-1}\hat{\eta})_{\mathbf{s}} = (\hat{\eta}'\,\widehat{\tau}'(\widehat{T}_{N-1}')^{-1}\cdots(\widehat{T}_1')^{-1}(\psi')^{-1}\hat{\eta})_{\mathbf{s}} \\
&= (\widehat{\tau}\,\widehat{T}_{N-1}\cdots\widehat{T}_1\,\psi)_{\mathbf{s}} = (\Phi(\widehat{\mathcal{Y}}_1)\,\psi)_{\mathbf{s}}.
\end{aligned}
$$

$\square$

Finally, we describe how Miki automorphism changes the grading.

Proposition 3.1.12. *If $X \in \mathcal{E}_{\mathbf{s}}$ is homogeneous, then*

$$
\begin{aligned}
\deg_\delta(\psi_{\mathbf{s}}(X)) &= -\deg_0(X), \\
\deg_i(\psi_{\mathbf{s}}(X)) &= \deg_\delta(X) + \deg_i(X) - \deg_0(X) \qquad (i \in \hat{I}).
\end{aligned}
\tag{3.1.22}
$$

Proof. The proposition follows from (1.3.25), (1.3.27), (2.1.21), and (2.1.23). $\square$

Note that, if $X \in U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$, then (3.1.22) reduces to (1.3.25). And, similarly, if $X \in U_q^{hor}\widehat{\mathfrak{sl}}_{\mathbf{s}}$, then (3.1.22) reduces to (1.3.26).

4. REPRESENTATIONS

A version of this chapter is pending publication in "Algebras and Representation Theory", see [6].

In this chapter, all superalgebras are considered with standard parity.

4.1 Level (1,0) modules, bosonic picture

In this section, we construct $\mathcal{E}_{m|n}$-modules of level $(1,0)$ using vertex operators.

It will be useful to consider the following notation. Let $\hat{I}^+ = \{1, \ldots, m-1\}$, $\hat{I}^- = \{m+1, \ldots, N-1\}$, $\hat{I}^1 = \{0, m\}$ and $\hat{I} = \hat{I}^+ \cup \hat{I}^- \cup \hat{I}^1$. In particular, if $n = 1$, we have $\hat{I}^- = \emptyset$, and if $m = 1$, $\hat{I}^+ = \emptyset$.

4.1.1 Heisenberg algebra

Let $\mathcal{H}$ be the associative algebra generated by $H_{i,r}$, $c_{j,r}$, $i \in \hat{I}$, $j \in \hat{I}^- \cup \{m\}$, $r \in \mathbb{Z}^\times$, satisfying

$$[H_{i,r}, H_{j,s}] = \delta_{r+s,0} \cdot \frac{[rA_{i,j}][r]}{r} d^{-rM_{i,j}}, \qquad (4.1.1)$$

$$[c_{i,r}, c_{j,s}] = \delta_{i,j}\delta_{r+s,0} \cdot \frac{[r]^2}{r},$$

$$[H_{i,r}, c_{j,s}] = 0.$$

Note that (4.1.1) is equivalent to equation (2.1.19) with $C = q$.

Denote by $\mathcal{H}^\pm$ the (commutative) subalgebra generated by $H_{i,r}, c_{j,r}$ with $\pm r > 0$, $i \in \hat{I}$, $j \in \hat{I}^- \cup \{m\}$.

Let $\mathcal{F}$ be the Fock space generated by a vector v_0 satisfying $H_{i,r}v_0 = c_{j,r}v_0 = 0$, for $r > 0$, $i \in \hat{I}$, $j \in \hat{I}^- \cup \{m\}$. Thus, $\mathcal{F}$ is a free $\mathcal{H}^-$-module of rank 1

$$\mathcal{F} = \mathcal{H}v_0 = \mathcal{H}^- v_0.$$

Moreover, since $\det\big([rA_{i,j}]d^{-rM_{i,j}}\big)_{i,j\in\hat{I}}\neq 0$, $\mathcal{F}$ is an irreducible $\mathcal{H}$-module.

4.1.2 Level (1,0) $\mathcal{E}_{m|n}$-modules

Let $Q_{m|n}$ be the $\mathfrak{sl}_{m|n}$ root lattice and let $\mathbb{C}\{Q_{m|n}\}$ be a twisted group algebra of $Q_{m|n}$ generated by invertible elements $e^{\alpha_i}, i \in I$, satisfying the relations

$$
e^{\alpha_i}e^{\alpha_j} = \begin{cases} (-1)^{\langle\alpha_i|\alpha_j\rangle}e^{\alpha_j}e^{\alpha_i} & (i,j\in\hat{I}^+\cup\{m\}), \\ e^{\alpha_j}e^{\alpha_i} & (i\text{ or }j\in\hat{I}^-). \end{cases}
$$

Define $\varepsilon : \hat{I}\times\hat{I} \to \{\pm 1\}$ by

$$
\varepsilon(i,j) = \begin{cases} (-1)^{\langle\alpha_i|\alpha_j\rangle} & (i,j\in\hat{I}^+\cup\{m\},\ i>j), \\ (-1)^{1+\delta_{m,1}} & (i=0,\ j\in\{1,m\}), \\ 1 & (\text{otherwise}). \end{cases}
$$

Note that

$$
e^{\alpha_i}e^{\alpha_j} = \varepsilon(i,j)\varepsilon(j,i)e^{\alpha_j}e^{\alpha_i} \qquad (i,j\in\hat{I}).
$$

Let Q_c be the integral lattice generated by elements c_i, $i\in\hat{I}^-\cup\{m\}$, with bilinear form given by

$$
\langle c_i|c_j\rangle = \delta_{i,j}.
$$

Define $Q = Q_{m|n}\oplus Q_c$ and extend the bilinear forms on $Q_{m|n}$ and Q_c to Q by requiring $\langle\alpha_i|c_j\rangle = 0$. Set also $\langle\Lambda|c_j\rangle = 0$, for any $\mathfrak{gl}_{m|n}$ weight Λ.

Let $\mathbb{C}[Q_c]$ be the (commutative) group algebra of Q_c, generated by elements e^{c_i}, $i\in\hat{I}^-\cup\{m\}$, and define $\mathbb{C}\{Q\} = \mathbb{C}\{Q_{m|n}\}\otimes\mathbb{C}[Q_c]$.

For $\alpha = \sum_{l\in I}r_l\alpha_l + \sum_{k\in\hat{I}^-\cup\{m\}}s_kc_k\in Q$, define

$$
e^\alpha = (e^{\alpha_1})^{r_1}\cdots(e^{\alpha_{m+n-1}})^{r_{m+n-1}}(e^{c_m})^{s_m}\cdots(e^{c_{m+n-1}})^{s_{m+n-1}}. \tag{4.1.2}
$$

Then, $\{e^\alpha \,|\, \alpha\in Q\}$ is a basis of $\mathbb{C}\{Q\}$.

Let $\tilde{Q} \subset Q$ be the sublattice of rank $N-1$ generated by $\alpha_i, \alpha_m + c_m, \alpha_j + c_j - c_{j-1}$, $i \in \hat{I}^+, j \in \hat{I}^-$, and let $\mathbb{C}\{\tilde{Q}\}$ be the subalgebra of $\mathbb{C}\{Q\}$ spanned by e^α, $\alpha \in \tilde{Q}$.

Following [20], a $\widehat{\mathfrak{sl}}_{m|n}$ weight Λ is a level 1 partially integrable weight if and only if $\Lambda = \Lambda_i$, $i \notin \hat{I}^1$, or $\Lambda = (1-a)\Lambda_0 + a\Lambda_m$, $a \in \mathbb{C}$.

Set

$$\tilde{\Lambda} = \Lambda_i \qquad\qquad (\Lambda = \Lambda_i, \ i \in \hat{I}^+),$$

$$\tilde{\Lambda} = \Lambda_j - \sum_{i=j}^{m+n-1} c_i \qquad\qquad (\Lambda = \Lambda_j, \ j \in \hat{I}^-),$$

$$\tilde{\Lambda} = a\Lambda_m - a \sum_{i=m}^{m+n-1} c_i \qquad (\Lambda = (1-a)\Lambda_0 + a\Lambda_m, \ a \in \mathbb{C}).$$

Given a level 1 partially integrable weight Λ, define the vector superspace

$$\mathcal{F}_\Lambda := \mathcal{F} \otimes \mathbb{C}\{\tilde{Q}\} e^{\tilde{\Lambda}}.$$

For $v \in \mathcal{F}$, $\alpha \in \tilde{Q}$, the parity of $v \otimes e^\alpha e^{\tilde{\Lambda}} \in \mathcal{F}_\Lambda$ is $|v \otimes e^\alpha e^{\tilde{\Lambda}}| = (1 - (-1)^{r_m})/2$, where r_m is the multiplicity of α_m in α as in (4.1.2).

Define an action of the algebras $\mathcal{H}$ and $\mathbb{C}\{\tilde{Q}\}$ on $\mathcal{F}_\Lambda$ as follows.

For $v \in \mathcal{F}$, $\alpha \in \tilde{Q}$, set

$$x(v \otimes e^\alpha e^{\tilde{\Lambda}}) = (xv) \otimes e^\alpha e^{\tilde{\Lambda}} \qquad\qquad (x \in \mathcal{H}),$$

$$e^\beta(v \otimes e^\alpha e^{\tilde{\Lambda}}) = v \otimes (e^\beta e^\alpha e^{\tilde{\Lambda}}) \qquad\qquad (\beta \in \tilde{Q}).$$

In particular, $\mathcal{F}_\Lambda$ is a free $\mathcal{H}^- \otimes \mathbb{C}\{\tilde{Q}\}$-module of rank 1.

Introduce the zero-mode linear operators $z^{\pm H_{i,0}}, q^{\pm \alpha_{i,0}}, z^{\pm c_{j,0}}$, $i \in \hat{I}$, $j \in \hat{I}^- \cup \{m\}$, acting on $\mathcal{F}_\Lambda$ as follows.

For $v \otimes e^\alpha e^{\tilde{\Lambda}} \in \mathcal{F}_\Lambda$, with $\alpha = \sum_{l \in I} r_l \alpha_l + \sum_{k \in \hat{I}^- \cup \{m\}} s_k c_k$, set

$$z^{\pm H_{i,0}}(v \otimes e^\alpha e^{\tilde{\Lambda}}) = z^{\pm \langle \alpha_i | \alpha + \tilde{\Lambda} \rangle} d^{\pm \frac{1}{2} \sum_{l \in I} r_l A_{i,l} M_{i,l}} v \otimes e^\alpha e^{\tilde{\Lambda}},$$

$$q^{\pm \alpha_{i,0}}(v \otimes e^\alpha e^{\tilde{\Lambda}}) = q^{\pm \langle \alpha_i | \alpha + \tilde{\Lambda} \rangle} v \otimes e^\alpha e^{\tilde{\Lambda}},$$

$$z^{\pm c_{j,0}}(v \otimes e^\alpha e^{\tilde{\Lambda}}) = z^{\pm \langle c_j | \alpha + \tilde{\Lambda} \rangle} v \otimes e^\alpha e^{\tilde{\Lambda}}.$$

For $i \in \hat{I}$ and $j \in \hat{I}^- \cup \{m\}$, let

$$H_i^{\pm}(z) = \sum_{r>0} \frac{H_{i,\pm r}}{[r]} z^{\mp r},$$

$$c_j^{\pm}(z) = \sum_{r>0} \frac{c_{j,\pm r}}{[r]} z^{\mp r},$$

and define

$$\Gamma_i^+(z) = z \exp\left(H_i^-(q^{-1}z)\right) \exp\left(-H_i^+(z)\right) e^{\alpha_i} z^{H_{i,0}},$$

$$\Gamma_i^-(z) = z \exp\left(-H_i^-(z)\right) \exp\left(H_i^+(q^{-1}z)\right) e^{-\alpha_i} z^{-H_{i,0}},$$

$$C_j^{\pm}(z) = \exp\left(\pm c_j^-(z)\right) \exp\left(\mp c_j^+(z)\right) e^{\pm c_j} z^{\pm c_{j,0}}.$$

Note that these currents act on the larger space $\mathcal{F} \otimes \mathbb{C}\{Q\}e^{\tilde{\Lambda}}$. However, their products considered on Theorem 4.1.3 below preserve the subspace $\mathcal{F}_\Lambda \subset \mathcal{F} \otimes \mathbb{C}\{Q\}e^{\tilde{\Lambda}}$.

The following is proved by a direct computation.

Lemma 4.1.1. *For $i, j \in \hat{I}$, $r \in \mathbb{Z}^\times$, we have*

$$[H_{i,r}, \Gamma_j^{\pm}(z)] = \pm \frac{[rA_{i,j}]}{r} d^{-rM_{i,j}} q^{-(r\pm|r|)/2} z^r \, \Gamma_j^{\pm}(z). \tag{4.1.3}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Define the normal ordering by

$$: x_r y_s := \begin{cases} x_r y_s & (r < 0) \\ y_s x_r & (r \geq 0) \end{cases} \qquad\qquad (x_r, y_r \in \{H_{i,r}, c_{i,r}\}),$$

$$: Ae^\alpha :=: e^\alpha A := e^\alpha A \qquad\qquad (\alpha \in Q, A \in \{z^{\pm H_{i,0}}, q^{\pm \alpha_{i,0}}, z^{\pm c_{j,0}}\}),$$

$$: e^{\alpha_i} e^{\alpha_j} := \varepsilon(i,j) e^{\alpha_i + \alpha_j} \qquad\qquad (i, j \in \hat{I}),$$

and extended inductively from right to left on larger products, e.g., $: abc :=: a(: bc :) :$.

Given two currents $X(z)$ and $Y(w)$, we say that the product $X(z)Y(w)$ has contraction $\langle X(z)Y(w)\rangle$ if

$$X(z)Y(w) = \langle X(z)Y(w)\rangle : X(z)Y(w) : .$$

In this text, all contractions $\langle X(z)Y(w)\rangle$ are Laurent series converging to rational functions in the region $|z| \gg |w|$.

Lemma 4.1.2. *For $i, j \in \hat{I}$, $k, l \in \hat{I}^- \cup \{m\}$ we have*

$$\left\langle \Gamma_i^\pm(z) \Gamma_i^\pm(w) \right\rangle = \left((z - w)(z - q^{\mp 2} w) \right)^{A_{i,i}/2}, \tag{4.1.4}$$

$$\left\langle \Gamma_i^\pm(z) \Gamma_j^\pm(w) \right\rangle = \varepsilon(i, j) \left(z - d^{-M_{i,j}} q^{\mp 1} w \right)^{A_{i,j}} d^{A_{i,j} M_{i,j}/2} \qquad (i \neq j), \tag{4.1.5}$$

$$\left\langle \Gamma_i^\pm(z) \Gamma_i^\mp(w) \right\rangle = \left((z - qw)(z - q^{-1} w) \right)^{-A_{i,i}/2}, \tag{4.1.6}$$

$$\left\langle \Gamma_i^\pm(z) \Gamma_j^\mp(w) \right\rangle = \varepsilon(i, j) \left(z - d^{-M_{i,j}} w \right)^{-A_{i,j}} d^{-A_{i,j} M_{i,j}/2} \qquad (i \neq j), \tag{4.1.7}$$

$$\left\langle C_k^\pm(z) C_l^\pm(w) \right\rangle = (z - w)^{\delta_{k,l}}, \tag{4.1.8}$$

$$\left\langle C_k^\pm(z) C_l^\mp(w) \right\rangle = (z - w)^{-\delta_{k,l}}. \tag{4.1.9}$$

Proof. Let $\epsilon \in \{\pm 1\}$.

Equations (4.1.4)-(4.1.7) follow from

$$\left\langle z^{\pm \epsilon H_{i,0}} e^{\epsilon \alpha_j} \right\rangle = z^{\pm A_{i,j}} d^{\pm A_{i,j} M_{i,j}/2},$$

$$\left\langle \exp\left(\pm \epsilon H_i^+(z)\right) \exp\left(\epsilon H_i^-(w)\right) \right\rangle = \left(1 - q \frac{w}{z} \right)^{\mp A_{i,i}/2} \left(1 - q^{-1} \frac{w}{z} \right)^{\mp A_{i,i}/2},$$

$$\left\langle \exp\left(\pm \epsilon H_i^+(z)\right) \exp\left(\epsilon H_j^-(w)\right) \right\rangle = \left(1 - d^{-M_{ij}} \frac{w}{z} \right)^{\mp A_{i,j}} \qquad (i \neq j).$$

The equations (4.1.8) and (4.1.9) follow from

$$\left\langle z^{\pm \epsilon c_{k,0}} e^{\epsilon c_l} \right\rangle = z^{\pm \delta_{k,l}},$$

$$\left\langle \exp\left(\pm \epsilon c_k^+(z)\right) \exp\left(\epsilon c_l^-(w)\right) \right\rangle = \left(1 - \frac{w}{z} \right)^{\mp \delta_{k,l}}.$$

These contractions are checked by a straightforward computation. $\qquad \square$

Let ∂_z be the q-difference operator

$$\partial_z f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}.$$

Theorem 4.1.3. *The following expressions define a graded admissible $\mathcal{E}_{m|n}-module$ structure of level $(1,0)$ on $\mathcal{F}_\Lambda$.*

$$C = q, \quad K_i^{\pm 1} = q^{\pm \alpha_{i,0}}, \quad H_{i,r} = H_{i,r} \qquad (i \in \hat{I}),$$

$$E_i(z) = \Gamma_i^+(z) \qquad (i \in \hat{I}^+),$$

$$F_i(z) = \Gamma_i^-(z) \qquad (i \in \hat{I}^+),$$

$$E_m(z) = d^m \Gamma_m^+(z) C_m^+(d^m z),$$

$$F_m(z) = \Gamma_m^-(z) \, \partial_z[C_m^-(d^m z)],$$

$$E_i(z) = d^{(2m-i)} \Gamma_i^+(z) : C_i^+(d^{2m-i} z) \, \partial_z[C_{i-1}^-(d^{2m-i} z)] : \qquad (i \in \hat{I}^-),$$

$$F_i(z) = d^{(2m-i)} \Gamma_i^-(z) : C_{i-1}^+(d^{2m-i} z) \, \partial_z[C_i^-(d^{2m-i} z)] : \qquad (i \in \hat{I}^-),$$

$$E_0(z) = \Gamma_0^+(z) \, \partial_z[C_{m+n-1}^-(d^{m-n} z)],$$

$$F_0(z) = d^{m-n} \Gamma_0^-(z) C_{m+n-1}^+(d^{m-n} z).$$

Proof. The C, K relations are clear.

The *H-H*, *H-E* and *H-F* relations follow from Lemma 4.1.1. Note that $H_{i,r}$ commutes with $c_j(z)$ for all possible i, j.

We now check the *E-E* relations.

If $i = j \in \hat{I}^+$, it follows from (4.1.4) that

$$\Gamma_i^+(z)\Gamma_i^+(w) = \Gamma_i^+(w)\Gamma_i^+(z) \left(\frac{z - q^{-2}w}{q^{-2}z - w} \right)^{A_{i,i}/2},$$

which is equivalent to the *E-E* relation.

If $i \in \hat{I}^+$, $j \in \hat{I}$ and $i \neq j$ we use (4.1.5) to get

$$\Gamma_i^+(z)\Gamma_j^+(w) = \Gamma_j^+(w)\Gamma_i^+(z) \left(\frac{d^{M_{i,j}} z - q^{-1}w}{w - d^{M_{i,j}} q^{-1}z} \right)^{A_{i,j}} \frac{\varepsilon(i,j)}{\varepsilon(j,i)},$$

but in this case $\varepsilon(i,j) = (-1)^{A_{i,j}}\varepsilon(j,i)$, which is the needed sign.

The cases with $i \in \hat{I}^-$ or $i = j \in \hat{I}^1$ follow from the above equations noting that $\varepsilon(i,j) = \varepsilon(j,i)$ and, by (4.1.8) and (4.1.9),

$$\left\langle C_i^{\pm}(z)C_j^{\pm}(w) \right\rangle = (-1)^{\delta_{i,j}} \left\langle C_j^{\pm}(w)C_i^{\pm}(z) \right\rangle,$$

$$(z - w) \left\langle C_i^{\pm}(z)C_j^{\mp}(w) \right\rangle = (-1)^{\delta_{i,j}}(z - w) \left\langle C_j^{\mp}(w)C_i^{\pm}(z) \right\rangle.$$

For example, if $i = m + 1$ and $j = m$, let $\epsilon \in \{\pm 1\}$, we have

$$(dqz - w) \langle \Gamma^+_{m+1}(w)\Gamma^+_m(z)\rangle \langle C^-_m(d^{m-1}q^\epsilon w)C^+_m(d^m z)\rangle = d^{-1/2}\frac{(dqz - w)(w - dq^{-1}z)}{(d^{m-1}q^\epsilon w - d^m z)}$$

$$= d^{1/2-m}(dz - q^{-\epsilon}w),$$

and

$$(dz - qw) \langle \Gamma^+_m(z)\Gamma^+_{m+1}(w)\rangle \langle C^+_m(d^m z)C^-_m(d^{m-1}q^\epsilon w)\rangle = d^{1/2}\frac{(dz - qw)(z - d^{-1}q^{-1}w)}{(d^m z - d^{m-1}q^\epsilon w)}$$

$$= d^{1/2-m}(dz - q^{-\epsilon}w).$$

This shows $(dz - qw)E_m(z)E_{m+1}(w) = (dqz - w)E_{m+1}(w)E_m(z)$.

If $i, j \in \hat{I}^1$ and $i \neq j$, we have $\varepsilon(i, j) = (-1)^{\delta_{m,1}+1}\varepsilon(j, i)$. Thus,

$$E_i(z)E_j(w) = (-1)^{\delta_{n,1}+\delta_{m,1}+1}E_j(w)E_i(z)\left(\frac{d^{M_{i,j}}z - q^{-1}w}{w - d^{M_{i,j}}q^{-1}z}\right)^{A_{i,j}}.$$

Therefore, the E-E relations hold for all $i, j \in \hat{I}$.

The F-F relations are analogous.

The E-F relations are trivial for $|i - j| > 1$. For i or $j \in \hat{I}^+$ with $i \neq j$, it follows directly from (4.1.7).

If $i \in \hat{I}^- \cup \{m\}$ and $j = i + 1$, we have

$$\langle E_i(z)F_{i+1}(w)\rangle = \langle F_{i+1}(w)E_i(z)\rangle$$

$$= d^{4m-2i-1}\langle \Gamma^+_i(z)\Gamma^-_{i+1}(w)\rangle \langle C^+_i(d^{2m-i}z)C^+_i(d^{2m-i-1}w)\rangle = d^{6m-3i-3/2}.$$

Thus, $[E_i(z), F_{i+1}(w)] = 0$.

The case $i \in \hat{I}^- \cup \{0\}$ and $j = i - 1$ is treated similarly. Due to the presence of the q-difference operators ∂_z and ∂_w in a non-trivial contraction, the expansion of $: E_i(z)F_j(w) :$ has four normal-ordered summands. However, the coefficient of each summand is a Laurent polynomial. Thus, $[E_i(z), F_{i-1}(w)] = 0$.

If $i = j \in \hat{I}^+$, we have

$$\langle \Gamma^+_i(z)\Gamma^-_i(w)\rangle = \left(\frac{1}{(z - qw)(z - q^{-1}w)}\right) \qquad (|z| \gg |w|),$$

$$\langle \Gamma^-_i(w)\Gamma^+_i(z)\rangle = \left(\frac{1}{(z - qw)(z - q^{-1}w)}\right) \qquad (|w| \gg |z|).$$

We can change the region where the second rational function is expanded to the same region as the first one by adding δ-functions

$$\left(\frac{1}{(z-qw)(z-q^{-1}w)}\right)_{(|w|\gg|z|)} = \tag{4.1.10}$$
$$= \left(\frac{1}{(z-qw)(z-q^{-1}w)}\right)_{(|z|\gg|w|)} - \frac{1}{qw^2(q-q^{-1})}\delta\left(q\frac{w}{z}\right) - \frac{1}{qz^2(q^{-1}-q)}\delta\left(q\frac{z}{w}\right).$$

Now,

$$\frac{1}{qw^2(q-q^{-1})}\delta\left(q\frac{w}{z}\right) : \Gamma_i^+(z)\Gamma_i^-(w) := \frac{1}{(q-q^{-1})}\delta\left(q\frac{w}{z}\right)K_i^+(w),$$
$$\frac{1}{qz^2(q^{-1}-q)}\delta\left(q\frac{z}{w}\right) : \Gamma_i^+(z)\Gamma_i^-(w) := -\frac{1}{(q-q^{-1})}\delta\left(q\frac{z}{w}\right)K_i^-(z).$$

Therefore, the E-F relations follow for $i = j \in \hat{I}^+$.

For $i = j = 0$ we have

$$E_0(z)F_0(w) = d^{m-n} : \Gamma_0^+(z)\Gamma_0^-(w) : \partial_z[C_{m+n-1}^-(d^{m-n}z)]C_{m+n-1}^+(d^{m-n}w),$$
$$F_0(w)E_0(z) = d^{m-n} : \Gamma_0^-(w)\Gamma_0^+(z) : C_{m+n-1}^+(d^{m-n}w)\,\partial_z[C_{m+n-1}^-(d^{m-n}z)].$$

By (4.1.9), we have

$$d^{m-n}C_{m+n-1}^-(d^{m-n}z)C_{m+n-1}^+(d^{m-n}w) = \frac{1}{z-w} \qquad (|z| \gg |w|),$$
$$d^{m-n}C_{m+n-1}^+(d^{m-n}w)C_{m+n-1}^-(d^{m-n}z) = \frac{1}{w-z} \qquad (|w| \gg |z|).$$

Then,

$$[E_0(z), F_0(w)] =: \Gamma_0^+(z)\Gamma_0^-(w) : \partial_z\left[\frac{1}{w}\delta\left(\frac{w}{z}\right) : C_{m+n-1}^+(d^{m-n}w)C_{m+n-1}^-(d^{m-n}z) :\right].$$

Now, for all $j \in \hat{I}^- \cup \{m\}$, we have

$$\delta\left(\frac{w}{z}\right) : C_j^+(w)C_j^-(z) := \delta\left(\frac{w}{z}\right)\exp\left(c_j^-(w) - c_j^-(z)\right)\exp\left(c_j^+(z) - c_j^+(w)\right) = \delta\left(\frac{w}{z}\right).$$

Thus,

$$[E_0(z), F_0(w)] =: \Gamma_0^+(z)\Gamma_0^-(w) : \partial_z\left[\frac{1}{w}\delta\left(\frac{w}{z}\right)\right]$$
$$= \frac{1}{zw(q-q^{-1})}\left(\delta\left(q\frac{w}{z}\right) - \delta\left(q\frac{z}{w}\right)\right) : \Gamma_0^+(z)\Gamma_0^-(w) : .$$

Therefore, the E-F relations also follow for $i = 0$. The case $i = m$ is analogous and the case $i \in \hat{I}^-$ is longer, but checked by the same procedure.

In any admissible representation, it is enough to check the quadratic relations, then the Serre relations follow automatically. Namely, the Serre relations are checked by commuting each summand and passing to a common region of convergence of the rational functions by adding suitable δ-functions. We check the quartic relation (2.1.13) with $i = m$ as an example.

Write the ten summands in (2.1.13) as follows. Let

$$E_m(z_1)E_{m+1}(w_1)E_m(z_2)E_{m-1}(w_2) = \mathbf{E}. \tag{4.1.11}$$

Then, using the E-E relations, write the remaining terms of (2.1.13) in the form

$$E_{m+1}(w_1)E_m(z_2)E_{m-1}(w_2)E_m(z_1) = -\frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)}{(dq^{-1}z_1 - w_2)(dqz_1 - w_1)}\mathbf{E}, \tag{4.1.12}$$

$$E_m(z_1)E_{m-1}(w_2)E_m(z_2)E_{m+1}(w_1) = \frac{(dz_2 - q^{-1}w_2)(dqz_2 - w_1)}{(dz_2 - qw_1)(dq^{-1}z_2 - w_2)}\mathbf{E}, \tag{4.1.13}$$

$$E_{m+1}(w_1)E_m(z_1)E_{m-1}(w_2)E_m(z_2) = \frac{(dz_2 - q^{-1}w_2)(dz_1 - qw_1)}{(dq^{-1}z_2 - w_2)(dqz_1 - w_1)}\mathbf{E}, \tag{4.1.14}$$

$$E_{m-1}(w_2)E_m(z_1)E_{m+1}(w_1)E_m(z_2) = \frac{(dz_2 - q^{-1}w_2)(dz_1 - q^{-1}w_2)}{(dq^{-1}z_2 - w_2)(dq^{-1}z_1 - w_2)}\mathbf{E}, \tag{4.1.15}$$

$$E_{m-1}(w_2)E_m(z_2)E_{m+1}(w_1)E_m(z_1) =$$
$$= -\frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)(dqz_2 - w_1)(dz_2 - q^{-1}w_2)}{(dq^{-1}z_1 - w_2)(dqz_1 - w_1)(dz_2 - qw_1)(dq^{-1}z_2 - w_2)}\mathbf{E}, \tag{4.1.16}$$

$$E_m(z_2)E_{m+1}(w_1)E_m(z_1)E_{m-1}(w_2) = -\frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)(dqz_2 - w_1)}{(dqz_1 - w_1)(dz_2 - qw_1)(dz_1 - q^{-1}w_2)}\mathbf{E}, \tag{4.1.17}$$

$$E_m(z_2)E_{m-1}(w_2)E_m(z_1)E_{m+1}(w_1) = -\frac{(dz_1 - q^{-1}w_2)(dqz_2 - w_1)(dqz_1 - w_1)}{(dq^{-1}z_1 - w_2)(dqz_1 - w_1)(dz_2 - qw_1)}\mathbf{E}, \tag{4.1.18}$$

$$- [2]\, E_m(z_1)E_{m+1}(w_1)E_{m-1}(w_2)E_m(z_2) = -[2]\frac{(dz_2 - q^{-1}w_2)}{(dq^{-1}z_2 - w_2)}\mathbf{E}, \tag{4.1.19}$$

$$- [2]\, E_m(z_2)E_{m+1}(w_1)E_{m-1}(w_2)E_m(z_1) = [2]\frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)(dqz_2 - w_1)}{(dq^{-1}z_1 - w_2)(dqz_1 - w_1)(dz_2 - qw_1)}\mathbf{E} \tag{4.1.20}$$

The rational functions of the r.h.s. of the equations (4.1.12)-(4.1.20) are expanded in the region given by the increasing order of appearance of the coordinates in the l.h.s.. For example, the rational function in equation (4.1.12) is expanded in the region $|w_1| \gg |z_2| \gg |w_2| \gg |z_1|$.

Summing up l.h.s. of equations (4.1.11)-(4.1.20) we get the expansion of (2.1.13). The sum of the rational functions in the r.h.s. vanishes as a rational function. However, similar to E-F relation, we must switch to the common convergence region and verify that the coefficients of the delta functions yielded also vanish at the respective support, cf., (4.1.10).

For example, we choose $|z_2| \gg |w_2| \gg |z_1| \gg |w_1|$ as a common region. The rational function in the r.h.s. (4.1.14) in this region becomes

$$
\frac{(dz_2 - q^{-1}w_2)(dz_1 - qw_1)}{(dq^{-1}z_2 - w_2)(dqz_1 - w_1)} + (1 - q^{-2})(1 - q^2)\delta\left(\frac{dqz_1}{w_1}\right)\delta\left(\frac{dz_2}{qw_2}\right) -
$$
$$
-\frac{(dz_2 - q^{-1}w_2)(1 - q^2)}{q(dq^{-1}z_2 - w_2)}\delta\left(\frac{dqz_1}{w_1}\right) - \frac{(1 - q^{-2})(dz_1 - qw_1)}{q^{-1}(dqz_1 - w_1)}\delta\left(\frac{dz_2}{qw_2}\right).
$$

Other terms are similar. After changing the region of convergence of all rational functions the δ-functions yielded are $\delta\left(\frac{dz_2}{qw_2}\right)$, $\delta\left(\frac{dqz_1}{w_1}\right)$, $\delta\left(\frac{dqz_1}{w_2}\right)$, $\delta\left(\frac{dqz_1}{w_2}\right)\delta\left(\frac{dqz_1}{w_1}\right)$ and $\delta\left(\frac{dz_2}{qw_2}\right)\delta\left(\frac{dqz_1}{w_1}\right)$, and the coefficient of each one vanishes at the respective suport.

Thus, (2.1.13) with $i = m$ is proved. □

4.1.3 Screenings

The $\mathcal{E}_{m|n}$-modules obtained in Theorem 4.1.3 are not irreducible in general. To find their irreducible quotient, we follow [23],[21], and introduce the following ξ–η system.

We set $\mathrm{Res}_z(\sum_{i\in\mathbb{Z}} a_i z^{-i}) = a_1$.

For $i \in \hat{I}^- \cup \{m\}$, introduce the screening operators

$$\xi_i = \mathrm{Res}_z \left(z^{-1} C_i^-(z) \right),$$

$$\eta_i = \mathrm{Res}_z \, C_i^+(z),$$

acting on $\mathcal{F}_\Lambda$, with $\Lambda = \Lambda_j$, $j \notin \hat{I}^1$, or $\Lambda = (1-a)\Lambda_0 + a\Lambda_m$, $a \in \mathbb{Z}$.

The odd operators ξ_i, η_i, satisfy

$$[\xi_i, \eta_j] = \delta_{i,j},$$

$$[\xi_i, \xi_j] = [\eta_i, \eta_j] = 0,$$

$$\mathcal{F}_\Lambda = \xi_i \eta_i \mathcal{F}_\Lambda \oplus \eta_i \xi_i \mathcal{F}_\Lambda,$$

for all $i, j \in \hat{I}^- \cup \{m\}$.

Define

$$\xi = \prod_{i \in \hat{I}^- \cup \{m\}} \xi_i, \quad \eta = \prod_{i \in \hat{I}^- \cup \{m\}} \eta_i.$$

Proposition 4.1.4. *If $\Lambda = \Lambda_i$, $i \notin \hat{I}^1$ or $\Lambda = (1-a)\Lambda_0 + a\Lambda_m, a \in \mathbb{Z}$, the screening operators $\eta_i, i \in \hat{I}^- \cup \{m\}$, (super)commute with the $\mathcal{E}_{m|n}$-action on $\mathcal{F}_\Lambda$ given by Theorem 4.1.3. Thus, $\ker \eta$ and $\mathrm{coker}\, \eta$ are $\mathcal{E}_{m|n}$-modules.*

Proof. It is enough to show $[\eta_i, C_i^+(w)] = [\eta_i, \partial_w C_i^-(w)] = 0$.

Using (4.1.8) we have

$$[\eta_i, C_i^+(w)] = \mathrm{Res}_z [C_i^+(z), C_i^+(w)] = 0,$$

and by (4.1.9)

$$[\eta_i, \partial_w C_i^-(w)] = \partial_w \left(\mathrm{Res}_z [C_i^+(z), C_i^-(w)] \right) = \partial_w (1) = 0.$$

$\square$

Level 1 partially integrable representations of $U_q \widehat{\mathfrak{sl}}_{m|n}$ with $m \neq n$ were constructed in [21] using the formulas in Theorem 4.1.3 for $i \in I$ and $d = 1$. Our space $\mathcal{F}_\Lambda$ differs from theirs by the extra current $H_0(z)$ present in $U_q^{ver} \widehat{\mathfrak{gl}}_{m|n}$. The conjectural identification given in [24] and [21] in our context is the following.

Conjecture 4.1.5. *We have the following identifications*

$$V(\Lambda_i) = \ker \eta = \eta\xi\mathcal{F}_{\Lambda_i} \qquad\qquad (i \in I),$$

$$V((1-a)\Lambda_0 + a\Lambda_m) = \mathcal{F}_{(1-a)\Lambda_0+a\Lambda_m} \qquad\qquad (a \in \mathbb{C} \setminus \mathbb{Z}),$$

$$V((1-a)\Lambda_0 + a\Lambda_m) = \operatorname{coker} \eta = \xi\eta\mathcal{F}_{(1-a)\Lambda_0+a\Lambda_m} \qquad\qquad (a \in \mathbb{Z}_{>0}),$$

$$V((1-a)\Lambda_0 + a\Lambda_m) = \ker \eta = \eta\xi\mathcal{F}_{(1-a)\Lambda_0+a\Lambda_m} \qquad\qquad (a \in \mathbb{Z}_{\leq 0}),$$

where $V(\Lambda)$ is the irreducible highest weight $U_q^{ver}\widehat{\mathfrak{gl}}_{m|n}$-module with highest weight Λ.

4.2 Evaluation homomorphism

In this section, we construct an evaluation map from $\mathcal{E}_{m|n}$ to a suitable completion $\widetilde{U}_q\widehat{\mathfrak{gl}}_{m|n}$ of $U_q\widehat{\mathfrak{gl}}_{m|n}$. We follow the strategy used in [15].

4.2.1 A completion of $U_q\widehat{\mathfrak{gl}}_{m|n}$

For $r \in \mathbb{Z}^\times$, let

$$\beta_{i,r} = \begin{cases} \dfrac{q^{(m-n-i)r} + q^{ir}}{q^r - q^{-r}} & (i \in \hat{I}^+ \cup \hat{I}^1), \\[3mm] \dfrac{q^{(i-m-n)r} + q^{(2m-i)r}}{q^r - q^{-r}} & (i \in \hat{I}^-). \end{cases} \qquad (4.2.1)$$

The coefficients $\beta_{i,r}$ are solutions of the system

$$\sum_{i\in\hat{I}} \beta_{i,r}[rA_{i,j}] = 0 \quad (j \in I).$$

Then, the elements $h_r = \sum_{i\in\hat{I}} \beta_{i,r}h_{i,r} \in U_q\widehat{\mathfrak{gl}}_{m|n}$ commute with $U_q\widehat{\mathfrak{sl}}_{m|n} \subset U_q\widehat{\mathfrak{gl}}_{m|n}$ and satisfy

$$[h_r, h_s] = \delta_{r+s,0}[(n-m)r]\frac{1}{r}\frac{c^r - c^{-r}}{q - q^{-1}}.$$

Set $h(z) = \sum_{k\in\mathbb{Z}^\times} h_k z^{-k}$.

We use a completion of $U_q\widehat{\mathfrak{gl}}_{m|n}$, denoted by $\widetilde{U}_q\widehat{\mathfrak{gl}}_{m|n}$, obtained by performing the following two steps.

Let $\widehat{Q}_{m|n}$ be the root lattice of $\widehat{\mathfrak{sl}}_{m|n}$. The algebra $U_q\,\widehat{\mathfrak{gl}}_{m|n}$ contains the group algebra $\mathbb{C}[\widehat{Q}_{m|n}]$ of the root lattice $\widehat{Q}_{m|n}$ generated by $k_i = q^{\alpha_i}, i \in \hat{I}$. As a first step, we extend it to the weight lattice in a straightforward way. Namely, let P be the $\widehat{\mathfrak{sl}}_{m|n}$ weight lattice and $\mathbb{C}[P]$ the corresponding group algebra spanned by $q^\Lambda, \Lambda \in P$. We have an inclusion of algebras $\mathbb{C}[\widehat{Q}_{m|n}] \subset \mathbb{C}[P]$. Let U_P be the superalgebra $U_P = U_q\,\widehat{\mathfrak{gl}}_{m|n} \otimes_{\mathbb{C}[\widehat{Q}_{m|n}]} \mathbb{C}[P]$ with the relations

$$q^\Lambda q^{\Lambda'} = q^{\Lambda+\Lambda'}, \quad q^0 = 1, \quad q^\Lambda x_i^\pm(z) q^{-\Lambda} = q^{\pm\langle\Lambda|\alpha_i\rangle} x_i^\pm(z) \quad (\Lambda, \Lambda' \in P).$$

For each $i \in I$, the superalgebra U_P has a $\mathbb{Z}$-grading given by

$$\deg_i(x_{j,k}^\pm) = \pm\delta_{i,j}, \qquad \deg_i(h_{j,r}) = \deg_i(q^\Lambda) = \deg_i(c) = 0 \quad (j \in I, k \in \mathbb{Z}, r \in \mathbb{Z}^\times).$$

There is also the *homogeneous $\mathbb{Z}$-grading* given by

$$\deg_\delta(x_{j,k}^\pm) = k, \quad \deg_\delta(h_{j,r}) = r, \quad \deg_\delta(q^\Lambda) = \deg_\delta(c) = 0 \quad (j \in I, k \in \mathbb{Z}, r \in \mathbb{Z}^\times).$$

Thus, the superalgebra U_P has a $\mathbb{Z}^{m+n}$-grading given by

$$\deg(X) = \big(\deg_1(X), \ldots, \deg_{m+n-1}; \deg_\delta(X)\big), \qquad X \in U_P.$$

As the second step, we define $\widetilde{U}_q\,\widehat{\mathfrak{gl}}_{m|n}$ to be the completion of U_P with respect to the homogeneous grading in the positive direction. The elements of $\widetilde{U}_q\,\widehat{\mathfrak{gl}}_{m|n}$ are series of the form $\sum_{j=s}^\infty g_j$, with $g_j \in U_P$, $\deg_\delta g_j = j$.

Lemma 4.2.1. *We have an embedding*

$$U_q\,\widehat{\mathfrak{gl}}_{m|n} \to \widetilde{U}_q\,\widehat{\mathfrak{gl}}_{m|n}.$$

$\square$

A $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-module V is *admissible* if for any $v \in V$ there exist $N = N_v > 0$ such that $xv = 0$ for all $x \in U_q\,\widehat{\mathfrak{gl}}_{m|n}$ with $\deg_\delta x > N$. Any admissible $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-module is also an $\widetilde{U}_q\,\widehat{\mathfrak{gl}}_{m|n}$-module.

A $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-module V is called *highest weight module* if V is generated by the highest weight vector v:

$$V = U_q\,\widehat{\mathfrak{gl}}_{m|n}v, \qquad e_i v = 0, \qquad k_0^+(z)v = q^{\lambda_0}v, \qquad t_j v = q^{\lambda_j}v, \qquad i \in \hat{I}, \; j \in I.$$

Highest weight $U_q\,\widehat{\mathfrak{gl}}_{m|n}$-modules are admissible.

4.2.2   Fused currents

Introduce the following fused currents in $\widetilde{U}_q \, \widehat{\mathfrak{gl}}_{m|n}$

$$
\mathsf{X}^+(z) = \left[\prod_{i=1}^{m+n-2} \left(1 - \frac{z_i}{z_{i+1}} \right) \right] x_{m+n-1}^+(q^{-m+n-1}c^{-1}z_{m+n-1}) \cdots x_{m+i}^+(q^{-m+i}c^{-1}z_{m+i}) \cdots
$$
$$
\cdots x_m^+(q^{-m}c^{-1}z_m) \cdots x_i^+(q^{-i}c^{-1}z_i) \cdots x_1^+(q^{-1}c^{-1}z_1) \Big|_{z_1=\cdots=z_{m+n-1}=z} \,,
$$
$$
\mathsf{X}^-(z) = \left[\prod_{i=1}^{m+n-2} \left(1 - \frac{z_{i+1}}{z_i} \right) \right] x_1^-(q^{-1}c^{-1}z_1) \cdots x_i^-(q^{-i}c^{-1}z_i) \cdots x_m^-(q^{-m}c^{-1}z_m) \cdots
$$
$$
\cdots x_{m+i}^-(q^{-m+i}c^{-1}z_{m+i}) \cdots x_{m+n-1}^-(q^{-m+n-1}c^{-1}z_{m+n-1}) \Big|_{z_1=\cdots=z_{m+n-1}=z} \,,
$$
$$
\mathsf{k}^\pm(z) = \prod_{i=1}^m k_i^\pm(q^{-i}c^{-1}z) \prod_{j=m+1}^{m+n-1} k_j^\pm(q^{-2m+j}c^{-1}z).
$$

See [14] for the details on fused currents.

The homomorphism $v = v_{\mathbf{s}}$, where $\mathbf{s}$ is standard, defined in the Lemma 2.1.2 maps the element $h_{0,r}$ in the following way

$$
v(h_{0,r}) = \frac{1}{\beta_{0,r}} \left(\gamma_{0,r} H_{0,r} + \sum_{i \in \hat{I}^+ \cup \{m\}} (\gamma_{i,r} - \beta_{i,r} d^{ir}) H_{i,r} + \sum_{j \in \hat{I}^-} (\gamma_{j,r} - \beta_{j,r} d^{(2m-j)r}) H_{j,r} \right),
$$

where $\{\gamma_{i,r}\}_{i \in \hat{I}}$ and $\{\beta_{i,r}\}_{i \in \hat{I}}$ are the fixed solutions of the systems (2.1.22) and (4.2.1), respectively.

For each $r \in \mathbb{Z}^\times$, define

$$
\tilde{h}_{0,r} = \frac{1}{\gamma_{0,r}} \left(\beta_{0,r} h_{0,r} + \sum_{i \in \hat{I}^+ \cup \{m\}} (\beta_{i,r} - \gamma_{i,r} d^{-ir}) h_{i,r} + \sum_{j \in \hat{I}^-} (\beta_{j,r} - \gamma_{j,r} d^{-(2m-j)r}) h_{j,r} \right).
$$

Thus, $v(\tilde{h}_{0,r}) = H_{0,r}$ for all $r \in \mathbb{Z}^\times$.

Define $A_{\pm r}, B_{\pm r}, r \in \mathbb{Z}_{>0}$ by

$$A_r = -\frac{q-q^{-1}}{c^r - c^{-r}} \left(\tilde{h}_{0,r} + \sum_{i=1}^{m} (c^2 q^i)^r h_{i,r} + \sum_{j=m+1}^{m+n-1} (c^2 q^{2m-j})^r h_{j,r} \right),$$

$$A_{-r} = \frac{q-q^{-1}}{c^r - c^{-r}} \, c^{-r} \left(\tilde{h}_{0,-r} + \sum_{i=1}^{m} q^{-ir} h_{i,-r} + \sum_{j=m+1}^{m+n-1} q^{(-2m+j)r} h_{j,-r} \right),$$

$$B_r = \frac{q-q^{-1}}{c^r - c^{-r}} \, c^r \left(\tilde{h}_{0,r} + \sum_{i=1}^{m} q^{ir} h_{i,r} + \sum_{j=m+1}^{m+n-1} q^{(2m-j)r} h_{j,r} \right),$$

$$B_{-r} = -\frac{q-q^{-1}}{c^r - c^{-r}} \left(\tilde{h}_{0,-r} + \sum_{i=1}^{m} (c^{-2} q^{-i})^r h_{i,-r} + \sum_{j=m+1}^{m+n-1} (c^{-2} q^{-2m+j})^r h_{j,-r} \right),$$

and let $A^{\pm}(z) = \sum_{r>0} A_{\pm r} z^{\mp r}$, $B^{\pm}(z) = \sum_{r>0} B_{\pm r} z^{\mp r}$.

Let also $\mathcal{K} = q^{-\Lambda_{m+n-1} - \Lambda_1}$. We have $\mathcal{K} x_i^{\pm}(z) \mathcal{K}^{-1} = q^{\mp(\delta_{1,i} + \delta_{m+n-1,i})} x_i^{\pm}(z)$.

Theorem 4.2.2. *Fix $u \in \mathbb{C}^{\times}$. The following map is a surjective homomorphism of superalgebras* $\mathrm{ev}_u : \mathcal{E}_{m|n} \to \widetilde{U}_q \widehat{\mathfrak{gl}}_{m|n}$ *with $C^2 = q_3^{m-n}$:*

$$K \mapsto 1, \quad C \mapsto c, \quad H^{ver}(z) \mapsto h(z),$$

$$E_i(z) \mapsto x_i^+(d^i z), \quad F_i(z) \mapsto x_i^-(d^i z), \quad K_i^{\pm}(z) \mapsto k_i^{\pm}(d^i z) \qquad (i \in \hat{I}^+ \cup \{m\}),$$

$$E_j(z) \mapsto x_j^+(d^{2m-j} z), \quad F_j(z) \mapsto x_j^-(d^{2m-j} z), \quad K_j^{\pm}(z) \mapsto k_j^{\pm}(d^{2m-j} z) \qquad (j \in \hat{I}^-),$$

$$E_0(z) \mapsto u^{-1} e^{A-(z)} \boldsymbol{X}^-(z) e^{A+(z)} \mathcal{K},$$

$$F_0(z) \mapsto u \mathcal{K}^{-1} e^{B-(z)} \boldsymbol{X}^+(z) e^{B+(z)}.$$

Moreover, the evaluation map ev_u *is graded. More precisely, if $X \in \mathcal{E}_{m|n}$ and* $\deg(X) = (d_0, d_1, \ldots, d_{m+n-1}; d_{\delta})$, *then* $\deg(\mathrm{ev}_u(X)) = (d_1 - d_0, \ldots, d_{m+n-1} - d_0; d_{\delta})$.

Proof. For simplicity, we fix $u = 1$ and write $\mathrm{ev}_1 = \mathrm{ev}$. The relations with no index 0 are clear.

For $i \in \hat{I}, r > 0$, we have

$$[h_{i,r}, e^{A^+(z)}] = 0,$$

$$[h_{i,r}, e^{A^-(z)}] = z^r e^{A^-(z)} c^{-r} \left(\sum_{j \in \hat{I}^+ \cup \hat{I}^1} \frac{[rA_{i,j}]}{r} q^{-jr} + \sum_{j \in \hat{I}^-} \frac{[rA_{i,j}]}{r} q^{-(2m-j)r} \right),$$

$$[h_{i,r}, \mathsf{X}^-(z)] = -z^r \mathsf{X}^-(z) c^{-r} \left(\sum_{j \in \hat{I}^+ \cup \{m\}} \frac{[rA_{i,j}]}{r} q^{-jr} + \sum_{j \in \hat{I}^-} \frac{[rA_{i,j}]}{r} q^{-(2m-j)r} \right).$$

Thus,

$$\mathrm{ev}([H_{i,r}, E_0(z)]) = z^r c^{-r} \frac{[rA_{i,0}]}{r} \mathrm{ev}(E_0(z)).$$

The H-E relations with $r < 0$ and the H-F relations can be checked in the same way.

To check the E-E relations we first use (A.1) and (A.2) to get

$$\mathrm{ev}(E_0(z)E_i(w)) = e^{A^-(z)}\mathsf{X}^-(z)\,\mathrm{ev}(E_i(w))e^{A^+(z)}\mathcal{K} \left(\frac{z - q_3^{-1}w}{z - q_1 w} \right)^{\delta_{i,1}} q^{-\delta_{i,m+n-1}-\delta_{i,1}},$$

$$\mathrm{ev}(E_i(w)E_0(z)) = e^{A^-(z)}\,\mathrm{ev}(E_i(w))\mathsf{X}^-(z)e^{A^+(z)}\mathcal{K} \left(\frac{w - q_3 z}{w - q_1^{-1}z} \right)^{\delta_{i,m+n-1}}.$$

For $i \neq 1, m+n-1$, $\mathrm{ev}([E_0(z), E_i(w)]) = 0$ by (A.13).

For $i = 1$, the E-E relation reduces to

$$\left(q^{-1}z - dw \right) e^{A^-(z)}[\mathsf{X}^-(z)x_1^+(dw)]e^{A^+(z)}\mathcal{K} = 0,$$

which follows from (A.18). The case $i = m+n-1$ is similar. The case $i = 0$ is checked using (A.3), (A.4) and (A.9).

The F-F relations are verified by the same argument.

For the E-F relations

$$\mathrm{ev}([E_0(z), F_i(w)]) = 0 \quad (i \neq 0),$$

we proceed as in the E-E case by bringing $A^-(z)$ to the left and $A^+(z)$ to the right using (A.3) and (A.4). The relations then follow from (A.13), (A.16) and (A.17). The same is done for $\mathrm{ev}([E_i(z), F_0(w)]) = 0 \quad (i \neq 0)$.

For the $i = 0$ case, using (A.1),(A.6) and (A.10), we get

$$\mathrm{ev}(E_0(z)F_0(w)) = e^{A^-(z)}e^{B^-(w)}\mathsf{X}^-(z)\mathsf{X}^+(w)e^{A^+(z)}e^{B^+(w)},$$

and similarly

$$\mathrm{ev}(F_0(w)E_0(z)) = e^{A^-(z)}e^{B^-(w)}\mathsf{X}^+(w)\mathsf{X}^-(z)e^{A^+(z)}e^{B^+(w)}.$$

By (A.20),

$$\mathrm{ev}([E_0(z), F_0(w)]) = \frac{e^{A^-(z)}e^{B^-(w)}}{q - q^{-1}}\left(\delta\left(c\frac{w}{z}\right)\mathsf{k}^-(w) - \delta\left(c\frac{z}{w}\right)\mathsf{k}^+(z)\right)e^{A^+(z)}e^{B^+(w)}.$$

The relation (2.1.7) with $i = j = 0$ follows from

$$e^{A^-(z)}e^{B^-(cz)} = \tilde{k}_0^-(z), \quad e^{A^-(cw)}e^{B^-(w)} = k_0(\mathsf{k}^-(w))^{-1},$$

$$e^{A^+(cw)}e^{B^+(w)} = \tilde{k}_0^+(w), \quad e^{A^+(z)}e^{B^+(cz)} = k_0^{-1}(\mathsf{k}^+(z))^{-1},$$

where $\tilde{k}_0^\pm = \exp\left(\pm(q - q^{-1})\sum_{r>0}\tilde{h}_{0,\pm r}z^{\mp r}\right)$.

Finally, we check the Serre relations.

For the relation

$$\mathrm{ev}\left(\mathrm{Sym}_{z_1,z_2}[E_1(z_1), [E_1(z_2), E_0(w)]_q]_{q^{-1}}\right) = 0$$

we use (A.1) and (A.18) to obtain

$$\mathrm{ev}\left(E_1(z_1)E_0(w)E_1(z_2)\right) = q\left(\frac{z_2 - q_3 w}{z_2 - q_1^{-1}w}\right)\mathrm{ev}\left(E_1(z_1)E_1(z_2)E_0(w)\right),$$

$$\mathrm{ev}\left(E_0(w)E_1(z_1)E_1(z_2)\right) = q^2\left(\frac{z_1 - q_3 w}{z_1 - q_1^{-1}w}\right)\left(\frac{z_2 - q_3 w}{z_2 - q_1^{-1}w}\right)\mathrm{ev}\left(E_1(z_1)E_1(z_2)E_0(w)\right).$$

Thus,

$$\mathrm{Sym}_{z_1,z_2}(\mathrm{ev}\,[E_1(z_1), [E_1(z_2), E_0(w)]_q]_{q^{-1}}) =$$

$$\frac{(1 - q^2)w}{q_3(w - q_1 z_1)(w - q_1 z_2)}\mathrm{Sym}_{z_1,z_2}\left((z_1 - q^2 z_2)\,\mathrm{ev}\,(E_1(z_1)E_1(z_2)E_0(w))\right) = 0,$$

where the last equality follows from the quadratic relation for $x_1^+(dz_1)x_1^+(dz_2)$.

The Serre relations in all the remaining cases are checked in the same way.

The statement about grading is straightforward. □

By Theorem 4.2.2, any admissible $U_q\widehat{\mathfrak{gl}}_{m|n}$-module of generic level c can be pulled back by ev_u to a representation of $\mathcal{E}_{m|n}$ with $q_3^{m-n} = c^2$ and $q_2 = q^2$. Such $\mathcal{E}_{m|n}$ modules are called *evaluation modules*.

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APPENDIX

Commutation relations

In this appendix, we collect some useful formulas for commutation relations of various currents.

Lemma A.0.1. *For $i \in I$, we have*

$$e^{A^+(z)} x_i^+(dw) e^{-A^+(z)} = x_i^+(dw) \left(\frac{z - q_3^{-1}w}{z - q_1 w} \right)^{\delta_{i,1}}, \tag{A.1}$$

$$e^{-A^-(z)} x_i^+(d^{m-n+1}w) e^{A^-(z)} = x_i^+(d^{m-n+1}w) \left(\frac{w - q_3 z}{w - q_1^{-1}z} \right)^{\delta_{i,m+n-1}}, \tag{A.2}$$

$$e^{A^+(z)} x_i^-(dw) e^{-A^+(z)} = x_i^-(dw) \left(\frac{z - cq_1 w}{z - cq_3^{-1}w} \right)^{\delta_{i,1}}, \tag{A.3}$$

$$e^{-A^-(z)} x_i^-(d^{m-n+1}w) e^{A^-(z)} = x_i^-(d^{m-n+1}w) \left(\frac{w - cq_1^{-1}z}{w - cq_3 z} \right)^{\delta_{i,m+n-1}}, \tag{A.4}$$

$$e^{B^+(z)} x_i^+(d^{m-n+1}w) e^{-B^+(z)} = x_i^+(d^{m-n+1}w) \left(\frac{z - c^{-1}q_1 w}{z - c^{-1}q_3^{-1}w} \right)^{\delta_{i,m+n-1}}, \tag{A.5}$$

$$e^{-B^-(z)} x_i^+(dw) e^{B^-(z)} = x_i^+(dw) \left(\frac{w - c^{-1}q_1^{-1}z}{w - c^{-1}q_3 z} \right)^{\delta_{i,1}}, \tag{A.6}$$

$$e^{B^+(z)} x_i^-(d^{m-n+1}w) e^{-B^+(z)} = x_i^-(d^{m-n+1}w) \left(\frac{z - q_3^{-1}w}{z - q_1 w} \right)^{\delta_{i,m+n-1}}, \tag{A.7}$$

$$e^{-B^-(z)} x_i^-(dw) e^{B^-(z)} = x_i^-(dw) \left(\frac{w - q_3 z}{w - q_1^{-1}z} \right)^{\delta_{i,1}}, \tag{A.8}$$

$$e^{A^+(z)} e^{A^-(w)} = e^{A^-(w)} e^{A^+(z)} \frac{(z-w)^2}{(z - q_2 w)(z - q_2^{-1}w)}, \tag{A.9}$$

$$e^{A^+(z)} e^{B^-(w)} = e^{B^-(w)} e^{A^+(z)} \frac{(z - cq_2 w)(z - c^{-1}q_2^{-1}w)}{(z - cw)(z - c^{-1}w)}, \tag{A.10}$$

$$e^{B^+(z)} e^{B^-(w)} = e^{B^-(w)} e^{B^+(z)} \frac{(z-w)^2}{(z - q_2 w)(z - q_2^{-1}w)}, \tag{A.11}$$

$$e^{B^+(w)} e^{A^-(z)} = e^{A^-(z)} e^{B^+(w)} \frac{(z - c^{-1}q_2 w)(z - cq_2^{-1}w)}{(z - cw)(z - c^{-1}w)}. \tag{A.12}$$

$\square$

Lemma A.0.2. *The fused currents* $X^{\pm}(z)$ *satisfy*

$$[x_i^{\pm}(z), X^{\pm}(w)] = [x_i^{\pm}(z), X^{\mp}(w)] = 0 \qquad (i \neq 1, m+n-1), \tag{A.13}$$

$$q(w - c^{-1}q_3z)X^+(z)x_1^+(dw) = (w - c^{-1}q_1^{-1}z)x_1^+(dw)X^+(z), \tag{A.14}$$

$$q(z - c^{-1}q_1w)X^+(z)x_{m+n-1}^+(d^{m-n+1}w) = (z - c^{-1}q_3^{-1}w)x_{m+n-1}^+(d^{m-n+1}w)X^+(z), \tag{A.15}$$

$$q(z - cq_1w)X^-(z)x_1^-(dw) = (z - cq_3^{-1}w)x_1^-(dw)X^-(z), \tag{A.16}$$

$$q(w - cq_3z)X^-(z)x_{m+n-1}^-(d^{m-n+1}w) = (w - cq_1^{-1}z)x_{m+n-1}^-(d^{m-n+1}w)X^-(z), \tag{A.17}$$

$$(z - q_3^{-1}w)[X^-(z), x_1^+(dw)] = (w - q_3z)[X^-(z), x_{m+n-1}^+(d^{m-n+1}w)] = 0, \tag{A.18}$$

$$(w - q_3z)[X^+(z), x_1^-(dw)] = (z - q_3^{-1}w)[X^+(z), x_{m+n-1}^-(d^{m-n+1}w)] = 0, \tag{A.19}$$

$$[X^+(w), X^-(z)] = \frac{1}{q - q^{-1}}\left(-\delta\left(c\frac{z}{w}\right)k^+(z) + \delta\left(c\frac{w}{z}\right)k^-(w)\right), \tag{A.20}$$

$$[X^{\pm}(z), X^{\pm}(w)] = 0. \tag{A.21}$$

$\square$