LOCAL LANGLANDS CORRESPONDENCE FOR ASAI L AND EPSILON FACTORS

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I would like to dedicate this thesis to all my friends at Purdue who made my time here so enjoyable.

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If I had to summarize what I learned from Freydoon about how to approach a challenging problem in number theory, I would say this: if you have an unshakable conviction that what you are trying to prove is true, nothing will stop you from proving it.

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ABSTRACT

Shankman, Daniel Ph.D., Purdue University, May 2020. Local Langlands Correspondence for Asai L and Epsilon Factors. Major Professor: Freydoon Shahidi.

Let E/F be a quadratic extension of p-adic fields. The local Langlands correspondence establishes a bijection between n-dimensional Frobenius semisimple representations of the Weil-Deligne group of E and smooth, irreducible representations of $\mathrm{GL}(n, E)$. We reinterpret this bijection in the setting of the Weil restriction of scalars $\mathrm{Res}(\mathrm{GL}(n), E/F)$, and show that the Asai L-function and epsilon factor on the analytic side match up with the expected Artin L-function and epsilon factor on the Galois side.

INTRODUCTION

Local Langlands correspondence for general reductive groups

Let k be a local field of characteristic zero, and W'_k the Weil-Deligne group of k. To each finite dimensional, complex, Frobenius semisimple representation ρ of W'_k, and each nontrivial unitary character ψ of k, there is an associated Artin L-function $L(s,\rho)$ and ϵ-factor $\epsilon(s,\rho,\psi)$, both meromorphic functions of the complex variable s [Ta79]. There is also the gamma factor

$$\gamma(s,\rho,\psi) = \frac{L(1-s,\rho^\vee)\epsilon(s,\rho,\psi)}{L(s,\rho)}$$

where $\rho^\vee$ is the contragredient of ρ.

Let $\mathbf{G}$ be a connected, reductive group over k. To each irreducible, admissible representation π of $\mathbf{G}(k)$, each continuous, finite dimensional complex representation r of the L-group ${}^L\mathbf{G}$ of $\mathbf{G}$ whose restriction to the connected component of ${}^L\mathbf{G}$ is complex analytic, and each nontrivial unitary character ψ of k, there are associated a conjectural L-function $L(s,\pi,r)$ and epsilon factor $\epsilon(s,\pi,r,\psi)$. The conjectural gamma factor $\gamma(s,\pi,r,\psi)$ is defined by

$$\gamma(s,\pi,r,\psi) = \frac{L(1-s,\pi^\vee,r)\epsilon(s,\pi,r,\psi)}{L(s,\pi,r)}.$$

These factors are defined in many special cases, in particular by the Langlands-Shahidi method ([Sh81], [Sh90]).

The conjectural local Langlands correspondence (LLC) predicts the following:

1. A partition of the classes of irreducible, admissible representations of $\mathbf{G}(k)$ into finite sets, called L-packets.

2. A bijection from the set of L-packets to the set of classes of admissible homomorphisms of W_k' into ${}^L\mathbf{G}$ (8.2 of [Bo79]).

3. For each representation r of ${}^L\mathbf{G}$, an equality of L and epsilon factors

$$L(s,\pi,r) = L(s, r\circ\rho),$$

$$\epsilon(s,\pi,r,\psi) = \epsilon(s, r\circ\rho,\psi),$$

whenever π is an element of an L-packet corresponding to ρ, and whenever the left hand sides can be defined.

Parts 1 and 2 of the LLC are notably established for archimedean groups [Kn94], tori [Yu09], and the general linear group. For archimedean groups, the left hand sides of part 3 are defined as the right hand sides. For GL_n, part 3 is established for the standard representation ([He00], [HaTa01], [Sc13]), and for the symmetric and exterior square representations [CoShTs17].

We remark that whenever the partition and bijection of parts 1 and 2 are established for $\mathbf{G}$, they are also established for the group $\operatorname{Res}_{k/k_0}\mathbf{G}$, where k_0 is a local field contained in k (8.4 of [Bo79]). Here $\operatorname{Res}_{k/k_0}$ denotes Weil restriction of scalars. This procedure is compatible with the existing correspondence for archimedean groups and for tori.

The Asai representation

Let E/F be a quadratic extension of characteristic zero local fields. Let $\mathbf{M}$ be the group $\operatorname{Res}_{E/F}\mathrm{GL}_n$ obtained by Weil restriction of scalars. Then $\mathbf{M}$ is a connected, reductive group over F, with $\mathbf{M}(F) = \mathrm{GL}_n(E)$. The L-group ${}^L\mathbf{M}$ can be identified with the semidirect product of $\mathrm{GL}(V)\times\mathrm{GL}(V)$ by $\mathrm{Gal}(E/F)$, where V is an n-dimensional complex vector space, and $\mathrm{Gal}(E/F)$ acts by $\sigma.(T,S) = (S,T)$, where σ

is the nontrivial element of $\mathrm{Gal}(E/F)$. We define the Asai representation $\mathscr{R} : {}^L\mathbf{M} \to \mathrm{GL}(V \otimes V)$ by

$$\mathscr{R}(T, S) = T \otimes S,$$

$$\mathscr{R}(\sigma)(v \otimes v') = v' \otimes v.$$

Now let π be an irreducible, admissible representation of $\mathrm{GL}_n(E)$, corresponding to the Frobenius semisimple representation $\rho : W'_E \to \mathrm{GL}(V)$ under the local Langlands correspondence. As explained in 8.4 of [Bo79], this corresponds to an admissible homomorphism

$$\underline{\rho} : W'_F \to {}^L\mathbf{M}$$

which we can explicitly describe as follows: identifying W'_E as a subgroup of W'_F, choose a $z \in W'_F$ which is not in W'_E. Then

$$\underline{\rho}(a) = \begin{cases} (\rho(a), \rho(zaz^{-1}, 1_E) & \text{if } a \in W'_E \\ (\rho(az^{-1}, za, \sigma) & \text{if } a \notin W'_E. \end{cases}$$

Thus $\pi \leftrightarrow \underline{\rho}$ is the local Langlands correspondence for $\mathbf{M}$. Our main result, which we formulate in various equivalent ways in (2.6.3), is that the equality of epsilon factors holds for the Asai representation:

Main Theorem. *If π is an irreducible, admissible representation of $\mathrm{GL}_n(E)$, and ρ is the n-dimensional Frobenius semisimple representation of W'_E corresponding to π, then*

$$\epsilon(s, \pi, \mathscr{R}, \psi) = \epsilon(s, \mathscr{R} \circ \underline{\rho}, \psi).$$

Remark. *The equality of L-functions*

$$L(s, \pi, \mathscr{R}) = L(s, \mathscr{R} \circ \underline{\rho})$$

is already a theorem due to Henniart (Theorem 5.2 of [He10]). Henniart proved the existence of a root of unity ζ *such that*

$$\zeta\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \mathscr{R} \circ \underline{\rho}, \psi).$$

As we explain in (2.7), since Henniart knew the gamma factors were equal up to a root of unity, he was able to deduce the equality of L-functions.

In our work, we show the exact equality of the gamma factors (Theorem 2.6.3.6). This is to say, we show that $\zeta = 1$*. Using Henniart's method, we recover the equality of the L-functions, and finally deduce the equality of epsilon factors (Corollary 2.6.3.7).*

The Asai L-function $L(s, \pi, \mathscr{R})$ and epsilon factor $\epsilon(s, \pi, \mathscr{R}, \psi)$ can be defined using the Langlands-Shahidi method [Go94]. Granting the local Langlands correspondence for archimedean groups, and its compatability with Weil restriction of scalars, our Main Theorem holds trivially in the archimedean case, because the left hand side is equal to the right hand side, by definition. Thus our main interest in the theorem is in the p-adic case.

Asai L-functions were originally considered by T. Asai in [As77]. He considered the case of a real quadratic extension K of $\mathbb{Q}$, and associated an L-function $L(f, s)$ to a Hilbert modular form f of $K/\mathbb{Q}$. This was central to the work of Harder, Langlands, and Rapoport in their work on Tate's conjecture for Hilbert modular surfaces [HaLaRa86]. When f is a normalized newform, $L(f, s)$ has a factorization over the places of $\mathbb{Q}$. The local factor of $L(f, s)$ at the rational primes p which do not split in K is of the type defined above.

Summary of the proof of the Main Theorem

As we mentioned in the Remark above, Henniart's argument (2.7.1) reduces the main theorem to the problem of showing the equality of gamma factors (Theorem 2.6.3.6) on both sides. This already holds in the case where F is archimedean, or when F is nonarchimedean and π has an Iwahori fixed vector.

Chapter One of this thesis is a review of the Weil and Weil-Deligne group for characteristic zero local and global fields. We define these groups and explain how L-functions may be attached to their representations.

In Chapter Two, we give a summary of the Langlands-Shahidi method, and explain how the Asai factors are defined using this method. We also prove a multiplicativity result for Asai factors. We summarize the Bernstein-Zelevinsky classification of irreducible, admissible representations of GL_n for a p-adic field, state our main theorem, and at the end of the chapter give an exposition of Henniart's proof of the equality of Asai L factors with the corresponding Artin L factors.

Our proof of Theorem 2.6.3.6 is global. One realizes the extension E/F as the completion of a quadratic extension of number fields K/k, and realizes various representations of $\mathrm{GL}_n(E)$ as local components of cuspidal automorphic representations Π of $\mathrm{GL}_n(K)$. On the Galois side, one realizes various representations of the local Weil group W_E as coming from representations Σ of the global Weil group W_K. The global functional equations of L-functions associated to Π and Σ are compared in a way that a finite number of local gamma factors on each side can be isolated and compared. This is the content of Chapter Three. The idea behind this global method was carried out successfully by Cogdell, Shahidi, and Tsai in their proof of the local Langlands correspondence for symmetric and exterior square epsilon factors in [CoShTs17].

One part of the global argument, called *stable equality* (Proposition 3.2.2.2), relies a purely local result, called *analytic stability* (Proposition 3.2.2.8). Analytic stability states that the Asai gamma factor of a supercuspidal representation only depends on its central character, up to highly ramified twist.

The proof of analytic stability is long, and occupies Chapters Four and Five. In Chapter Five, we apply Shahidi's local coefficient formula to write the Asai gamma factor of a supercuspidal representation as a Mellin transform of a partial Bessel integral. Chapter Four is a detailed analysis of the partial Bessel integral, in particular its *asymptotic expansion*. It is through the asymptotic expansion of partial Bessel integrals that the analytic stability result falls out. A similar asymptotic expansion was developed by Cogdell, Shahidi, and Tsai in [CoShTs17], and our approach in Chapter Four closely follows theirs.

Normalization of Langlands-Shahidi local factors

It should be pointed out that the Langlands parameterization of the semisimple conjugacy classes in the L-group given in Langlands' paper *Euler Products* [La71] is different from the one we use in this thesis. The parameterization in *Euler Products* leads to the appearance of contragredients in the local factors occurring in the Langlands-Shahidi method. Our change in the parameterization will remove the contragredients throughout the theory and is farily formal.

Thus in our notation, the Langlands-Shahidi gamma factor $\gamma(s,\pi,r,\psi)$ denotes what is normally written in the literature as $\gamma(s,\pi,r^\vee,\overline{\psi})$. The same goes for the other local factors. See Theorem 2.2.20.1 and the remarks below it.

The discrepancy comes from the definition of Langlands' L-function $L(s,\pi,r)$ in the unramified case and the choice of Harish-Chandra map: if $\mathbf{G}$ is a split group over a p-adic field k, and π is an unramified representation of $\mathbf{G}(k)$, then the Satake isomorphism attaches to π an unramified character χ_π of the k-points of a maximal k-split torus $\mathbf{T}$ of $\mathbf{G}$.

If we let ${}^L\mathbf{T}$ be a maximal torus in ${}^L\mathbf{G}$, and $X({}^L\mathbf{T})$ the additive group of rational characters of ${}^L\mathbf{T}$, the Harish-Chandra map

$$\Lambda : \mathbf{T}(k) \to \mathrm{Hom}_{\mathbb{Z}}(X(\mathbf{T}),\mathbb{Z}) = X({}^L\mathbf{T})$$

induces an isomorphism of $\mathbf{T}(k)/\mathbf{T}(\mathcal{O}_k)$ with $X({}^L\mathbf{T})$, and therefore identifies χ_π with an element A_π of $\mathrm{Hom}_{\mathrm{grp}}(X({}^L\mathbf{T}),\mathbb{C}^*) = {}^L\mathbf{T}$. Then Langlands defines

$$L(s,\pi,r) = \det(1-q^{-s}r(A_\pi))^{-1}.$$

Here q is the number of elements in the residue field of the ring of integers $\mathcal{O}_k$ of k. There are two ways to define Λ. One way is

$$\langle \eta, \Lambda(t) \rangle = -\log_q |\eta(t)|. \qquad (\eta \in X(\mathbf{T}))$$

This is how Langlands defines it in *Problems in Automorphic Forms* [La70]. The other way is

$$\langle \eta, \Lambda(t) \rangle = \log_q |\eta(t)|.$$

This is how Langlands defines it in [La71]. Shahidi defines his local factors so that his local L-functions $L(s, \pi, r)$ agree with those of Langlands in the case where π is unramified and Λ is defined in the second way: $\langle \eta, \Lambda(t) \rangle = \log_q |\eta(t)|$.

We take the opposite definition of Λ, following the convention of [La70]. We do this so that the Langlands-Shahidi gamma factors will agree with the Artin factors under the version of the local Langlands correspondence which we state in (2.3.8).

We remark that *Euler Products*, although it was published a year later than *Problems in Automorphic Forms*, is actually earlier material, as it was a monograph based on lectures given by Langlands at Yale in the spring of 1967.

These adjustments are not serious and do not affect the main results of the Langlands-Shahidi method in any significant way, nor their proofs. The main difference is that Shahidi states many results in terms of subrepresentations of parabolically induced representations, and with our adjustment we must use instead use quotients. We remark that if H_T is the Harish-Chandra map defined in (2.2.16), then our choice of Λ satisfies $\Lambda = -H_T$.

1. THE ARITHMETIC THEORY

In this chapter, we review the theory of local and global Weil groups and their representations. Our primary references are [Ta79] and [BuHe06].

1.1 The Weil group of a p-adic field

1.1.1 The basic setting

Let F be a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_F$ be the integral closure of $\mathbb{Z}_p$ in F; it is a discrete valuation ring whose unique maximal ideal we will denote by $\mathfrak{p}_F$. Let $\varpi = \varpi_F$ denote a uniformizer for F. The residue field $\kappa = \kappa_F$ of $\mathcal{O}_F$ is a finite field of characteristic p. Let $q = q_F$ be the number of elements of κ. We fix an absolute value $|\cdot| = |\cdot|_F$ on F, normalized so that a uniformizer has value q^{-1}.

1.1.2 The maximal unramified extension

Let $\overline{F}$ be an algebraic closure of F. Let F^{ur} be the maximal unramified extension of F inside $\overline{F}$. The integral closure of $\mathcal{O}_F$ in F^{ur} is a local ring, whose residue field $\overline{\kappa}$ identifies as an algebraic closure of κ. The Galois group of F^{ur} over F can be identified with the Galois group over $\overline{\kappa}$ over κ.

1.1.3 Definition of the Weil group and geometric Frobenius

Let I_F be the inertia group of F. It is the Galois group of $\overline{F}$ over F^{ur}. Since the Galois group of F^{ur} over F identifies with that of $\overline{\kappa}$ over κ (1.1.2), we have an exact sequence of topological groups

$$1 \to I_F \to \mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{\kappa}/\kappa) \to 1.$$

The local Weil group W_F is defined to be the subgroup of those $\tau \in \mathrm{Gal}(\overline{F}/F)$ which induce on $\overline{\kappa}$ an automorphism of the form $x \mapsto x^{q^n}$ for some $n \in \mathbb{Z}$. Any element of W_F which induces an automorphism of the form $x \mapsto x^{q^{-1}}$ will be called a **geometric Frobenius**. If Φ is a geometric Frobenius of W_F, then Φ has infinite order, and W_F is the semidirect product of the inertia group and the cyclic group generated by Φ. The choice of Φ yields a split exact sequence of groups

$$1 \to I_F \to W_F \to \mathbb{Z} \to 0$$

and therefore a bijection of sets $W_F \to I_F \times \mathbb{Z}$. We give W_F the product topology, where I_F is assumed to have its usual profinite topology, and $\mathbb{Z}$ is given the discrete topology. Then W_F is a locally profinite topological group, i.e. a Hausdorff topological group in which every neighborhood of the identity contains a compact open subgroup. The topology on W_F is defined independently of the choice of Φ.

1.1.4 Galois representations as representations of the Weil group

The topology on W_F defined in (1.1.3) is not the induced topology from $\mathrm{Gal}(\overline{F}/F)$. However, the inclusion of W_F into $\mathrm{Gal}(\overline{F}/F)$ is continuous with dense image. In fact, $\mathrm{Gal}(\overline{F}/F)$ is the profinite completion of W_F. Therefore, if (ρ, V) is a continuous, finite dimensional, complex representation of $\mathrm{Gal}(\overline{F}/F)$, then ρ is determined up to isomorphism by its restriction to W_F. Moreover, ρ is irreducible if and only if $\rho|_{W_F}$ is.

For the rest of (1.1), "representation" will mean "continuous, finite dimensional, complex representation," unless specified otherwise. If (ρ, V) is a representation of W_F, we will say that ρ is a **Galois representation** if it is the restriction to W_F of a representation of $\mathrm{Gal}(\overline{F}/F)$.

By an "abstract representation" of a group H, we will mean a group homomorphism of π of H into the group $\mathrm{GL}(V)$ of linear automorphisms of a complex vector space V, where V is not necessarily finite dimensional.

Note that if H is a locally profinite group, for example W_F, I_F, or $\mathrm{Gal}(\overline{F}/F)$, and (ρ, V) is an abstract representation of H whose underlying space V is finite dimensional, then ρ is continuous if and only if the kernel of ρ is an open subgroup of H. This follows from the fact that $\mathrm{GL}(V)$ has a neighborhood of the identity which contains no nontrivial subgroups.

1.1.5 Weil group norm

If $w \in W_F$, then there is a unique integer n such that w induces the automorphism $x \mapsto x^{q^n}$ on $\overline{\kappa}$. We define the norm $||w||$ of w to be q^n. This norm is multiplicative ($||ww'|| = ||w|| \cdot ||w'||$). The inertia group is the group of norm one elements of W_F, and an element is a geometric Frobenius (1.1.3) if and only if it has norm q^{-1}.

The norm map is a continuous homomorphism of W_F into $\mathbb{C}^*$, i.e. a one dimensional representation. If (ρ, V) is a representation of W_F, and $s \in \mathbb{C}$, we may then define a representation $\rho|| \cdot ||^s$ of W_F with underlying space V by $w \mapsto \rho(w)||w||^s$. If ρ is irreducible, there always exists an s such that $\rho|| \cdot ||^s$ is a Galois representation (1.1.4) ([BuHe06], Proposition 28.6).

A *character* of a topological group will always mean a continuous homomorphism of that group into the multiplicative group of complex numbers. A character χ of W_F will be called **unramified** if it is trivial on the inertia group. If χ is an unramified character of W_F, then there exists a complex number s such that $\chi(w) = ||w||^s$. The real part of s is uniquely determined, while the imaginary part of s is determined up to an integer multiple of $\frac{2\pi i}{\log q}$.

1.1.6 The relative setting

Let E be a finite extension of F. Then $\overline{F}$ is also an algebraic closure of E, and we have $E^{\mathrm{ur}} = EF^{\mathrm{ur}}$. In this way, we can define the Weil group W_E of E in the same way that we have defined the Weil group of F. In fact, we will have $W_E = W_F \cap \mathrm{Gal}(\overline{F}/E)$, and the inclusion of W_F into $\mathrm{Gal}(\overline{F}/F)$ induces a bijection of W_F/W_E onto $\mathrm{Gal}(\overline{F}/F)/\,\mathrm{Gal}(\overline{F}/E)$.

In particular, $[E : F] = [W_F : W_E]$, and if E is a Galois extension of F, then W_F/W_E can be identified with the Galois group of E over F.

Just as have we have defined the norm $||\cdot||$ for W_F (1.1.5), we also define the norm for W_E. This can be done by simply restricting the norm on W_F to W_E. Thus there is no need to distinguish between different norms, e.g. $||\cdot||_F$ or $||\cdot||_E$.

1.1.7 Definition of local Artin L-functions

Let (ρ, V) be a representation of W_F. Then V^{I_F}, the set of $v \in V$ which are fixed pointwise by I_F, is a subrepresentation of ρ. If Φ is a geometric Frobenius of W_F, then $\rho(\Phi)$ is well defined as a linear map on V^{I_F}, independent of the choice of Φ. The **local Artin L-function** is defined by

$$L(s,\rho) = \det(1 - q_F^{-s}\rho(\Phi)|_{V^{I_F}})^{-1}.$$

It is a nonzero meromorphic function of the complex variable s. In fact, its inverse is a polynomial in q^{-s}. If ρ is irreducible, then $L(s,\rho) = 1$ unless ρ is a nontrivial unramified character, i.e. ρ is one dimensional and trivial on the inertia group.

We remark that Artin originally defined his L-functions the other way, replacing Φ by its inverse, the *arithmetic Frobenius*.

1.1.8 Inductivity and additivity of L-functions

Let H be a subgroup of a group G. If (ρ, V) is an abstract representation of H, let $\operatorname{Ind}_H^G \rho$ be the representation of G induced by ρ. This is the representation of G whose underlying space consists of all functions $f : G \to V$ satisfying $f(hg) = \rho(h)f(g)$ for all $g \in G$ and $h \in H$, and G acts on these functions by right translation.

Let E be a finite extension of F. Identify the Weil group of E as a subgroup of the Weil group of F as in (1.1.6). If (ρ, V) is a representation of W_E, then $\operatorname{Ind}_{E/F} \rho = \operatorname{Ind}_{W_E}^{W_F} \rho$ is continuous as a representation of W_F, and we have

$$L(s, \operatorname{Ind}_{E/F} \rho) = L(s, \rho).$$

That is, L-functions are inductive ([Ta79], equation (3.3)). Also, L-functions are additive, which is to say that if

$$0 \to \rho' \to \rho \to \rho'' \to 0$$

is an exact sequence of representations of W_F, then

$$L(s, \rho) = L(s, \rho')L(s, \rho'').$$

1.1.9 The abelianized Weil group

Let W_F^{ab} denote the abelianization of W_F, i.e. the quotient of W_F by the closure of its derived group. The continuous inclusion of W_F into $\operatorname{Gal}(\overline{F}/F)$ induces a continuous injection of W_F^{ab} into $\operatorname{Gal}(\overline{F}/F)^{\mathrm{ab}} = \operatorname{Gal}(F^{\mathrm{ab}}/F)$, where F^{ab} denotes the maximal abelian extension of F inside $\overline{F}$.

Actually, the abelianization of W_F^{ab} can be defined directly, without reference to W_F itself. Recall that the Galois group of F^{ur} over F identifies with the Galois group of $\overline{\kappa}$ over κ, where κ is the residue field of F (1.1.2). We have an exact sequence of topological groups

$$1 \to \mathrm{Gal}(F^{\mathrm{ab}}/F^{\mathrm{ur}}) \to \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(\overline{\kappa}/\kappa) \to 1$$

and W_F^{ab} (or rather, its image inside $\mathrm{Gal}(F^{\mathrm{ab}}/F)$) can be defined to be the subgroup of those $\tau \in \mathrm{Gal}(F^{\mathrm{ab}}/F)$ which induce on $\overline{\kappa}$ an automorphism of the form $x \mapsto x^{q^n}$ for some $n \in \mathbb{Z}$. Any element of W_F^{ab} which induces an automorphism of the form $x \mapsto x^{q^{-1}}$ will be called a **geometric Frobenius** of W_F^{ab}. If Φ is a geometric Frobenius of W_F^{ab}, then Φ has infinite order, and W_F^{ab} is the direct product of $\mathrm{Gal}(F^{\mathrm{ab}}/F^{\mathrm{ur}})$ and the cyclic group generated by Φ. The choice of Φ yields a split exact sequence of abelian groups

$$1 \to \mathrm{Gal}(F^{\mathrm{ab}}/F^{\mathrm{ur}}) \to W_F^{\mathrm{ab}} \to \mathbb{Z} \to 0$$

and therefore an isomorphism of abelian groups $W_F^{\mathrm{ab}} \to \mathrm{Gal}(F^{\mathrm{ab}}/F^{\mathrm{ur}}) \times \mathbb{Z}$. We give W_F^{ab} the product topology. Then W_F^{ab} is a locally profinite group, and its topology is defined independently of the choice of Φ. In fact, the topology which we have just defined on W_F^{ab} is the same as the quotient topology coming from W_F.

The topology on W_F^{ab} is not the induced topology from $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. However, the inclusion of W_F^{ab} into $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is continuous with dense image, and $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is the profinite completion of W_F^{ab}.

1.1.10 Local class field theory

By the main results of local class field theory, there exists a unique isomorphism of topological groups $\mathrm{Art} = \mathrm{Art}_F : F^* \to W_F^{\mathrm{ab}}$ which sends uniformizers to geometric Frobenius elements, and such that if E is a finite abelian extension of F, the kernel of the induced map

$$F^* \to W_F^{\mathrm{ab}} \to \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(E/F)$$

is $N_{E/F}(E^*)$, the image of the norm. We will call this isomorphism the **local Artin map**. In more traditional treatments of local class field theory, the local Artin map is defined to be the composition $F^* \xrightarrow{\mathrm{Art}} W_F^{\mathrm{ab}} \to \mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{x \mapsto x^{-1}} \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

1.1.11 Identification under the local Artin map

If E is a finite extension of F (not necessarily Galois or abelian), then the homomorphism $W_E^{\mathrm{ab}} \to W_F^{\mathrm{ab}}$ induced by inclusion $W_E \subset W_F$ corresponds to the norm under the Artin reciprocity map:

$$\begin{array}{ccc} E^* & \xrightarrow{\mathrm{Art}_E} & W_E^{\mathrm{ab}} \\ \downarrow{\scriptstyle N_{E/F}} & & \downarrow \\ F^* & \xrightarrow{\mathrm{Art}_F} & W_F^{\mathrm{ab}}. \end{array} \tag{1.1.11.1}$$

The norm $||\cdot||$ on W_F (1.1.5), being continuous, is well defined as a continuous homomorphism on W_F^{ab}. One advantage of defining the local Artin map in the way we have is that the norm on W_F^{ab} is compatible with the normalized absolute value on F:

$$||\,\mathrm{Art}_F(x)|| = |x|_F. \qquad (x \in F^*)$$

Since the normalized absolute value on E can be defined in terms of the normalized absolute value on F by $|y|_E = |N_{E/F}(y)|_F$, the commutativity of the diagram in (1.1.11) gives a proof that the restriction to W_E of the norm $||\cdot||$ on W_F is equal to the norm on W_E, as we have remarked in (1.1.6).

Assume E is a Galois extension of F. Let $z \in W_F$, and let τ be the image of z in $W_F/W_E = \mathrm{Gal}(E/F)$. Then the diagram

$$\begin{array}{ccc} E^* & \xrightarrow{\mathrm{Art}_E} & W_E^{\mathrm{ab}} \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \iota_z} \\ E^* & \xrightarrow{\mathrm{Art}_F} & W_E^{\mathrm{ab}} \end{array} \tag{1.1.11.2}$$

is commutative, where ι_z denotes conjugation by z ($w \mapsto zwz^{-1}$).

1.1.12 Self dual measures and Fourier transform

Let dx be a Haar measure on F. Let ψ be a nontrivial additive character of F, i.e. a continuous homomorphism of F into $\mathbb{C}^*$. Note that ψ is automatically unitary, because F is the union of its compact open subgroups. Define $\mathscr{C}_c^\infty(F)$ (resp. $\mathscr{C}_c^\infty(F^*)$) to be the algebra of locally constant and complex valued functions on F (resp. F^*) which vanish outside a compact set. If $f \in \mathscr{C}_c^\infty(F)$, the **Fourier transform** $\hat{f}$ is an element of $\mathscr{C}_c^\infty(F)$ defined by

$$\hat{f}(x) = \int_F f(y)\psi(xy)\,dy.$$

There is a unique choice of Haar measure on F (depending on ψ) such that the Fourier inversion formula

$$\hat{\hat{f}}(x) = f(-x)$$

holds for all $f \in \mathscr{C}_c^\infty(F)$. We will call such a measure **self dual** (with respect to ψ). Given ψ, we will always assume that the Haar measure dx on F is chosen to be self dual.

1.1.13 Local factors for nonarchimedean GL_1

Let χ be character of $F^* = \mathrm{GL}_1(F)$ (assumed always to be continuous, but not necessarily unitary). We say that χ is **unramified** if χ is trivial on $\mathcal{O}_F^*$, and otherwise we say χ is **ramified**. We define the **local analytic L-function**

$$L(s,\chi) = \begin{cases} (1 - q^{-s}\chi(\varpi))^{-1} & \chi \text{ unramified} \\ 1 & \chi \text{ ramified.} \end{cases}$$

It is a meromorphic function of the complex variable s. Note that when χ is unramified, $\chi(\varpi)$ is independent of the choice of uniformizer ϖ.

Let ψ be a nontrivial character of F. Assume the Fourier transform on F is defined according to the character ψ and the Haar measure dx which is self dual with respect to ψ (1.1.12). Let d^*x be any Haar measure on F^*, for example $\frac{dx}{|x|}$. There is a meromorphic function $\epsilon(s,\chi,\psi)$ of the complex variable s, with the property that for any $f \in \mathscr{C}_c^\infty(F^*)$,

$$\frac{\int_{F^*} \hat{f}(x)\chi(x)^{-1}|x|^{1-s}\, d^*x}{L(1-s,\chi^{-1})} = \epsilon(s,\chi,\psi)\frac{\int_{F^*} f(x)\chi(x)|x|^s\, d^*x}{L(s,\chi)}$$

([Ta79], equation (3.21)). We call $\epsilon(s,\chi,\psi)$ the **local epsilon factor**. It is a monomial in the variable q^{-s}. We also define the **local gamma factor** $\gamma(s,\chi,\psi)$ by

$$\gamma(s,\chi,\psi) = \frac{\epsilon(s,\chi,\psi)L(1-s,\chi^{-1})}{L(s,\chi)}.$$

The **conductor** of ψ is defined to be the largest integer d such that ψ is trivial on $\mathfrak{p}_F^{-d}$. The **conductor** of χ is defined to be 0 if χ is unramified, and the smallest natural number f such that χ is trivial on $1+\mathfrak{p}_F^f$ if χ is ramified.

Let d and f be the conductors of ψ and χ. Assume that χ is ramified. If we take the Haar measure $d^*x = \frac{dx}{|x|}$, then

$$\int\limits_{\substack{x\in F^* \\ \mathrm{ord}_{\mathfrak{p}}(x)=k}} \chi(x)|x|^s\psi(x)d^*x = \begin{cases} \gamma(s,\chi,\psi)^{-1} & \text{if } k=-d-f \\ 0 & \text{if } k\neq -d-f. \end{cases} \tag{1.1.13.1}$$

Thus the gamma factor can be calculated formally as an integral

$$\gamma(s,\chi,\psi)^{-1} = \int\limits_{F^*} \chi(x)|x|^s\psi(x)d^*x$$

even though the right hand side does not converge absolutely.

1.1.14 Definition of local factors for characters of the Weil group

Let ρ be one dimensional representation of W_F, i.e. a character. Then ρ can be identified with a character of W_F^{ab}, so that $\chi = \mathrm{Art}^{-1}(\rho)$ is a character of F^*. Here Art is the local Artin map of (1.1.10). Then we have

$$L(s,\rho) = L(s,\chi)$$

where the left hand side was defined in (1.1.7), and the right hand side was defined in (1.1.13). We define

$$\epsilon(s,\rho,\psi) = \epsilon(s,\chi,\psi)$$

$$\gamma(s,\rho,\psi) = \gamma(s,\chi,\psi)$$

so that

$$\gamma(s,\rho,\psi) = \frac{\epsilon(s,\rho,\psi)L(1-s,\rho^{-1})}{L(s,\rho)}.$$

1.1.15 The Grothendieck group of W_F

If (ρ, V) is a representation of W_F, then ρ need not be semisimple, i.e. V need not decompose into a direct sum of irreducible representations. However, ρ always has a composition series, i.e. a sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$ of subrepresentations of ρ such that each quotient representation V_i/V_{i-1} is irreducible. We define the **semisimplification** ρ_{ss} of ρ to be the semisimple representation $V_1/V_0 \oplus \cdots \oplus V_r/V_{r-1}$. Up to isomorphism, it does not depend on the choice of composition series, and ρ is semisimple if and only if ρ is isomorphic to its semisimplification.

A representation (ρ, V) of W_F is semisimple if and only if the image of a geometric Frobenius of W_F in $\mathrm{GL}(V)$ is diagonalizable ([BuHe06], Proposition 28.7). In fact, since we are dealing only with representations in characteristic zero, it suffices for a nonzero power of the image of a geometric Frobenius to be diagonalizable.

Let $R(W_F)$ be the Grothendieck group of W_F, which is the free abelian group on the isomorphism classes of (continuous, finite dimensional) irreducible representations of W_F. If ρ is a representation of W_F, let $[\rho]$ be the image of ρ in $R(W_F)$: by definition, this is the sum of the classes of the composition factors of ρ. We have $[\rho\oplus\rho'] = [\rho]+[\rho']$, and more generally if

$$0 \to \rho' \to \rho \to \rho'' \to 0$$

is an exact sequence of representations of W_F, then $[\rho] = [\rho'] + [\rho'']$. It follows that $[\rho] = [\rho_{\mathrm{ss}}]$.

A typical element of $R(W_F)$ is $n_1[\rho_1] + \cdots + n_r[\rho_r]$ for irreducible representations ρ_i of W_F and integers n_i. We define the **degree** of such an element to be $n_1+\cdots+n_r$.

Let E be a finite extension of F. Since $\mathrm{Ind}_{E/F}$ is an exact functor from the category of representations of W_E to representations of W_F, it is well defined as a function $R(W_E) \to R(W_F)$. Then for any representation ρ of W_E, we have

$$\mathrm{Ind}_{E/F}[\rho] = [\mathrm{Ind}_{E/F}(\rho)].$$

1.1.16 Local factors for representations of W_F

We have defined the epsilon factor $\epsilon(s, \rho, \psi)$ for one dimensional representations of W_F (1.1.14). More generally, Deligne has shown the existence of epsilon factors for general representations of W_F ([De72]):

Theorem 1.1.16.1. *(Deligne) There is a unique function ϵ which associates to each finite extension E of F, each nontrivial character ψ_E of E, and each representation ρ of W_E a meromorphic function $\epsilon(s, \rho, \psi_E)$ such that:*

(i): $\epsilon(s, \rho, \psi_E) = \epsilon(s, \chi, \psi_E)$ if ρ is a one dimensional representation corresponding to a character χ of W_E as in (1.1.14).

(ii): $\epsilon(s, \rho, \psi_E)$ is additive in the variable ρ of representations of W_E (hence well defined on $R(W_E)$).

(iii): If $F \subset E \subset E'$ are finite extensions of F, and $[\rho] \in R(W_{E'})$ has degree zero, then

$$\epsilon(s, [\rho], \psi_E \circ \mathrm{Tr}_{E'/E}) = \epsilon(s, \mathrm{Ind}_{E/E'}[\rho], \psi_E).$$

(iv): If χ is the character $w \mapsto ||w||^{s_0}$ of W_E for some complex number s_0, and ρ is a representation of W_E, then

$$\epsilon(s, \rho\chi, \psi_E) = \epsilon(s + s_0, \rho, \psi_E).$$

The epsilon factor $\epsilon(s, \rho, \psi_E)$ turns out to always be a monomial in q_E^{-s}. Although epsilon factors are not inductive in general (only in degree zero), there is an induction formula for them:

$$\lambda(E/F, \psi)^{\dim \rho} \epsilon(s, \rho, \psi \circ \mathrm{Tr}_{E/F}) = \epsilon(s, \mathrm{Ind}_{E/F}(\rho), \psi)$$

where $\lambda(E/F, \psi)$ is the Langlands lambda function ([BuHe06], Section 30.4). We finally define the gamma function $\gamma(s, \rho, \psi)$ by

$$\gamma(s, \rho, \psi) = \frac{\epsilon(s, \rho, \psi) L(1 - s, \rho^\vee)}{L(s, \rho)}$$

where $\rho^\vee$ is the contragredient of ρ. Like epsilon factors, gamma factors are additive, and satisfy the inductivity formula:

$$\lambda(E/F, \psi)^{\dim \rho} \gamma(s, \rho, \psi \circ \mathrm{Tr}_{E/F}) = \gamma(s, \mathrm{Ind}_{E/F}(\rho), \psi) \tag{1.1.16.1}$$

for all finite extensions E of F and all representations ρ of W_E.

1.1.17 Tensor induction

Let G be a group, and let H be a subgroup of G which is of index 2. Let z be an element of G which is not in H. If (ρ, V) is a representation of H, let $(\sigma, V \oplus V)$ be the representation of G given by

$$\sigma(g)(v, v') = \begin{cases} (\rho(g)v, \rho(zgz^{-1})v') & \text{if } g \in H \\ (\rho(gz^{-1})v', \rho(zg)v) & \text{if } g \notin H. \end{cases}$$

Up to isomorphism, this representation does not depend on the choice of z, and in fact this representation is isomorphic to the induced representation $\operatorname{Ind}_H^G \rho$ (1.1.8). An isomorphism is given by sending a function $f : G \to V$ satisfying $f(hg) = \rho(h)f(g)$ to the pair $(f(1), f(z))$.

Similarly, we define a representation of G with underlying space $V \otimes V$ by the formula

$$g.(v \otimes v') = \begin{cases} \rho(g)v \otimes \rho(zgz^{-1})v' & \text{if } g \in H \\ \rho(gz^{-1})v' \otimes \rho(zg)v & \text{if } g \notin H. \end{cases}$$

We call this the representation of G obtained from ρ by **tensor induction**, and denote it by $\otimes \operatorname{Ind}_H^G \rho$. Up to isomorphism, it does not depend on the choice of z.

1.1.18 Tensor induction and composition series

Let G, H, and z be as in the last section.

Lemma 1.1.18.1. *Let (ρ_1, V) and (ρ_2, W) be representations of H. Define a representation δ of G with underlying space $(V \otimes W) \oplus (V \otimes W)$ by*

$$\delta(g).(v \otimes w, v' \otimes w') = \begin{cases} (\rho_1(g)v \otimes \rho_2(zgz^{-1})w, \rho_1(zgz^{-1})v' \otimes \rho_2(g)w') & \textit{if } g \in H \\ (\rho_1(gz^{-1})v' \otimes \rho_2(zg)w', \rho_1(zg)v \otimes \rho_2(gz^{-1})w) & \textit{if } g \notin H. \end{cases}$$

Then $\delta \cong \operatorname{Ind}_H^G \rho_1 \otimes (\rho_2 \circ \iota_z)$, where ι_z denotes conjugation by z ($g \mapsto zgz^{-1}$).

Proof: Consider the restriction of δ to H. Then $\delta|_H = (\rho_1 \otimes (\rho_2 \circ \iota_z)) \oplus ((\rho_1 \circ \iota_z) \otimes \rho_2)$. Since $z^2 \in H$, we have $\rho \circ \iota_{z^2} \cong \rho$ for any representation ρ of H, and therefore

$$\begin{aligned}(\rho_1 \circ \iota_z) \otimes \rho_2 &\cong ((\rho_1 \circ \iota_z) \otimes \rho_2) \circ \iota_{z^2} \\ &= ((\rho_1 \circ \iota_z) \otimes \rho_2) \circ \iota_z \circ \iota_z \\ &= ((\rho_1 \circ \iota_{z^2}) \otimes (\rho_2 \circ \iota_z)) \circ \iota_z \\ &\cong (\rho_1 \otimes (\rho_2 \circ \iota_z)) \circ \iota_z.\end{aligned}$$

This implies that

$$\delta|_H \cong (\rho_1 \otimes \rho_2 \circ \iota_z) \oplus (\rho_1 \otimes \rho_2 \circ \iota_z) \circ \iota_z$$

which is exactly the restriction of $\operatorname{Ind}_H^G \rho_1 \otimes (\rho_2 \circ \iota_z)$ to H (given in the form (1.1.7)). One checks that these isomorphisms actually intertwine the action of G, not just H. $\square$

If V is a complex vector space which is equal to a direct sum $V_1 \oplus \cdots \oplus V_r$, then $V \otimes V$ is equal to a direct sum

$$[\bigoplus_{i=1}^{r} V_i \otimes V_i] \oplus [\bigoplus_{1 \leq i < j \leq r} (V_i \otimes V_j) \oplus (V_i \otimes V_j)].$$

The same procedure allows us to decompose a representation obtained by tensor induction:

Lemma 1.1.18.2. *Suppose that (ρ, V) is a representation of H which decomposes as a direct sum of subrepresentations $(\rho_1, V_1) \oplus \cdots \oplus (\rho_r, V_r)$. Then $(\otimes \operatorname{Ind}_H^G \rho, V \otimes V)$ decomposes as a direct sum of subrepresentations*

$$\otimes \operatorname{Ind}_H^G \rho = \bigoplus_{i=1}^{r} \otimes \operatorname{Ind}_H^G \rho_i \oplus \bigoplus_{1 \leq i < j \leq r} \operatorname{Ind}_H^G \rho_i \otimes (\rho_j \circ \iota_z).$$

Proof: This follows from direct computation and applying Lemma 1.1.18.1. $\square$

If instead of a direct sum of subrepresentations, we have a filtration of subrepresentations of ρ, then we can find a corresponding filtration of subrepresentations $\otimes \operatorname{Ind}_H^G \rho$ as in Lemma 1.1.18.2.

Lemma 1.1.18.3. *Suppose that (ρ, V) is a finite dimensional representation of H. Let $(V_1, \rho_1), ..., (V_r, \rho_r)$ be the composition factors of a composition series of ρ. There is a sequence of subrepresentations of $\otimes \operatorname{Ind}_H^G \rho$*

$$0 = L_0 \subset L_1 \subset \cdots \subset V \otimes V$$

for which the following representations show up as the quotients L_i/L_{i+1}:

$$\otimes \operatorname{Ind}_H^G \rho_i : 1 \leq i \leq r$$

$$\operatorname{Ind}_H^G \rho_i \otimes (\rho_j \circ \iota_z) : 1 \leq i < j \leq r.$$

Proof: Let W be a subrepresentation of V such that V/W is irreducible. Let π be the quotient map $V \to V/W$, let $\overline{\rho}$ be the corresponding representation of H on V/W, and let ρ_1 be the representation of H on W. Then $(\otimes \operatorname{Ind}_H^G, V \otimes V)$ has $W \otimes W$ as a G-stable subspace, with $W \otimes W \cong \otimes \operatorname{Ind}_H^G \rho_1$.

We will investigate the representation $(V \otimes V)/(W \otimes W)$ of G. First, note that the quotient map $\pi \otimes \pi : (V \otimes V, \otimes \operatorname{Ind}_H^G \rho) \to (V/W \otimes V/W, \otimes \operatorname{Ind}_H^G \overline{\rho})$ is a G-linear map with kernel $W \otimes V + V \otimes W$.

We next define a linear map

$$T : W \otimes V + V \otimes W \to (W \otimes V/W) \oplus (V/W \otimes W)$$

as follows: if $x \in W \otimes V$, and $y \in V \otimes W$, then we set

$$T(x + y) = ((1_W \otimes \pi)(x), (\pi \otimes 1_W)(y)).$$

This is well defined, since $W \otimes V \cap V \otimes W = W \otimes W$. Furthermore, T is surjective with kernel $W \otimes W$, and the induced representation on the image of T is easily seen to be isomorphic to

$$\operatorname{Ind}_H^G \overline{\rho} \otimes (\rho_1 \circ \iota_z).$$

The computation is the same as that for Lemma 1.1.18.1.

We have found subrepresentations

$$W \otimes W \subsetneq W \otimes V + V \otimes V \subsetneq V \otimes V$$

whose quotients are of the form in the statement of the lemma. If W is irreducible, we are done. If W is not irreducible, then we take a subrepresentation W_0 of W such that W/W_0 is irreducible, and iterate the same procedure with the inclusion $W_0 \otimes W_0 \subseteq W \otimes W$. $\square$

We finally note that tensor induction commutes with taking the contragredient.

Lemma 1.1.18.4. *Suppose that (ρ, V) is a representation of H. Let $(\rho^\vee, V^*)$ be the contragredient representation of H, where $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual space of V. Then*

$$\otimes \operatorname{Ind}_H^G(\rho^\vee) \cong \left(\otimes \operatorname{Ind}_H^G \rho \right)^\vee.$$

Proof: The underlying space of the first representation is $V^* \otimes V^*$, while the underlying space of the second representation is $(V \otimes V)^*$. An isomorphism of these vector spaces is defined by sending an elementary tensor $v^* \otimes w^*$ to the linear functional on $V \otimes V$ defined by

$$v \otimes w \mapsto \langle v^*, v \rangle \langle w^*, w \rangle.$$

It is immediate that this isomorphism intertwines the action of G. $\square$

1.1.19 Tensor induction and twisting by characters

If η is a character of H, and ρ is a representation of G, then $\otimes \operatorname{Ind}_H^G \eta$ is a character of G, and

$$\otimes \operatorname{Ind}_H^G(\rho\eta) = (\otimes \operatorname{Ind}_H^G \rho)(\otimes \operatorname{Ind}_H^G \chi).$$

Let E/F be a quadratic extension of p-adic fields. Identify the Weil group W_E of E as a subgroup of index two of W_F (1.1.6). If ρ is a representation of W_E, we will write $\otimes \operatorname{Ind}_{E/F} \rho$ instead of $\otimes \operatorname{Ind}_{W_E}^{W_F} \rho$.

Lemma 1.1.19.1. *Let E/F be a quadratic extension of p-adic fields. Let ρ be a character of the Weil group W_E.*

(i): If ρ is an unramified character of W_E, then $\otimes \operatorname{Ind}_{E/F} \rho$ is an unramified character of W_F. More specifically, if $||\cdot||$ is the Weil group norm, and $\rho = ||\cdot||^{s_0}$ for a complex number s_0, then $\otimes \operatorname{Ind}_{E/F} \rho = ||\cdot||^{2s_0}$.

(ii): The character $\otimes \operatorname{Ind}_{E/F} \rho$ can be made highly ramified by choosing ρ to be highly ramified.

Proof: (i) is a straightforward computation, and (ii) can be seen by identifying W_F^{ab} and W_E^{ab} with F^* and E^* via local class field theory. One uses the fact that the homomorphism $W_E^{\text{ab}} \to W_F^{\text{ab}}$ coming from the inclusion map $W_E \subset W_F$ identifies with the norm $E^* \to F^*$. $\square$

1.2 The Weil-Deligne group of a p-adic field

As in the previous section, every representation of a locally profinite group will be assumed to be finite dimensional, complex, and continuous. Let F be a p-adic field. Recall that a representation of $\operatorname{Gal}(\overline{F}/F)$ is completely determined by its restriction to the local Weil group W_F. The class of representations of the Galois group is therefore a subset of the class of representations of the Weil group.

In order to formulate the local Langlands correspondence, which relates the finite dimensional representations of this chapter to (usually infinite dimensional) representations of reductive groups, it will be necessary to further expand the class of finite dimensional representations which we will consider. The natural way to do this is by extending the Weil group W_F to a larger group called the Weil-Deligne group.

1.2.1 Definition of the Weil-Deligne group

Let G be a profinite group, and let R be the ring of locally constant functions from G to $\mathbb{Q}$. Then R is a zero dimensional ring, and $g \mapsto \{f \in R : f(g) = 0\}$ defines a homeomorphism of G onto $\operatorname{Spec} R$. In this way, G is naturally an affine group scheme over $\mathbb{Q}$ with global section R.

Let F be a p-adic field, and let W_F be the Weil group of F. Each coset of the inertia group I_F in W_F is a scheme over $\mathbb{Q}$, being isomorphic to I_F. Since W_F is the disjoint union of these cosets, we can regard W_F as a scheme over $\mathbb{Q}$. In fact, the group structure of W_F makes W_F into a group scheme over $\mathbb{Q}$.

Let $\mathbb{G}_a$ be the additive group over $\mathbb{Q}$. We define the **Weil-Deligne group** W_F' of F to be the $\mathbb{Q}$-group scheme $\mathbb{G}_a \rtimes W_F$, where W_F acts on $\mathbb{G}_a$ by $w.x = ||w||x$. The Weil-Deligne group is neither affine nor of finite type.

1.2.2 Representations of the Weil-Deligne group

The group of $\mathbb{C}$-rational points $W_F'(\mathbb{C}) = \operatorname{Hom}_{\mathbb{Q}}(\operatorname{Spec} \mathbb{C}, W_F')$ can be identified with the semidirect product of $\mathbb{C}$ by W_F', where W_F acts on $\mathbb{C}$ by $w.x = ||w||x$. Actually, we will always identify W_F' with its $\mathbb{C}$-rational points.

Let V be a finite dimensional complex vector space. The following three objects are in bijective correspondence, and can be naturally identified:

- A morphism of group schemes $W_F' \times_{\mathbb{Q}} \operatorname{Spec} \mathbb{C} \to \mathrm{GL}(V)$.
- A group homomorphism $\mathbb{C} \rtimes W_F \to \mathrm{GL}(V)$ whose restriction to W_F is continuous, and whose restriction to $\mathbb{C}$ is a morphism of varieties.

- A pair (ρ, N), where ρ is a (continuous, finite dimensional, complex) representation of W_F with underlying space V, and N is a nilpotent linear operator on V such that $\rho(w)N\rho(w)^{-1} = ||w||N$ for all $w \in W_F$.

Any of these three objects will be called a **representation** of W_F'. We will explain the equivalence of the second and third notions. Given ρ and N of the third description, we extend ρ to a homomorphism on $W_F' = \mathbb{C} \rtimes W_F$ by sending (x, w) to $\exp(xN)\rho(w)$. Given a homomorphism $\rho' : W_F' \to \mathrm{GL}(V)$ of the second description, we first let ρ be the restriction of ρ' to W_F. The restriction of ρ' to $\mathbb{C}$, being a one parameter subgroup of the complex Lie group $\mathrm{GL}(V)$, must be of the form $x \mapsto \exp(xN)$ for a unique linear operator N of V. But being a morphism of varieties, the series defining $\exp(xN)$ must also be a polynomial in x, so N must be nilpotent.

In light of these equivalences, we can say that a representation of W_F is the same thing as a representation of W_F' whose nilpotent operator is zero. Since the kernel of the nilpotent operator is W_F-stable, it is clear that a representation of W_F' is never irreducible unless it is actually just a representation of W_F, i.e. its nilpotent operator is zero.

1.2.3 Operations on Weil-Deligne representations

If (ρ', V) is a representation of $W_F' = \mathbb{C} \rtimes W_F$, we have shown in the previous section that we can identify ρ' with the triple (ρ, V, N), where $\rho = \rho'|_{W_F}$, and N is the derivative at $x = 0$ of the map $x \mapsto \rho'|_{\mathbb{C}}(x)$. If an operation is applied to ρ' (direct sum, tensor product, contragredient), it is useful to to know the underlying Weil representation and nilpotent operator.

Let $\rho_1' = (\rho_1, V_1, N_1)$ and $\rho_2' = (\rho_2, V_2, N_2)$ be two representations of W_F'. Then we have

$$\rho_1' \oplus \rho_2' = (V_1 \oplus V_2, \rho_1 \oplus \rho_2, N_1 \oplus N_2)$$

$$\rho_1' \otimes \rho_2' = (V_1 \otimes V_2, \rho_1 \otimes \rho_2, N_1 \otimes 1 + 1 \otimes N_2)$$

$$\rho_1'^{\vee} = (\rho_1^{\vee}, V^*, -N_1^{\vee})$$

where $^{\vee}$ denotes the contragredient.

1.2.4 The special representations Sp(m)

For $m \geq 1$, define Sp(m) to be the following m-dimensional representation of W_F': the underlying space is the m dimensional vector space $\mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{m-1}$, the underlying representation of W_F is given by $w.e_i = ||w||^i e_i$, and the nilpotent operator N is given by

$$Ne_i = \begin{cases} e_{i+1} & \text{if } i \neq m-1 \\ 0 & \text{if } i = m-1. \end{cases}$$

If $(\rho, V) = (\rho, V, 0)$ is a representation of W_F, then $\rho \otimes \mathrm{Sp}(m)$ can be thought of in the following way: the underlying Weil representation is the direct sum

$$(\rho \oplus \rho|| \cdot || \oplus \cdots \oplus \rho|| \cdot ||^{m-1}, \bigoplus_{i=0}^{m-1} V)$$

and the nilpotent operator is given by

$$(v_0, ..., v_{m-2}, v_{m-1}) \mapsto (0, v_1, ..., v_{m-2}).$$

1.2.5 Classification of Weil-Deligne representations

A representation of W_F' is said to be **Frobenius semisimple** if its restriction to W_F is semisimple. A representation of W_F' is said to be **indecomposable** if it cannot be written as a direct sum of two proper subrepresentations.

Every indecomposable, Frobenius semisimple representation ρ' of W_F' is isomorphic to $\rho \otimes \mathrm{Sp}(m)$ for an irreducible representation ρ of W_F and a positive integer m. It is a represention of W_F (in the sense that the nilpotent operator is zero) if and only if $m = 1$. The class of ρ and the integer m are determined by ρ'.

If ρ' is a representation of W'_F which is Frobenius semisimple, then there exist irreducible representations $\rho_1, ..., \rho_r$ of W_F and positive integers $m_1, ..., m_r$ such that

$$\rho' = \rho_1 \otimes \mathrm{Sp}(m_1) \oplus \cdots \oplus \rho_r \otimes \mathrm{Sp}(m_r).$$

The pairs (ρ_i, m_i) are uniquely determined.

1.2.6 Induced representations

Let E be a finite extension of F. As usual, we will consider W_E as a subgroup of W_F (1.1.6). Since the norm on W_F restricts to the norm on W_E, the inclusion homomorphism of W_E into W_F induces an inclusion homomorphism of W'_E into W'_F. We have $[W'_F : W'_E] = [E : F]$.

Let $\rho' = (\rho, V, N)$ be a representation of W'_E. Consider the induced representation $\mathrm{Ind}_{W'_E}^{W'_F} \rho'$ of W'_F, and an element $f : W'_F \to V$ of the underlying space of this representation. Since for $x \in \mathbb{C}$, we have

$$f(x) = \exp(xN) f(1_{W'_F})$$

we see immediately that $f \mapsto f|_{W_F}$ defines an isomorphism of W_F-representations

$$\mathrm{Ind}_{W'_E}^{W'_F} \rho' \to \mathrm{Ind}_{W_E}^{W_F} \rho.$$

We can then identify $\mathrm{Ind}_{W'_E}^{W'_F} \rho'$ with the representation $\mathrm{Ind}_{E/F} \rho$ of W_F, together with the nilpotent operator T given by $(T \cdot f)(w) = \exp(||w||N) f(w)$. We will write $\mathrm{Ind}_{E/F} \rho'$ instead of $\mathrm{Ind}_{W'_F}^{W'_E} \rho'$.

1.2.7 Local factors for Weil-Deligne representations

Let $\rho' = (\rho, V, N)$ be a Frobenius semisimple representation of W'_F. Then $\rho_N = \operatorname{Ker} N$ is a subrepresentation of ρ. We define the **L-function** and **epsilon factor** by

$$L(s, \rho') = L(s, \rho_N)$$

$$\epsilon(s, \rho', \psi) = \epsilon(s, \rho, \psi) \frac{L(1-s, \rho^\vee)}{L(s, \rho)} \frac{L(s, \rho')}{L(1-s, \rho'^\vee)}.$$

Here ψ is a nontrivial character of F, and the factors on the right hand side were defined in (1.1.7) and (1.1.16). The epsilon factor remains a monomial in q^{-s}. The **gamma factor** is again defined by

$$\gamma(s, \rho', \psi) = \frac{\epsilon(s, \rho', \psi) L(1-s, \rho'^\vee)}{L(s, \rho')}.$$

Notice that $\gamma(s, \rho', \psi) = \gamma(s, \rho, \psi)$. That is, the gamma factor depends only on the underlying Weil representation.

Example 1.2.7.1. *Let ρ be a representation of W_F. Consider the representation $\rho \otimes \mathrm{Sp}(m)$ of W'_F. The underlying space of this representation is the direct sum of m copies of V, as in (1.2.4). We see that the kernel of the nilpotent operator is the representation $(\rho || \cdot ||^{m-1}, V)$ of W_F, so*

$$L(s, \rho \otimes \mathrm{Sp}(m)) = L(s, \rho || \cdot ||^{m-1}) = L(s + m - 1, \rho).$$

1.2.8 Inductivity and additivity for Weil-Deligne representations

If ρ'_1, ρ'_2 are representations of W'_F, then

$$L(s, \rho'_1 \oplus \rho'_2) = L(s, \rho'_1) L(s, \rho'_2)$$

$$\epsilon(s, \rho'_1 \oplus \rho'_2), \psi) = \epsilon(s, \rho'_1, \psi) \epsilon(s, \rho'_2, \psi).$$

However, these factors are not additive over short exact sequences. On the other hand, the gamma factors are additive, i.e. if

$$0 \to \rho_1' \to \rho_2' \to \rho_3' \to 0$$

is an exact sequence of representations of W_F', then

$$\gamma(s, \rho_2', \psi) = \gamma(s, \rho_1', \psi)\gamma(s, \rho_3', \psi).$$

This is because the gamma factor only depending on the underlying representation of W_F (1.2.7), and the gamma factor of a Weil representation is additive. L-functions of Weil-Deligne representations are inductive, and gamma and epsilon factors of Weil-Deligne representations satisfy the same inductivity rule as Weil representations. If ρ' is a representation of W_E' for a finite extension E of F, then

$$L(s, \operatorname{Ind}_{E/F} \rho') = L(s, \rho')$$

$$\lambda(E/F, \psi)^{\dim \rho'} \epsilon(s, \rho', \psi \circ \operatorname{Tr}_{E/F}) = \epsilon(s, \operatorname{Ind}_{E/F}(\rho'), \psi)$$

$$\lambda(E/F, \psi)^{\dim \rho'} \gamma(s, \rho', \psi \circ \operatorname{Tr}_{E/F}) = \gamma(s, \operatorname{Ind}_{E/F}(\rho'), \psi)$$

where $\lambda(E/F, \psi)$ is the Langlands lambda function (1.1.6).

1.2.9 Weil-Deligne representations as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$

There is another way to think of Frobenius semisimple Weil-Deligne representations which is useful in considering tempered representations (see 1.2.10). If $\rho' = (\rho, V, N)$ is a Frobenius semisimple representation of W_F', there are unique elements $h, f \in \operatorname{End}(V)$ such that $\rho(w) f \rho(w)^{-1} = ||w||^{-1} f$ and $\rho(w) h \rho(w)^{-1} = h$ for all $w \in W_F$, and such that N, h, f form an $\mathfrak{sl}_2$-triple, which is to say:

$$[h, N] = 2N \quad [h, f] = -2f \quad [N, f] = h$$

where $[X,Y] = XY - YX$. We construct a representation τ of $W_F \times \mathrm{SL}_2(\mathbb{C})$ by defining

$$\tau(w) = \exp(\frac{-v(w)}{2}\log(q)h)\rho(w) \qquad (w \in W_F)$$

where $v(w) = -\log_q ||w||$, and defining $\tau|_{\mathrm{SL}_2(\mathbb{C})}$ to be the unique complex analytic representation of $\mathrm{SL}_2(\mathbb{C})$ whose tangent space map is given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto N$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f.$$

If τ is a representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ whose restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic, then we can recover $\rho' = (\rho, V, N)$ by

$$\rho(w) = \tau(w, \begin{pmatrix} ||w||^{\frac{1}{2}} & \\ & ||w||^{-\frac{1}{2}} \end{pmatrix}) \ , \quad N = d\tau \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Proposition 1.2.9.1. *$\rho' \mapsto \tau$ defines a bijection from Frobenius semisimple representations of W'_F to representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$ whose restriction to W_F is continuous, whose restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic, and for which the image of a geometric Frobenius is semisimple.*

Proof: This is essentially the Proposition of Section 6 of [Ro94]. □

1.2.10 Tempered and square integrable Weil-Deligne representations

Let $\rho' = \rho \otimes \mathrm{Sp}(m)$ be an indecomposable Frobenius semisimple representation of W'_F, for ρ an irreducible representation of W_F and $m \geq 1$. We will say that ρ' is **square integrable** if $\det(\rho||\cdot||^{\frac{m-1}{2}})$ is a unitary character of W_F.

If ρ' is an arbitrary Frobenius semisimple representation of W'_F, we will say that ρ' is **tempered** if all its indecomposable constituents are square integrable.

Lemma 1.2.10.1. *If ρ' is tempered, then so is its contragredient, and $L(s,\rho')$ has no poles for $Re(s) > 0$.*

Proof: We may assume that ρ' is square integrable, and write $\rho' = \rho \otimes \mathrm{Sp}(m)$ for an irreducible representation (ρ, V) of W_F, and an integer $m \geq 1$. If we write $\rho'^\vee$ in the form $\underline{\rho} \otimes \mathrm{Sp}(n)$ for some representation $\underline{\rho}$ of W_F, we see that $n = m$ and $\underline{\rho}$ must be equal to $\rho^\vee||\cdot||^{-(m-1)}$. Then

$$\underline{\rho}||\cdot||^{\frac{m-1}{2}} = \rho^\vee||\cdot||^{-\frac{m-1}{2}} = (\rho||\cdot||^{\frac{m-1}{2}})^\vee$$

which has unitary composition with the determinant since its contragredient $\rho||\cdot||^{-\frac{m-1}{2}}$ has unitary composition with the determinant by assumption. This shows that $\rho'^\vee$ is square integrable and hence tempered. As for the L-function, we have

$$L(s,\rho') = L(s, \rho||\cdot||^{m-1}) = L(s + \frac{m-1}{2}, \Sigma)$$

where $\Sigma = \rho||\cdot||^{\frac{m-1}{2}}$ is a representation of W_F such that $\det \circ \Sigma$ is unitary. If Σ has dimension greater than one, or if Σ has dimension one but is not trivial on the inertia group, the L-function is identically 1 and we are done. Otherwise, Σ is an unramified character of W_F, $\Sigma(\Phi)$ lies on the unit circle for Φ a geometric Frobenius, and

$$L(s,\Sigma)^{-1} = (1 - q^{-s}\Sigma(\Phi))$$

We see that $L(s,\Sigma)^{-1}$ cannot have a zero with $\mathrm{Re}(s) > 0$. Hence $L(s,\Sigma)$ has no pole with $\mathrm{Re}(s) > 0$. This implies $L(s,\rho') = L(s+\frac{m-1}{2},\Sigma)$ has no pole with $\mathrm{Re}(s) > -\frac{m-1}{2}$, and hence no pole with $\mathrm{Re}(s) > 0$. □

There is a criterion for a representation to be tempered using representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$.

Theorem 1.2.10.2. *Let ρ' be a Frobenius semisimple representation of W_F', and let τ be the corresponding representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$. The following are equivalent:*

(i): ρ' is tempered. (ii): The image of W_F under τ is bounded. (iii): The image of a geometric Frobenius under τ has all its eigenvalues on the unit circle.

Proof: This is proved in 5.2.2 of [Ku94]. □

1.2.11 Tensor induction preserves temperedness

Assume that E is a quadratic extension of F. Let z be an element of W_F which is not in W_E, so z is an element of W_F' which is not in W_E'.

Let $\rho' = (\rho, V, N)$ be a Frobenius semisimple representation of W_E'. Recall the definition of tensor induction (1.1.17). Then $\otimes \mathrm{Ind}_{E/F}\, \rho' := \otimes \mathrm{Ind}_{W_E'}^{W_F'}\, \rho'$ is a representation of W_F'. Its underlying Weil representation is $(\otimes \mathrm{Ind}_{E/F}\, \rho, V \otimes V)$, and the nilpotent operator T is easily seen to be $N \otimes 1 + 1 \otimes ||z||N$, where z is the chosen element of W_F, not in W_E, which defines $\otimes \mathrm{Ind}_{E/F}\, \rho$.

Proposition 1.2.11.1. *The tensor induced representation $\otimes \mathrm{Ind}_{E/F}\, \rho'$ is Frobenius semisimple. If ρ' is tempered, then so is $\otimes \mathrm{Ind}_{E/F}\, \rho'$.*

Proof: Recall that for a Weil-Deligne representation to be Frobenius semisimple, it is necessary and sufficient that a nonzero power of a geometric Frobenius element define a semisimple linear operator (1.1.15). If Φ is a geometric Frobenius of W_F, then Φ^2 is a power of the geometric Frobenius of W_E, and it suffices to show that Φ^2 defines a semisimple linear operator of $V \otimes V$. The representation $\otimes \mathrm{Ind}_{E/F}\, \rho'$ of W_F applied to Φ^2 is

$$\rho(\Phi^2) \otimes \rho(z\Phi^2 z^{-1})$$

which is semisimple because ρ and hence $\rho(\Phi^2)$ and $\rho(z\Phi^2z^{-1})$ are semisimple.

Suppose that ρ' is tempered. There exist $h, f \in \mathrm{End}(V)$ as in (1.2.9) such that N, h, f form an $\mathfrak{sl}_2$-triple, where N is the nilpotent operator of ρ'. Let Ψ be a geometric Frobenius of W_E. By the Theorem of 1.2.10, we know that all the eigenvalues of

$$\exp(-\frac{1}{2}\log(q_E)h)\rho(\Psi)$$

lie on the unit circle.

The nilpotent operator of $\otimes \mathrm{Ind}_{E/F}\,\rho'$ is $T = N \otimes 1 + 1 \otimes ||z||N$, and it is straightforward to check that if we define

$$H = h \otimes 1 + 1 \otimes h$$

$$F = f \otimes 1 + 1 \otimes ||z||^{-1}h \in \mathrm{End}(V \otimes V)$$

then T, H, F form an $\mathfrak{sl}_2$ triple in $\mathrm{End}(V \otimes V)$ and satisfy the required relations of (1.2.9). Then if Φ is any geometric Frobenius of W_F, we are done by the Theorem of 1.2.10 if we can show that all the eigenvalues of the operator

$$\exp(-\frac{1}{2}\log(q_F)H)\Phi$$

on $V \otimes V$ lie on the unit circle. Since H and Φ commute with each other, it will actually suffice to show that the eigenvalues of

$$\exp(-\frac{1}{2}\log(q_F)H)^l\Phi^l$$

lie on the unit circle, where l is a positive integer. If E/F is ramified, then $q_E = q_F$ and we can take $\Phi = \Psi$ and $l = 1$, so that

$$\exp(-\frac{1}{2}\log(q_F)H)^l\Phi^l = \exp(-\frac{1}{2}\log(q_E)h)\Psi \otimes \exp(-\frac{1}{2}\log(q_E)h)\Psi$$

which is unitary. If E/F is unramified, then $q_E = q_F^2$, $\Psi = \Phi^2$ is a geometric Frobenius for W_E, and we take $l = 4$:

$$\begin{aligned}
\exp(-\frac{1}{2}\log(q_F)H)^l\Phi^l &= \exp(-\frac{1}{2}\log(q_E)H)^2\Psi^2 \\
&= (\exp(-\frac{1}{2}\log(q_E)H)\Psi)^2 \\
&= \left(\exp(-\frac{1}{2}\log(q_E)h)\Psi \otimes \exp(-\frac{1}{2}\log(q_E)h)\Psi\right)^2
\end{aligned}$$

which is unitary. □

1.3 The Weil group of an archimedean local field

1.3.1 Definition of the Weil group

Suppose that k is an archimedean local field, so that $\overline{k} \cong \mathbb{C}$. If k is real, we take the local Weil group W_k to be the topological group $\overline{k}^* \cup j\overline{k}^*$, where $j^2 = -1$, and $jxj^{-1} = \sigma(x)$, where σ is the nontrivial element of $\mathrm{Gal}(\overline{k}/k)$. We have a continuous surjective homomorphism $W_k \to \mathrm{Gal}(\overline{k}/k)$ sending j to σ, and everything else to $1_{\overline{k}}$.

If k is complex, then we take W_k to be $\overline{k}^*$. The Galois group $\mathrm{Gal}(\overline{k}/k)$ is trivial, and we have a continuous surjective homomorphism $W_k \to \mathrm{Gal}(\overline{k}/k)$ defined in the only possible way, taking everything to $1_{\overline{k}}$.

This defines the Weil group of an archimedean local field k. There is no analogue of a Weil-Deligne group in the archimedean case. For uniformity of notation, we will define the Weil-Deligne group W_k' of an archimedean local field to just be the Weil group W_k.

1.3.2 Archimedean and nonarchimedean Weil groups

If k is any local field of characteristic zero, and $\overline{k}$ is an algebraic closure of k, then we have defined in all cases (k nonarchimedean, real, complex) a local Weil group W_k together with a continuous homomorphism $W_k \to \mathrm{Gal}(\overline{k}/k)$ with dense image.

If K is a finite extension of k (if k is archimedean, then either $K = k$ or $K = \overline{k}$), and we can define a Weil group of W_K by taking the preimage of $\mathrm{Gal}(\overline{k}/K)$ under $W_k \to \mathrm{Gal}(\overline{k}/K)$.

1.3.3 Local factors for representations of the archimedean Weil group

Suppose again that k is archimedean. The (continuous, finite dimensional, complex) irreducible representations of W_k are completely classified. When k is complex, every irreducible representation of $W_k \cong \mathbb{C}^*$ is just a character of $\mathbb{C}^*$. When k is real, the irreducible representations of W_k are either one or two dimensional.

The irreducible representations of W_k being completely classified, one then obtains a classification of all semisimple representations of W_k.

We refer to Knapp ([Kn94]) for the definition of the L-function $L(s, \rho)$ and epsilon factor $\epsilon(s, \rho, \psi)$ associated to a semisimple representation ρ and nontrivial additive character ψ of k. These factors are additive over short exact sequences of semisimple representations and satisfy inductivity formulas: if K is a finite extension of k, ρ is a representation of W_K, and ψ is a nontrivial unitary character of k, then

$$L(s, \rho) = L(s, \mathrm{Ind}_{K/k}\, \rho)$$

$$\epsilon(s, \rho, \psi \circ \mathrm{Tr}_{K/k})\lambda(K/k, \psi)\epsilon(s, \mathrm{Ind}_{K/k}\, \rho, \psi)$$

where $\lambda(K/k, \psi)$ is the Langlands lambda function for real groups. Just as in the nonarchimedean case, the local gamma factor is defined by

$$\gamma(s, \rho, \psi) = \frac{\epsilon(s, \rho, \psi)L(1 - s, \rho^\vee)}{L(s, \rho)}. \qquad (1.3.3.1)$$

1.4 The Weil group of a number field

Our main reference for this section is [Ta79].

1.4.1 Essential properties of the global Weil group

Let k be a number field. The construction of the global Weil group W_k is more complicated than the construction of a local Weil group, and is not important for our purposes. We refer to [Ta79] for a proof of this construction.

The essential property of the global Weil group is that it is a Hausdorff topological group W_k, together with a continuous homomorphism $\varphi : W_k \to \mathrm{Gal}(\overline{k}/k)$ with dense image, and an isomorphism $r_k : \mathbb{A}_k^*/k^* \to W_k^{\mathrm{ab}}$, where $\mathbb{A}_k$ is the ring of adeles of k, and W_k^{ab} is the abelianization of W_k. For each finite extension K of k, $W_K = \varphi^{-1}\,\mathrm{Gal}(\overline{k}/K)$ is a Weil group of K, which is equipped with an isomorphism $r_K : \mathbb{A}_K^*/K^* \to W_K^{\mathrm{ab}}$ which is compatible with r_k via the "transfer homomorphism" (see the first section of [Ta79]).

1.4.2 Connection between the global Weil group W_k and the local Weil groups W_{k_v}

For each place v of k, choose an algebraic closure $\overline{k_v}$ of k_v and an embedding $i_v : \overline{k} \to \overline{k_v}$ of algebraic closures. Then i_v induces an injection $\mathrm{Gal}(\overline{k_v}/k_v) \to \mathrm{Gal}(\overline{k}/k)$ defined by $\tau \mapsto i_v^{-1} \circ \tau \circ i_v$.

Define the Weil group W_{k_v} as in either (1.3.1) or (1.1.3), depending on whether v is archimedean or not, together with the map $W_{k_v} \to \mathrm{Gal}(\overline{k_v}/k_v)$.

There exists a continuous homomorphism $\theta_v : W_{k_v} \to W_k$, unique up to inner isomorphism by an element of $\mathrm{Ker}\,\varphi$, such that the diagram

$$\begin{tikzcd} W_k \arrow[r, "\varphi"] & \mathrm{Gal}(\overline{k}/k) \\ W_{k_v} \arrow[u, "\theta_v"] \arrow[r] & \mathrm{Gal}(\overline{k_v}/k_v) \arrow[u] \end{tikzcd}$$

is commutative ([Ta79], Proposition 1.6.1).

1.4.3 L and epsilon factors for the global Weil group

For each place v of k, let θ_v be as in (1.4.2). If $\Sigma : W_k \to \mathrm{GL}(V)$ is a representation of W_k (assumed as usual to be continuous, finite dimensional, and complex), let $\Sigma_v = \Sigma \circ \theta_v$, the "restriction of Σ to W_{k_v}." Define the global L-function and epsilon factor

$$L(s, \Sigma) = \prod_v L(s, \Sigma_v)$$

$$\epsilon(s, \Sigma) = \prod_v \epsilon(s, \Sigma_v, \Psi_v)$$

where $\Psi = \otimes \Psi_v$ is any nontrivial character of $\mathbb{A}_k/k$. For a given character Ψ, the local epsilon factor $\epsilon(s, \Sigma_v, \Psi_v)$ will be equal to 1 at almost all places v, and their product $\epsilon(s, \Sigma)$ will not depend on the choice of Ψ.

The infinite product defining the global L-function will converge to an analytic function of s in some right half plane. Moreover, $L(s, \Sigma)$ admits a meromorphic continuation to the entire complex plane satisfying the functional equation

$$L(s, \Sigma) = \epsilon(s, \Sigma) L(1 - s, \Sigma^\vee)$$

where $\Sigma^\vee$ is the contragredient of Σ ([Ta79], Theorem (3.5.3)). Note that $L(s, \Sigma^\vee) = \prod_v L(s, \Sigma_v^\vee)$.

1.4.4 Global Weil groups in the relative setting

Suppose that K is a quadratic extension of k. Then $W_K = \varphi^{-1} \mathrm{Gal}(\overline{k}/K)$ is a Weil group of K, W_K is a normal subgroup of W_k, and we can identify the quotient W_k/W_K with the Galois group $\mathrm{Gal}(K/k)$. For the rest of this section, let us choose once and for all an element Z in W_k which is not in W_K, and let $\tilde{\sigma} = \varphi(Z)$.

For each place v of k, the embedding $i_v : \overline{k} \to \overline{k_v}$ determines a place w of K which lies over v: the completion K_w is the composite field $i_v(K)k_v$, in which i_v embeds K as a dense subfield.

If k_v is properly contained in K_w, then w is the only place of K lying over v. If not, then there is another place w' which also lies over v, which we now describe: just as $K_w = k_v$, together with the embedding $i_v : K \to K_w$, is the completion of K at the place w, we also have that $K_{w'} = k_v$, together with the embedding $i_v \circ \tilde{\sigma}^{-1} : K \to K_{w'}$, is the completion of K at the place w'.

1.4.5 Connection between the global Weil group W_K and the local Weil groups W_{K_w}

For each place v of k, we have a place w of K lying over v, determined by the embedding $i_v : \overline{k_v} \to \overline{k}$. Given our Weil group $W_{k_v} \to \mathrm{Gal}(\overline{k_v}/k_v)$ (1.3.2), we can define the Weil group W_{K_w} of K_w as usual to be the preimage of $\mathrm{Gal}(\overline{k_v}/K_w)$ under this last map, so that $[K_w : k_v] = [W_{k_v} : W_{K_w}]$.

1.4.6 Splitting of places in K/k

We had fixed a continuous homomorphism $\theta_v : W_{k_v} \to W_k$, defined in (1.4.2), satisfying a compatability property and unique up to inner isomorphism by $\operatorname{Ker}\theta$. Now for the place w lying over v, determined by the embedding $i_v : \overline{k} \to \overline{k_v}$ (1.4.4), we do the same for W_{K_w} and W_K. We do this simply by taking $\theta_w : W_{K_w} \to W_K$ to be the restriction of θ_v to W_{K_v}.

Suppose that w is not the only place of K which lies over v. Let w' be the other one. Then the map $i_{w'} = i_v \circ \tilde{\sigma}^{-1} : \overline{k} \to \overline{k_v}$ gives an embedding of algebraic closures of $K, K_{w'}$ respectively, through which we obtain a homomorphism $\mathrm{Gal}(\overline{k_v}/K_{w'}) \to \mathrm{Gal}(\overline{k}/K)$. Then we may take $\theta_{w'} : W_{K_{w'}} \to W_K$ to be the homomorphism given by $\theta_{w'}(x) = Z\theta_w(x)Z^{-1}$. We see immediately that the diagram

$$\begin{tikzcd} W_K \arrow[r, "\varphi"] & \mathrm{Gal}(\overline{k}/K) \\ W_{K_{w'}} \arrow[u, "\theta_{w'}"] \arrow[r] & \mathrm{Gal}(\overline{k_v}/K_{w'}) \arrow[u] \end{tikzcd}$$

is commutative. Of course, under our identifications, $K_w = K_{w_1} = k_v$, and $W_{K_w} = W_{K_{w_1}} = W_{k_v}$.

The point of doing all of this is that in terms of the fixed homomorphisms θ_v : v a place of k, and the choice of $Z \in W_k - W_K$, we now have for *every* place of K a homomorphism $\theta_w : W_{K_w} \to W_K$ as in (1.4.2).

1.4.7 Tensor induction when K/k does not split at v

Now suppose that (Σ, V) is a representation of the global Weil group W_K. We will consider the tensor induced representation $(\otimes \operatorname{Ind}_{K/k} \Sigma, V \otimes V)$ of the global Weil group W_k (1.1.7) defined by our choice of $Z \in W_k$ (1.4.4).We recall that $\otimes \operatorname{Ind}_{K/k} \Sigma$ is defined by

$$w.(v \otimes v') = \begin{cases} \Sigma(w)v \otimes \Sigma(ZwZ^{-1})v' & \text{if } w \in W_K \\ \Sigma(wZ^{-1})v' \otimes \Sigma(Zw)v & \text{if } w \notin W_K. \end{cases}$$

We will consider the restriction $(\otimes \operatorname{Ind}_{K/k} \Sigma)_v$ of this representation of W_k to various W_{k_v}.

Let v be a place of k, and let w be the place of K lying over v as in (1.4.4). Suppose first that $k_v = K_w$, so that there is another place w' of K which lies over k. Then the image of θ_v is contained in W_K, and the representation $(\otimes \operatorname{Ind}_{K/k} \Sigma)_v$ of W_{k_v} is given by

$$a.(v \otimes v') = \Sigma(\theta_v(a))v \otimes \Sigma(Z\theta_v(a)Z^{-1})v'$$

for all $a \in W_{k_v}$. But recall that under our identifications (1.4.6), $W_{k_v} = W_{K_w} = W_{K_{w'}}$, $\theta_v = \theta_w$, and $\iota_Z \circ \theta_v = \theta_{w'}$, where ι_Z denotes conjugation by Z. Therefore, we have

$$(\otimes \operatorname{Ind}_{K/k} \Sigma)_v = \Sigma_w \otimes \Sigma_{w'}$$

whenever there are two distinct places w and w' lying over v.

1.4.8 Tensor induction when K/k splits at v

We remain in the setting of (1.4.7), but now suppose that w is the only place of K lying over v, so that $[K_w : k_v] = 2$. If $a \in W_{k_v}$, then $\theta_v(a)$ lies in W_K if and only if a is in W_{K_w}. Consequently, $(\otimes \operatorname{Ind}_{K/k} \Sigma)_v$ is given for $a \in W_{k_v}$ by

$$a.(v \otimes v') = \begin{cases} \Sigma(\theta_w(a))v \otimes \Sigma(Z\theta_w(a)Z^{-1})v' & \text{if } a \in W_{K_w} \\ \Sigma(\theta_v(a)Z^{-1})v' \otimes \Sigma(Z\theta_v(a))v & \text{if } a \notin W_{K_w}. \end{cases}$$

Let z be any element of W_{k_v} which is not in W_{K_w}. Then $\theta_v(z)$ is an element of W_k which is not in W_K. Consequently, $\theta_v(z)Z^{-1}$ lies in W_K, and the map $v \otimes v' \mapsto v \otimes \Sigma(\theta_v(z)Z^{-1})v'$ defines an isomorphism

$$(\otimes \operatorname{Ind}_{K/k} \Sigma)_v \xrightarrow{\cong} \otimes \operatorname{Ind}_{K_w/k_v} \Sigma_w.$$

2. THE ANALYTIC THEORY

In the previous chapter we summarized what we needed from the Galois side. In this chapter, we do the same on the analytic side. In Section 1, we define the Langlands dual group of a reductive group, and explain the conjectural objects associated to its representations. In Section 2, we give a summary of what we will need from the Langlands-Shahidi method. In Section 3, we give an exposition of the Bernstein-Zelevinsky (BZ) classification of smooth, irreducible representations of $\mathrm{GL}_n(k)$, where k is a p-adic field. We also summarize the local Langlands correspondence for GL_n and explain how arbitrary Frobenius semisimple representations of the Weil-Deligne group are built out of irreducible representations in the same way that arbitrary smooth, irreducible representations of GL_n are built out of supercuspidals from the BZ-classification.

In Section 4, we apply the Langlands-Shahidi method in the setting of Weil restriction of scalars. As far as we know, the content of Section 4 has not been written down anywhere in published form, although it is surely known to the experts. Section 5 is an application of the result of Section 4, where we show that certain local factors showing up in the multiplicativity formula for Asai representations are really Rankin products.

Finally in Section 6 we define the Asai representation and state our main result, the equality of the local Asai epsilon factor as defined by the Langlands-Shahidi method, with the expected epsilon factor on the Galois side via the local Langlands correspondence. We prove the multiplicativity formula for the Asai gamma factors.

We remark that the equality of the local Asai L-function with the expected L-function on the Galois side is already a theorem due to Henniart. In fact, Henniart had shown that the Asai gamma factors were equal up to a root of unity. In Section 7 we explain Henniart's argument. Our approach to proving our main theorem will be to show the exact equality of the gamma factors. On account of Henniart's result of the equality of L-factors, we will get the equality of epsilon factors as a consequence.

2.1 The Langlands dual group

The Langlands dual group, or L-group, is a complex Lie group associated to a connected, reductive group. It is used in defining L-functions, epsilon factors, and gamma factors on the analytic side. In this section, we explain how the L-group is defined and state some of its associated properties and conjectures. Our primary reference for this section is [Bo79].

2.1.1 Based root data

Let $\mathbf{G}$ be a connected, reductive group, defined and quasi split over a field k. Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$, $\mathbf{T} = Z_{\mathbf{G}}(\mathbf{S})$ a maximal torus of $\mathbf{G}$ which is defined over k, and $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ which is defined over k and contains $\mathbf{T}$.

Let $X(\mathbf{T})$ be the group of rational characters of $\mathbf{T}$, and $X^\vee(\mathbf{T})$ the group of rational cocharacters of $\mathbf{T}$. Let $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ be the set of roots of $\mathbf{T}$ in $\mathbf{G}$, and $\Phi^\vee = \Phi^\vee(\mathbf{G}, \mathbf{T})$ the set of coroots of $\mathbf{T}$ in $\mathbf{G}$. Then the quadruple $(X(\mathbf{T}), \Phi, X^\vee(\mathbf{T}), \Phi^\vee)$ is a root datum. The group $\mathbf{G}$ is determined up to $\overline{k}$-isomorphism by its root datum.

The choice of the Borel subgroup $\mathbf{B}$ determines a set Δ of simple roots for Φ and a set $\Delta^\vee$ of simple coroots of $\Phi^\vee$. Then the sextuple $R = (X(\mathbf{T}), \Phi, \Delta, X^\vee(\mathbf{T}), \Phi^\vee, \Delta^\vee)$ is a based root datum.

2.1.2 Root vectors and splitting

For each root $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the corresponding root subgroup. A **root vector** for α is an isomorphism $\mathbf{x}_\alpha : \mathbb{G}_a \to \mathbf{U}_\alpha$ of algebraic groups over $\overline{k}$.

A **splitting** is a collection $\mathbf{x}_\alpha : \alpha \in \Delta$ of simple root vectors. The group $\mathrm{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T})$ of $\overline{k}$-automorphisms of $\mathbf{G}$ which stabilize $\mathbf{B}$ and $\mathbf{T}$ acts on the set of splittings. Given a splitting $\mathbf{x}_\alpha : \alpha \in \Delta$, the group $\mathrm{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T}, \mathbf{x}_\alpha : \alpha \in \Delta)$ of automorphisms in $\mathrm{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T})$ which fix the splitting is isomorphic to the group of automorphisms of the based root datum R.

Furthermore, $\mathrm{Aut}(\mathbf{G})$ is the semidirect product of the group of inner automorphisms of $\mathbf{G}$ and $\mathrm{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T}, x_\alpha : \alpha \in \Delta)$ ([Sp79], 2.14).

2.1.3 Definition of the L-group

Let Γ be the Galois group $\mathrm{Gal}(k_s/k)$, where k_s is a separable closure of k. For each finite Galois extension K of k, let $\Gamma_K = \mathrm{Gal}(K/k)$. Then Γ acts as a group of automorphisms on $X(\mathbf{T})$ and $X^\vee(\mathbf{T})$, and in fact acts as a group of automorphisms on the based root datum R and its dual $R^\vee = (X^\vee(\mathbf{T}), \Phi^\vee, \Delta^\vee, X(\mathbf{T}), \Phi, \Delta)$.

Let $\mathbf{G}^\vee$ be a connected, reductive group over $\mathbb{C}$, with Borel subgroup $\mathbf{B}^\vee$ containing a maximal torus $\mathbf{T}^\vee$, whose based root datum is isomorphic to $R^\vee$. Let $\mathbf{x}_{\alpha^\vee} : \alpha^\vee \in \Delta^\vee$ be a splitting for this based root datum. The choice of this splitting gives an injection of $\mathrm{Aut}\, R^\vee$ into $\mathrm{Aut}(\mathbf{G}^\vee, \mathbf{B}^\vee, \mathbf{T}^\vee)$, as explained in the previous section. In this way, the Galois group Γ acts as a group of automorphisms of $\mathbf{G}^\vee$ which stabilize $\mathbf{B}^\vee$ and $\mathbf{T}^\vee$.

The **Langlands dual group**, or **L-group**, ${}^L\mathbf{G}$ is defined to be the semidirect product of $\mathbf{G}$ by Γ according to this action. It is a complex Lie group with connected component ${}^L\mathbf{G}^\circ = \mathbf{G}^\vee$. Up to Γ-isomorphism, it does not depend on any of the choices we have made for $\mathbf{B}, \mathbf{T}, \mathbf{G}^\vee, \mathbf{B}^\vee, \mathbf{T}^\vee$ or the splitting for $(\mathbf{G}^\vee, \mathbf{B}^\vee, \mathbf{T}^\vee)$.

If $\mathbf{T}$ splits over a Galois extension K of k, then $\mathrm{Gal}(k_s/K)$ acts trivially on $X(\mathbf{T})$ and $X^\vee(\mathbf{T})$, hence trivially on ${}^L\mathbf{G}^\circ$. Then the L-group of $\mathbf{G}$ is often identified with the semidirect product ${}^L\mathbf{G}^\circ \rtimes \mathrm{Gal}(K/k)$.

2.1.4 Representations of the L-group

By a **representation** of ${}^L\mathbf{G}$, we will mean a continuous, finite dimensional complex representation of ${}^L\mathbf{G}$ whose restriction to the complex Lie group ${}^L\mathbf{G}^\circ$ is holomorphic.

Suppose that $\mathbf{G}$ is defined over a number field k. For each place v of k, and algebraic closure $\overline{k_v}$ of k_v, the choice of k-embedding $\overline{k} \to \overline{k_v}$ defines an injection $\mathrm{Gal}(\overline{k_v}/k_v) \to \mathrm{Gal}(\overline{k}/k)$.

Let $\mathbf{G}_v = \mathbf{G} \times_k k_v$. Since a based root datum for $\mathbf{G}$ is also one for $\mathbf{G}_v$, we can arrange that ${}^L\mathbf{G}^\circ = {}^L\mathbf{G}_v^\circ$, so that the inclusion of $\mathrm{Gal}(\overline{k_v}/k_v)$ into $\mathrm{Gal}(\overline{k}/k)$ defines an inclusion ${}^L\mathbf{G}_v$ into ${}^L\mathbf{G}$. If r is a representation of ${}^L\mathbf{G}$, we let r_v be the restriction of r to ${}^L\mathbf{G}_v$.

2.1.5 Identification of L-groups

Let $\varphi : \mathbf{G} \to \mathbf{G}'$ be an isomorphism of algebraic groups which is defined over k. If we identify $(\mathbf{G}', \varphi\mathbf{B}, \varphi\mathbf{T})$ with $(\mathbf{G}, \mathbf{B}, \mathbf{T})$, we can define the L-group of $\mathbf{G}'$ together with a corresponding isomorphism $\varphi^\vee : {}^L\mathbf{G}' \to {}^L\mathbf{G}$.

2.1.6 Conjectural local factors

Assume that k is a local field of characteristic zero. To each representation r of ${}^L\mathbf{G}$, each irreducible, admissible representation π of $\mathbf{G}(k)$, and each nontrivial unitary character ψ of k, there is a conjectural **local analytic L-function** $L(s, \pi, r)$ and a conjectural **local analytic epsilon factor** $\epsilon(s, \pi, r, \psi)$. There is also the **gamma factor**

$$\gamma(s, \pi, r, \psi) = \frac{L(1 - s, \pi^\vee, r)\epsilon(s, \pi, r, \psi)}{L(s, \pi, r)}. \tag{2.1.6.1}$$

These are defined in many special cases, in particular by the Langlands-Shahidi method. We give a summary of the method in the next section. There are two important representations of L-groups which we will define. They are the standard representation, and the tensor product. First, if $\mathbf{G} = \mathrm{GL}_n$, then the L-group of $\mathbf{G}$ can be identified with $\mathrm{GL}(V)$, where V is an n-dimensional complex vector space. The identity map r on $\mathrm{GL}(V)$ is called the **standard representation**. The **standard L-function** $L(s, \pi, r)$ is written as just $L(s, \pi)$ with the r omitted, and the same goes for the standard epsilon and gamma factor.

Next, suppose $\mathbf{G} = \mathrm{GL}_n \times \mathrm{GL}_m$. Then the L-group of $\mathbf{G}$ can be identified with $\mathrm{GL}(V) \times \mathrm{GL}(W)$, where V and W are complex vector spaces of dimensions n and m. Let r be the tensor product representation ${}^L\mathbf{G} \to \mathrm{GL}(V \otimes W)$ given by $(T, S) \mapsto T \otimes S$. An irreducible, admissible representation π of $\mathbf{G}(k)$ factors as a tensor product $\pi_1 \boxtimes \pi_2$, where π_1 and π_2 are irreducible admissible representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$, respectively. Their isomorphism classes are determined uniquely by that of π. The **Rankin product L-function** $L(s, \pi_1 \boxtimes \pi_2, r)$ is written as $L(s, \pi_1 \times \pi_2)$, and the epsilon and gamma factors are similarly written, e.g. $\epsilon(s, \pi_1 \times \pi_2, \psi)$.

2.1.7 Conjectural local Langlands correspondence

Assume that k is a local field of characteristic zero, and let W_k' be the Weil-Deligne group of k. Recall that for k archimedean, W_k' is defined to just be the Weil group. In ([Bo79], 8.1), Borel gives a definition of an *admissible homomorphism* $\varphi : W_k' \to {}^L\mathbf{G}$. Two admissible homomorphisms are equivalent if they differ by an inner automorphism of ${}^L\mathbf{G}^\circ$.

The conjectural local Langlands correspondence (as stated by Borel in [Bo79], Chap. III) hopes to associate, to a general reductive group $\mathbf{G}$, the following:

- A partition of the set of isomorphism classes of irreducible, admissible representations of $\mathbf{G}(k)$ into finite sets, called L-packets

- A bijection from the set of equivalence classes of admissible homomorphisms $\rho' : W_k' \to {}^L\mathbf{G}$ to the set of L-packets such that if π is an element of the L-packet corresponding to ρ', then

$$L(s, r \circ \rho') = L(s, \pi, r) \tag{2.1.7.1}$$

$$\epsilon(s, r \circ \rho', \psi) = \epsilon(s, \pi, r, \psi) \tag{2.1.7.2}$$

whenever the right hand sides can be defined.

2.1.8 Some special cases of the local Langlands correspondence

Let k be a local field of characteristic zero. For k archimedean, the conjectural partition and bijection for the local Langlands correspondence for $\mathbf{G}$ have been established in terms of Langlands classification ([La89], see also [Kn94]). One then typically *defines* $L(s, \pi, r)$ and $\epsilon(s, \pi, r, \psi)$ to be the left hand sides of equations (2.1.7.1) and (2.1.7.2), effectively establishing the archimedean local Langlands correspondence for general reductive groups.

For k nonarchimedean, and $\mathbf{G} = \mathrm{GL}_n$, the conjectural partition and bijection have been established. The L-packets are singleton sets, and the bijection is the celebrated "Local Langlands correspondence for GL_n" proved independently in by Henniart [He00] in 2000, and Harris and Taylor [HaTa01] in 2001. A new proof was given by Scholze [Sch10] in 2010. The function field version was proved in 1993 by Laumon, Rapoport, and Stuhler [LaRaSt93].

For $\mathbf{G} = \mathrm{GL}_n$, the L-group ${}^L\mathbf{G}$ can be identified with $\mathrm{GL}_n(\mathbb{C})$, and the L and epsilon factors are shown to agree for $r = 1_{\mathrm{GL}_n(\mathbb{C})}$. Establishing the equality of L and epsilon factors for general r, when the local analytic L and epsilon factors are defined, is an ongoing task, although the equality has been established in many special cases.

For example, it has been established for $r = \mathrm{Sym}^2$ and $r = \Lambda^2$ in by Cogdell, Shahidi, and Tsai [CoShTs17] and G. Henniart [He10]. Henniart proved the equality of L-functions for these r; he did this by proving first that the gamma factors were equal up to a root of unity, and then deducing the equality of L-functions as a consequence. Later, Cogdell, Shahidi, and Tsai proved a crucial "analytic stability" result which, combined with some global arguments, allowed them to get an exact equality of the gamma factors.

When $\mathbf{G}$ is a torus, the conjectural partition and bijection have been established by Langlands ([La97], [Yu09]). This is the "local Langlands correspondence for tori." In this case, the L-packets are again singleton sets.

2.1.9 The L-group and restriction of scalars

Suppose that E/F is a finite extension of local fields of characteristic zero. Let $\mathbf{G}$ be a connected, reductive group over E, and let $\underline{\mathbf{G}} = \mathrm{Res}_{E/F}\,\mathbf{G}$, the Weil restriction of scalars of E/F (see (2.4.1)). Then we can identify $\mathbf{G}(E) = \underline{\mathbf{G}}(F)$. As Borel explains in 8.1 of [Bo79], there is a canonical bijection from the set of equivalence classes of admissible homomorphisms of W'_E into ${}^L\mathbf{G}$, and the set of equivalence classes of admissible homomorphisms of W'_F into ${}^L\underline{\mathbf{G}}$. So if a conjectural local Langlands correspondence is established for $\mathbf{G}$, then we have also a conjectural local Langlands correspondence for ${}^L\underline{\mathbf{G}}$. This bijection is compatible with the local Langlands correspondence for tori and for archimedean groups.

2.2 Langlands-Shahidi method

Let $\mathbf{M}$ be a connected, reductive group, quasi split over a characteristic zero local field k. Let ${}^L\mathbf{M}$ be the Langlands dual group of $\mathbf{M}$ (2.1.3). In this section, we give a summary of the Langlands-Shahidi method which defines the L-function $L(s, \pi, r)$ and epsilon factor $\epsilon(s, \pi, r, \psi)$ for a nontrivial unitary character ψ of k, certain irreducible, admissible representations π of $\mathbf{M}(k)$, and certain representations r of ${}^L\mathbf{M}$. Our primary references for this section are [Sh81], [Sh90], and [Sh10].

2.2.1 The setting

Let $\mathbf{G}$ be a connected, reductive group which is quasi split over the local field k. Let $\mathbf{B}, \mathbf{T}, \mathbf{S}$ be as in (2.1.1). Assume that $\mathbf{M}$ is a standard Levi subgroup of a parabolic k-subgroup $\mathbf{P}$ of $\mathbf{G}$ which contains $\mathbf{B}$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Then $\mathbf{B_M} = \mathbf{B} \cap \mathbf{M}$ is a Borel subgroup of $\mathbf{M}$ with unipotent radical $\mathbf{U_M} = \mathbf{U} \cap \mathbf{M}$.

The choice of $\mathbf{B}$ defines a set of simple absolute roots $\Delta \subset \Phi = \Phi(\mathbf{G}, \mathbf{T})$ and a set $\Delta_k \subset \Phi_k = \Phi(\mathbf{G}, \mathbf{S})$ of simple relative roots. The parabolic subgroup $\mathbf{P}$ is parameterized by a subset θ of Δ, as well as by a subset θ_k of Δ_k.

2.2.2 LS-representations I

The L-group ${}^L\mathbf{M}$ of $\mathbf{M}$ can be chosen in a natural way as a subgroup of the L-group ${}^L\mathbf{G}$, since ${}^L\mathbf{M}^\circ$ is isomorphic to the standard Levi subgroup of $({}^L\mathbf{G}^\circ, {}^L\mathbf{B}^\circ, {}^L\mathbf{T}^\circ)$ which corresponds to the subset $\theta^\vee$ of $\Delta^\vee$.

If ${}^L\mathbf{P}^\circ$ is the corresponding standard parabolic subgroup of ${}^L\mathbf{G}^\circ$, let ${}^L\mathbf{N}^\circ$ be its unipotent radical, and let ${}^L\mathfrak{n}$ be the Lie algebra of ${}^L\mathbf{N}^\circ$. Then ${}^L\mathbf{M}$ acts on ${}^L\mathfrak{n}$ by the adjoint representation

$$\mathrm{Ad} : {}^L\mathbf{M} \to \mathrm{GL}({}^L\mathfrak{n}).$$

The Langlands-Shahidi method defines gamma factors, and consequently L and epsilon factors for those representations r which are isomorphic to irreducible constituents of an adjoint representation as above. We shall call such a representation of ${}^L\mathbf{M}$ an **LS-representation**. The factors will depend only on the isomorphism class of r, not on the choice of $\mathbf{G}$ or the way in which $\mathbf{M}$ sits inside $\mathbf{G}$ as a Levi subgroup.

More generally, suppose that $\mathbf{H}$ is a connected, reductive quasi split group over k, r is an irreducible representation of ${}^L\mathbf{H}$, and suppose there exists a Levi subgroup $\mathbf{M}$ of a group $\mathbf{G}$ as above together with an isomorphism $\varphi : \mathbf{H} \to \mathbf{M}$ of algebraic groups over k. Then φ induces an isomorphism of L-groups $\varphi^\vee : {}^L\mathbf{M} \to {}^L\mathbf{H}$. If

$r \circ \varphi^\vee$ is an LS-representation of $\mathbf{M}$ in the above sense, then we will also call r an LS-representation.

If π is an irreducible, admissible representation of $\mathbf{H}(k)$, then $\pi \circ \varphi^{-1}$ is one of $\mathbf{M}(k)$. Suppose that r is an LS-representation of ${}^L\mathbf{H}$, and the gamma factor, L-function, and epsilon factor are defined for $(\pi \circ \varphi^{-1}, r \circ \varphi^\vee)$. Then we can define the corresponding factors for $\mathbf{H}$ by

$$\gamma(s, \pi, r, \psi) = \gamma(s, \pi \circ \varphi^{-1}, r \circ \varphi^\vee, \psi)$$

$$L(s, \pi, r) = L(s, \pi \circ \varphi^{-1}, r \circ \varphi^\vee)$$

$$\epsilon(s, \pi, r, \psi) = \epsilon(s, \pi \circ \varphi^{-1}, r \circ \varphi^\vee, \psi).$$

These definitions are independent of the choice of isomorphism φ as well as the way in which $\mathbf{M}$ sits inside $\mathbf{G}$ as a Levi subgroup.

2.2.3 LS-representations II

The Langlands-Shahidi method actually defines gamma factors for certain representations r which are not necessarily irreducible. If $r_1, ..., r_t$ are the irreducible constituents of such a representation r, then the gamma factor for r will be the product of the corresponding gamma factors for the r_i. For example, suppose that k is a number field, $\mathbf{M}$ is a maximal Levi, and the adjoint representation of ${}^L\mathbf{M}$ on ${}^L\mathbf{N}$ is irreducible. Then r_v, the restriction of r to ${}^L\mathbf{M}_v$ (see 2.1.4), is not necessarily irreducible, but the gamma factor is still defined.

If r is an arbitrary representation of ${}^L\mathbf{M}$ whose irreducible constituents r_i are LS-representations, one expects that the L, gamma, or epsilon factor for r should be the product of the corresponding factors of the r_i. But there is at present no way to ensure that such factors are well defined.

2.2.4 Splitting and generic characters

Let Γ be the Galois group of $\overline{k}/k$, and let $\mathbf{x}_\alpha : \alpha \in \Delta$ be a splitting for $\mathbf{G}, \mathbf{B}, \mathbf{T}$ (2.1.2). Then Γ acts on the set of splittings. We will say that a splitting is **defined over** k if it is fixed by Γ. Since $\mathbf{G}$ is quasi split, there always exists a splitting which is defined over k.

Given a splitting $\mathbf{x}_\alpha : \alpha \in \Delta$ which is defined over k, every element u of $\mathbf{U}(k)$ can be written as

$$u = \prod_{\alpha \in \Delta} \mathbf{x}_\alpha(a_\alpha) u'$$

for $a_\alpha \in \overline{k}$ and $u' \in \mathbf{U}(k)_{\mathrm{der}}$. Changing the order of the product will change the element u', but it will not change any a_α. The sum of the a_α will lie in k. Given a nontrivial unitary character ψ of k (automatically unitary when k is nonarchimedean), define

$$\chi(u) = \psi\Big(\sum_{\alpha \in \Delta} a_\alpha\Big).$$

Then χ is a character of $\mathbf{U}(k)$, called a **generic character**. It depends on ψ and the given splitting. For every unitary character χ of $\mathbf{U}(k)$ which is nontrivial on each simple root subgroup, and every choice of ψ, there exists a splitting which defines χ in this way.

Also, $\mathbf{x}_\alpha : \alpha \in \theta$ is a splitting for $(\mathbf{M}, \mathbf{B}_\mathbf{M}, \mathbf{T})$ which is defined over k, and the corresponding generic character of $\mathbf{U}_\mathbf{M}(k)$ with respect to ψ is $\chi|_{\mathbf{U}_\mathbf{M}(k)}$.

2.2.5 Generic representations

Given a nontrivial unitary character ψ of k, and the corresponding generic character χ of $\mathbf{U}_\mathbf{M}(k)$, an irreducible, admissible representation (π, V) of $\mathbf{M}(k)$ is said to be **generic** with respect to χ, or **χ-generic**, if there exists a nonzero linear functional $\lambda : V \to \mathbb{C}$ such that $\lambda(\pi(u)v) = \chi(u)\lambda(v)$ for all $u \in \mathbf{U}_\mathbf{M}(k)$ and $v \in V$. When k is archimedean, λ must be a bounded linear functional.

Note that a given representation of $\mathbf{M}(k)$ may be generic with respect to one generic character, but not with respect to another. However, when $\mathbf{M} = \mathrm{GL}_n$, or more generally when $\mathbf{M}$ is a product of copies of restriction of scalars of general linear groups, a representation is generic with respect to one generic character if and only if it is generic with respect to all generic characters. Furthermore, all supercuspidal representations of such groups are generic.

Assume k is nonarchimedean. The Langlands-Shahidi method will define L-functions and epsilon factors for generic representations (with respect to a given character). But in the case when $\mathbf{M} = \mathrm{GL}_n$, or more generally when $\mathbf{M}$ is a finite product of copies of restriction of scalars of general linear groups, we will be able to define L- and epsilon factors for general irreducible admissible representations (see 2.2.9). When k is archimedean, the Langlands-Shahidi method defines the gamma factors for generic representations, but not the L and epsilon factors. The L and epsilon factors in the archimedean case are defined to correspond directly to the factors on the Galois side under the local Langlands correspondence for archimedean groups (2.1.8). It is known that the gamma factors defined by the Langlands-Shahidi method coincide with the corresponding Artin gamma factors ([Sh90], Theorem 3.5, (1)).

Assume k is nonarchimedean. Let r be an LS-representation of ${}^L\mathbf{M}$ (2.2.2), and π a generic representation of ${}^L\mathbf{M}$ (that is, generic with respect to a given character ψ). The Langlands-Shahidi method defines the gamma factor $\gamma(s, \pi, r, \psi)$ first. The L and epsilon factors are consequently defined in terms of the gamma factor. We will not explain how the gamma factors are defined in general, but we will explain how the L- and epsilon factors are consequently defined in terms of gamma factors (2.2.9).

2.2.6 Multiplicativity of gamma factors

We return to k being an arbitrary local field of characteristic zero. Let π be an irreducible, admissible, χ-generic representation of $\mathbf{M}(k)$. Let $\mathbf{M}_*$ be a standard Levi subgroup of $\mathbf{M}$, and suppose there exists an irreducible, admissible representation

π_* of $\mathbf{M}_*(k)$ such that π is isomorphic to a quotient of $I_{\mathbf{M}_*}^{\mathbf{M}}\pi_*$. Here $I_{\mathbf{M}_*}^{\mathbf{M}}$ denotes *normalized parabolic induction.*

Then a gamma factor for π can be expressed as a product of gamma factors of π_* in a process called "multiplicativity." Here is one way of stating multiplicativity. Let r be an LS-representation of ${}^L\mathbf{M}$. The L-group of $\mathbf{M}_*$ is contained in the L-group of $\mathbf{M}$. Let $r_1, ..., r_t$ be the irreducible constituents of the restriction of r to ${}^L\mathbf{M}_*$. Then each r_i is an LS-representation of ${}^L\mathbf{M}_*$, and

$$\gamma(s, \pi, r, \psi) = \prod_{i=1}^{t} \gamma(s, \pi_*, r_i, \psi).$$

2.2.7 Removing nonrelevant groups

Suppose that $\mathbf{M}$ is a product of quasi split groups $\mathbf{M}_1 \times_k \cdots \times_k \mathbf{M}_t$. Let $\hat{\mathbf{M}}_i$ be the product of the $\mathbf{M}_j$ with the ith term omitted. A Borel subgroup/maximal torus for $\mathbf{M}$ can be obtained by taking a product of Borel subgroups/maximal tori for the $\mathbf{M}_i$, so we can identify the dual group of $\mathbf{M}$ with the product of the dual groups of the $\mathbf{M}_i$. Consequently, the L-groups of $\mathbf{M}_i$ and $\hat{\mathbf{M}}_i$ can be identified as subgroups of the L-group of $\mathbf{M}$.

Let π be an irreducible, admissible representation of $\mathbf{M}(k)$. It factors as a tensor product $\pi_1 \boxtimes \cdots \boxtimes \pi_t$ of irreducible, admissible representations π_i of the groups $\mathbf{M}_i(k)$, their isomorphism classes being determined by that of π. Let $\hat{\pi}_i$ be the representation of $\hat{\mathbf{M}}_i(k)$ obtained by deleting π_i.

Suppose that r is an LS-representation of ${}^L\mathbf{M}$ whose restriction to ${}^L\mathbf{M}_i^\circ$ is trivial. Let $\hat{r}_i$ be the restriction of r to ${}^L\hat{\mathbf{M}}_i$. Then $\hat{r}_i$ is also an LS-representation, and

$$\gamma(s, \pi, r, \psi) = \gamma(s, \hat{\pi}_i, \hat{r}_i, \psi)$$

and the same goes for the L and epsilon factors.

Example 2.2.7.1. *Consider the group* $\mathbf{H} = \mathrm{GL}_n \times \mathrm{GL}_m$, *whose L-group we can identify with* $\mathrm{GL}(V) \times \mathrm{GL}(W)$, *where* V *and* W *are complex vector spaces of dimen-*

sions n and m. Let $\pi_1 \boxtimes \pi_2$ be an irreducible, admissible representation of $\mathbf{H}(k) = \mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$*, and suppose π_2 is isomorphic to a quotient of* $I^{\mathrm{GL}_m}_{\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}} \sigma_1 \boxtimes \sigma_2$ *for a Levi subgroup* $\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}$ *of* GL_m*. Then $\pi_1 \boxtimes \pi_2$ is isomorphic to a quotient of*

$$\pi_1 \boxtimes I^{\mathrm{GL}_m}_{\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}}(\sigma_1 \boxtimes \sigma_2) \cong I^{\mathrm{GL}_n \times \mathrm{GL}_m}_{\mathrm{GL}_n \times \mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}}(\pi_1 \boxtimes \sigma_1 \boxtimes \sigma_2).$$

Let $\mathbf{H}_*$ *be the Levi subgroup* $\mathrm{GL}_n \times \mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}$ *of* $\mathbf{H}$*. The L-group of* $\mathbf{H}_*$ *can be identified with* $\mathrm{GL}(V) \times \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$*, where W_1, W_2 are complex vector spaces of dimensions m_1 and m_2 with $W_1 \oplus W_2 = W$.*

The tensor product representation $r : {}^L\mathbf{H} = \mathrm{GL}(V) \times \mathrm{GL}(W) \to \mathrm{GL}(V \otimes W)$ *sending (T, S) to $T \otimes S$ gives the Rankin product L-function. If we restrict r to* ${}^L\mathbf{H}_*$*, then r breaks up into two irreducible representations $r_1 \oplus r_2$, where*

$$r_1 : {}^L\mathbf{H}_* \to \mathrm{GL}(V \otimes W_1), (T, S_1, S_2) \mapsto T \otimes S_1$$

$$r_1 : {}^L\mathbf{H}_* \to \mathrm{GL}(V \otimes W_2), (T, S_1, S_2) \mapsto T \otimes S_2.$$

Multiplicativity tells us that

$$\gamma(s, \pi_1 \times \pi_2, \psi) = \gamma(s, \pi_1 \boxtimes \sigma_1 \boxtimes \sigma_2, r_1, \psi)\gamma(s, \pi_1 \boxtimes \sigma_1 \boxtimes \sigma_2, r_2, \psi)$$

and the above principle tells us that $\gamma(s, \pi_1 \boxtimes \sigma_1 \boxtimes \sigma_2, r_i, \psi) = \gamma(s, \pi_1 \times \sigma_i, \psi)$*. Consequently, multiplicativity tells us that*

$$\gamma(s, \pi_1 \times \pi_2, \psi) = \gamma(s, \pi_1 \times \sigma_1, \psi)\gamma(s, \pi_1 \times \sigma_2, \psi).$$

2.2.8 Langlands classification

Suppose that k is nonarchimedean. If r is an LS-representation, and π is χ-generic, the definition of $L(s, \pi, r)$ relies on the Langlands classification for p-adic groups ([Si78], [Ko03]).

There are two versions of the Langlands classification, one for subrepresentations and one for quotients. Let $\mathbf{M}_*$ be a standard Levi subgroup of $\mathbf{M}$, and let θ_* be the subset of θ defining $\mathbf{M}_*$. The restriction map $X(\mathbf{M}_*)_k \to X(\mathbf{S})$ induces an injection of $X(\mathbf{M}_*)_k \otimes_{\mathbb{Z}} \mathbb{R}$ into $X(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$. An element $\nu \in X(\mathbf{M}_*)_k \otimes_{\mathbb{Z}} \mathbb{R}$ is said to lie in the positive (resp. negative) Weyl chamber of $\mathbf{M}_*$ if $\langle \nu, \beta^\vee \rangle$ is positive (negative) for all roots $\beta \in \theta - \theta_*$.

Theorem. *(Langlands classification) Let $\mathbf{M}_*$ be a standard Levi subgroup of $\mathbf{M}$. If π_* is a smooth, irreducible representation of $\mathbf{M}_*(k)$ which is tempered (see 2.3.5), and ν lies in the positive (resp. negative) Weyl chamber of $\mathbf{M}_*$, then*

$$I_{\mathbf{M}_*}^{\mathbf{M}} \pi_* q^{\langle \nu, H_{\mathbf{M}_*}(-) \rangle}$$

has a unique irreducible quotient (resp. subrepresentation) π. If we consider the set of triples $(\mathbf{M}_*, \nu, \pi_*)$, where $\mathbf{M}_*$ is a standard Levi subgroup of $\mathbf{M}$, ν is in the positive (resp. negative) Weyl chamber of $\mathbf{M}_*$, and π_* is a tempered representation of $\mathbf{M}_*(k)$, then $(\mathbf{M}_*, \nu, \pi_*) \mapsto \pi$ is bijective (with π_* and π each being taken up to isomorphism).

If π is given in the Langlands classification for quotients as a triple $(\mathbf{M}_*, \nu, \pi_*)$, then its contragredient $\pi^\vee$ is given in the Langlands classification for subrepresentations as a triple $(\mathbf{M}_*, -\nu, \pi_*^\vee)$.

2.2.9 Definition of L and epsilon factors in terms of gamma factors

Suppose that k is nonarchimedean. Once the gamma factors have been defined by the Langlands-Shahidi method, the L-functions and the epsilon factors are consequently defined by a process which we now explain. Suppose first that π is a χ-generic representation of $\mathbf{M}(k)$ which is tempered. If r is an LS-representation of ${}^L\mathbf{M}$, the gamma factor $\gamma(s,\pi,r,\psi)$ is a rational function in $q^{-s} = q_k^{-s}$. It is independent of χ. Let $R(X) \in \mathbb{C}(X)$ be such that $R(q^{-s}) = \gamma(s,\pi,r,\psi)$. We can write R uniquely as

$$R(X) = aX^{\alpha}\frac{P(X)}{Q(X)}$$

where $a \in \mathbb{C}^*, \alpha \in \mathbb{Z}$, and $P, Q \in \mathbb{C}[X]$ are relatively prime with $P(0) = Q(0) = 1$. Then we define

$$L(s,\pi,r) = P(q^{-s})^{-1}.$$

This is independent of ψ and χ.

If π is a χ-generic representation of $\mathbf{M}(k)$ which is quasi-tempered, then there exists an unramified character μ of $\mathbf{M}(k)$ such that $\pi\mu$ is tempered. There exists a complex number s_0 such that $\gamma(s+s_0,\pi\mu,r,\psi) = \gamma(s,\pi,r,\psi)$. We then define

$$L(s,\pi,r) = L(s+s_0,\pi\mu,r).$$

Since the gamma factor and L-function for quasi-tempered χ-generic representations, the epsilon factor is consequently defined by equation (2.1.6.1).

Now let π be an arbitrary χ-generic representation of $\mathbf{M}(k)$. By the Langlands classification for p-adic reductive groups (2.2.8), there exists a standard Levi subgroup $\mathbf{M}_*$ of $\mathbf{M}$ and a quasi-tempered representation π_* of $\mathbf{M}_*(k)$ with positive Langlands parameter such that π is a quotient of

$$I_{\mathbf{M}_*}^{\mathbf{M}}\pi_*.$$

Since π is χ-generic, so is π_*. If $r_1, ..., r_t$ are the irreducible constituents of the restriction of r to ${}^L\mathbf{M}_*$, then multiplicativity tells us that

$$\gamma(s,\pi,r,\psi) = \prod_{i=1}^{t}\gamma(s,\pi_*,r_i,\psi). \tag{2.2.9.1}$$

We consequently define

$$L(s,\pi,r)=\prod_{i=1}^{t}L(s,\pi_*,r_i). \tag{2.2.9.2}$$

Then the epsilon factor $\epsilon(s,\pi,r,\psi)$ is defined by equation (2.1.6.1), and we have

$$\epsilon(s,\pi,r,\psi)=\prod_{i=1}^{t}\epsilon(s,\pi_*,r_i,\psi). \tag{2.2.9.3}$$

2.2.10 Local factors for nongeneric representations

Suppose that k is nonarchimedean, and every quasi-tempered representation of every Levi subgroup of $\mathbf{M}$ is generic with respect to every generic character. This is the case for GL_n, or more generally for a finite product of restriction of scalars of general linear groups. Then for every LS-representation r of ${}^L\mathbf{M}$, we can define the gamma factor, epsilon factor, and L-function for every irreducible, admissible representation π of $\mathbf{M}(k)$. One uses the Langlands classification to realize π as a quotient of $I_{\mathbf{M}_*}^{\mathbf{M}}\pi_*$ as in (2.2.8), and then one defines $L(s,\pi,r)$ and $\epsilon(s,\pi,r,\psi)$ by means of equations (2.2.9.2) and (2.2.9.3) above. Then equation (2.2.9.1) holds for $\gamma(s,\pi,r,\psi)$.

Furthermore, multiplicativity as stated in (2.2.6) and the property (2.2.7) hold for $\gamma(s,\pi,r,\psi)$.

2.2.11 Local Langlands correspondence for tori and Langlands-Shahidi gamma factors

Assume that k is nonarchimedean. The group $\mathbf{T}(k)$ has a unique maximal compact subgroup which is open in $\mathbf{T}(k)$. A character χ of $\mathbf{T}(k)$ is said to be **unramified** if it is trivial on this subgroup. Suppose π is an irreducible, admissible generic representation of $\mathbf{M}(k)$. Assume that π has a nonzero Iwahori fixed vector. Equivalently, π is isomorphic to a subquotient of $I_{\mathbf{T}}^{\mathbf{M}}\chi$ for some unramified character χ of $\mathbf{T}(k)$.

Let $\rho : W_k \to {}^L\mathbf{T}$ be the admissible homomorphism corresponding to χ by the local Langlands correspondence for tori. If r is an LS-representation of ${}^L\mathbf{M}$, we may consider the restriction of r to a representation of ${}^L\mathbf{T}$. Then ([Sh90], Theorem 3.5, (1))

$$\gamma(s, \pi, r, \psi) = \gamma(s, r \circ \rho, \psi).$$

Under the conjectural local Langlands correspondence for general reductive groups (2.1.7), it is expected that π should be parameterized by a homomorphism $\rho' : W_k' \to {}^L\mathbf{M}$. It is also expected that the restriction of ρ' to W_k should equal ρ. Since the gamma factor is determined by its underlying Weil representation, this will imply

$$\gamma(s, \pi, r, \psi) = \gamma(s, r \circ \rho', \psi).$$

2.2.12 Local Langlands correspondence for real groups and Langlands-Shahidi gamma factors

Suppose that k is archimedean, π is an irreducible, admissible representation of $\mathbf{M}(k)$, and r is an LS-representation. Let $\rho' : W_k' = W_k \to {}^L\mathbf{M}$ be the Langlands parameterization for π (2.1.8). Suppose that π is generic, so that the gamma factor $\gamma(s, \pi, r, \psi)$ is defined by the Langlands-Shahidi method. Then this gamma factor agrees with the one on the Galois side ([Sh90], Theorem 3.5, (1)):

$$\gamma(s, \pi, r, \psi) = \gamma(s, r \circ \rho', \psi).$$

We then define

$$L(s, \pi, r) = L(s, r \circ \rho'), \epsilon(s, \pi, r, \psi) = \epsilon(s, r \circ \rho', \psi).$$

2.2.13 Local factors and contragredients

Let k be a local field of characteristic zero. In all situations where the gamma factor $\gamma(s,\pi,r,\psi)$ is defined by Shahidi's method, the gamma factors $\gamma(s,\pi,r^\vee,\psi)$ and $\gamma(s,\pi^\vee,r,\psi)$ are also defined, and we have

$$\gamma(s,\pi,r^\vee,\psi)=\gamma(s,\pi^\vee,r,\psi)$$

$$L(s,\pi^\vee,r)=L(s,\pi,r^\vee)$$

$$\epsilon(s,\pi^\vee,r,\psi)=\epsilon(s,\pi,r^\vee,\psi).$$

2.2.14 The global functional equation

Suppose that k is a number field, and $\mathbf{M}$ is a quasi split group over k. Let $\mathbf{T}$ be a maximal torus of $\mathbf{M}$ which is defined over k, and $\mathbf{B}$ a Borel subgroup of $\mathbf{M}$ containing $\mathbf{T}$ which is defined over k. There is a global analogue of a generic character χ of $\mathbf{U_M}(\mathbb{A}_k)$, where $\mathbf{U_M}$ is the unipotent radical of $\mathbf{B_M}$. The generic character is defined in terms of a nontrivial character $\Psi=\otimes\psi_v$ of $\mathbb{A}_k/k$. Then χ factors as a tensor product of characters χ_v of $\mathbf{U_M}(k_v)$, generic with respect to ψ_v and a given global splitting.

Let K be the smallest field containing k over which $\mathbf{T}$ splits. Let $w_0 \mid v_0$ be an extension of places of K/k for which $[K:k]=[K_{w_0}:k_{v_0}]$, and suppose that $\mathbf{M}_{v_0}$ does not split over any proper subfield of K_{w_0}. Then we can identify ${}^L\mathbf{M}={}^L\mathbf{M}_v$. Let r be an LS-representation of ${}^L\mathbf{M}_v$, regarded as a representation of ${}^L\mathbf{M}$.

Let Π be a cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_k)$ which is globally generic with respect to χ. Then Π factors as a tensor product of unitary, χ_v-generic, irreducible, admissible representations π_v of $\mathbf{M}(k_v)$. We then set

$$L(s,\Pi,r)=\prod_v L(s,\pi_v,r_v).$$

The right hand side converges to an analytic function of s in some right half plane, and $L(s, \Pi, r)$ admits a meromorphic continuation to the entire complex plane satisfying a global functional equation

$$L(s, \Pi, r) = \epsilon(s, \Pi, r) L(1 - s, \Pi^\vee, r)$$

where $\epsilon(s, \Pi, r) = \prod_v \epsilon(s, \pi_v, r_v, \psi_v)$ ([Sh10], Theorem 8.4.5). The global epsilon factor $\epsilon(s, \Pi, r)$ is actually a finite product, since $\epsilon(s, \pi_v, r_v, \psi_v) = 1$ whenever π_v and ψ_v are unramified.

2.2.15 A special case of the Langlands-Shahidi method

Since we will need it later, we will explain in this section a special case of how gamma factors are defined. Suppose throughout this section that k is nonarchimedean, and $\mathbf{G}, \mathbf{T}, \mathbf{S}, \mathbf{M}, \mathbf{P}, \mathbf{N}, \mathbf{B}, \mathbf{U}, \mathbf{B_M}, \mathbf{U_M}$ are as before. Let $W(\mathbf{G}, \mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the Weyl group of $\mathbf{T}$ in $\mathbf{G}$, and $W(\mathbf{G}, \mathbf{S}) = N_{\mathbf{G}}(\mathbf{S})/\mathbf{T} = N_{\mathbf{G}(k)}(\mathbf{S}(k))/\mathbf{T}(k)$ the relative Weyl group.

We also will assume that:

- $\mathbf{P}$ is a maximal k-parabolic.
- The adjoint action r of ${}^L\mathbf{M}$ on ${}^L\mathfrak{n}$ is irreducible.
- The root system $\Phi(\mathbf{G}, \mathbf{S})$ is reduced.

2.2.16 Harish-Chandra map

The maximal k-parabolic subgroup $\mathbf{P}$ is defined by a simple root $\alpha \in \Delta \subset X(\mathbf{S})$. Let $(-, -)$ be a symmetric, positive definite bilinear form on $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ which is both $\Gamma = \mathrm{Gal}(\overline{k}/k)$ and $W(\mathbf{G}, \mathbf{T})$-invariant. Let ρ be half the sum of the roots of $\mathbf{T}$ in $\mathbf{N}$, let α' be any root of $\mathbf{T}$ in $\mathbf{G}$ whose restriction to $X(\mathbf{S})$ is α, and define

$$\tilde{\alpha} = \frac{(\alpha', \alpha')}{2(\rho, \alpha')}\rho.$$

This will not depend on the choice of form $(-,-)$, nor on the choice of α'. We may restrict $\tilde{\alpha}$ to an element of $X(A_{\mathbf{M}}) \otimes_{\mathbb{Z}} \mathbb{R}$, where $A_{\mathbf{M}}$ is the split component of $\mathbf{M}$, and then identify $\tilde{\alpha}$ as an element of $\mathfrak{a}_{\mathbf{M}}^* = X(\mathbf{M})_k \otimes_{\mathbb{Z}} \mathbb{R}$, where $X(\mathbf{M})_k$ is the group of rational characters of $\mathbf{M}$ which are defined over k. This is on account of the fact that restriction of rational characters of $\mathbf{M}$ to $A_{\mathbf{M}}$ defines an injection from $X(\mathbf{M})_k$ onto a subgroup of finite index of $X(A_{\mathbf{M}})$, and consequently an isomorphism $\mathfrak{a}_{\mathbf{M}}^* \to X(A_{\mathbf{M}}) \otimes_{\mathbb{Z}} \mathbb{R}$.

We can identify the dual vector space $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_{\mathbf{M}}^*, \mathbb{R})$ of $\mathfrak{a}_{\mathbf{M}}^*$ with

$$\mathfrak{a}_{\mathbf{M}} = \mathrm{Hom}_{\mathbb{Z}}(X(\mathbf{M})_k, \mathbb{R}).$$

The pairing $\langle -,- \rangle : \mathfrak{a}_{\mathbf{M}}^* \times \mathfrak{a}_{\mathbf{M}} \to \mathbb{R}$ extends to a pairing $\langle -,- \rangle : \mathfrak{a}_{\mathbf{M},\mathbb{C}}^* \times \mathfrak{a}_{\mathbf{M}} \to \mathbb{C}$, where $\mathfrak{a}_{\mathbf{M},\mathbb{C}}^* = \mathfrak{a}_{\mathbf{M}}^* \otimes_{\mathbb{R}} \mathbb{C}$. If $H_{\mathbf{M}} : \mathbf{M}(k) \to \mathfrak{a}_{\mathbf{M}}$ is the Harish-Chandra map defined by

$$H_{\mathbf{M}}(g)(\chi) = \log_q |\chi(g)|$$

then for any $\nu \in \mathfrak{a}_{\mathbf{M},\mathbb{C}}^*$, we can define a (continuous, complex valued) character of $\mathbf{M}(k)$ by $q_k^{\langle \nu, H_{\mathbf{M}}(-) \rangle}$. A character of this form on $\mathbf{M}(k)$ is called **unramified**. This generalizes the definition of an unramified character of the torus $\mathbf{T}(k)$ defined in (2.2.11).

2.2.17 Canonical Weyl group representatives in terms of a splitting

Let $\beta \in \Delta$ be a simple root of $\mathbf{S}$ in $\mathbf{G}$. For each root $\tilde{\beta}$ of $\mathbf{T}$ in $\mathbf{G}$ restricting to β, we have a root vector $x_{\tilde{\beta}} : \mathbb{G}_a \to \mathbf{U}_{\tilde{\beta}}$ from the splitting. There exists a unique root vector $x_{-\tilde{\beta}} : \mathbb{G}_a \to \mathbf{U}_{-\tilde{\beta}}$ such that

$$m_{\tilde{\beta}} = x_{\tilde{\beta}}(1)x_{-\tilde{\beta}}(1)x_{\tilde{\beta}}(1)$$

lies in the derived group of $\mathbf{G}(\overline{k})$ and normalizes $\mathbf{T}$. Then all the $m_{\tilde{\beta}}$ commute with each other, and we define

$$w_\beta = \prod_{\tilde{\beta}|_{\mathbf{S}}=\beta} m_{\tilde{\beta}} \in N_{\mathbf{G}}(\mathbf{S})(k).$$

The image of w_β in $W(\mathbf{G}, \mathbf{S})$ is the simple reflection $\tilde{w}_\beta$ corresponding to β.

Now for any element $\tilde{w} \in W(\mathbf{G}, \mathbf{S})$, we take a reduced decomposition $(\tilde{w}_{\beta_1}, ..., \tilde{w}_{\beta_t})$ and define a representative

$$w = w_{\beta_1} \cdots w_{\beta_t}$$

of $\tilde{w}$. This will be independent of the choice of reduced decomposition. In this way, we have a canonical Weyl group representative w of every element $\tilde{w}$ of $W(\mathbf{G}, \mathbf{S})$.

2.2.18 Self dual measures

Let β be a simple root of $\mathbf{S}$ in $\mathbf{G}$. Let $\tilde{\beta}$ be a root of $\mathbf{T}$ in $\mathbf{G}$ which restricts to β. The roots of $\mathbf{T}$ in $\mathbf{G}$ which restrict to β are all simple, and form a Galois orbit. Let $K_{\tilde{\beta}}$ be the splitting field of $\tilde{\beta}$, the intersection of all subfields of $\overline{k}$ over which $\tilde{\beta}$ splits. The roots of $\mathbf{T}$ which restrict to $\mathbf{S}$ are $\tau.\tilde{\beta}$, as τ runs through the k-embeddings of $K_{\tilde{\beta}}$ into $\overline{k}$.

The root subgroup $\mathbf{U}_\beta$ is the product of all the $\mathbf{U}_{\tau.\tilde{\beta}}$, and in fact $\mathbf{U}_\beta$ is isomorphic to $\mathrm{Res}_{K_{\tilde{\beta}}/k}\,\mathbf{U}_{\tilde{\beta}}$. Consequently, our splitting chosen earlier gives an isomorphism of

$\mathbf{U}_\beta(k)$ onto the additive group $K_{\tilde{\beta}}$, and we can choose a Haar measure on $\mathbf{U}_\beta(k)$ corresponding to the self dual Haar measure on $K_{\tilde{\beta}}$ with respect to $\psi \circ \mathrm{Tr}_{K_{\tilde{\beta}}/k}$. This Haar measure will not depend on the choice of root $\tilde{\beta}$ restricting to β.

Now suppose β is *any* root of $\mathbf{S}$ in $\mathbf{G}$. There exists a $\tilde{w} \in W(\mathbf{G}, \mathbf{S})$ such that $\tilde{w}.\beta$ is a simple root. Let w be the canonical representative of $\tilde{w}$ defined in (2.2.17). Then $w\mathbf{U}_\beta(k)w^{-1} = \mathbf{U}_{\tilde{w}.\beta}(k)$, and we transfer on $\mathbf{U}_\beta(k)$ the self dual measure on $\mathbf{U}_{\tilde{w}.\beta}(k)$ just defined. This measure does not depend on the choice of $\tilde{w}$, only on ψ and the initial splitting $\mathbf{x}_{\tilde{\beta}} : \beta \in \Delta$.

Consider a Zariski-closed subgroup $\mathbf{U}_0$ of $\mathbf{U}$ which is defined over k and normalized by $\mathbf{T}$. Then as a variety over k, $\mathbf{U}_0$ is the product of the root subgroups it contains. We take products of measures just defined and obtain a Haar measure on $\mathbf{U}_0(k)$.

2.2.19 Whittaker functionals for induced representations

There is a unique element $\tilde{w_0} \in W(\mathbf{G}, \mathbf{S})$ which maps $\Delta - \alpha$ into Δ and sends α to a negative root. Let w_0 be a representative of $\tilde{w_0}$. The group $\mathbf{M}' = w_0 \mathbf{M} w_0^{-1}$ is a standard Levi subgroup of $\mathbf{G}$. Let $\mathbf{P}'$ be the corresponding standard parabolic, and $\mathbf{N}'$ its unipotent radical. We call $\mathbf{P}'$ the parabolic subgroup associated to $\mathbf{P}$. If $\mathbf{P} = \mathbf{P}'$, then $\mathbf{P}$ is called **self associate**.

Suppose that π is an irreducible, admissible χ-generic representation of $\mathbf{M}(k)$. Let $\pi_s = \pi q_k^{\langle s\tilde{\alpha}, H_{\mathbf{M}}(-)\rangle}$, and let $I(s,\pi) = I_{\mathbf{M}}^{\mathbf{G}} \pi_s$.

If λ is a χ-Whittaker functional of π, then $\lambda_\chi(s,\pi)$ is one for $I(s,\pi)$, defined by

$$\lambda_\chi(s,\pi)f = \int_{\mathbf{N}'(k)} \langle f(xn), \lambda \rangle \, dn' \tag{2.2.19.1}$$

where x is a representative of the inverse of $\tilde{w_0}$, and the Haar measure dn' on $\mathbf{N}'(k)$ is defined as in (2.2.18).

Some explanation of the formula (2.2.19.1) is in order. First, the integral as given does not converge for general f (it does converge if f is supported inside the open set

$\mathbf{P}(k)\tilde{w_0}^{-1}\mathbf{N}'(k)$ of $\mathbf{G}(k)$); rather the integration is really defined over a suitably large open compact subgroup N_0 of $\mathbf{N}'(k)$ (depending on f), with the property that the value of the integral does not change if N_0 is replaced with any larger open compact subgroup ([Sh10], Theorem 3.4.7).

Second, in order for (2.2.19.1) to actually define a Whittaker functional, the representative x must be **compatible** with χ in the sense that $\chi(xux^{-1}) = \chi(u)$ for any $u \in \mathbf{U}_{\mathbf{M}'}(k)$, where $\mathbf{U}_{\mathbf{M}'} = \mathbf{U} \cap \mathbf{M}'$. The canonical Weyl group representative of $\tilde{w_0}^{-1}$ from (2.2.18) is compatible with χ in this sense.

In the following section (2.2.20), we will take x to be the canonical representative of $\tilde{w_0}^{-1}$ (which is generally not the same as the inverse of the canonical representative w_0 of $\tilde{w_0}$). In the case that $\mathbf{P}$ is self associate, we have $\tilde{w_0} = \tilde{w_0}^{-1}$, and both x and x^{-1} are compatible with χ.

2.2.20 Definition of the Shahidi local coefficient

Let $w_0(\pi)$ be the representation of $\mathbf{M}'(k)$ given by $w_0(\pi)(m') = \pi(w_0^{-1}m'w_0)$. Since $\mathbf{M}'$ is also a maximal k-parabolic subgroup of $\mathbf{G}$, we can define the analogous representation $I(s, w_0(\pi))$ of $\mathbf{G}(k)$ obtained by normalized induction from $\mathbf{P}'(k)$ to $\mathbf{G}(k)$, and its Whittaker functional $\lambda_\chi(s, w_0(\pi))$.

We have an intertwining operator

$$A(s,\pi) : I(s,\pi) \to I(-s, w_0(\pi))$$

defined for $\mathrm{Re}(s)$ sufficiently large by a Gelfand-Pettis integral

$$A(s,\pi)f(g) = \int\limits_{\mathbf{N}'(k)} f(w_0^{-1}n'g)\, dn'.$$

There exists a meromorphic function $C_\chi(s,\pi)$ on $\mathbb{C}$, called the **Shahidi local coefficient**, such that

$$C_\chi(s,\pi)\lambda_\chi(-s,w_0(\pi))\circ A(s,\pi)=\lambda_\chi(s,\pi).$$

For each root β of $\mathbf{S}$ in $\mathbf{N}'$, choose a root $\tilde{\beta}$ of $\mathbf{T}$ restricting to β, and let $K_{\tilde{\beta}}$ be the splitting field of $\tilde{\beta}$. The Langlands lambda function $\lambda(K_{\tilde{\beta}}/k,\psi)$ is independent of the choice of $\tilde{\beta}$. Define

$$\lambda(w_0,\psi)=\prod_{\beta\in\Phi(\mathbf{N},\mathbf{S})}\lambda(K_{\tilde{\beta}}/k,\psi).$$

Theorem 2.2.20.1. *(Shahidi) The gamma factor and local coefficient are related by*

$$C_\chi(s,\pi)=\lambda(w_0,\psi)^{-1}\gamma(s,\pi,r,\psi).$$

This is a special case of Theorem 3.5 of [Sh90], and can be considered a definition of $\gamma(s,\pi,r,\psi)$. We remark that we are using an unconventional normalization of Shahidi's gamma factors. Our term $\gamma(s,\pi,r,\psi)$ is what is normally written in the literature as $\gamma(s,\pi,r^\vee,\overline{\psi})$.

The change from $\overline{\psi}$ to ψ is done to agree with Proposition 3.4 of [KeSh88], as well as Theorem 3.1 of [Sh85] when the presence of lambda functions were first noticed.

The removal of the contragredient from r comes from the way Langlands' L-function $L(s,\pi,r)$ is defined in the unramified case. We refer to [La70] or [La71] for the details. In this situation, k is p-adic, $\mathbf{M}$ and π are unramified, and π is attached to a semisimple conjugacy class A_π in ${}^L\mathbf{M}$. Langlands defines

$$L(s,\pi,r)=\det(1-r(A_\pi)q^{-s})^{-1}.$$

The assignment $\pi\mapsto A_\pi$ depends on a choice of Harish-Chandra map. Our choice of Harish-Chandra map in this situation would be that of [La70], page 7, which Langlands refers to as v. Shahidi's choice is that of [La71], page 7, which Langlands refers to as λ. The relationship between these choices is $\lambda=-v$. Consequently, the semisimple conjugacy class A_π we assign to π is the inverse of the one used by Langlands in [La71], and the unramified L-function $L(s, \pi, r)$ in [La70] would be called $L(s, \pi, r^\vee)$ in [La71].

The Langlands-Shahidi local factors are defined to coincide with those of Langlands in the unramified case. Following [La71] leads to the appearance of contragredients in the Langlands-Shahidi method, while following [La70] removes them.

The reason we follow [La70] is that the resulting Langlands-Shahidi factors agree with the Artin factors under the version of the local Langlands correspondence stated in (2.3.7).

Example 2.2.20.2. *Suppose k is p-adic, $\mathbf{G} = \mathrm{GL}_2$, $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{GL}_1$, and $\mathbf{N}$ is the group of upper triangular matrices in $\mathbf{G}$ with 1s on the diagonal. If π is the representation of $\mathbf{M}(k)$ given by*

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto \chi_1(t_1)\chi_2(t_2)$$

for two characters χ_i of k^*, then the local coefficient turns out to be

$$C_\chi(s, \pi) = \gamma(s, \chi_1\chi_2^{-1}, \psi)$$

where the right hand side is the gamma factor defined in (1.1.13).

2.2.21 Unramified twists and gamma factors

Assume k is nonarchimedean. Let s_0 be a complex number, let π be a generic representation of $\mathbf{M}(k)$, and let $\pi_0 = \pi q^{\langle s_0\tilde{\alpha}, H_{\mathbf{M}}(-)\rangle}$. Then ([Sh90], Theorem 3.5, (2))

$$\gamma(s, \pi_0, r, \psi) = \gamma(s + s_0, \pi, r, \psi). \tag{2.2.21.1}$$

Note that since we are using a different normalization of Shahidi's local factors (2.2.20), this formula must also be adjusted from its original statement.

When every quasi-tempered representation of $\mathbf{M}(k)$ is generic (for example, when $\mathbf{M} = \mathrm{GL}_n$), the gamma factors, L-functions, and epsilon factors can be defined for arbitrary irreducible, admissible representations π of $\mathbf{M}(k)$, not just generic ones (2.2.10). Then the same formula (2.2.21.1) holds for arbitrary π. We also have

$$L(s, \pi_0, r) = L(s + s_0, \pi, r).$$

2.3 Classification of smooth, irreducible representations of $\mathrm{GL}_n(k)$

Throughout this section, k is a nonarchimedean local field of characteristic zero. If $\mathbf{H}$ is a connected, reductive group over k, then an irreducible, admissible representation of $\mathbf{H}(k)$ is the same thing as an irreducible, smooth representation of $\mathbf{H}(k)$.

In this section, we review the local Langlands correspondence (LLC) for $\mathrm{GL}_n(k)$, which gives a bijection between smooth irreducible representations of $\mathrm{GL}_n(k)$ and n-dimensional Frobenius semisimple representations of the Weil-Deligne group W_k'. In order to do this, we need to state the Bernstein-Zelevinsky classification theorems, which explain how arbitrary smooth irreducible representations of $\mathrm{GL}_n(k)$ are built out of supercuspidal representations of smaller GLs. Our primary references for the Bernstein-Zelevinsky classification are [Ze80], [Rod82], and Chapter 14.5 of [GoHu11].

We can summarize the LLC as follows: there is a bijection between supercuspidal representations of $\mathrm{GL}_n(k)$ and irreducible representations of the local Weil group W_k. Granting this bijection, the correspondence for general smooth irreducible representations of $\mathrm{GL}_n(k)$ follows from the fact that arbitrary representations of $\mathrm{GL}_n(k)$ are built out of supercuspidals in a compatible manner in which arbitrary Frobenius semisimple representations of W_k' are built from irreducibles.

2.3.1 Normalized parabolic induction for GL_n

Let M be a standard Levi subgroup of $\mathrm{GL}_n(k)$, i.e. a subgroup of the form $\mathrm{GL}_{n_1}(k) \times \cdots \times \mathrm{GL}_{n_r}(k)$, where the n_i are positive integers whose sum is n. Let P

be the standard parabolic corresponding to M, and N the unipotent radical of P. If π is a smooth, irreducible representation of M, then π decomposes as a tensor product $\pi_1 \boxtimes \cdots \boxtimes \pi_r$, where π_i is a smooth, irreducible representation of $\mathrm{GL}_{n_i}(k)$. The isomorphism classes of the π_i are uniquely determined by π. The representation π is supercuspidal if and only if all the π_i are supercuspidal.

We may extend π to a representation of P by making it trivial on N. Let $\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi_1 \boxtimes \cdots \boxtimes \pi_r$, or just $\mathrm{Ind}\, \pi_1 \boxtimes \cdots \boxtimes \pi_r$, designate the smooth representation of $\mathrm{GL}_n(k)$ obtained from P by normalized parabolic induction. In (2.2) we used $I_M^{\mathrm{GL}_n}$ to denote normalized parabolic induction, and in the more traditional notation,

$$\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi_1 \boxtimes \cdots \boxtimes \pi_r = \mathrm{Ind}_P^{\mathrm{GL}_n(k)} \pi \delta_P^{\frac{1}{2}}$$

where δ_P is the modulus character of P. Explicitly, $\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi$ consists of all locally constant functions f from $\mathrm{GL}_n(k)$ to the underlying space of π which satisfy

$$f(mng) = \pi(m)\delta_P(m)^{\frac{1}{2}} f(g)$$

for all $m \in M, n \in N, g \in G$, and $\mathrm{GL}_n(k)$ acts on these functions by right translation.

2.3.2 A basic classification result

Here is a basic classification theorem of smooth, irreducible representations of $\mathrm{GL}_n(k)$.

Theorem 2.3.2.1. *([Ze80], Proposition 1.10) Let π be a smooth, irreducible representation of $\mathrm{GL}_n(k)$. Then there exists a partition $(n_1, ..., n_r)$ of n and irreducible, supercuspidal representations $\pi_i, 1 \leq i \leq r$ of $\mathrm{GL}_{n_i}(k)$ such that π is a subquotient of $\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi_1 \boxtimes \cdots \boxtimes \pi_r$.*

If $(m_1, ..., m_{r'})$ is another partition of n, and $\pi_i', 1 \leq i \leq r'$ are irreducible, supercuspidal representations of $\mathrm{GL}_{m_i}(k)$, such that π is a subquotient of $\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi_1' \boxtimes \cdots \boxtimes \pi_{r'}'$, then $r = r'$, and after some permutation, $n_i = m_i$ and $\pi_i \cong \pi_i'$. Moreover, *the n_i can be permuted in such a way that π is a subrepresentation or quotient of* $\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi_1 \boxtimes \cdots \boxtimes \pi_r$.

The unordered tuple $(\pi_1, ..., \pi_r)$ of supercuspidal representations is called the **supercuspidal support** of π.

The Bernstein-Zelevinsky classification gives a more detailed classification of smooth, irreducible representations of $\mathrm{GL}_n(k)$ in terms of supercuspidal representations. We will formulate it in order to state the local Langlands correspondence (2.3.4).

2.3.3 The unique irreducible quotient associated to an interval

If π is a smooth representation of $\mathrm{GL}_n(k)$, and z is a complex number, let $\pi(z)$ be the smooth representation $\pi|\det(-)|^z$ of $\mathrm{GL}_n(k)$. It is of course irreducible, or supercuspidal, if and only if π is, and for π irreducible and z_i real, we have $\pi(z_1) \cong \pi(z_2)$ if and only if $z_1 = z_2$. This is on account of the fact that if $z_1 \neq z_2$, then the central characters of $\pi(z_1)$ and $\pi(z_2)$ will be different.

By an **interval**, we will mean a set of isomorphism classes of supercuspidal representations of GL_n of the form $\{\pi, \pi(1), ..., \pi(m-1)\}$. We define the **length** of such an interval to be m, and the **degree** of such an interval to be nm.

Proposition 2.3.3.1. *([Rod82], Proposition 9) Let $\Delta = \{\pi, \pi(1), ..., \pi(m-1)\}$ be an interval of length m and degree n. Then $\mathrm{Ind}^{\mathrm{GL}_n(k)} \pi \boxtimes \pi(1) \boxtimes \cdots \boxtimes \pi(m-1)$ has a unique irreducible quotient and a unique irreducible subrepresentation. The supercuspidal support of this irreducible quotient is $(\pi, \pi(1), ..., \pi(m-1))$.*

We will denote this unique irreducible quotient by $Q(\Delta)$.

2.3.4 The Bernstein-Zelevinsky classification theorem

Let $\Delta = \{\pi, \pi(1), ..., \pi(m-1)\}$ and $\Delta' = \{\pi', \pi'(1), ..., \pi'(m'-1)\}$ be two intervals of lengths m and m', and degrees nm and $n'm'$, so that π and π' are representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_{n'}(k)$.

We define a partial ordering on the set of isomorphism of supercuspidal representations by saying that σ_1 is less than or equal to σ_2 if $\sigma_2 \cong \sigma_1(m)$ for some nonnegative integer m. If σ_1 and σ_2 are representations of different size general linear groups, then σ_1 and σ_2 are incomparable.

We will say that Δ and Δ' are **linked** if $n = n'$, neither of Δ or Δ' is a subset of the other, and $\Delta \cup \Delta'$ is an interval. We say that Δ **precedes** Δ' if Δ and Δ' are linked, and the minimal element of Δ is smaller than the minimal element of Δ'.

If $\Delta_1, ..., \Delta_r$ are intervals, we say that $\Delta_1, ..., \Delta_r$ satisfy the "**does not precede condition**" if whenever $i < j$, Δ_i does not precede Δ_j. By permuting the indices, it is always possible to ensure that an ordered list of intervals satisfies the "does not precede" condition.

Theorem 2.3.4.1. *(Zelevinksy)*

(i): Suppose that $\Delta_1, ..., \Delta_r$ are intervals of degrees $d_1, ..., d_r$ which satisfy the "does not precede" condition. Then

$$\mathrm{Ind}^{\mathrm{GL}_{d_1+\cdots+d_r}(k)} Q(\Delta_1) \boxtimes \cdots \boxtimes Q(\Delta_r)$$

has a unique irreducible quotient, which we denote by $Q(\Delta_1, ..., \Delta_r)$. This induced representation is irreducible if and only if no two of the intervals Δ_i, Δ_j are linked.

(ii): Every smooth, irreducible representation of a general linear group is isomorphic to a representation $Q(\Delta_1, ..., \Delta_r)$ of the form (i).

(iii): If $\Delta'_1, ..., \Delta'_{r'}$ is another collection of intervals which satisfy the "does not precede" condition, and $Q(\Delta_1, ..., \Delta_r) \cong Q(\Delta'_1, ..., \Delta'_{r'})$, then $r = r'$, and the indices can be permuted so that $\Delta_i = \Delta'_i$.

If $\Delta_i = \{\pi_i, \pi_i(1), ..., \pi_i(m_i - 1)\}$, then the supercuspidal support of $Q(\Delta_1, ..., \Delta_r)$ is the unordered tuple $(\pi_i(j) : 1 \leq i \leq r, 1 \leq j \leq m_i - 1)$.

The original formulation of this theorem, using subrepresentations instead of quotients, is Theorem 6.1 of [Ze80]. This version, formulated by Rodier, can be found as Theorem 3 in [Rod82].

2.3.5 Tempered and square integrable representations

Let π be a representation of the group G of rational points of a connected, reductive group over k, and let r be a positive real number. We say that π is L^r if the central character of π is unitary, and if for any matrix coefficient f of π, the integral $\int_{G/Z} |f(g)|^r \, dg$ is finite, where Z is the center of G. We say that π is **square integrable** (resp. **tempered**) if it is L^2 (resp. $L^{2+\epsilon}$ for every $\epsilon > 0$). We say that π is **essentially square integrable** (resp. **quasi-tempered**) if some twist of π by an unramified character is square integrable (tempered).

Under the Bernstein-Zelevinsky classification, a representation of $\mathrm{GL}_n(k)$ is essentially square integrable if and only if it is of the form $Q(\Delta)$. A representation $\pi = Q(\Delta_1, ..., \Delta_r)$ is tempered if and only if each $Q(\Delta_i)$ is square integrable ([Rod83], Propositions 11-13). If this is the case, then no two of the intervals Δ_i or Δ_j are linked, so π is fully induced by the $Q(\Delta_i)$. This follows from the fact that if $Q(\Delta_i)$ is square integrable, then it has a unitary central character, which means that the central character of $\pi_i \boxtimes \cdots \boxtimes \pi_i(m_i - 1)$ must be unitary as well.

2.3.6 Connection with the Langlands classification

In this section, we relate the Bernstein-Zelevinsky classification to the Langlands classification (2.2.8). We will rely on two facts about induced representations. First, if π_i is a smooth representation of $\mathrm{GL}_{n_i}(k)$ for $1 \leq i \leq r$, $n = n_1 + \cdots + n_r$, and Π is the representation $\mathrm{Ind}\, \pi_1 \boxtimes \cdots \boxtimes \pi_r$ of $\mathrm{GL}_n(k)$, then

$$\Pi(z) = \mathrm{Ind}\, \pi_1(z) \boxtimes \cdots \boxtimes \pi_r(z)$$

for all $z \in \mathbb{C}$. Second, suppose that each π_i is itself induced from a smooth representation $\sigma_{1,i} \boxtimes \cdots \boxtimes \sigma_{j_i,i}$ of a Levi subgroup of $\mathrm{GL}_{n_i}(k)$, Then

$$\begin{aligned}\operatorname{Ind} \pi_1 \boxtimes \cdots \boxtimes \pi_r &= \operatorname{Ind}\left(\operatorname{Ind}\left(\sigma_{1,1} \boxtimes \cdots \boxtimes \sigma_{j_1,1}\right) \boxtimes \cdots \boxtimes \operatorname{Ind}\left(\sigma_{1,r} \boxtimes \cdots \boxtimes \sigma_{j_r,r}\right)\right) \\ &\cong \operatorname{Ind} \sigma_{1,1} \boxtimes \cdots \boxtimes \sigma_{j_r,r}.\end{aligned}$$

Suppose that π is a representation of $\mathrm{GL}_n(k)$, given in the Bernstein-Zelevinsky classification by intervals $\Delta_1, ..., \Delta_r$. For each $1 \leq i \leq r$, there is a unique $x_i \in \mathbb{R}$ such that $Q(\Delta_i)(-x_i)$ is square integrable. We arrange the order on the intervals Δ_i so that

$$y_1 = x_1 = \cdots = x_{n_1} > y_2 = x_{n_1+1} = \cdots = x_{n_2} > \cdots = y_s.$$

One can check that this arrangement of the intervals satisfies the "does not precede condition," so π is the unique irreducible quotient of $\operatorname{Ind} Q(\Delta_1) \boxtimes \cdots \boxtimes Q(\Delta_r)$. Now for $1 \leq i \leq s$, and $n_{i-1} + 1 \leq j \leq n_i$ (taking $n_0 = 0$), let Δ'_j be the interval obtained by twisting each element of Δ_j by $|\det(-)|^{-y_i}$. Then

$$Q(\Delta_j)(-y_i) = Q(\Delta'_j)$$

is square integrable, and by (2.3.5), none of the intervals Δ'_j for $n_{i-1}+1 \leq j \leq n_i$ are linked, so if we set

$$\pi_i = \operatorname{Ind} \boxtimes_{j=n_{i-1}+1}^{n_i} Q(\Delta'_j)$$

then π_i is irreducible and tempered. Now

$$\pi_* = \pi_1 \boxtimes \cdots \boxtimes \pi_s$$

is a tempered representation of a standard Levi subgroup M of $\mathrm{GL}_n(k)$. If we let $D_1, ..., D_s$ be the determinant maps on the blocks of M, then $\nu = y_1 D_1 + \cdots + y_s D_s$ lies in the positive Weyl chamber of M. The Langlands classification tells us that

$$\mathrm{Ind}\, \pi_* q^{\langle \nu, H_M(-)\rangle}$$

has a unique irreducible quotient. But since

$$\pi_i(y_i) = \mathrm{Ind} \boxtimes_{j=n_{i-1}+1}^{n_i} Q(\Delta_j')(y_i) = \mathrm{Ind} \boxtimes_{j=n_{i-1}+1}^{n_i} Q(\Delta_j)$$

we have

$$\begin{aligned} \mathrm{Ind}\, \pi_* q^{\langle \nu, H_M(-)\rangle} &= \mathrm{Ind}\, \pi_1(y) \boxtimes \cdots \boxtimes \pi_s(y) \\ &= \mathrm{Ind} \boxtimes_{i=1}^{s} \mathrm{Ind} \boxtimes_{j=n_{i-1}+1}^{n_i} Q(\Delta_j) \\ &= \mathrm{Ind}\, Q(\Delta_1) \boxtimes \cdots \boxtimes Q(\Delta_r). \end{aligned}$$

We have expressed π as a unique irreducible quotient in the Langlands classification.

2.3.7 Local Langlands correspondence for GL_n

Recall the local Artin map $\mathrm{Art} = \mathrm{Art}_k : k^* \to W_k^{\mathrm{ab}}$ (1.1.10), which sends a uniformizer to a geometric Frobenius. The local Artin map gives the "local Langlands correspondence for $\mathrm{GL}_1(k) = k^*$." That is, $\chi \mapsto \chi \circ \mathrm{Art}^{-1}$ defines a bijection between smooth irreducible representations (characters) of $\mathrm{GL}_1(k) = k^*$ and characters of W_k. Notice that a one dimensional Frobenius semisimple representation of W_k' is the same thing as a one dimensional represenation of W_k.

More generally, there is the local Langlands correspondence for $\mathrm{GL}_n(k)$, which gives a bijection between smooth, irreducible representations of $\mathrm{GL}_n(k)$ and n-dimensional Frobenius semisimple representations of W_k'. Let $\mathcal{A}_n$ be the set of isomorphism classes of smooth, irreducible representations of $\mathrm{GL}_n(k)$, and let $\mathcal{G}_n$ be the set of isomorphism classes of n-dimenional Frobenius semisimple representations of W_k'.

Theorem 2.3.7.1. *(Local Langlands Correspondence) There is a unique collection of bijections $\mathcal{A}_n \to \mathcal{G}_n : n \geq 1$ satisfying the following properties:*

1. *The bijection for $n = 1$ is the correspondence $\chi \mapsto \chi \circ \mathrm{Art}^{-1}$ given just above by the local Artin map.*

2. *If π_i corresponds to ρ'_i $(i = 1, 2)$, then*

$$L(s, \pi_1 \times \pi_2) = L(s, \rho'_1 \otimes \rho'_2)$$

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = \epsilon(s, \rho'_1 \otimes \rho'_2, \psi)$$

   for all nontrivial characters ψ of k.

3. *If π corresponds to ρ', then the contragredient $\pi^\vee$ of π corresponds to the contragredient $\rho'^\vee$ of ρ'.*

4. *If π corresponds to ρ', then the central character ϖ_π of π corresponds to $\det \rho$ under the local Artin map.*

5. *If π corresponds to ρ', and a character η of k^* corresponds to a character χ of W_k, then $\pi(\chi \circ \det)$ corresponds to $\rho' \otimes \eta$.*

The theorem was proved for p-adic fields independently by Henniart [He00] and Harris and Taylor [HaTa01]. A new proof was given later by Scholze [Sc10].

2.3.8 A description of the local Langlands correspondence

Under the local Langlands correspondence, irreducible supercuspidal representations correspond to irreducible representations of the Weil group. It is difficult to describe the local Langlands correspondence explicitly, but granting a correspondence between the supercuspidals and the irreducibles, the remaining representations may be described as follows:

Suppose that π is a smooth, irreducible representation of $\mathrm{GL}_n(k)$ which occurs in the Bernstein-Zelevinsky classification as $Q(\Delta_1, ..., \Delta_r)$, where $\Delta_i = (\pi_i, \pi_i(1), ..., \pi_i(m_i - 1))$ is an interval of length m_i, and π_i is an irreducible supercuspidal representation. Suppose that π_i corresponds to an irreducible representation ρ_i of the Weil group W_k. Then π corresponds to

$$\rho' = \bigoplus_{i=1}^{r} \rho_i \otimes \mathrm{Sp}(m_i).$$

Furthermore, π is tempered in the sense of (2.3.5) if and only if ρ' is tempered in the sense of (1.2.10).

2.3.9 Galois representations and the local Langlands correspondence

Suppose that ρ and ρ' are Galois representations (1.1.4) of dimensions n_1 and n_2. Let $n = n_1 + n_2$, and let Π, π, π' be the smooth, irreducible representations of $\mathrm{GL}_n(k), \mathrm{GL}_{n_1}(k), \mathrm{GL}_{n_2}(k)$ which correspond to $\rho \oplus \rho', \rho$, and ρ' respectively.

Write ρ and ρ' as a direct sum of irreducible Galois representations:

$$\rho = \rho_1 \oplus \cdots \oplus \rho_s$$

$$\rho' = \rho'_1 \oplus \cdots \oplus \rho'_t$$

and let π_i, π'_i be the supercuspidal representations corresponding to ρ_i, ρ'_i. Since the ρ_i and ρ'_i and Galois representations, none of the singleton intervals $\{\pi_i\}$ or $\{\pi'_i\}$ are linked, so Π, π, π' are fully induced from their supercuspidal supports (2.3.5):

$$\Pi = \mathrm{Ind}^{\mathrm{GL}_n(k)} \pi_1 \boxtimes \cdots \boxtimes \pi_s \boxtimes \pi'_1 \boxtimes \cdots \boxtimes \pi'_t$$

$$\pi = \mathrm{Ind}^{\mathrm{GL}_{n_1}(k)} \pi_1 \boxtimes \cdots \boxtimes \pi_s$$

$$\pi' = \mathrm{Ind}^{\mathrm{GL}_{n_2}(k)} \pi'_1 \boxtimes \cdots \boxtimes \pi'_t.$$

Then

$$\begin{aligned}
\mathrm{Ind}^{\mathrm{GL}_n(k)}\,\pi\boxtimes\pi' &= \mathrm{Ind}^{\mathrm{GL}_n(k)}\left(\mathrm{Ind}^{\mathrm{GL}_{n_1}(k)}\,\pi_1\boxtimes\cdots\boxtimes\pi_s\right)\boxtimes\left(\mathrm{Ind}^{\mathrm{GL}_{n_2}(k)}\,\pi_1'\boxtimes\cdots\boxtimes\pi_t'\right)\\
&= \mathrm{Ind}^{\mathrm{GL}_n(k)}\,\pi_1\boxtimes\cdots\boxtimes\pi_s\boxtimes\pi_1'\boxtimes\cdots\boxtimes\pi_t'\\
&= \Pi.
\end{aligned}$$

2.3.10 Local Langlands correspondence and field automorphisms

Suppose that σ is an automorphism of $\overline{k}$ which fixes $\mathbb{Q}_p$, so that σ defines an isomorphism of p-adic fields $k\to\sigma k$. We have the Weil groups W_k and $W_{\sigma k}$, and the local Langlands correspondence of (2.3.7) holds for each.

Then σ defines an isomorphism of Weil groups $W_{\sigma k}\to W_k, w\mapsto\tau\sigma\tau^{-1}$. This isomorphism extends naturally to an isomorphism of Weil-Deligne groups. If ρ' is an n-dimensional, Frobenius semisimple representation of W_k' which corresponds to a smooth, irreducible representation π of $\mathrm{GL}_n(k)$, then $\rho\circ\iota_\sigma$ is an n-dimensional representation of $W_{\sigma k}'$ which corresponds to the smooth, irreducible representation $\pi\circ\sigma^{-1}$ of $\mathrm{GL}_n(\sigma k)$.

In particular, suppose E/F is a quadratic extension of p-adic fields, σ is the nontrivial element of $\mathrm{Gal}(E/F)$, and z is any element of W_F which is not in W_E. If ρ' is an n-dimensional, Frobenius semisimple representation of W_E, and π is the smooth, irreducible representation of $\mathrm{GL}_n(E)$ corresponding to ρ' under the local Langlands correspondence, then $\rho'\circ\iota_z$ corresponds to $\pi\circ\sigma$.

2.3.11 Example: principal series representations of GL_2

It is interesting to interpret the non-supercuspidal representations of $\mathrm{GL}_2(k)$ in terms of two dimensional Frobenius semisimple representations of W_k'. Suppose that π is an irreducible representation of $\mathrm{GL}_2(k)$ which is not supercuspidal. Then according to the Bernstein-Zelevinsky classification, it must be a constituent of a *principal* *series.* This is to say, there exist characters χ_1 and χ_2 of k^* such that π is isomorphic to a subquotient of $\operatorname{Ind}^{\mathrm{GL}_2(k)} \chi_1 \boxtimes \chi_2$. Such a realization of π is almost unique. To describe the possibilities without repetition, we can reduce to three cases.

1. The first case is where π is isomorphic to $Q(\Delta_1, \Delta_2)$, where the singleton intervals $\Delta_i = \{\chi_i\}$ are not linked. In other words, $\chi_1 \neq \chi_2(\pm 1)$. In this case, $\pi = \operatorname{Ind} \chi_1 \boxtimes \chi_2$ is irreducible and infinite dimensional, and we have

$$L(s, \pi) = L(s, \chi_1 \oplus \chi_2) = L(s, \chi_1) L(s, \chi_2).$$

2. The second case is where π is isomorphic to $Q(\Delta_1, \Delta_2)$ as before, but the intervals Δ_i are linked. In order to arrange the "does not precede" condition, we must order the intervals so that $\chi_1 = \chi_2(1)$. If we set $\chi = \chi_1$, then π is the unique irreducible quotient of $\operatorname{Ind} \chi(1) \boxtimes \chi$. Actually, this quotient turns out to be one dimensional, so π is a character of $\mathrm{GL}_2(k)$. In terms of L-functions, we have

$$L(s, \pi) = L(s, \chi(1) \oplus \chi) = L(s+1, \chi) L(s, \chi).$$

3. The last case is where π is isomorphic to $Q(\Delta)$, where $\Delta = \{\chi, \chi(1)\}$ for a character χ of k^*. Here π is the unique irreducible quotient of $\operatorname{Ind} \chi \boxtimes \chi(1)$. It is infinite dimensional and essentially square integrable, and we have

$$L(s, \pi) = L(s, \chi \otimes \mathrm{Sp}(2)) = L(s+1, \chi).$$

It is also interesting to recall in this situation how the L-function can be read from the gamma factor in the tempered case (see (2.2.9) and (2.7.3)). Let us consider the third case above, where π is essentially square integrable and hence quasi-tempered. Even if π is not tempered in this case, the L-function can be read from the gamma factor. The underlying representation of the Weil group of $\chi \otimes \mathrm{Sp}(2)$ is $\chi \oplus \chi(1)$, and since the gamma factor only depends on the restriction to W_k, we have

$$\gamma(s,\pi,\psi) = \gamma(s,\chi\oplus\chi(1),\psi) = \epsilon(s,\chi\oplus\chi(1),\psi)\frac{L(1-s,\chi^{-1})L(-s,\chi^{-1})}{L(s,\chi)L(s+1,\chi)}.$$

Then $L(-s,\chi^{-1})/L(s,\chi)$ cancels out to become a constant times a monomial in q^{-s}, and can be absorbed into the epsilon factor. The L-function $L(s+1,\chi)$ can then be read off from here as the inverse of the numerator of $\gamma(s,\pi,\psi)$, realized as a simplified rational function in q^{-s}.

In the second and third cases, the underlying representations of W_k are the same, and so the gamma factors are identical. But in the second case, the L-function cannot be read from the gamma factor, because characters of $\mathrm{GL}_2(k)$ are never quasi-tempered.

To recover the L-functions from the gamma factors in the first and second cases as in (2.2.9), one must realize π in the Langlands classification and possibly apply multiplicativity, if π cannot be twisted by an unramified character to become tempered.

2.4 Weil restriction and local coefficients

In the next few sections, we will show the compatibility of gamma factors defined by the Langlands-Shahidi method in the setting of restriction of scalars (Theorem 2.4.9.1). Although it should be possible to prove compatibility in much greater generality than we do, we will work in a very special case, since it is all that we will need.

What we will do is follow the arguments from (2.2.15) to (2.2.20) and show that the local coefficient is independent of whether it is calculated in the setting of restriction of scalars or not. Since the gamma factor is related to the local coefficient by Shahidi's theorem (Theorem 2.2.20.1), we will then establish our desired result on gamma factors.

2.4.1 Definition of Weil restriction of scalars

Let K/k be a finite extension of fields of characteristic zero. The fiber product functor $\mathbf{H} \mapsto \mathbf{H} \times_k K$ from the category of linear algebraic groups over k to the category of linear algebraic groups over K has a right adjoint $\mathrm{Res}_{K/k}$, called the Weil restriction of scalars. That is, $\mathrm{Res}_{K/k}$ is a functor from the category of linear algebraic groups over K to the category of linear algebraic groups over k, such that for any linear algebraic groups $\mathbf{G}$ over k and $\mathbf{H}$ over K, there is a bijection

$$\mathrm{Hom}_{K\text{-grp}}(\mathbf{G} \times_k K, \mathbf{H}) \to \mathrm{Hom}_{k\text{-grp}}(\mathbf{G}, \mathrm{Res}_{K/k}\,\mathbf{H})$$

natural in $\mathbf{G}$ and $\mathbf{H}$. When $[K : k] = 2$, and $\mathbf{G}$ is a linear algebraic group over k, there is a particularly nice construction of $\underline{\mathbf{G}} = \mathrm{Res}_{K/k}(\mathbf{G} \times_k K)$ (which we abbreviate as simply $\mathrm{Res}_{K/k}(\mathbf{G})$) which will be used. Let $\overline{k}$ be an algebraic closure of k containing K, and let $\Gamma = \mathrm{Gal}(\overline{k}/k)$. As a group over $\overline{k}$, $\underline{\mathbf{G}}$ is given on closed points by

$$\underline{\mathbf{G}}(\overline{k}) = \mathbf{G}(\overline{k}) \times \mathbf{G}(\overline{k})$$

with Γ acting on $\underline{\mathbf{G}}(\overline{k})$ by

$$\tau.(x, y) = \begin{cases} (\tau(x), \tau(y)) & \text{if } \tau \in \mathrm{Gal}(\overline{k}/K) \\ (\tau(y), \tau(x)) & \text{if } \tau \notin \mathrm{Gal}(\overline{k}/K). \end{cases}$$

Thus $\underline{\mathbf{G}}(K) = \mathbf{G}(K) \times \mathbf{G}(K)$, and $\underline{\mathbf{G}}(k)$ identifies with $\mathbf{G}(K)$.

2.4.2 Some hypotheses on our groups

Let E/F be a quadratic extension of p-adic fields with nontrivial automorphism $\sigma \in \mathrm{Gal}(E/F)$. Let $\mathbf{G}$ be a split reductive group over F, regarded as a group over E. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ which is defined over F, and let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Let $\mathbf{P}$ be a maximal parabolic subgroup of $\mathbf{G}$ containing

$\mathbf{B}$, with Levi decomposition $\mathbf{P} = \mathbf{MN}$. Suppose that the adjoint action r of ${}^L\mathbf{M}$ on ${}^L\mathfrak{n}$ is irreducible.

For each algebraic variety Z which is defined over E, let $\underline{Z} = \mathrm{Res}_{E/F} Z$. Then $\underline{\mathbf{G}}$ is a quasi split group over F with Borel subgroup $\underline{\mathbf{B}}$ and maximal torus $\underline{\mathbf{T}}$, and $\underline{\mathbf{P}}$ is a maximal F-parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\underline{\mathbf{P}} = \underline{\mathbf{MN}}$.

2.4.3 Maximal and maximal split tori

Over an algebraic closure, we can identify the maximal torus $\underline{\mathbf{T}}$ of $\underline{\mathbf{G}}$ with the product $\mathbf{T} \times \mathbf{T}$. The image $\underline{\mathbf{S}}$ of the diagonal embedding $\mathbf{T} \to \underline{\mathbf{T}}$ is a maximal F-split torus of $\underline{\mathbf{G}}$, and under this isomorphism we have a natural identification of the character lattices $X(\mathbf{T})$ and $X(\underline{\mathbf{S}})$. This identifies the roots (resp. positive roots) (resp. simple roots) of $\mathbf{T}$ in $\mathbf{G}$ with respect to $\mathbf{B}$, with those of $\underline{\mathbf{S}}$ in $\underline{\mathbf{G}}$ with respect to $\underline{\mathbf{B}}$.

The diagonal embedding of $N_{\mathbf{G}}(\mathbf{T})$ into $\underline{\mathbf{G}}$ induces an isomorphism of the Weyl group $W(\mathbf{T}, \mathbf{G})$ of $\mathbf{T}$ in $\mathbf{G}$ with the Weyl group $W(\underline{\mathbf{S}}, \underline{\mathbf{G}})$ of $\underline{\mathbf{S}}$ in $\underline{\mathbf{G}}$, and under this isomorphism, the action of $W(\mathbf{T}, \mathbf{G})$ on $X(\mathbf{T})$ identifies with that of $W(\underline{\mathbf{S}}, \underline{\mathbf{G}})$ on $X(\underline{\mathbf{S}})$.

2.4.4 Identifying characters of M and $\underline{\mathbf{M}}$

The maximal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is defined by the choice of a simple root α of $\mathbf{T}$ in $\mathbf{B}$. If we identify α with a simple root of $\underline{\mathbf{S}}$ in $\underline{\mathbf{B}}$, then $\underline{\mathbf{P}} = \underline{\mathbf{MN}}$ is also defined by α.

We can identify $X(\underline{\mathbf{M}}) = X(\mathbf{M}) \times X(\mathbf{M})$, and for a character $(\chi_1, \chi_2) \in X(\underline{\mathbf{M}})$, we have that (χ_1, χ_2) is defined over F if and only if $\chi_2 = \sigma.\chi_1$. Thus we can identify $X(\underline{\mathbf{M}})_F = X(\mathbf{M})$, and hence $\mathfrak{a}^*_{\mathbf{M},\mathbb{C}} = \mathfrak{a}^*_{\underline{\mathbf{M}},\mathbb{C}}$.

We have $\underline{\mathbf{M}} = \mathbf{M} \times \mathbf{M}$, with $\underline{\mathbf{M}}(F) = \{(m, \sigma(m)) : m \in \mathbf{M}(E)\}$. Let $(m, \sigma(m)) \in \underline{\mathbf{M}}(F)$. and let $(\chi, \sigma.\chi)$ be an element of $X(\underline{\mathbf{M}})_F$. Then

$$\begin{aligned} q_F^{\langle(\chi,\sigma.\chi), H_{\underline{\mathbf{M}}}((m,\sigma.(m)))\rangle} &= |\chi(m)\sigma.\chi(\sigma(m)))|_F \\ &= |\chi(m)\sigma\chi(m)|_F \\ &= |N_{E/F}(\chi(m))|_F \\ &= |\chi(m)|_E \\ &= q_E^{\langle\chi, H_{\mathbf{M}}(m)\rangle}. \end{aligned}$$

From here we can immediately conclude the following. Suppose we identify:

- $\underline{\mathbf{G}}(F) = \mathbf{G}(E)$
- $\underline{\mathbf{P}}(F) = \mathbf{P}(E)$
- $\underline{\mathbf{M}}(F) = \mathbf{M}(E)$
- $\mathfrak{a}^*_{\mathbf{M},\mathbb{C}} = \mathfrak{a}^*_{\underline{\mathbf{M}},\mathbb{C}}$

If π is a smooth, irreducible representation of $\underline{\mathbf{M}}(F) = \mathbf{M}(E)$, and ν is an element of $\mathfrak{a}^*_{\mathbf{M},\mathbb{C}} = \mathfrak{a}^*_{\underline{\mathbf{M}},\mathbb{C}}$, then the representations $I_{\mathbf{M}}^{\mathbf{G}}\pi q_E^{\langle\nu, H_{\mathbf{M}}(-)\rangle}$ and $I_{\underline{\mathbf{M}}}^{\underline{\mathbf{G}}}\pi q_F^{\langle\nu, H_{\underline{\mathbf{M}}}(-)\rangle}$ of $\underline{\mathbf{G}}(F) = \mathbf{G}(E)$ obtained by normalized parabolic induction are equal.

Note that normalized parabolic induction is the same in both cases, because the modulus characters of $\underline{\mathbf{P}}(F) = \mathbf{P}(E)$ are equal.

2.4.5 Identifying Weyl group invariant forms

From any symmetric, nondegenerate, $W(\mathbf{G}, \mathbf{T})$-invariant bilinear form $(-,-)$ on $X(\mathbf{T})$, we get a symmetric, nondegenerate, $W(\underline{\mathbf{G}}, \underline{\mathbf{T}})$ and $\mathrm{Gal}(E/F)$-invariant form $(-,-)_1$ on $X(\underline{\mathbf{T}}) = X(\mathbf{T}) \oplus X(\mathbf{T})$ by

$$((\chi_1, \chi_1'), (\chi_2, \chi_2'))_1 = (\chi_1, \chi_2) + (\chi_1', \chi_2').$$

If ρ is half the sum of the roots of $\mathbf{T}$ in $\mathbf{B}$, then (ρ, ρ) is half the sum of the roots of $\underline{\mathbf{T}}$ in $\underline{\mathbf{B}}$. If α is the simple root of $\mathbf{T}$ in $\mathbf{G}$ which defines $\mathbf{P}$, then $(\alpha, 0)$ is a root of $\underline{\mathbf{T}}$ in $\underline{\mathbf{G}}$ whose restriction to the maximal split torus $\underline{\mathbf{S}}$ of $\mathbf{T}$ is α.

It is then straightforward to check that under the identification $\mathfrak{a}^*_{\mathbf{M},\mathbb{C}} = \mathfrak{a}^*_{\underline{\mathbf{M}},\mathbb{C}}$, the element $\tilde{\alpha}$ of (2.2.16) is the same whether it is calculated for $\underline{\mathbf{G}}$ or for $\mathbf{G}$.

2.4.6 Identifying splitting and Weyl group representatives

For each root β of $\mathbf{T}$ in $\mathbf{B}$, we have the root subgroup $\mathbf{U}_\beta$ of $\mathbf{U}$. If we interpret β as a root of $\underline{\mathbf{S}}$ in $\underline{\mathbf{B}}$, then the root subgroup of β in $\underline{\mathbf{U}}$ is the restriction of scalars $\underline{\mathbf{U}_\beta}$.

The simple roots of $\underline{\mathbf{T}}$ in $\underline{\mathbf{B}}$ come in pairs, as $\alpha_1 = (\alpha, 0)$ and $\alpha_2 = (0, \alpha)$, where α is a simple root of $\mathbf{T}$ in $\mathbf{G}$. Given a splitting $\mathbf{x}_\alpha : \mathbb{G}_a \to \mathbf{U}_\alpha : \alpha \in \Delta$ which is defined over E (that is, all the x_α are defined over E), we get a splitting for the simple roots of $\underline{\mathbf{T}}$ by taking the pairs

$$\mathbf{x}_{\alpha_1} : \mathbb{G}_a \xrightarrow{x_\alpha} \mathbf{U}_\alpha \xrightarrow{x \mapsto (x,1)} \underline{\mathbf{U}}_{\alpha_1}$$

$$\mathbf{x}_{\alpha_2} : \mathbb{G}_a \xrightarrow{\sigma . x_\alpha} \mathbf{U}_\alpha \xrightarrow{x \mapsto (1,x)} \underline{\mathbf{U}}_{\alpha_2}.$$

This splitting is defined over F. Thus we have two related splittings for $\mathbf{G}$ and $\underline{\mathbf{G}}$, through which we define canonical Weyl group representatives and generic characters. It follows from (2.2.4) that if we identify the normalizer of $\mathbf{T}(F)$ in $\mathbf{G}(F)$ with the normalizer of $\underline{\mathbf{S}}$ in $\underline{\mathbf{G}}(F)$ via $n \mapsto (n, n)$, then the canonical Weyl group representatives (2.2.17) for $W(\mathbf{G}, \mathbf{T})$ are the same as the canonical Weyl group representatives for $W(\underline{\mathbf{G}}, \underline{\mathbf{S}})$.

It also follows from (2.2.4) that if ψ is a nontrivial character of F, and we identify $\underline{\mathbf{U}}(F) = \mathbf{U}(E)$, then the generic character $\underline{\chi}$ of $\underline{\mathbf{U}}(F)$ coming from ψ and the splitting for $\underline{\mathbf{G}}$, is equal to the generic character χ of $\mathbf{U}(E)$ coming from $\psi \circ \mathrm{Tr}_{E/F}$ and the splitting for $\mathbf{G}$.

2.4.7 Local coefficients are unchanged by restriction of scalars

Now let π be a smooth, irreducible representation of $\underline{\mathbf{M}}(F) = \mathbf{M}(E)$. Then π is generic with respect to χ if and only if it is generic with respect to $\underline{\chi}$. The induced representation $I(s,\pi)$ is the same whether we are considering it as a representation of $\underline{\mathbf{G}}(F)$ or $\mathbf{G}(E)$.

The self dual Haar measure on $\mathbf{N}'(F) = \underline{\mathbf{N}'}(E)$ of (2.2.18), is the same, regardless of the field over which the group is considered, a χ-Whittaker functional for π is the same as a $\underline{\chi}$-Whittaker functional for π. The canonical Weyl group representative w_0 is same regardless of the field over which the groups are considered. So is the intertwining operator $A(s,\pi)$ and the Whittaker functionals $\lambda_\chi(s,\pi) = \lambda_{\underline{\chi}}(s,\pi)$. It follows from here that

$$C_\chi(s,\pi) = C_{\underline{\chi}}(s,\pi). \tag{2.4.7.1}$$

That is, the local coefficient does not depend on whether we consider π as a generic representation of $\mathbf{M}(E)$ or of $\underline{\mathbf{M}}(F)$.

2.4.8 Adjoint action of the L-group

Let r be the adjoint action of ${}^L\mathbf{M}$ on ${}^L\mathfrak{n}$. Since $\mathbf{M}$ is split, we will identify ${}^L\mathbf{M} = {}^L\mathbf{M}^\circ$. Recall we are assuming that r is irreducible. The L-group of $\underline{\mathbf{M}}$ can be identified with the semidirect product of ${}^L\mathbf{M} \times {}^L\mathbf{M}$ by $\mathrm{Gal}(E/F)$, where σ acts by $\sigma.(x,y) = (y,x)$.

The Lie algebra ${}^L\underline{\mathfrak{n}}$ identifies with ${}^L\mathfrak{n} \oplus {}^L\mathfrak{n}$, and the adjoint action $\underline{r}$ of ${}^L\underline{\mathbf{M}}$ on ${}^L\underline{\mathfrak{n}}$ is then given by

$$\underline{r}(x,y,1)(X,Y) = (r(x)X, r(y)Y)$$

$$\underline{r}(\sigma)(X,Y) = (Y,X).$$

Then $\underline{r}$ is clearly irreducible.

2.4.9 Gamma factors and Weil restriction of scalars

For π a χ-generic representation of $\mathbf{M}(E)$, the local coefficient and gamma factor are related by Shahidi's theorem (2.2.20)

$$C_\chi(s,\pi) = \gamma(s,\pi,r,\psi\circ\mathrm{Tr}_{E/F}).$$

Since $\mathbf{M}$ is split over E, the number $\lambda(w_0,\psi\circ\mathrm{Tr}_{E/F})$ showing up in the formula of Shahidi's theorem in (2.3.6) will be 1.

If we consider π as a $\underline{\chi}$-generic representation of $\underline{\mathbf{M}}(F)$, then the number $\lambda(w_0,\psi)$ is equal to $\lambda(E/F,\psi)^{\mathrm{Dim}\,\mathbf{N}}$. Then Shahidi's formula tells us that

$$C_{\underline{\chi}}(s,\pi) = \lambda(E/F,\psi)^{-\mathrm{Dim}\,\mathbf{N}'}\gamma(s,\pi,\underline{r},\psi).$$

Putting this together with equation (2.4.7.1), we have

$$\lambda(E/F,\psi)^{-\mathrm{Dim}\,\mathbf{N}'}\gamma(s,\pi,\underline{r},\psi) = \gamma(s,\pi,r,\psi\circ\mathrm{Tr}_{E/F}).$$

We state this as a theorem:

Theorem 2.4.9.1. *Let E/F be a quadratic extension of p-adic fields. Let $\mathbf{M}$ be a split reductive group over F, base changed to E. Let r be an LS-representation of ${}^L\mathbf{M}$, and let $\underline{r}$ be the representation of (2.4.8). Suppose that $\mathbf{M}$ is isomorphic to a Levi subgroup $\mathbf{M}_0$ of a maximal parabolic subgroup $\mathbf{P}_0 = \mathbf{M}_0\mathbf{N}_0$ of some reductive group, and that under this isomorphism the representation r of ${}^L\mathbf{M}$ identifies with the adjoint action of ${}^L\mathbf{M}_0$ on ${}^L\mathfrak{n}_0$. Then $\underline{r}$ is an LS-representation of $\underline{\mathbf{M}} = \mathrm{Res}_{E/F}\,\mathbf{M}$, and*

$$\gamma(s,\pi,\underline{r},\psi) = \lambda(E/F,\psi)^{\mathrm{Dim}\,r}\gamma(s,\pi,r,\psi\circ\mathrm{Tr}_{E/F})$$

for any generic representation π of $\mathbf{M}(E)$.

2.5 Weil restriction and Rankin products

Still E/F is a quadratic extension of p-adic fields, and ψ is a nontrivial character of F.

2.5.1 Some computations with the L-group of $\operatorname{Res}_{E/F}\mathrm{GL}_n \times \mathrm{GL}_m$

We will apply Theorem 2.4.9.1 in particular to the group $\mathbf{M} = \mathrm{GL}_n \times \mathrm{GL}_m$. Let V and W be complex vector spaces of dimensions n and m. We can identify the L-group of $\mathbf{M}$ with $\mathrm{GL}(V) \times \mathrm{GL}(W)$. Let R be the tensor product representation of ${}^L\mathbf{M}$:

$$R : {}^L\mathbf{M} \to \mathrm{GL}(V \otimes W)$$

$$R(T, S) = T \otimes S.$$

This representation satisfies the hypothesis of Theorem 2.4.9.1, since

$$(g, h) \mapsto \begin{pmatrix} g & \\ & {}^t h^{-1} \end{pmatrix}$$

is an isomorphism of $\mathbf{M}$ onto a maximal Levi subgroup of GL_{n+m}, under which R identifies with corresponding adjoint action.

We can identify the L-group of $\underline{\mathbf{M}} = \operatorname{Res}_{E/F}\mathrm{GL}_n \times \mathrm{GL}_m$ with

$$\mathrm{GL}(V) \times \mathrm{GL}(W) \times \mathrm{GL}(V) \times \mathrm{GL}(W) \rtimes \mathrm{Gal}(E/F)$$

where $\mathrm{Gal}(E/F)$ acts by $\sigma.(T_1, S_1, T_2, S_2) = (T_2, S_2, T_1, S_1)$.

The corresponding representation $\underline{R}$ of (2.4.8) has underlying space $(V \otimes W) \oplus (V \otimes W)$, and is given by

$$(T_1, S_1, T_2, S_2).(v \otimes w, v' \otimes w') = (T_1 v \otimes S_1 w, T_2 v' \otimes S_2 w')$$

$$\sigma.(v \otimes w, v' \otimes w') = (v' \otimes w', v \otimes w).$$

Theorem 2.4.9.1 tells us that if π_1 and π_2 are generic representations of $\mathrm{GL}_n(E)$ and $\mathrm{GL}_m(E)$, then

$$\begin{aligned}\gamma(s, \pi_1 \boxtimes \pi_2, \underline{R}, \psi) &= \lambda(E/F, \psi)^{nm} \gamma(s, \pi_1 \boxtimes \pi_2, R, \psi \circ \mathrm{Tr}_{E/F}) \\ &= \lambda(E/F, \psi)^{nm} \gamma(s, \pi_1 \times \pi_2, \psi \circ \mathrm{Tr}_{E/F}).\end{aligned}$$

If π_1 and π_2 are not necessarily generic, then this last equation still holds, since π_1 and π_2 have generic inducing data. One applies multiplicativity on both sides ((2.2.6) and Example 2.2.7.1).

2.5.2 The peculiar representation $\mathfrak{R}$

In this section we describe the properties of a particular L-group representation which will show up in the multiplicativity of Asai gamma factors.

Again let $\underline{\mathbf{M}} = \mathrm{Res}_{E/F} \mathrm{GL}_n \times \mathrm{GL}_m$. Let V and W be complex vector spaces of dimensions n and m, so that the L-group of $\underline{\mathbf{M}}$ again identifies with the semidirect product of $\mathrm{GL}(V) \times \mathrm{GL}(W) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ by $\mathrm{Gal}(E/F)$, with $\mathrm{Gal}(E/F)$ acting by sending (T_1, S_1, T_2, S_2) to (T_2, S_2, T_1, S_1).

Define a representation $\mathfrak{R}$ of ${}^L\underline{\mathbf{M}}$ with underlying space $(V \otimes W) \oplus (V \otimes W)$, by

$$\mathfrak{R}(T_1, S_1, T_2, S_2)(v \otimes w, v' \otimes w') = (T_1 v \otimes S_2 w, T_1 v' \otimes S_1 w')$$

$$\mathfrak{R}(\sigma)(v \otimes w, v' \otimes w') = (v' \otimes w', v \otimes w)$$

Define an automorphism φ of $\underline{\mathbf{M}}$ by $(x, y, x', y') \mapsto (x,\, y', x',\, y)$. Then $\varphi = \varphi^{-1}$ is defined over F, and if we identify $\underline{\mathbf{M}}(F) = \mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$, then φ is defined on points by

$$\varphi(x, y) = (x,\, \sigma(y)).$$

Let us state this as a lemma.

Lemma 2.5.2.1. *If $\pi = \pi_1 \boxtimes \pi_2$ is a smooth, irreducible representation of $\underline{\mathbf{M}}(F) = \mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$, then $\pi \circ \varphi$ is isomorphic to the representation $\pi_1 \boxtimes (\pi_2 \circ \sigma)$, where σ is the nontrivial element of $\mathrm{Gal}(E/F)$.*

Now φ induces an automorphism $\varphi^\vee$ of ${}^L\underline{\mathbf{M}}$, which sends (T_1, S_1, T_2, S_2) to (T_1, S_2, T_2, S_1).

Recall the LS-representation $\underline{R}$ of the last section. It is straightforward to check that $\underline{R} \circ \varphi^\vee \cong \mathfrak{R}$, and therefore by (2.2.2), for any representation $\pi = \pi_1 \boxtimes \pi_2$ of $\underline{\mathbf{M}}(F)$, we have

$$\gamma(s, \pi \boxtimes \pi_2, \mathfrak{R}, \psi) = \gamma(s, \pi_1 \boxtimes (\pi_2 \circ \sigma), \underline{R}, \psi).$$

Putting this together with (2.5.1), we have:

Proposition 2.5.2.2. *Suppose that $\pi_1 \boxtimes \pi_2$ is an irreducible, admissible representation of $\underline{\mathbf{M}}(F) = \mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$. Then*

$$\gamma(s, \pi_1 \boxtimes \pi_2, \mathfrak{R}, \psi) = \lambda(E/F, \psi)^{nm} \gamma(s, \pi_1 \times (\pi_2 \circ \sigma), \psi \circ \mathrm{Tr}_{E/F}).$$

Corollary 2.5.2.3. *Suppose that $\pi_1 \boxtimes \pi_2$ is an irreducible, admissible representation of $\underline{\mathbf{M}}(F) = \mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$. Then*

$$L(s, \pi_1 \boxtimes \pi_2, \mathfrak{R}) = L(s, \pi_1 \times (\pi_2 \circ \sigma)).$$

Proof: The equality of gamma factors $\gamma(s, \pi_1 \boxtimes \pi_2, \mathfrak{R}, \psi)$ and $\gamma(s, \pi_1 \times \pi_2 \circ \sigma, \psi \circ \mathrm{Tr}_{E/F})$, up to a root of unity, suffices to deduce the equality of the L-functions. The L-functions on both sides are initially defined for tempered representations. Since a polynomial in q_E^{-s} is also a polynomial in q_F^{-s}, we see from the definition of the L-function in the tempered case that the L-functions agree when $\pi_1 \boxtimes \pi_2$ is tempered.

We also get the equality of L-functions in the quasi-tempered case, since a twist by an unramified character affects both L-functions in the same way.

We then get the equality of L-functions for general π_1 and π_2, since Langlands classification is the same whether we formulate it for $\underline{\mathbf{M}}(F)$ or for $\mathrm{GL}_n(E) \times \mathrm{GL}_m(E)$, and multiplicativity is analogous for both groups. $\square$

2.6 The Asai representation

We finally state our main result in this section. In (2.6.1), we define the Asai representation $\mathscr{R}$, which is an LS-representation of the L-group of $\operatorname{Res}_{E/F} \mathrm{GL}_n$, where E/F is a quadratic extension of characteristic zero local fields. The gamma, epsilon, and L factors as defined by the Langlands-Shahidi method will be called **Asai factors**.

In (2.6.2), we state the local Langlands correspondence for $\operatorname{Res}_{E/F} \mathrm{GL}_n$ in terms of the local Langlands correspondence for GL_n, using 8.1 of [Bo79]. If $\underline{\rho}'$ is an admissible homomorphism from the Weil-Deligne group W_F' into the L-group of $\operatorname{Res}_{E/F} \mathrm{GL}_n$, then the Frobenius semisimple representation $\mathscr{R} \circ \underline{\rho}'$ will be called the **Asai representation on the Galois side**, and the corresponding Artin gamma, epsilon, and L-functions will be called **Asai factors on the Galois side**.

In (2.6.3), we state our main theorem, which is the equality of the Asai epsilon factor with the corresponding Asai epsilon factor on the Galois side. We also cite two of Henniart's results in [He10]. First, Henniart proved the equality of the Asai gamma factors on both sides, up to a root of unity. Next, Henniart used this equality to sketch a proof of the equality of the Asai L-functions on both sides. In Section 2.7, we will flesh out Henniart's argument in detail and reprove the equality of the Asai L-functions, granting the equality of Asai gamma factors up to a root of unity.

We remark that, since we know that the Asai L-functions on both sides are equal, an equivalent formulation of our main theorem is the exact equality of the Asai gamma factors on both sides. This is the theorem which we will actually prove.

In (2.6.4), we state how our factors are affected unramified twists. Finally in (2.6.5), we prove a formula for the multiplicativity of the Asai gamma factor.

2.6.1 Definition of the Asai representation

Let E/F be a quadratic extension of characteristic zero local fields, and let $\mathbf{M} = \operatorname{Res}_{E/F} \mathrm{GL}_n$. As in (2.4.1), we identify $\mathbf{M}(\overline{F}) = \mathrm{GL}_n(\overline{F}) \times \mathrm{GL}_n(\overline{F})$. Let

$\Gamma_E = \mathrm{Gal}(E/F)$, with nontrivial automorphism σ. We take the Borel subgroup and maximal torus of $\mathbf{M}$ to be the product of the usual ones of GL_n.

The L-group ${}^L\mathbf{M}$ can be identified with the semidirect product of ${}^L\mathbf{M}^\circ = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ by Γ_E, where σ acts on $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ by $\sigma.(x, y) = (y, x)$. In fact, let V be an n-dimensional complex vector space. Then we can identify ${}^L\mathbf{M}^\circ$ with $\mathrm{GL}(V) \times \mathrm{GL}(V)$.

The **Asai representation** $\mathscr{R} = \mathscr{R}_n$ is defined to be the following representation of ${}^L\mathbf{M}$ with underlying space $V \otimes V$:

$$\mathscr{R}(T, S, \sigma^n) v \otimes v' = \begin{cases} Tv \otimes Sv & \text{for } n = 0 \\ Tv' \otimes Sv & \text{for } n = 1. \end{cases}$$

It is an LS-representation, corresponding to the adjoint action of $\mathbf{M}$ embedded as a maximal Siegel levi inside the unitary group $U(n, n)$ of E/F (5.1.2).

2.6.2 Restriction of scalars and tensor induction

Since we have a local Langlands correspondence for GL_n ((2.3.7) in the nonarchimedean case and (2.1.8) in the archimedean), we also have one for $\mathbf{M}$ by (2.1.9). Specifically, let V be an n-dimensional complex vector space, and let

$$\rho' : W'_E \to \mathrm{GL}(V) = {}^L\mathrm{GL}_{n,E}$$

be an admissible homomorphism of W'_E (2.1.7). Choose a $z \in W_F$ which is not in W_E. Define a homomorphism

$$\underline{\rho'} : W'_F \to \mathrm{GL}(V) \times \mathrm{GL}(V) \rtimes \mathrm{Gal}(E/F) = {}^L\mathbf{M}$$

by

$$\underline{\rho'}(a) = \begin{cases} (\rho'(a), \rho'(zaz^{-1}), 1) & \text{if } a \in W'_E \\ (\rho'(az^{-1}), \rho'(za), \sigma) & \text{if } a \notin W'_E. \end{cases}$$

Up to equivalence, this does not depend on the choice of z. This association $\rho' \mapsto \underline{\rho'}$ is exactly the bijection defined by Borel in (8.1, [Bo79]).

Proposition 2.6.2.1. *The representation $\mathscr{R} \circ \underline{\rho'} : W'_F \to {}^L\mathbf{M}$ is isomorphic to $\otimes \operatorname{Ind}_{E/F} \rho'$, the representation of W'_F obtained from ρ' by tensor induction (1.1.17).*

Proof: Direct computation. □

2.6.3 Statement of our main results

We now state our main theorem. Suppose E/F is a quadratic extension of characteristic zero local fields, and $\mathbf{M} = \operatorname{Res}_{E/F} \operatorname{GL}_{n,E}$. If π is an irreducible, admissible representation of $\mathbf{M}(F) = \operatorname{GL}_n(E)$, and ψ is a nontrivial unitary character of F (there is no need to specify that ψ is unitary when F is nonarchimedean), the gamma factor $\gamma(s, \pi, \mathscr{R}, \psi)$, L-function $L(s, \pi, \mathscr{R})$, and epsilon factor $\epsilon(s, \pi, \mathscr{R}, \psi)$ are all defined by the Langlands Shahidi method, where $\mathscr{R}$ is the Asai representation of (2.6.1).

If ρ' is the n dimensional representation of W'_E corresponding to π under the local Langlands correspondence for $\operatorname{GL}_n(E)$, define $\underline{\rho'}$ as in (2.6.2). Then $\pi \mapsto \underline{\rho'}$ is the local Langlands correspondence for $\mathbf{M}$.

Let us state formally what we want to prove with our Asai factors. As we explain below, some of this has already been proved by Henniart in [He10], and some of this is new.

Theorem 2.6.3.1. *Let π be an irreducible, admissible representation of $\operatorname{GL}_n(E)$, and let $\underline{\rho'}$ correspond to π as above. Then:*

(i): (Equality of Asai epsilon factors)

$$\epsilon(s, \pi, \mathscr{R}, \psi) = \epsilon(s, \mathscr{R} \circ \underline{\rho'}, \psi)$$

(ii): (Equality of Asai gamma factors)

$$\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \mathscr{R} \circ \underline{\rho}', \psi)$$

(iii): (Equality of Asai L-functions)

$$L(s, \pi, \mathscr{R}) = L(s, \mathscr{R} \circ \underline{\rho}').$$

From the general results of the Langlands-Shahidi method, we can already conclude that Theorem 2.6.3.1 holds in two particular cases.

Remark 2.6.3.2. *Theorem 2.6.3.1 holds in the archimedean case. It also holds in the p-adic case when π has an Iwahori fixed vector.*

Proof: In either case, the passage from ρ' to $\underline{\rho}'$ is compatible with the existing local Langlands correspondence for archimedean groups or tori (2.1.9). As we remarked in (2.12), in the archimedean case, the equality of the Langlands-Shahidi gamma factor with the corresponding Artin gamma factor is a theorem of Shahidi ([Sh90], Theorem 3.5, (1)). The Langlands-Shahidi L and epsilon factors are by definition the corresponding Artin factors.

In the nonarchimedean case when π has an Iwahori fixed vector, [Sh90], Theorem 3.5 (1) also takes care of the equality of gamma factors on both sides. The equality of L factors in this case is implied by Henniart's result which we state below. The equality of epsilon factors falls out as a result of the equality of gamma and L factors. $\square$

Now we cite two of Henniart's results (Theorem 5.2 of [He10]).

Theorem 2.6.3.3. *(Henniart) Assume F is p-adic. There exists a root of unity ζ such that*

$$\zeta\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \mathscr{R} \circ \underline{\rho}', \psi).$$

Theorem 2.6.3.4. *(Henniart) Assume F is p-adic. The equality of L-functions ((iii) of Theorem 2.6.3.1) holds.*

We have stated our main results in a way that is amenable to the local Langlands correspondence for general reductive groups. In light of Proposition 2.6.2.1 and the isomorphism $\mathscr{R}\circ\underline{\rho'} \cong \otimes \operatorname{Ind}_{E/F} \rho'$, we can restate our main theorem in a more practical way.

Theorem 2.6.3.5. *(Equivalent formulation of Theorem 2.6.3.1) Let π be an irreducible, admissible representation of $\mathrm{GL}_n(E)$, and let ρ' be the representation of W'_E corresponding to π under the local Langlands correspondence for GL_n. Then: (i): (Equality of Asai epsilon factors)*

$$\epsilon(s, \pi, \mathscr{R}, \psi) = \epsilon(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi)$$

(ii): (Equality of Asai gamma factors)

$$\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi)$$

(iii): (Equality of Asai L-functions)

$$L(s, \pi, \mathscr{R}) = L(s, \otimes \operatorname{Ind}_{E/F} \rho')$$

where $\otimes \operatorname{Ind}_{E/F} \rho'$ is the representation of W'_F obtained by tensor induction.

As we remarked above, Henniart had proved the equality of Asai gamma factors up to a root of unity. The main result of this thesis is the exact equality of the Asai gamma factors:

Main Theorem 2.6.3.6. *The equality of gamma factors ((ii) of Theorem 2.6.3.1, or equivalently its reformulation in Theorem 2.6.3.5), holds. That is,*

$$\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi).$$

As a consequence of our main theorem, combined with Henniart's result of the equality of L factors, we get the equality of the epsilon factors.

Corollary 2.6.3.7. *The equality of epsilon factors ((i) of Theorem 2.6.3.1, or equivalently its reformulation in Theorem 2.6.3.5), holds. That is,*

$$\epsilon(s, \pi, \mathscr{R}, \psi) = \epsilon(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi).$$

Following the corollary our result, together with Henniart's results, finally establishes the entirety of Theorem 2.6.3.1.

2.6.4 Unramified twists and Asai factors

Assume that E/F is p-adic. As we will show in Chapter Five, the Asai representation is defined by embedding $\mathbf{M} = \operatorname{Res}_{E/F} \mathrm{GL}_n$ as a Siegel Levi subgroup of the unitary group $\mathbf{G} = U(n,n)$. If $\tilde{\alpha}$ is as in (2.2.16), it is a consequence of Lemma 5.1.3.1 (i) that

$$q_F^{\langle s\tilde{\alpha}, H_{\mathbf{M}}(m)\rangle} = |\det(m)|_E^{\frac{s}{2}}$$

for all $m \in \mathrm{GL}_n(E) = \mathbf{M}(F)$. It follows from (2.2.21) that

$$\gamma(s, \pi|\det(-)|_E^{s_0}, \mathscr{R}, \psi) = \gamma(s + 2s_0, \pi, \mathscr{R}, \psi)$$

$$L(s, \pi|\det(-)|_E^{s_0}, \mathscr{R}) = L(s + 2s_0, \pi, \mathscr{R}).$$

Let ρ' be a representation of W_E'. By Lemma 1.1.9.1 and Theorem 1.1.16.1, part (iv), we see that the same effect from an unramified twist occurs on the Artin side:

$$\gamma(s, \otimes \operatorname{Ind}_{E/F} \rho'||\cdot||^{s_0}, \psi) = \gamma(s + 2s_0, \otimes \operatorname{Ind}_{E/F} \rho', \psi)$$

$$L(s, \otimes \operatorname{Ind}_{E/F} \rho'||\cdot||^{s_0}, \psi) = L(s + 2s_0, \otimes \operatorname{Ind}_{E/F} \rho', \psi).$$

2.6.5 Multiplicativity of Asai gamma factors

Again assume that E/F is p-adic. Let $\mathbf{M}_* = \operatorname{Res}_{E/F} \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ be a standard Levi subgroup of $\mathbf{M}$. Let π be an irreducible, admissible representation of $\mathbf{M}_*(F) = \mathrm{GL}_n(E)$, and suppose that π is isomorphic to a quotient of $I_{\mathbf{M}_*}^{\mathbf{M}} \pi_1 \boxtimes \pi_2$, for irreducible admissible representations π_i of $\mathrm{GL}_{n_i}(E)$.

In order to apply multiplicativity, we must consider the restriction of the Asai representation $\mathscr{R} = \mathscr{R}_n$ to ${}^L\mathbf{M}_*$.

Let V be an n dimensional complex vector space, and let V_1 and V_2 be n_1 and n_2 dimensional subspaces of V such that $V_1 \oplus V_2 = V$. Then we can identify

$$ {}^L\mathbf{M} = \mathrm{GL}(V) \times \mathrm{GL}(V) \rtimes \mathrm{Gal}(E/F) $$

$$ {}^L\mathbf{M}_* = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \rtimes \mathrm{Gal}(E/F). $$

Now we can break up $V \otimes V$ into three subspaces: $V_1 \otimes V_1$, $V_2 \otimes V_2$, $(V_1 \otimes V_2) \oplus (V_1 \otimes V_2)$. We see that each subspace is stable under the action of ${}^L\mathbf{M}_*$, and:

- The subrepresentation $(V_1 \otimes V_2) \oplus (V_1 \otimes V_2)$ of ${}^L\mathbf{M}_*$ is isomorphic to the representation $\mathfrak{R}$ of (2.5.2).
- The restriction of $\mathscr{R}|_{{}^L\mathbf{M}_*}$ to $\mathrm{GL}(V_1) \times \mathrm{GL}(V_1) \rtimes \mathrm{Gal}(E/F)$ is trivial, and the restriction to $\mathrm{GL}(V_2) \times \mathrm{GL}(V_2) \rtimes \mathrm{Gal}(E/F)$ is isomorphic to $\mathscr{R}_{n_2}$.
- The restriction of $\mathscr{R}|_{{}^L\mathbf{M}_*}$ to $\mathrm{GL}(V_2) \times \mathrm{GL}(V_2) \rtimes \mathrm{Gal}(E/F)$ is trivial, and the restriction to $\mathrm{GL}(V_1) \times \mathrm{GL}(V_1) \rtimes \mathrm{Gal}(E/F)$ is isomorphic to $\mathscr{R}_{n_1}$.

Therefore, by (2.2.6) and (2.2.7), we can conclude that

$$ \gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \pi_1, \mathscr{R}_{n_1}, \psi)\gamma(s, \pi_2, \mathscr{R}_{n_2}, \psi)\gamma(s, \pi_1 \boxtimes \pi_2, \mathfrak{R}, \psi). $$

But on account of Proposition 2.5.2.2, this last gamma factor is really that of a Rankin product:

$$\gamma(s,\pi,\mathscr{R},\psi)=\gamma(s,\pi_1,\mathscr{R}_{n_1},\psi)\gamma(s,\pi_2,\mathscr{R}_{n_2},\psi)\lambda(E/F,\psi)^{n_1n_2}\gamma(s,\pi_1\times(\pi_2\circ\sigma),\psi\circ\mathrm{Tr}_{E/F})$$

where σ is the nontrivial element of $\mathrm{Gal}(E/F)$. Applying the same reasoning to the general case, we have:

Theorem 2.6.5.1. *Suppose that π is an irreducible, admissible representation of $\mathrm{Res}_{E/F}\,\mathrm{GL}_n(F)=\mathrm{GL}_n(E)$, and π is isomorphic to a quotient of $\mathrm{Ind}^{\mathrm{GL}_n(E)}\,\pi_1\boxtimes\cdots\boxtimes\pi_r$. Then*

$$\gamma(s,\pi,\mathscr{R},\psi)=\prod_{i=1}^{r}\gamma(s,\pi_i,\mathscr{R}_{n_i},\psi)\prod_{1\le i<j\le r}\lambda(E/F,\psi)^{n_in_j}\gamma(s,\pi_i\times(\pi_j\circ\sigma),\psi\circ\mathrm{Tr}_{E/F})$$

where π_i is a representation of $\mathrm{GL}_{n_i}(E)$.

Thus multiplicativity allows us to write an Asai gamma factor as a product of smaller dimension Asai gamma factors, and a product of Rankin product gamma factors.

Remark 2.6.5.2. *In the case where $\pi_1\boxtimes\cdots\boxtimes\pi_r$ is quasi-tempered with positive Langlands parameter, Goldberg calculates this same multiplicativity formula for $\gamma(s,\pi,\mathscr{R},\psi)$ (Section 5 of [Go94]).*

2.7 Equality of gamma factors implies equality of L-functions

Again E/F is p-adic. In this section we give a detailed exposition of Henniart's proof of the equality of L-functions (Theorem 2.6.3.4), granting his result that the gamma factors are equal, up to a root of unity (Theorem 2.6.3.3).

The idea is to first prove the equality of L-functions for tempered representations, making use of the general result that the L-functions for tempered representations (on either the analytic or Galois side) can be read off from the gamma factors.

The equality for L-functions of arbitrary representations following from the tempered case using Langlands classification.

2.7.1 The tempered case I

Let $\rho' = (\rho, V, N)$ be a representation of the Weil-Deligne group W'_F. Recall that (1.2.7)

$$L(s, \rho') = \det(1 - q^{-s}\rho_N(\Phi))^{-1}$$

where ρ_N is the kernel of N, and $q = q_F$. The function $L(s, \rho')^{-1}$ is a polynomial in the variable q^{-s}. Let us write $L(s, \rho')^{-1} = F(q^{-s})$, for a unique polynomial $F(X) \in \mathbb{C}[X]$. It is clear that for $\sigma \in \mathbb{R}$,

$$\lim_{\sigma \to \infty} L(\sigma, \rho') = 1.$$

Therefore, we must have $F(0) = 1$.

2.7.2 The tempered case II

Suppose that H is a polynomial in $\mathbb{C}[X]$ with $H(0) = 1$. There is a unique rational function $G_1 \in \mathbb{C}(X)$ such that $H(q^{s-1}) = G_1(q^{-s})$. If t is the degree of H, then there is a unique polynomial $G \in \mathbb{C}[X]$ and a unique $c_1 \in \mathbb{C}$ such that $G_1(X) = c_1 X^{-t} G(X)$, with $G(0) = 1$.

2.7.3 The tempered case III

Let π be a tempered representation of $\mathrm{GL}_n(E)$, and let ρ' be the corresponding representation of W'_E. By Theorem 2.6.3.3 of Henniart, we know that

$$\zeta\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi) \tag{2.2.7.1}$$

for some root of unity ζ. We will show that $L(s,\pi,\mathscr{R}) = L(s,\otimes \operatorname{Ind}_{E/F}\rho')$. There exists a rational function $f \in \mathbb{C}(X)$ such that

$$\gamma(s,\pi,\mathscr{R},\psi) = f(q^{-s}).$$

We can find unique $\alpha \in \mathbb{C}, a \in \mathbb{Z}$, and $P, Q \in \mathbb{C}[X]$ such that $f(X) = \alpha X^a \frac{P(X)}{Q(X)}$, with $P(0) = Q(0) = 1$. Since π is tempered, we know that $L(s,\pi,\mathscr{R}) = P(q^{-s})^{-1}$ (2.2.9).

Now, let F and H be polynomials for which

$$L(s,\otimes \operatorname{Ind}_{E/F}\rho',\psi) = F(q^{-s})^{-1}, L(s,\otimes \operatorname{Ind}_{E/F}\rho'^{\vee},\psi) = H(q^{-s})^{-1}.$$

We know that $F(0) = H(0) = 1$ (2.7.1). We also know that, since π is tempered, so are ρ' and $\rho'^{\vee}$ (2.3.8), and so are $\otimes \operatorname{Ind}_{E/F}\rho'$ and $\otimes \operatorname{Ind}_{E/F}\rho'^{\vee}$ (Proposition 1.2.11.1). Hence $F(q^{-s})$ and $H(q^{-s})$ have no zeroes for $\operatorname{Re}(s) > 0$ (Lemma 1.2.10.1). This implies that $F(q^{-s})$ and $H(q^{s-1})$ have no zeroes in common whatsoever.

Let c, t, G_1, G be as in (2.7.2). Then

$$H(q^{s-1}) = G_1(q^{-s}) = c_1 q^{ts} G(q^{-s})$$

and we see that $G(q^{-s})$ and $F(q^{-s})$ have no zeroes in common. Therefore, the polynomials $G(X)$ and $F(X)$ must be relatively prime.

We also have $\epsilon(s,\otimes \operatorname{Ind}_{E/F}\rho',\psi) = cq^{ns}$ for some $c \in \mathbb{C}$ and $n \in \mathbb{Z}$. Writing

$$\begin{aligned}\gamma(s,\otimes \operatorname{Ind}_{E/F}\rho',\psi) &= \frac{\epsilon(s,\otimes \operatorname{Ind}_{E/F}\rho',\psi)L(1-s,\otimes \operatorname{Ind}_{E/F}\rho'^{\vee})}{L(s,\otimes \operatorname{Ind}_{E/F}\rho')} \\ &= cc_1^{-1}q^{(n-t)s}\frac{F(q^{-s})}{G(q^{-s})}\end{aligned}$$

and putting this together with equation (2.2.7.1) we get

$$\zeta\alpha q^{-as}\frac{P(q^{-s})}{Q(q^{-s})} = cc_1^{-1}q^{(n-t)s}\frac{F(q^{-s})}{G(q^{-s})}$$

and hence

$$\zeta\alpha X^a \frac{P(X)}{Q(X)} = cc_1^{-1} X^{(t-n)} \frac{F(X)}{G(X)}.$$

The pairs of polynomials P, Q and F, G are relatively prime, and satisfy $P(0) = Q(0) = F(0) = G(0) = 1$. We must conclude that $cc_1^{-1} = \alpha, t - n = a$, and $F(X) = P(X), G(X) = Q(X)$. In particular,

$$L(s, \pi, \mathscr{R}) = P(q^{-s})^{-1} = F(q^{-s})^{-1} = L(s, \otimes \operatorname{Ind}_{E/F} \rho').$$

This completes the proof of Theorem 2.2.6.4 in the case where π is tempered.

2.7.4 Proof of the quasi-tempered case

Suppose that π is a quasi-tempered representation of $\mathrm{GL}_n(E)$. Let ρ' be the corresponding representation of W'_E. There exists a complex number s_0 such that $\pi|\det(-)|^{s_0}$ is tempered. Then by (2.6.4) and (2.7.3), we have

$$\begin{aligned} L(s, \pi, \mathscr{R}) &= L(s - 2s_0, \pi|\det(-)|^{s_0}, \mathscr{R}) \\ &= L(s - 2s_0, \otimes \operatorname{Ind}_{E/F} \rho' || \cdot ||^{s_0}) \\ &= L(s, \otimes \operatorname{Ind}_{E/F} \rho'). \end{aligned}$$

This completes the proof of Theorem 2.2.7.4 in the case when π is quasi-tempered.

2.7.5 The general case

Finally, suppose that π is an arbitrary smooth, irreducible representation of $\mathrm{GL}_n(E)$, and ρ' is the corresponding representation of W'_E. Recall that the L-function $L(s, \pi, \mathscr{R})$ is defined using Langlands classification for quotients ((2.2.8) and (2.2.9)). We use the relationship between Bernstein-Zelevinsky classification and Langlands classification given in (2.3.6).

Let $\Delta_1, ..., \Delta_r$ be intervals such that $\pi = Q(\Delta_1, ..., \Delta_r)$ in the Bernstein-Zelevinsky classification. Let

$$\Delta_j = \{\pi_j, \pi_j(1), ..., \pi_j(m_j - 1)\}$$

for irreducible supercuspidal representations π_j and integers $m_j \geq 1$. Let ρ_j be the irreducible representation of W_E corresponding to π_j, and let $\rho_j' = \rho_j \otimes \mathrm{Sp}(m_j)$, so that π corresponds to $\rho_1' \oplus \cdots \oplus \rho_r'$ under the local Langlands correspondence.

For each $1 \leq j \leq r$, there is a unique real number x_j such that $Q(\Delta_j)(-x_j) = Q(\Delta_j')$ is square integrable, where

$$\Delta_j' = \{\pi_j(-x_j), \pi_j(-x_j + 1), ..., \pi_j(-x_j + m_j - 1)\}.$$

We may assume that the intervals Δ_j are ordered so that

$$y_1 = x_1 = \cdots = x_{n_1} > y_2 = x_{n_1+1} = \cdots = x_{n_2} > \cdots = y_s.$$

If for $1 \leq i \leq s$ and $n_{i-1} + 1 \leq j \leq n_i$ (taking $n_0 = 0$), we set

$$\delta_i = Q(\Delta_{n_{i-1}+1}) \boxtimes \cdots \boxtimes Q(\Delta_{n_i})$$

$$\delta_i' = Q(\Delta_{n_{i-1}+1}') \boxtimes \cdots \boxtimes Q(\Delta_{n_i}')$$

then $\tau_i' = \mathrm{Ind}\, \delta_i'$ is irreducible and tempered, and coincides with $Q(\Delta_{n_{i-1}+1}', ..., \Delta_{n_i}')$ in the Bernstein-Zelevinsky classification. Hence $\tau_i = \tau_i'(y_i) = \mathrm{Ind}\, \delta_i$ is irreducible and quasi-tempered, and coincides with $Q(\Delta_{n_{i-1}+1}, ..., \Delta_{n_i})$ in the Bernstein-Zelevinsky clasification.

Now $\tau = \tau_1 \boxtimes \cdots \boxtimes \tau_s$ (resp. $\tau' = \tau_1' \boxtimes \cdots \boxtimes \tau_s'$) is a quasi-tempered (resp. tempered) representation of a standard Levi subgroup M of $\mathrm{GL}_n(E)$. If we let D_i be the determinant on the ith block of M, and let ν be the unramified character $y_1 D_1 + \cdots + y_s D_s$ of M, then ν lies in the positive Weyl chamber of M, and $\tau =$

$\tau' q^{\langle \nu, H_M(-)\rangle}$. As we explained in (2.3.6), π is the unique irreducible quotient of $\operatorname{Ind}\tau = \operatorname{Ind}\tau' q^{\langle \nu, H_M(-)\rangle}$.

Then (2.2.9), Theorem 2.6.5.1, and Corollary 2.5.2.3 give us a multiplicativity formula for the Asai L-function $L(s,\pi,\mathscr{R})$ analogous to that of the Asai gamma factor:

$$L(s,\pi,\mathscr{R}) = \prod_{i=1}^{s} L(s,\tau_i,\mathscr{R}) \prod_{1\le i<j\le s} L(s,\tau_i \times (\tau_j \circ \sigma))$$

where σ is the nontrivial element of $\mathrm{Gal}(E/F)$. On the other hand, if for $1 \le i \le s$ we set

$$\dot{\rho}_i = \rho'_{n_{i-1}+1} \oplus \cdots \oplus \rho'_{n_i}$$

then $\dot{\rho}_i$ corresponds to the quasi-tempered representation τ_i under the local Langlands correspondence, with

$$\rho = \dot{\rho}_1 \oplus \cdots \oplus \dot{\rho}_s.$$

Then we have

$$\begin{aligned} L(s, \otimes \operatorname{Ind}_{E/F}\rho') &= \prod_{i=1}^{s} L(s, \otimes \operatorname{Ind}_{E/F}\dot{\rho}_i) \prod_{1\le i<j\le s} L(s, \dot{\rho}_i \times (\dot{\rho}_j \circ \iota_z)) \\ &= \prod_{i=1}^{s} L(s,\tau_i,\mathscr{R}) \prod_{1\le i<j\le s} L(s,\tau_i \otimes (\tau_j \circ \sigma)) \\ &= L(s,\pi,\mathscr{R}) \end{aligned}$$

where z is an element of W_F that is not in W_E. We have used Lemma 1.1.18.2, the fact that L-functions of Weil-Deligne representations are additive over direct sums, the fact that Rankin products go to tensor products under the local Langlands correspondence (2.3.7), and that composition by σ corresonds to conjugation by z under the local Langlands correspondence (2.3.10). This completes the proof of Theorem 2.6.4 in general.

3. PROOF OF THE MAIN THEOREM

In this chapter we prove our main theorem, Theorem 2.6.3.6, which is the equality of the Asai gamma factor, as defined by the Langlands-Shahidi method, with the corresponding gamma factor on the Galois side. By Remark 2.6.3.1, the theorem already holds in the archimedean case. Our proof uses global methods, and is very general. The methods of proof we use were originally carried out successfully in [CoShTs17] to prove the equality of the symmetric and exterior square gamma factors, as defined by Langlands-Shahidi method, with the corresponding factors on the Galois side.

The method of proof in this chapter depends critically on a highly technical proposotion, called "analytic stability" (Proposition 3.2.2.8). The proposition says that if two supercuspidal representations of $\mathrm{GL}_n(E)$ have the same central character, then their corresponding Asai gamma factors become equal if the representations are twisted by a sufficiently highly ramified character. Our proof of analytic stability is completely local, and we postpone it to Chapters Four and Five. Our method of proof of stability was originally carried out in [CoShTs17].

3.1 Preliminaries

Before we state the proof of the main theorem, we give an overview of the methods involved. We also make a remark on the choice of the additive character occurring in the gamma factors.

3.1.1 Summary of the proof of the main theorem

We give a summary of the content of this chapter. Our quadratic extension of p-adic local fields E/F will be realized as the respective completions of a quadratic extension of number fields K/k at an extension of places $w_0 \mid v_0$. We will encounter certain representations Σ of the global Weil group W_K, and consider their restrictions Σ_w to the local Weil groups W_{K_w}. If Π_w is the representation of $\mathrm{GL}_n(K_w)$ corresponding to Σ_w via the local Langlands correspondence, our tensor product $\Pi = \otimes_w \Pi_w$ will be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$.

We will compare the Artin L-function $L(s, \otimes \operatorname{Ind}_{W_K}^{W_k} \Sigma)$ of the tensor induced representation of Σ to W_k, and the global Asai L-function $L(s, \Pi, R)$, where R is the "global Asai representation" and Π is realized as a cuspidal automorphic representation of $\operatorname{Res}_{K/k} \mathrm{GL}_n(\mathbb{A}_k)$.

In considering the global L-functions and their respective functional equations, we will be able to match up the local factors on both sides at almost all places. What we will end up with is a finite set S of places of k containing v_0, such that each place $v \in S$ has exactly one place w of K lying over it, and such that

$$\prod_{v \in S} \gamma(s, \otimes \operatorname{Ind}_{W_{K_w}}^{W_{k_v}} \Sigma_w, \Psi_v) = \prod_{v \in S} \gamma(s, \Pi_w, \mathscr{R}, \Psi_v)$$

where $\mathscr{R}$ is the Asai representation of the L-group of $\operatorname{Res}_{K_w/k_v} \mathrm{GL}_n$, and $\Psi = \otimes_v \Psi_v$ is a nontrivial character of $\mathbb{A}_k/k$.

Our first main result is a "stable version" of our main theorem (Proposition 3.2.2.2): we show that our main theorem holds for supercuspidal representations up to highly ramified twist. This result uses the idea above of matching up a global Weil representation with a cuspidal automorphic representation. It also relies the analytic stability result (Proposition 3.2.2.8) which will be proved later in Chapters Four and Five. Proposition 3.2.2.2 is done by induction on n, the induction hypothesis being that stable equality holds for *all* quadratic extensions of p-adic local fields, not just the extension E/F we started with.

Using multiplicativity, we readily remove the supercuspidal assumption in Corollary 3.2.2.9 for the stable version of our main theorem. We next use another global argument to show that our main theorem holds for monomial representations on the Galois side (Proposition 3.2.3.1). From there, Brauer's theorem gives us the main theorem for Galois representations (Proposition 3.2.4.1). Since irreducible representations of the local Weil group are unramified twists of Galois representations, we get the main theorem for supercuspidals. Finally, multiplicativity gives us our main theorem for arbitrary representations.

3.1.2 On the choice of additive character

Let E/F be a quadratic extension of p-adic fields. The gamma factors $\gamma(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi)$ and $\gamma(s, \pi, \mathscr{R}, \psi)$ depend on a choice of nontrivial character ψ of F. At several places in this chapter, F will be taken to be the completion of a number field k at a place v_0, and ψ will be assumed to be a local constituent of a nontrivial character $\Psi = \otimes_v \Psi_v$ of $\mathbb{A}_k/k$ such that $\Psi_{v_0} = \psi$. The extensions E/F and the global field k are variable throughout the chapter.

Not all characters ψ of $F = k_{v_0}$ occur as a local constituent of a global character. Consequently, a given global argument in this chapter will appear to give an equality of gamma factors only for certain characters ψ. Fortunately, the density of k in F takes care of this issue.

If $a \in F$, let ψ_a be the character $x \mapsto \psi(ax)$ of F. Similarly, if $a \in k$, let Ψ_a be the character $x \mapsto \Psi(ax)$ of $\mathbb{A}_k/k$. Every nontrivial character of F is equal to ψ_a for a unique $0 \neq a \in F$, and every nontrivial character of $\mathbb{A}_k/k$ is equal to Ψ_a for a unique $0 \neq a \in k$. In this way, the characters of F of the form Ψ_{v_0} form a dense set of all characters.

For fixed s, π, and ρ', the factors $\gamma(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi_a)$ and $\gamma(s, \pi, \mathscr{R}, \psi_a)$ are continuous functions of $a \in F$. Therefore during this chapter, if for a given quadratic extension E/F, a given number field k, and a given place v_0 such that $k_{v_0} = F$, any equality of the form

$$\gamma(s, \otimes \operatorname{Ind}_{E/F} \rho', \psi) = \gamma(s, \pi, \mathscr{R}, \psi) \tag{3.1.2.1}$$

is proved to hold for characters ψ of the form Ψ_{v_0}, then the equality also holds for arbitrary ψ. Therefore there is no loss of generality in always assuming that ψ takes the form Ψ_{v_0}.

3.2 Proof of the main theorem

3.2.1 Template of the global argument

We return to the notation and conventions of (1.4). In particular, K/k is a quadratic extension of number fields. In this section we go through a global argument which will be used repeatedly in the proof of Theorem 2.6.3.6. Let Σ be an n-dimensional representation of the global Weil group W_K, and let $\Psi = \otimes\Psi_v$ be a nontrivial character of $\mathbb{A}_k/k$ with $\Psi_{v_0} = \psi$. Consider the representation $\otimes \operatorname{Ind}_{K/k} \Sigma$ of W_k obtained from Σ by tensor induction (1.4.7). We have the global L-function and epsilon factor (1.4.3)

$$L(s, \otimes \operatorname{Ind}_{K/k} \Sigma) = \prod_v L(s, (\otimes \operatorname{Ind}_{K/k} \Sigma)_v)$$

$$\epsilon(s, \otimes \operatorname{Ind}_{K/k} \Sigma) = \prod_v \epsilon(s, (\otimes \operatorname{Ind}_{K/k} \Sigma)_v, \Psi_v).$$

The global epsilon factor is actually a finite product. The L-function is initially only well defined in some right half plane, but admits a meromorphic continuation to all of $\mathbb{C}$ satisfying the global functional equation

$$L(s, \otimes \operatorname{Ind}_{K/k} \Sigma) = \epsilon(s, \otimes \operatorname{Ind}_{K/k} \Sigma) L(1-s, (\otimes \operatorname{Ind}_{K/k} \Sigma)^\vee). \tag{3.2.1.1}$$

Let Π_w be the representation of $\mathrm{GL}_n(K_w)$ corresponding to the semisimplification of Σ_w under the local Langlands correspondence. Suppose that $\Pi = \otimes\Pi_w$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$. We will let $\mathbf{H} = \mathrm{Res}_{K/k}\,\mathrm{GL}_n$, and consider $\Pi = \otimes_v \pi_v$ as a cuspidal automorphic representation of $\mathbf{H}(\mathbb{A}_k)$. The L-group of $\mathbf{H}$ is the semidirect product of $\mathrm{GL}(V)\times\mathrm{GL}(V)$ by $\mathrm{Gal}(K/k)$, where V is an n-dimensional complex vector space, and we can define the global Asai representation R of ${}^L\mathbf{H}$ exactly as in the local case (2.6.1). We have the global L-function and epsilon factor

$$L(s,\Pi,R) = \prod_v L(s,\pi_v,R_v)$$

$$\epsilon(s,\Pi,R) = \prod_v \epsilon(s,\pi_v,R_v,\Psi_v)$$

where R_v is the composition of ${}^L\mathbf{H}_{k_v} \to {}^L\mathbf{H}$ and R (2.2.14). The global epsilon factor is a finite product, while the global L-function admits a meromorphic continuation to all of $\mathbb{C}$ and satisfies the global functional equation

$$L(s,\Pi,R) = \epsilon(s,\Pi,R)L(1-s,\Pi^\vee,R). \tag{3.2.1.2}$$

Proposition 3.2.1.1. *If v is a place of k, and any of the following conditions is met:*

(i): There are two places w and w' of K lying over v

(ii): v is archimedean

(iii): v is nonarchimedean and π_v has an Iwahori fixed vector

then

$$\gamma(s,\pi_v,R_v,\Psi_v) = \gamma(s,(\otimes\,\mathrm{Ind}_{K/k}\,\Sigma)_v,\Psi_v)$$

$$L(s,\pi_v,R_v) = L(s,(\otimes\,\mathrm{Ind}_{K/k}\,\Sigma)_v)$$

$$L(s,\pi_v^\vee,R_v) = L(s,(\otimes\,\mathrm{Ind}_{K/k}\,\Sigma)_v^\vee)$$

$$\epsilon(s,\pi_v,R_v,\Psi_v) = \epsilon(s,(\otimes\,\mathrm{Ind}_{K/k}\,\Sigma)_v,\Psi_v).$$

Proof: Assume we are in the case (i). Then we can identify $k_v = K_w = K_{w'}$, and we have

$$\mathbf{H}_{k_v} = \mathrm{GL}_n \times \mathrm{GL}_n$$

$${}^L\mathbf{H}_{k_v} = \mathrm{GL}(V) \times \mathrm{GL}(V)$$

$$\pi_v = \Pi_w \boxtimes \Pi_{w'}$$

$$\gamma(s, \pi_v, R_v, \Psi_v) = \gamma(s, \Pi_w \times \Pi_{w'}, \Psi_v)$$

$$(\otimes \operatorname{Ind}_{K/k} \Sigma)_v = \Sigma_w \otimes \Sigma_{w'}$$

where this last equality is from (1.4.7). We then have the equality of gamma factors, because tensor products go to Rankin products under the local Langlands correspondence. Note that the gamma factor on the Artin side only depends on the semisimplification of the representation.

Assume that we are not in the case (i), so there is only one place w of k lying over v. Then

$$\mathbf{H}_{k_v} = \operatorname{Res}_{K_w/k_v} \mathrm{GL}_n$$

$$\pi_v = \Pi_w$$

$$R_v = \mathscr{R}$$

$$(\otimes \operatorname{Ind}_{K/k} \Sigma)_v = \otimes \operatorname{Ind}_{K_w/k_v} \Sigma_w$$

where the last equality is (1.4.8). The equality of gamma factors is then exactly the assertion of Theorem 2.6.3.6, which we have remarked is valid in the cases (ii) and (iii) (see Remark 2.6.3.2).

Note that for all places v, we have the equality of L factors, since these are either Rankin products (Rankin product L factors are equal by the local Langlands correspondence for GL_n), archimedean factors (the analytic L-factors are equal to the Galois ones by definition), or nonarchimedean factors (since the gamma factors for Iwahori fixed representations match up, so do the L-factors by Henniart's argument

(2.7)). Thus in the cases (i), (ii), (iii) we have also the equality of epsilon factors by equations (1.3.3.1) and (2.1.6.1). □

Corollary 3.2.1.2. *With the notation as above, we have*

$$\prod_{v\in S}\gamma(s,\pi_v,\mathscr{R},\Psi_v)=\prod_{v\in S}\gamma(s,\otimes\operatorname{Ind}_{K_w/k_v}\Sigma_w,\Psi_v)$$

where S is a finite set of finite places v of k, each of which has only one place w of K lying over it. We may exclude from S any place v at which π_v has an Iwahori fixed vector.

Proof: By Proposition 3.2.1.1, the local L and epsilon factors match up at all places outside S. We may therefore divide equation (3.2.1.1) by equation (3.2.1.2) to obtain

$$\prod_{v\in S}\frac{L(s,\pi_v,R_v)}{L(s,(\otimes\operatorname{Ind}_{K/k}\Sigma)_v)}=\prod_{v\in S}\frac{\epsilon(s,\pi_v,R_v,\Psi_v)}{\epsilon(s,(\otimes\operatorname{Ind}_{K/k}\Sigma)_v,\Psi_v)}\frac{L(1-s,\pi_v^\vee,R_v)}{L(1-s,(\otimes\operatorname{Ind}_{K/k}\Sigma)_v^\vee)}.$$

At the places v in S, we know that $R_v=\mathscr{R}$ is the Asai representation, and $(\otimes\operatorname{Ind}_{K/k}\Sigma)_v$ is the representation $\otimes\operatorname{Ind}_{K_w/k_v}\Sigma_w$ (1.4.8). Clearing the denominators of this last equation gets us the desired equality. □

3.2.2 The stable equality

From now on, E/F will be a quadratic extension of p-adic fields. Let π be a smooth, irreducible representation of $\mathrm{GL}_n(E)$, corresponding to an n-dimensional Frobenius semisimple representation ρ of W_E'. Let η be a character of W_E, which we identify as a character of E^* by local class field theory. Then define $\pi\eta$ to be $\pi(\eta\circ\det)$. Then $\pi\eta$ corresponds to $\rho\eta$ under the local Langlands correspondence (LLC).

The following lemma is a restatement of (2.6.4). Since we use it repeatedly in this chapter, we include it here.

Lemma 3.2.2.1. *Let $\eta = ||\cdot||^{s_0}$ be an unramified character of W_E, and π a smooth, irreducible representation of $\mathrm{GL}_n(E)$. Then*

$$\gamma(s, \pi\eta, \mathscr{R}, \psi) = \gamma(s + 2s_0, \pi, \mathscr{R}, \psi)$$

$$\gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho\eta), \psi) = \gamma(s + 2s_0, \otimes \operatorname{Ind}_{E/F} \rho, \psi).$$

It follows from Lemma 3.2.2.1 that if Theorem 2.6.3.6 holds for a given representation π, it also holds for unramified twists of π.

In this section, we prove the following stable version of Theorem 2.6.3.6 for supercuspidal representations:

Proposition 3.2.2.2. *(Stable equality for supercuspidals) Let π be a supercuspidal representation of $\mathrm{GL}_n(E)$, corresponding to an irreducible representation ρ of W_E. Then for all sufficiently highly ramified characters η of W_E, we have*

$$\gamma(s, \pi\eta, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho\eta), \psi).$$

We first prove Proposition 3.2.2.2 in the case $n = 1$. For this, we require a simple lemma on idelic characters:

Lemma 3.2.2.3. *Let K be a number field, $w_1, ..., w_r$ finite places of K, and $\eta_1, ..., \eta_r$ characters of $K^*_{w_1}, ..., K^*_{w_r}$. There exists a unitary character $\mathscr{X} = \otimes_w \mathscr{X}_w$ of $\mathbb{A}^*_K/K^*$ such that $\mathscr{X}_{w_i}$ and η_i agree on $\mathcal{O}^*_{w_i}$ for $1 \leq i \leq r$, and $\mathscr{X}_w$ is unramified at all other finite places w.*

Proof: Consider the compact subgroup $\prod_{w<\infty} \mathcal{O}^*_w$ of $\mathbb{A}^*_K$. Define a unitary character $\mathscr{X} = \otimes \mathscr{X}_w$ of this compact subgroup by setting $\mathscr{X}_{w_i} = \eta_i$ for $1 \leq i \leq r$, and $\mathscr{X}_w = 1$ for all other finite places w. Since this compact subgroup has trivial intersection with K^*, we can extend $\mathscr{X}$ to a unitary character of $K^* \prod_{w<\infty} \mathcal{O}^*_w$ by making it trivial on K^*. Now $K^* \prod_{w<\infty} \mathcal{O}^*_w$ is closed $\mathbb{A}^*_K$, being the product of a closed set and a compact set. Hence $\mathscr{X}$ can be extended to a unitary character of $\mathbb{A}^*_K$ by Pontryagin duality. It is trivial on K^*, and satisfies the requirements of the lemma. $\square$

Lemma 3.2.2.4. *Theorem 2.6.3.6 (and hence Proposition 3.2.2.2) holds for the case* $n = 1$.

Note that Proposition 3.2.2.2 follows from Theorem 2.6.3.6, because the local Langlands correspondence is compatible with twisting by characters.

Proof: In this case, π is a character χ of $E^* = \operatorname{Res}_{E/F} \mathrm{GL}_1(F)$, and the ρ corresponding to χ is a character of W_E. We globalize the situation by finding:

1. A quadratic extension of number fields K/k, with places $w_0 \mid v_0$, such that $K_{w_0} = E, k_{v_0} = F$
2. A unitary character $\mathscr{X} = \otimes \mathscr{X}_w$ of $\mathbb{A}_K/K^*$ such that $\mathscr{X}_{w_0}$ agrees with χ on $\mathcal{O}_E^*$, and $\mathscr{X}_w$ is unramified for finite $w \neq w_0$ (Lemma 3.2.2.3)
3. A character $\Psi = \otimes \Psi_v$ of $\mathbb{A}_k/k$ such that $\Psi_{v_0} = \psi$.

We identify $\mathscr{X}$ as a one dimensional representation of the global Weil group, so that for each place w of K, the one dimensional representation $(\mathscr{X})_w$ of W_{K_w} (defined in (1.4.3)) identifies with the character of $\mathscr{X}_w$ of K_w^* via the local Artin map. Now we identify $\mathscr{X} = \otimes_v \mathfrak{X}_v$ as a cuspidal automorphic representation of $\operatorname{Res}_{K/k} \mathrm{GL}_1(\mathbb{A}_k)$. Note that if a place v of k has only one place w of K lying over it, then $\mathfrak{X}_v = \mathscr{X}_w$. Using the template of the global argument (3.2.1), we have

$$\prod_{v \in S} \gamma(s, \mathscr{X}_w, \mathscr{R}, \Psi_v) = \prod_{v \in S} \gamma(s, \otimes \operatorname{Ind}_{K_w/k_v} \mathscr{X}_w, \Psi_v)$$

where S is a finite set of finite places containing v_0, such that each place v of S has only one place w of K lying over it. But since $\mathscr{X}_w$ is unramified at every place other than w_0, the situation (iii) of the Proposition 3.2.1.1 applies to allow us to cancel off all the gamma factors other than at v_0, giving us

$$\gamma(s, \mathscr{X}_{w_0}, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F} \mathscr{X}_{w_0}, \psi).$$

Now $\pi = \chi$ agrees with $\mathscr{X}_{w_0}$ on $\mathcal{O}_E^*$, hence they differ by an unramified character $||\cdot||^{s_0}$ for some complex number s_0. Since both gamma factors are compatible with twisting by unramified characters (Lemma 3.2.2.1), we get the required equality. $\square$

Now, we will finish the proof of the stability equality theorem by induction on n. Let us state formally our induction hypothesis:

Induction Hypothesis. *Let $n \geq 2$. For each integer $m < n$, each quadratic extension E/F of p-adic local fields, and each supercuspidal representation π of $\mathrm{GL}_m(E)$, corresponding to an m-dimensional irreducible representation ρ of the Weil group W_E, we have*

$$\gamma(s, \pi\eta, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho\eta), \psi)$$

for all sufficiently highly ramified characters η of W_E, with necessary degree of ramification depending on E/F, π, and ψ.

Assume for the rest of this section that the induction hypothesis holds for a given n. Let E/F be a quadratic extension of p-adic fields, π a supercuspidal representation of $\mathrm{GL}_n(E)$, and ρ the n-dimensional irreducible representation of W_E corresponding to π. Our first step in showing that the stable equality theorem holds for $n+1$ is to show "equality at a base point."

Propositon 3.2.2.5. *(Equality at a base point) Let $n \geq 2$ be an integer, and assume the induction hypothesis for n. Let ω_0 be any character of E^*. Then exists an n-dimensional irreducible representation ρ_0 of W_E, with $\det \rho_0$ corresponding to ω_0 by local class field theory, such that if π_0 is the supercuspidal representation of $\mathrm{GL}_n(E)$ corresponding to ρ_0, and η is any character of E^*, then*

$$\gamma(s, \pi_0\eta, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho_0\eta), \psi).$$

Our proof of equality at a base point will be a global argument. We start with a lemma of Henniart.

Lemma 3.2.2.6. *Let ω_0 be the character in Proposition 3.2.2.5. There exists a quadratic extension of number fields K/k, and an n-dimensional complex representation Σ of W_K, with the following properties:*

(i): There exist places w_0 and v_0 of K and k, respectively, such that $k_{v_0} = F$ and $K_{w_0} = E$.

(ii): The representation $\rho_0 := \Sigma_{w_0}$ of $W_{K_{w_0}} = W_E$ is irreducible, and $\det \Sigma_{w_0}$ corresponds to ω_0 by local class field theory.

(iii): At all finite places w of K with $w \neq w_0$, the representation Σ_w is not irreducible.

(iv): If π_w is the representation of $\mathrm{GL}_n(K_w)$ corresponding to the semisimplication $(\Sigma_w)_{ss}$ of Σ_w by the local Langlands correspondence, then $\Pi = \bigotimes_w \pi_w$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$.

Proof: This is essentially Lemma 3.1 of [CoShTs17]. The only difference is the lemma only mentions the field K, not the field k. In the proof, one fixes the unramified extension M of E of degree n, and produces a degree n extension of number fields $\mathbb{M}/K$ and an extension of places w'/w for which $\mathbb{M}_{w'} = M, K_w = E$. We can choose $k, \mathbb{M}$, and K at the same time to satisfy the hypotheses of our modified lemma. $\square$

Proof: (of Proposition 3.2.2.5) Following the global template (3.2.1), we have

$$\prod_{v \in S} \gamma(s, \pi_v, \mathscr{R}, \Psi_v) = \prod_{v \in S} \gamma(s, \otimes \operatorname{Ind}_{K_w/k_v} \Sigma_w, \Psi_v)$$

for a finite set S of finite places of k containing v_0, where each place v of S has only one place w of K lying over it. Let $T = S - \{v_0\}$. For each place v of T, we know that Σ_w is not irreducible, so the semisimplification $(\Sigma_w)_{\mathrm{ss}}$ decomposes as $\Sigma_{w,1} \oplus \cdots \oplus \Sigma_{w,r_w}$ for an irreducible representation $\Sigma_{w,i}$ of W_{K_w} of dimension $n_{w,i} < n$. Let $\Pi_{w,i}$ be the supercuspidal representation $\mathrm{GL}_{n_{w,i}}(K_w)$ corresponding to $\Sigma_{w,i}$. By the induction hypothesis, we have for all sufficiently highly ramified characters $\mathscr{X}_w$ of W_{K_w}, that

$$\gamma(s, \Pi_{w,i} \mathscr{X}_w, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{K_w/k_v} \Sigma_{w,i} \mathscr{X}_w, \psi). \tag{3.2.2.1}$$

We apply Lemma 3.2.2.3 to find a unitary character $\mathscr{X} = \otimes \mathscr{X}_w$ of $\mathbb{A}_K^*/K^*$, such that $\mathscr{X}_w$ is unramified for all finite places w with $w \mid v$ and $v \notin S$, $\mathscr{X}_{w_0}$ agrees with ω_0 on $\mathcal{O}_E^*$, and for each $v \in T$ with $w \mid v$, $\mathscr{X}_w$ is sufficiently highly ramified such that (3.2.2.1) holds for all $1 \leq i \leq r_w$.

We now apply the global template again, replacing Σ by $\Sigma\mathscr{X}$, and Π by $\Pi\mathscr{X}$, so that

$$\prod_{v \in S} \gamma(s, \Pi_w \mathscr{X}_w, \mathscr{R}, \Psi_v) = \prod_{v \in S} \gamma(s, \otimes \operatorname{Ind}_{K_w/k_v}(\Sigma_w \mathscr{X}_w), \Psi_v). \tag{3.2.2.2}$$

We have the same set S as before, since $\mathscr{X}_w$ is unramified for all finite places $v \notin S$, $w \mid v$. For $v \in T$, and $w \mid v$, we have that $\Pi_{w,i}\mathscr{X}_w, 1 \leq i \leq r_w$, is the supercuspidal support of $\Pi_w \mathscr{X}_w$, so that after relabeling the $\Pi_{w,i}\mathscr{X}_w$,

$$\Pi_w \mathscr{X}_w \subset \operatorname{Ind}^{\mathrm{GL}_n(E)}(\Pi_{w,1}\mathscr{X}_w) \boxtimes \cdots \boxtimes (\Pi_{w,r_w}\mathscr{X}_w)$$

and therefore multiplicativity (Theorem 2.6.5.1) gives

$$\begin{aligned}\gamma(s, \Pi_w \mathscr{X}_w, \mathscr{R}, \Psi_v) = \prod_{i=1}^{r_w} \gamma(s, \Pi_{w,i}\mathscr{X}_w, \mathscr{R}, \Psi_v) &\prod_{1 \leq i < j \leq r_w} \lambda(K_w/k_v, \Psi_v)^{n_{w_i} n_{w_j}} \\ &\gamma(s, (\Pi_{w,i}\mathscr{X}_w) \times (\Pi_{w,j}\mathscr{X}_w \circ \sigma_w), \Psi_v \circ \operatorname{Tr}_{w/v})\end{aligned}$$

where σ_w is the nontrivial element of $\operatorname{Gal}(K_w/k_v)$. On the other hand, $\Sigma_w \mathscr{X}_w$ has the irreducible representations $\Sigma_{w,i}\mathscr{X}_w$ as the factors of its composition series, so by Lemma 1.1.18.3, we have

$$\begin{aligned}\gamma(s, \otimes \operatorname{Ind}_{K_w/k_v} \Sigma_w \mathscr{X}_w, \Psi_v) = &\prod_{i=1}^{r_w} \gamma(s, \otimes \operatorname{Ind}_{K_w/k_v} \Sigma_{w,i}\mathscr{X}_w, \Psi_v) \\ &\prod_{1 \leq i < j \leq r_w} \gamma(s, \operatorname{Ind}_{K_w/k_v} \Sigma_{w,i}\mathscr{X}_w \otimes (\Sigma_{w,j}\mathscr{X}_w \circ \iota_z), \Psi_v)\end{aligned}$$

where z is an element of W_{k_v} which is not in W_{K_w}. By equation (1.1.16.1), (2) of Theorem 2.3.7.1, and (2.3.10), we have

$$\lambda(K_w/k_v, \Psi_v)^{n_{w_i} n_{w_j}} \gamma(s, (\Pi_{w,i} \mathscr{X}_w) \times (\Pi_{w,j} \mathscr{X}_w \circ \sigma_w), \Psi_v \circ \mathrm{Tr}_{w/v})$$
$$= \gamma(s, \mathrm{Ind}_{K_w/k_v} \Sigma_{w,i} \mathscr{X}_w \otimes (\Sigma_{w,j} \mathscr{X}_w \circ \iota_z), \Psi_v).$$

Combining this with equation (3.2.2.1), we see that the equality of gamma factors

$$\gamma(s, \Pi_w \mathscr{X}_w, \mathscr{R}, \Psi_v) = \gamma(s, \otimes \mathrm{Ind}_{K_w/k_v}(\Sigma_w \mathscr{X}_w), \Psi_v)$$

holds for every place $v \in T$. Thus from equation (3.2.2.2), we get

$$\gamma(s, \pi_0 \mathscr{X}_{w_0}, \mathscr{R}, \psi) = \gamma(s, \otimes \mathrm{Ind}_{E/F}(\rho_0 \mathscr{X}_{w_0}), \psi).$$

Since $\mathscr{X}_{w_0}$ and ω_0 agree on $\mathcal{O}_E^*$, they differ by an unramified character, so Lemma 3.2.2.1 completes the proof of equality at a base point. □

To proceed with the proof of stable equality, we will need the following stability results on both sides.

Proposition 3.2.2.7. *(Arithmetic stability) Let ρ_1, ρ_2 be two representations of W_E with $\det \rho_1 = \det \rho_2$. Then for all sufficiently highly ramified characters η of W_E, we have*

$$\gamma(s, \otimes \mathrm{Ind}_{E/F}(\rho_1 \eta), \psi) = \gamma(s, \otimes \mathrm{Ind}_{E/F}(\rho_2 \eta), \psi).$$

Proof: It is a consequence of Deligne's proof of the existence of the local epsilon factors that if $\det \rho_1 = \det \rho_2$, then $\gamma(s, \rho_1 \eta, \Psi) = \gamma(s, \rho_2 \eta, \Psi)$ for all characters Ψ of E and all sufficiently highly ramified characters of η of W_E (Lemma 4.16 of [De72]). We need only observe that under our hypothesis, we have $\det \otimes \mathrm{Ind}_{E/F} \rho_1 = \det \otimes \mathrm{Ind}_{E/F} \rho_2$, $\otimes \mathrm{Ind}_{E/F}(\rho_i \eta) = (\otimes \mathrm{Ind}_{E/F} \rho_i)(\otimes \mathrm{Ind}_{E/F} \eta)$, and that $(\otimes \mathrm{Ind}_{E/F} \eta)$ is highly ramified if η is (Lemma 1.1.19.1 (ii)). □

The stability result on the analytic side, Proposition 3.2.2.8, is much more difficult. Our proof, which is the content of Chapters Four and Five, is purely local, and mirrors the approach taken by Shahidi, Cogdell, and Tsai in [CoShTs17].

Proposition 3.2.2.8. *(Analytic stability) Let π_1, π_2 be supercuspidal representations of $\mathrm{GL}_n(E)$ with the same central character. Then for all sufficiently highly ramified characters η of E^*, we have*

$$\gamma(s, \pi_1\eta, \mathscr{R}, \psi) = \gamma(s, \pi_2\eta, \mathscr{R}, \psi).$$

Proof: This proposition will be shown to be equivalent to Theorem 5.1.3.3, whose proof will occupy the entirety of Chapters Four and Five. □

Granting the analytic stability result, Proposition 3.2.2.8, we can now finally finish the proof of stable equality (Proposition 3.2.2.2). Let π be a supercuspidal representation of $\mathrm{GL}_n(E)$, and ρ the corresponding n-dimensional irreducible representation of W_E. Let ω_0 be the central character of π, identified with a character of W_E, so that $\det\rho = \omega_0$. By Proposition 3.2.2.5, there exists an n-dimensional irreducible representation ρ_0 of W_E with $\det\rho_0 = \det\rho$, such that if π_0 is the supercuspidal representation $\mathrm{GL}_n(E)$ corresponding to ρ_0, then

$$\gamma(s, \pi_0\eta, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho_0\eta), \psi)$$

for all characters η of E^*. Now π and π_0 have the same central character ω_0. Taking η to be very highly ramified, we have by Propositions 3.2.2.7 and 3.2.2.8,

$$\begin{aligned}
\gamma(s, \pi\eta, \mathscr{R}, \psi) &= \gamma(s, \pi_0\eta, \mathscr{R}, \psi) \\
&= \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho_0\eta), \psi) \\
&= \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho\eta), \psi).
\end{aligned}$$

This completes the proof of the induction step, and the proof of Proposition 3.2.2.2.

Corollary 3.2.2.9. *(Stable equality for general representations) Let* π *be a smooth, irreducible representation of* $\mathrm{GL}_n(E)$*, corresponding to a Frobenius semisimple representation* ρ *of* W'_E*. Then for all sufficiently highly ramified characters* η *of* W_E*, we have*

$$\gamma(s, \pi\eta, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho\eta), \psi).$$

Proof: This follows from Proposition 3.2.2.2 and the following facts:

1. If $\pi_1, ..., \pi_r$ are supercuspidal representations of smaller GLs with π isomorphic to a subquotient of $\operatorname{Ind}^{\mathrm{GL}_n(E)} \pi_1 \boxtimes \cdots \boxtimes \pi_r$, and ρ_i is the irreducible Weil representation corresponding to π_i, then the underlying Weil representation of ρ is the direct sum of the ρ_i.

2. The gamma factor on the Artin side depends only on the underlying Weil representation.

3. The local Langlands correspondence is compatible by twisting of characters.

4. Multiplicativity of gamma factors on both sides (Lemma 1.1.18.3 and Theorem 2.6.5.1), as well as equation (1.1.16.1).

□

3.2.3 Equality for monomial representations

In this section, we prove the equality of gamma factors when ρ is a monomial representation, which is to say a representation of W_E which is induced from a finite order character of a finite Galois extension of E.

Proposition 3.2.3.1. *(Equality for monomial representations) Let* $E \subseteq L \subseteq M \subseteq \overline{F}$ *be fields with* M *a finite Galois extension of* E*, and* $n = [L : E]$*. Let* χ *be a character of* $\mathrm{Gal}(M/L)$*, and let* $\rho = \operatorname{Ind}_{L/E}(\chi)$*. Let* π *be the representation of* $\mathrm{GL}_n(E)$ *corresponding to* ρ*. Then*

$$\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho, \psi).$$

The representation ρ need not be irreducible, but being a Galois representation, it decomposes into a direct sum of irreducible representations $\rho_1, ..., \rho_r$ with degrees $n_1, ..., n_r$. Let π_i be the supercuspidal representation of $\mathrm{GL}_{n_i}(E)$ corresponding to ρ_i. Then since ρ is a Galois representation, π is fully induced from $\pi_1, ..., \pi_r$.

We will require the following lemma of Henniart:

Lemma 3.2.3.2. *There exist number fields*

$$k \subset K \subset \mathbb{L} \subset \mathbb{M}$$

together with a character $\mathscr{Y}$ of $\mathbb{A}_{\mathbb{L}}^*/\mathbb{L}^*$, and a place w_0'' of $\mathbb{M}$, such that:

1. *If w_0', w_0, v are the places of $\mathbb{L}, K, k$ over which w_0'' lies, then $F = k_{v_0}, E = K_{w_0}, L = \mathbb{L}_{w_0'}, M = \mathbb{M}_{w_0''}$.*
2. *$\mathbb{M}$ is Galois over K, with $[\mathbb{M} : K] = [\mathbb{M}_{w_0''} : K_{w_0}] = [M : E]$. Hence w_0'' is the only place of $\mathbb{M}$ lying over w_0.*
3. *$\mathscr{Y}_{w_0'} = \chi$ under local class field theory.*
4. *Let $\Sigma = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{M}/\mathbb{L})}^{\mathrm{Gal}(\mathbb{M}/K)}(\mathscr{Y})$, so that $\rho = \Sigma_{w_0}$. Then there is a cuspidal automorphic representation $\Pi = \otimes_w \Pi_w$ of $\mathrm{GL}_m(\mathbb{A}_K)$ (where $m = [L : E]$) such that Σ_w corresponds to π_w under the LLC at each place w of K. In particular, $\pi = \Pi_{w_0}$.*

This is Lemma 3.2 of [CoShTs17]. Like Lemma 3.2.2.6, the original statement of this lemma did not include the field k, but we can easily modify the construction to include it.

Following the template of the global argument (3.2.1), we have as usual

$$\prod_{v \in S} \gamma(s, \Pi_w, \mathscr{R}, \Psi_v) = \prod_{v \in S} \gamma(s, \otimes \mathrm{Ind}_{K_w/k_v} \Sigma_w, \Psi_v)$$

where S is a finite set of finite places v of k containing v_0, each of which has only one place w of K lying over it. Note that in the notation of (3.2.1), we have $\Pi_w = \pi_v$ for each $v \in S$. Let $\mathscr{X} = \otimes \mathscr{X}_w$ be a character of $\mathbb{A}_K^*/K^*$ which is very highly ramified at $v \in S - \{v_0\}$ and unramified at all other finite places. We replace Σ by $\Sigma\mathscr{X}$ in the template of the global argument, giving us

$$\prod_{v \in S} \gamma(s, \Pi_w \mathscr{X}_w, \mathscr{R}, \Psi_v) = \prod_{v \in S} \gamma(s, \otimes \operatorname{Ind}_{K_w/k_v}(\Sigma_w \mathscr{X}_w), \Psi_v)$$

for the same set S. Note that in our previous notation of writing $\mathscr{X} = \otimes_v \mathfrak{X}_v$ as a character of $\operatorname{Res}_{K/k} \operatorname{GL}_1(\mathbb{A}_k)$, we have $\mathfrak{X}_v = \mathscr{X}_w$ for each place $v \in S$, which has only one place w of K lying over it.

By Corollary 3.2.2.9, and the fact that each $\mathscr{X}_w$ for $v \in S - \{v_0\}$ is highly ramified, we have equality of the gamma factors at all places $v \neq v_0$, and therefore

$$\gamma(s, \pi \mathscr{X}_{w_0}, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F}(\rho \mathscr{X}_{w_0}), \psi).$$

Now $\mathscr{X}_{w_0}$ is unramified and therefore of the form $|| \cdot ||^{s_0}$ for some $s_0 \in \mathbb{C}$, so Lemma 3.2.2.1 concludes the proof of the proposition.

3.2.4 Equality for Galois representations

Using the equality of gamma factors for monomial representations and Brauer's theorem, we will prove the equality of gamma factors for all irreducible Galois representations.

Proposition 3.2.4.1. *(Equality for Galois representations) Let ρ be an irreducible n-dimensional Galois representation, and let π be the corresponding supercuspidal representation of $\operatorname{GL}_n(E)$. Then*

$$\gamma(s, \pi, \mathscr{R}, \psi) = \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho, \psi).$$

Proof: Choose $z \in W_F$ which is not in W_E. By Brauer's theorem, there exist monomial representations $\rho_1, ..., \rho_r, \rho'_1, ..., \rho'_t$, not necessarily all distinct, such that

$$\Sigma := \rho \oplus \bigoplus_{i=1}^{t} \rho'_i = \bigoplus_{i=1}^{r} \rho_i.$$

We can compute $\gamma(s, \otimes \operatorname{Ind}_{E/F} \Sigma, \psi)$ in two ways using Lemma 1.1.18.2. First,

$$\gamma(s, \otimes \operatorname{Ind}_{E/F} \Sigma, \psi) = \prod_{i=1}^{r} \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho_i, \psi) \prod_{1 \leq i < j \leq r} \gamma(s, \operatorname{Ind}_{E/F} \rho_i \otimes (\rho_j \circ \iota_z), \psi).$$

Second, we can write $\gamma(s, \otimes \operatorname{Ind}_{E/F} \Sigma, \psi)$ as

$$\begin{aligned}\gamma(s, \otimes \operatorname{Ind}_{E/F} \Sigma, \psi) = & \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho, \psi) \prod_{i=1}^{t} \gamma(s, \otimes \operatorname{Ind}_{E/F} \rho'_i, \psi) \\ & \prod_{1 \leq j \leq t} \gamma(s, \operatorname{Ind}_{E/F} \rho \otimes (\rho'_j \circ \iota_z), \psi) \prod_{1 \leq i < j \leq t} \gamma(s, \operatorname{Ind}_{E/F} \rho'_i \otimes (\rho'_j \circ \iota_z), \psi).\end{aligned}$$

Let $\pi_1, ..., \pi_r, \pi'_1, ..., \pi'_t$ be the smooth, irreducible representations corresponding to the Galois representations $\rho_1, ..., \rho_r, \rho'_1, ..., \rho'_t$. Let n_i, n'_i be the degrees of π_i, π'_i. Let Π be the smooth, irreducible representation of $\mathrm{GL}_n(E)$ corresponding to Σ. Applying (2.3.9), on the analytic side we have

$$\Pi = \operatorname{Ind}^{\mathrm{GL}_n(E)} \pi \boxtimes \pi_1 \boxtimes \cdots \boxtimes \pi_r = \operatorname{Ind}^{\mathrm{GL}_n(E)} \pi'_1 \boxtimes \cdots \boxtimes \pi'_t.$$

Then we can apply multiplicativity of gamma factors (Theorem 3.6.5.1) in two different ways:

$$\begin{aligned}\gamma(s, \Pi, \mathscr{R}, \psi) = & \prod_{i=1}^{t} \gamma(s, \pi'_i, \mathscr{R}, \psi) \\ \prod_{1 \leq i < j \leq t} & \lambda(E/F, \psi)^{n'_i n'_j} \gamma(s, \pi'_i \times (\pi'_j \circ \sigma), \psi \circ \operatorname{Tr}_{E/F})\end{aligned}$$

and

$$\gamma(s,\Pi,\mathscr{R},\psi) = \gamma(s,\pi,\mathscr{R},\psi)\prod_{i=1}^{r}\gamma(s,\pi_i,\mathscr{R},\psi)$$
$$\prod_{j=1}^{r}\lambda(E/F,\psi)^{nn_i}\gamma(s,\pi\times(\pi_j\circ\sigma),\psi\circ\mathrm{Tr}_{E/F})$$
$$\prod_{1\le i<j\le r}\lambda(E/F,\psi)^{n_in_j}\gamma(s,\pi_i\times(\pi_j\circ\sigma),\psi\circ\mathrm{Tr}_{E/F}).$$

Considering the first way we have applied multiplicativity on both sides, we have that

$$\lambda(E/F,\psi)^{n_i'n_j'}\gamma(s,\pi_i'\times(\pi_j'\circ\sigma),\psi\circ\mathrm{Tr}_{E/F}) = \gamma(s,\mathrm{Ind}_{E/F}(\rho_i\otimes(\rho_j\circ\iota_z)),\psi)$$

and we also have the equality of $\gamma(s,\pi_i',\mathscr{R},\psi)$ and $\gamma(s,\otimes\,\mathrm{Ind}_{E/F}\,\rho_i',\psi)$ by Proposition 3.2.3.1. Thus

$$\gamma(s,\Pi,\mathscr{R},\psi) = \gamma(s,\otimes\,\mathrm{Ind}_{E/F}\,\Sigma,\psi).$$

Now we write out the left and right hand sides of this last equation in the second way that we have described each. We match up the terms again using the same arguments, giving us

$$\gamma(s,\pi,\mathscr{R},\psi) = \gamma(s,\otimes\,\mathrm{Ind}_{E/F}\,\rho,\psi).$$

This completes the proof of equality for Galois representations. □

We now complete the proof of Theorem 2.6.3.6. Let π be a supercuspidal representation of $\mathrm{GL}_n(E)$, and ρ the corresponding n-dimensional irreducible representation of W_E. Then there exists an unramified character η of W_E such that $\rho\eta$ is a Galois representation (1.1.5). Identifying η as a character of $\mathrm{GL}_n(E)$ through the determinant, we know that $\pi\eta$ corresponds to $\rho\eta$ under the local Langlands correspondence. Proposition 3.2.4.1 tells us that Theorem 2.6.3.6 holds for $\pi\eta$ and $\rho\eta$. Since the gamma factors on both sides are compatible by twisting by unramified characters (Lemma 3.2.2.1), we see that Theorem 2.6.3.6 holds for supercuspidal representations.

Having established Theorem 2.6.3.6 for supercuspidals, we have Theorem 2.6.3.6 for general representations by the same reductions as in Corollary 3.2.2.9.

4. ANALYSIS OF PARTIAL BESSEL INTEGRALS

Throughout, E/F is a quadratic extension of p-adic fields, with nontrivial automorphism σ. If $x \in E$, then we will write $\overline{x}$ for $\sigma(x)$, and if g is a matrix with entries in E, then $\overline{g}$ will denote the entrywise application of σ to g.

Let $G = \mathrm{GL}_n(E)$, A the group of diagonal matrices of G, U the group of upper triangular unipotent matrices in G, and $B = AU$ the usual Borel subgroup of G. Let $W(G)$ be the Weyl group of G. Associated to A and U is the set Δ of simple roots of A in U. In this chapter we will fix once and for all an irreducible, supercuspidal representation π of G with central character ω_π.

The goal of this chapter is to establish an asymptotic expansion formula for partial Bessel integrals (Theorem 4.0.2). It is through this formula that we will be able to establish the stability of the Asai gamma factor for supercuspidal representations (Proposition 3.2.2.8, equivalently Theorem 5.1.3.3). This stability result is necessary for the proof of our main result (Theorem 2.6.3.6) given in Chapter Three.

Our asymptotic expansion formula is extremely similar to the one developed by Cogdell, Shahidi, and Tsai in their proof of stability for exterior square gamma factors in [CoShTs17]. The proof given in this chapter is essentially the same as theirs. Their field is F, and our field is E. Their use of the transpose ${}^t g$ must be replaced by the use of the conjugate transpose ${}^t\overline{g}$.

4.1 Preliminaries

4.1.1 Characters and Weyl group representatives

Let ψ be a nontrivial character of F, fixed once and for all. Note that ψ is automatically unitary. We define a character χ of U by the formula

$$\chi\begin{pmatrix} 1 & a_1 & & & * \\ & 1 & a_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & a_{n-1} \\ & & & & 1 \end{pmatrix} = \psi\Big(\sum_{i=1}^{n-1} \mathrm{Tr}_{E/F}(a_i)\Big).$$

Let $e_1, ..., e_n$ be the standard basis of rational characters of A. Then $\Delta = \{e_1 - e_2, ..., e_{n-1} - e_n\}$ is the set of simple roots of A in G corresponding to U. To each $\alpha = e_i - e_{i+1} \in \Delta$, define the canonical representative

$$\dot{w}_\alpha = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix} \in N_G(T)$$

of the corresponding simple reflection $w_\alpha \in W(G)$. The representatives $\dot{w}_\alpha : \alpha \in \Delta$ give us representatives for every $w \in W(G)$: if $w \in W(G)$, let $(w_{\alpha_1}, ..., w_{\alpha_t})$ be a reduced decomposition of w, for $\alpha_i \in \Delta$. Then define

$$\dot{w} = \dot{w}_{\alpha_1} \cdots \dot{w}_{\alpha_t}.$$

This will be independent of the choice of reduced decomposition. The Weyl group representatives are compatible with χ in the sense that if $w \in W(G), u \in U$, and $\dot{w}u\dot{w}^{-1} \in U$, then $\chi(\dot{w}u\dot{w}^{-1}) = \chi(u)$.

If w_G is the long element of $W(G)$, then the representative $\dot{w}_G$ attached to w_G is the matrix

$$J = J_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

Each parabolic subgroup of G containing B has a unique Levi subgroup M containing A, with Borel subgroup $B \cap M$. The Weyl group $W(M)$ of A in M is a subgroup of $W(G)$, and has a corresponding long element w_M. The group M is a block diagonal sum $\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_t}(E)$, where $n_1 + \cdots + n_t = n$, and the representative $\dot{w}_M$ is equal to

$$\dot{w}_M = \begin{pmatrix} J_{n_1} & & & \\ & J_{n_2} & & \\ & & \ddots & \\ & & & J_{n_t} \end{pmatrix}.$$

4.1.2 Bruhat decomposition

If $w \in W(G)$, let $C(w)$ be the **Bruhat cell** BwB. Then G is the disjoint union of the cells $C(w) : w \in W(G)$. This is the Bruhat decomposition for G. Each cell is open in its closure. The only closed cell is $C(1_W) = B$, and the only open cell is $C(w_G)$. We call $C(w_G)$ the **big cell**. The big cell is dense in G.

In order describe the elements of the Bruhat cells with uniqueness of expression, we will need to introduce certain subgroups of U. Let U^- be the group of lower triangular unipotent matrices in G. For $w \in W(G)$, let $U_w^+ = U \cap w^{-1}Uw$, and $U_w^- = U \cap w^{-1}U^-w$. Then U_w^+ (resp. U_w^-) is directly spanned by the root subgroups of those positive roots which remain positive (resp. which are made negative) by w. Then

$$C(w) = U\dot{w}AU_w^-$$

with uniqueness of expression. In fact, the map from $U \times A \times U_w^-$ to $C(w)$ which sends (u,a,n) to $u\dot{w}an$ is a homeomorphism.

4.1.3 Bruhat decomposition for Levi subgroups

If M is a standard Levi subgroup of G, there is also a Bruhat decomposition for M. Let $B_M = B \cap M$, and $U_M = U \cap M$. If for $w \in W(M)$, we let $C_M(w) = B_M w B_M$, then M is the disjoint union of the cells $C_M(w) : w \in W(M)$, each cell is open in its closure, the only closed cell is $C_M(1_W) = B_M$, and the open cell is $C_M(w_M)$. We call $C(w_M)$ the big cell of M. It is dense in M.

We have

$$C_M(w) = U_M \dot{w} A U_w^-$$

again with uniqueness of expression. In fact, the map from $U_M \times A \times U_w^-$ to $C_M(w)$ seneding (u,a,n) to $u\dot{w}an$ is a homeomorphism. We remark that U_w^- is already contained in U_M; there is no need to intersect it with U_M.

4.1.4 Functions which are compactly supported modulo the center of G

Let Z be the center of G. If S is any locally closed subset of G containing Z, let $\mathscr{C}_c^\infty(S;\omega_\pi)$ be the space of locally constant functions $f : S \to \mathbb{C}$ which satisfy $f(zg) = \omega_\pi(z)f(g)$ for all $z \in Z$ and $g \in G$, and which are compactly supported modulo Z. This last condition means that there exists a compact set $\Omega \subset S$ such that if $f(g) \neq 0$, then $zg \in \Omega$ for some $z \in Z$.

If S' is an open subset of S containing Z, we can identify $\mathscr{C}_c^\infty(S';\omega_\pi)$ as the subspace of $\mathscr{C}_c^\infty(S;\omega_\pi)$ consisting of those functions which vanish on $S - S'$. If E is a closed subset of S containing Z, the restriction map $\mathscr{C}_c^\infty(S;\omega_\pi) \to \mathscr{C}_c^\infty(E;\omega_\pi)$ is surjective. Thus the sequence

$$0 \to \mathscr{C}_c^\infty(S';\omega_\pi) \to \mathscr{C}_c^\infty(S;\omega_\pi) \to \mathscr{C}_c^\infty(S - S';\omega_\pi) \to 0$$

is exact.

4.1.5 Stripping off the center of G

Let

$$A' = \{\begin{pmatrix} 1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} : a_i \in E^*\}.$$

Then A is the direct product of the center Z of G and A', algebraically and topologically. Let $C'(w) = U\dot{w}A'U_w^-$, so that $C(w) = ZC'(w)$, again with uniqueness of expression. For $g \in C(w)$, let g' be the component of g in $C'(w)$. Thus $g = zg'$, with $z \in Z$.

Lemma 4.1.5.1. *Let $w \in W(G)$. Consider the homeomorphism*

$$U \times A \times U_w^- \to C(w)$$

sending (u, a, u^-) to $u\dot{w}au^-$. If $f \in \mathscr{C}_c^\infty(C(w); \omega_\pi)$, then there exist compact open subgroups $U_1 \subset U, U_2 \subset U_w^-$, and a compact set $K' \subseteq A'$ such that if $f(u\dot{w}au^-) \neq 0$, then $u \in U_1, u^- \in U_2$, and $a' \in K'$.

Here we are uniquely writing $a \in A$ as a product za', with $z \in Z$ and $a' \in A'$.

Proof: There exists a compact set $\Omega \subseteq C(w)$ such that if $f(g) \neq 0$, then $zg \in \Omega$ for some $z \in Z$. Via the homeomorphism $U \times Z \times A' \times U_w^-$ sending (u, z, a', u^-) to $u\dot{w}za'u^-$, let K_1, K', K_2 be the projections of Ω onto U, A', U_w^-. Choose open compact subgroups U_1 and U_2 of U and U_w^- containing K_1 and K_2, respectively. Assume $g = u\dot{w}za'u^- \in C(w)$ with $f(g) \neq 0$. Then there exists a $z_1 \in Z$ such that $gz_1 = u\dot{w}zz_1a'u^- \in \Omega$. This implies $u \in K_1 \subseteq U_1, a' \in K'$, and $u^- \in K_2 \subseteq U_2$. $\square$

4.1.6 The Bruhat order and cell closures

We begin with a lemma from group theory.

Lemma 4.1.6.1. *Let (W, S) be a Coxeter system. For $w_1, w_2 \in W$, the following are equivalent:*

(i): There is a reduced decomposition of w_2 containing a subexpression which is a reduced decomposition of w_1.

(ii): Every reduced decomposition of w_2 contains a subexpression which is a reduced decomposition of w_1.

Proof: This is Corollary 2.2.3 of [BjBr05]. □

If these conditions are satisfied, we write $w_1 \leq w_2$. This defines a partial ordering on (W, S) called the **Bruhat order**. If W has a long element w_l, then multiplication by w_l reverses the Bruhat order, i.e. $w_1 \leq w_2$ if and only if $w_l w_1 \geq w_l w_2$ (Corollary 2.2.5 of [BjBr05]).

The Weyl group $W(G)$, together with the set of simple reflections w_α corresponding to elements of Δ, is a Coxeter system. If M is a standard Levi subgroup of G corresponding to the subset θ of Δ, then so is $W(M)$, together with the simple reflections $w_\alpha : \alpha \in \theta$.

The Bruhat order on $W(G)$ is related to the closure of Bruhat cells in G. For $w_1, w_2 \in W(G)$, we have $w_1 \leq w_2$ if and only if $C(w_1) \subset \overline{C(w_2)}$ ([Bo91], Theorem 21.26). Therefore, we have

$$\overline{C(w)} = \bigcup_{\substack{w' \in W(G) \\ w' \leq w}} C(w'). \tag{4.1.6.1}$$

If M is a standard Levi subgroup of G, the same goes for the Bruhat order on $W(M)$ and the closure of the Bruhat cells of M.

4.1.7 Weyl group elements which support Bessel functions

Let $\mathbf{\Phi}$ be a root system with base $\mathbf{\Delta}$ and Weyl group $\mathbf{W}$. For each $\theta \subseteq \mathbf{\Delta}$, there is one and only one element w_0 of $\mathbf{W}$ which satisfies the following equivalent conditions:

1. w_0 maps θ into $\mathbf{\Delta}$, and maps $\mathbf{\Delta} - \theta$ into the negative roots.

2. We have $\ell(w_0 w) = \ell(w_0) + \ell(w)$ for all w in the Weyl group $\mathbf{W}_\theta$ of the root system spanned by θ, and w_0 is of maximal length with respect to this property.

3. $w_0 = w_\ell w_{\ell,\theta}$, where w_ℓ and $w_{\ell,\theta}$ are the long elements of $\mathbf{W}$ and $\mathbf{W}_\theta$.

If w_0 satisfies these conditions for some θ, then w_0 is said to **support a Bessel function**, and we can recover θ as $\{\alpha \in \Delta : w_0(\alpha) > 0\}$. The number of elements of $\mathbf{W}$ which support Bessel functions is $2^{|\Delta|}$; there is one for each subset of Δ.

We will let $B(G)$ be the set of $w \in W(G)$ which support Bessel functions, and for M a standard Levi subgroup of G, we will let $B(M)$ be the set of $w \in W(M)$ which support Bessel functions.

Let $w \in B(G)$, and let M be the standard Levi subgroup corresponding to w. That is, let θ be the subset of Δ to which w corresponds, let P be the parabolic subgroup of G containing B and corresponding to θ, and let M be the unique Levi subgroup of P containing A. If we set

$$A_w = \bigcap_{\alpha \in \theta} \operatorname{Ker} \alpha$$

then $A_w = Z_M$, the center of M. Furthermore we have

$$U_M = U_w^+ \qquad N = U_w^-$$

where N is the unipotent radical of P, $U_M = U \cap M$, and U_w^+, U_w^- are as in (4.1.2). The Bruhat order reverses the containment of the Levi subgroups: if $w' \in B(G)$ corresponds to M', then $w \leq w'$ if and only if $M' \subseteq M$, if and only if $A_w = Z_M$ is contained in $A_{w'} = Z_{M'}$.

If $w \in B(M)$ corresponds to a standard Levi subgroup L of M, we often write $w = w_L^M$. In terms of the canonical representatives defined in (4.1.1), it follows from the second of the three equivalent conditions of w_L^M given above that $\dot{w}_L^M = \dot{w}_M \dot{w}_L^{-1}$. Note that $w_L^G = w_G^M w_M^L$, and even $\dot{w}_L^G = \dot{w}_G^M \dot{w}_M^L$.

4.1.8 The open sets Ω_w

If $w \in W(G)$, let

$$\Omega_w = \bigcup_{w' \geq w} C(w').$$

It follows formally from equation (4.1.6.1) that Ω_w is open in G, and $C(w)$ is closed in Ω_w. We will be particularly interested in Ω_w in the case where w supports a Bessel function. In this case, the cell structure of Ω_w will parallel that of the cell structure of the Levi subgroup to which w corresponds.

Lemma 4.1.8.1. *Let $w = w_G^M \in B(G)$, corresponding to the Levi subgroup M. Let $E = \{w' \in W(G) : w' \geq w\}$. Then $x \mapsto wx$ defines a bijection $W(M) \to E$. This bijection preserves the Bruhat order.*

Proof: If $x \in W(M)$, then $\ell(wx) = \ell(w) + \ell(x)$, where ℓ is the length function. This implies that a reduced decomposition for wx can be obtained by concatenating one of w with one of x. In particular, $wx \geq w$ in the Bruhat order, and $x \mapsto wx$ preserves the Bruhat order.

Conversely, suppose that $w' \geq w$. Since $w = w_G w_M$, and multiplication by w_G reverses the Bruhat order, we have $w_G w' \leq w_M$. This implies that a reduced decomposition of $w_G w'$ consists of simple reflections from M. Thus $w_G w' \in W(M)$, and if we set $x = w_M w_G w' \in W(M)$, then $w' = wx$. $\square$

Proposition 4.1.8.2. *Let $w \in B(G)$ correspond to a Levi subgroup M. Then*

$$U_{w^{-1}}^- \times \{\dot{w}\} \times M \times U_w^- \to \Omega_w$$

$$(x, m, u) \mapsto x\dot{w}mu$$

is a homeomorphism.

The proposition shows that the Bruhat decomposition in Ω_w is the same as that in M with unipotent groups $U_{w^{-1}}$ and U_w^- tacked on the side of each of the cells $C_M(x) : x \in W(M)$.

Proof: By the Bruhat decomposition for M, we have that M is the disjoint union of the direct products $U_M \times \{x\} \times A \times U_x^-$ for $x \in W(M)$. Letting $w' = wx$, we leave it to the reader to check that

$$wU_M w^{-1} = U_{w^{-1}}^+$$

$$U_x^- U_w^- = U_{w'}^-.$$

Therefore the Bruhat cell $C_M(x)$ is

$$U_M x A U_x^- = w^{-1} w U_M w^{-1} w x A U_x^- = w^{-1} U_{w^{-1}}^+ w' A U_x^-$$

and since $U_{w^{-1}}^- U_{w^{-1}}^+ = U$ as a direct product of topological spaces, we have

$$U_{w^{-1}}^- w C_M(x) U_w^- = U_{w^{-1}}^- U_{w^{-1}}^+ w' A U_x^- U_w^- = U w' A U_{w'}^- = C(w')$$

and we can conclude that we have a homeomorphism

$$U_{w^{-1}} \times \{w\} \times C_M(x) \times U_w^- \cong C(w').$$

The proposition is now a consequence of Lemma 4.1.8.1. $\square$

It follows from equation (4.1.6.1) that the Bruhat order reverses the containment of the open sets Ω_w. That is, if $w_1, w_2 \in B(G)$, then $w_1 \leq w_2$ if and only if $\Omega_{w_2} \subset \Omega_{w_1}$. Note that $\Omega_e = G$.

4.1.9 Transferring from $\mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$ to $\mathscr{C}_c^\infty(M;\omega\pi)$

Let M be a standard Levi subgroup of G. Then $\mathscr{C}_c^\infty(M;\omega_\pi)$ was defined to be the space of locally constant, complex valued functions f on M which are compactly supported modulo the center of G (not the center of M) and which satisfy $f(zm) = \omega_\pi(z)f(m)$ for all z in the center Z of G and all m in M.

Let $w \in B(G)$ correspond to M. Proposition 4.1.8.2 shows that Ω_w has a very similar structure to M, just with unipotent groups tacked on both sides. If $f \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$, define $h_f \in \mathscr{C}_c^\infty(M;\omega_\pi)$ by

$$h_f(m) = \int\limits_{U_{w^{-1}}^-} \int\limits_{U_w^-} f(x\dot{w}mu)\overline{\chi(xu)}dudx.$$

Lemma 4.1.9.1. *The assignment $f \mapsto h_f$ defines a surjection $\mathscr{C}_c^\infty(\Omega_w;\omega_\pi) \to \mathscr{C}_c^\infty(M;\omega_\pi)$.*

Proof: Let us first verify the convergence of this integral. Since f is compactly supported modulo Z, it is easy to see that there are compact open subgroups U_0 and U_1 of $U_{w^{-1}}-$ and U_w^- such that for every $m \in M$, $f(x\dot{w}mu) \neq 0$ implies $x \in U_0$ and $u \in U_1$. Thus $h_f(m)$ is really an integral over the compact set $U_0 \times U_1$. Thus $h_f(m)$ converges absolutely.

To see that h_f lies in $\mathscr{C}_c^\infty(M;\omega_\pi)$, note that since M is closed in Ω_w, we have that for each $x \in U_{w^{-1}}^-$ and $u \in U_w^-$, the mapping $m \mapsto f(x\dot{w}mu)$ lies in $\mathscr{C}_c^\infty(M;\omega_\pi)$. Since f and χ are locally constant, and the integrand vanishes for x and u outside U_0 and U_1, and we see that h_f is a finite linear combination of functions in $\mathscr{C}_c^\infty(M;\omega_\pi)$, so it too lies there.

To show that $f \mapsto h_f$ is surjective, let $h \in \mathscr{C}_c^\infty(M;\omega_\pi)$. Let U_0 and U_1 be small open compact subgroups of $U_{w^{-1}}^-$ and U_w^- which are contained in the kernel of χ. Setting

$$f(x\dot{w}mu) = \frac{1}{\operatorname{meas}(U_0)\operatorname{meas}(U_1)} h(m)$$

it is straightforward to see that f lies in $\mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$, and $h_f = h$. $\square$

4.2 Bessel Integrals on Levi subgroups

In this section, we fix a standard Levi subgroup M of G. Recall that $U_M = U \cap M$, and w_M is the long element of M.

Let $h \in \mathscr{C}_c^\infty(M; \omega_\pi)$. We formally define the Bessel integral by the formula

$$m \mapsto \int\limits_{U_M} \int\limits_{U_M} h(xmu)\overline{\chi(xu)}dxdu. \qquad (m \in M)$$

We can understand this formula as an integral over part of the Bruhat cell $B_M m B_M$. However, this integral generally does not converge. In order to deal with a convergent integral, we modify the definition of the Bessel integral by introducing a cutoff function

4.2.1 Definition of the partial Bessel integral

Define a right action of U_M on $\mathrm{Mat}_n(E)$ by $x.u = \dot{w}_M\, {}^t\overline{u}\dot{w}_M^{-1} x u$. If $m \in M$, we will let $U_{M,m}$ be the stabilizer of m under this action.

Let $X \subset \mathrm{Mat}_n(E)$ be an open compact set, and let φ be the characteristic function of X. Define the **partial Bessel integral** $B_\varphi^M(m,h)$ by

$$B_\varphi^M(m,h) = \int\limits_{U_{M,m}\backslash U_M} \int\limits_{U_M} h(xmu)\varphi({}^t\overline{u}\dot{w}_M^{-1} m' u)\overline{\chi(xu)}dxdu.$$

Here m' is the element of M obtained by "stripping off the center" as in (4.1.5). Note that $U_{M,m} = U_{M,m'}$. In defining the partial Bessel integral we have removed some redundancy in u by taking U_M modulo its centralizer, and we are only integrating over those u for which ${}^t\overline{u}\dot{w}_M m' u$ lies within a certain bound.

In the next section, we will prove the absolute convergence of $B_\varphi(m,h)$. For now, let us show formally that the integral is well defined.

Lemma 4.2.1.1. *If $u \in U_M$, then*

$$u \mapsto \int\limits_{U_M} h(xmu)\varphi({}^t\overline{u}\dot{w}_M^{-1}m'u)\overline{\chi(xu)}dx$$

is well defined as a function on $U_{M,m}\backslash U_M$.

Proof: We need to show that if $u_1 \in U_{M,m}$, then the expression in the statement of the lemma is unchanged when we replace u by u_1u. So we begin with the expression

$$\int\limits_{U_M} h(xmu_1u)\varphi({}^t(\overline{u_1u})\dot{w}_M^{-1}m'u_1u)\overline{\chi(xu_1u)}dx. \tag{4.2.1.1}$$

Since $u_1 \in U_{M,m} = U_{M,m'}$, we have $m' = \dot{w}_M\,{}^t\overline{u_1}\dot{w}_M^{-1}m'u_1$, or $\dot{w}_M^{-1}m' = {}^t\overline{u_1}\dot{w}_M^{-1}m'uu_1$. Therefore

$$\varphi({}^t\overline{u_1u}\dot{w}_M^{-1}m'u_1u) = \varphi({}^t\overline{u}({}^t\overline{u_1}\dot{w}_M^{-1}m'u_1)u) = \varphi({}^t\overline{u}\dot{w}_M^{-1}m'u).$$

Thus (4.2.1.1) is equal to

$$\int\limits_{U_M} h(xmu_1u)\varphi({}^t\overline{u}\dot{w}_M^{-1}m'u)\overline{\chi(xu_1u)}dx.$$

We also have $mu_1 = \dot{w}_M\,{}^t\overline{u_1^{-1}}\dot{w}_M^{-1}m$, and so (4.2.1.1) is equal to

$$\int\limits_{U_M} h(x(\dot{w}_M\,{}^t\overline{u_1^{-1}}\dot{w}_M^{-1})mu)\varphi({}^t\overline{u}\dot{w}_M^{-1}m'u)\overline{\chi(xu_1u)}dx.$$

We replace x by $x(\dot{w}_M\,{}^t\overline{u_1}\dot{w}_M^{-1})$ in the integral. We are done if we verify that

$$\chi(\dot{w}_M\,{}^t\overline{u_1}\dot{w}_M^{-1}) = \overline{\chi(u_1)}.$$

We observe that replacing $\overline{u_1}$ by u_1 does not change the value of χ, since χ is obtained by taking the trace from E to F. We finally use the fact that u_1 is a block combination of upper triangular unipotent matrices and $\dot{w}_M$ is a block combination of antidiagonal matrices. The verification then reduces to the case of $M = G$, which is straightforward. □

4.2.2 Convergence of the partial Bessel integral

We now prove the absolute convergence of the partial Bessel function $B_\varphi^M(m, h)$. We begin by showing the convergence of the inner integral.

Proposition 4.2.2.1. *If $u \in U$ and $m \in M$, then the integral*

$$\int_{U_M} h(xmu)\overline{\chi(xu)}dx$$

converges absolutely.

Proof: Replacing mu by m and taking $\overline{\chi(u)}$ out of the integral, it suffices to prove the absolute convergence of

$$\int_{U_M} h(xm)\overline{\chi(x)}dx. \tag{4.2.2.1}$$

Since h is compactly supported modulo Z, there is a compact set $\Omega \subset M$ such that $h(m') \neq 0$ implies $m'z \in \Omega$ for some $z \in Z$. Since $U_M Z$ is a closed direct product in M, the projection Ω_0 of $\Omega m^{-1} \cap U_M Z$ to U_M is compact. Clearly Ω_0 consists of those $x \in U_M$ for which $xmz \in \Omega$ for some $z \in Z$.

It follows that if $h(xm) \neq 0$, then x lies in Ω_0. This shows that the integrand of (4.2.2.1) vanishes outside a compact set, and implies the required absolute convergence. □

The absolute convergence of the outer integral is more subtle and relies on the fact that orbits of unipotent algebraic groups on affine varieties are closed.

We will briefly review some facts about analytic manifolds over p-adic fields which we will need. For a general reference, see [Se65]. If k is a p-adic field, and $\mathbf{X}$ is a smooth affine variety over k, then $\mathbf{X}(k)$ is an analytic manifold. If $\mathbf{H}$ is a linear algebraic group over k, then $\mathbf{H}(k)$ is a p-adic Lie group.

Suppose that $\mathbf{H}$ acts on $\mathbf{X}$ as a morphism of varieties, and that this action is defined over k. Then this induces an analytic action of the Lie group $\mathbf{H}(k)$ on the manifold $\mathbf{X}(k)$. The orbits of $\mathbf{X}(k)$ under $\mathbf{H}(k)$ are locally closed.

Lemma 4.2.2.2. *Let $x \in \mathbf{X}(k)$, and let $\mathbf{H}(k)_x$ be the stabilizer of x in $\mathbf{H}(k)$. The orbit $\mathbf{H}(k).x$ is a locally closed submanifold of $\mathbf{X}$, and the natural bijection*

$$\mathbf{H}(k)/\mathbf{H}(k)_x \to \mathbf{H}(k).x$$

induced by $h \mapsto h.x$ is an isomorphism of analytic manifolds.

Proof: This is generally true when H is an analytic Lie group over a characteristic zero local field acting on an analytic manifold X. The result holds as long as H is a countable union of compact sets and the orbits of X under the action of H are known to be locally closed. See [Se65], Chapter IV, § 5, Theorem 4. □

The proof of the absolute convergence of the partial Bessel function will use Lemma 4.2.2.2 as well as some nonabelian Galois cohomology. We refer to [Se02] for the definition and basic properties of nonabelian Galois cohomology. We fix an algebraic closure $\overline{k}$ of k and let $\Gamma = \mathrm{Gal}(\overline{k}/k)$.

Lemma 4.2.2.3. *Let $\mathbf{H}$ be a unipotent linear algebraic group over k. Then $\mathbf{H}$ is connected, and $H^1(\Gamma, \mathbf{H}) = 1$.*

Proof: Since we are in characteristic zero, the exponential map is well defined on upper triangular unipotent matrices, in which all unipotent groups can be embedded. The Baker-Campbell-Hausdorff formula applies and gives a bijection between Lie subalgebras of upper triangular nilpotent matrices and closed subgroups of upper triangular unipotent matrices. In particular, every unipotent linear algebraic group is connected in characteristic zero.

So $\mathbf{H}$ is connected and therefore split in the sense of § 15 of [Bo91]. Then $\mathbf{H}$ has a composition series $1 = \mathbf{N}_0 \subseteq \cdots \subseteq \mathbf{N}_s = \mathbf{H}$ of closed, connected k-subgroups of $\mathbf{N}$ such that each quotient $\mathbf{N}_i/\mathbf{N}_{i-1}$ is k-isomorphic to the additive group $\mathbb{G}_a$.

We prove inductively that each $H^1(\Gamma, \mathbf{N}_i)$ is trivial. It is a standard result that $H^1(\Gamma, \mathbb{G}_a) = H^1(\Gamma, \bar{k}) = 1$ (Chapter X, § 1, Proposition 1 of [Se91]), so this takes care of the case $i = 1$.

For general i, we take cohomology of the short sequence sequence

$$1 \to \mathbf{N}_{i-1} \to \mathbf{N}_i \to \mathbb{G}_a \to 0$$

to obtain a long exact sequence

$$\cdots \to H^1(\Gamma, \mathbf{N}_{i-1}) \to H^1(\Gamma, \mathbf{N}_i) \to H^1(\Gamma, \mathbb{G}_a) \to \cdots .$$

with $H^1(\Gamma, \bar{k}) = 1$, and also $H^1(\Gamma, \mathbf{N}_{i-1}) = 1$ by induction. This implies that $H^1(\Gamma, \mathbf{N}_i) = 1$. $\square$

If $x \in \mathbf{X}(k)$, let $\mathbf{H}.x$ be the orbit of x under $\mathbf{H}$. It is a locally closed subvariety of $\mathbf{X}$ which is defined over k. It is not generally true that the set $[\mathbf{H}.x](k)$ of k-rational points in this orbit is equal to the orbit $\mathbf{H}(k).x$ of x under $\mathbf{H}(k)$. We only have a containment $\mathbf{H}(k).x \subset [\mathbf{H}.x](k)$. On the other hand, if $\mathbf{H}_x$ is the stabilizer of x in $\mathbf{H}$, then $\mathbf{H}_x$ is defined over k, and it is obvious that $\mathbf{H}_x(k) = \mathbf{H}(k)_x$.

Proposition 4.2.2.4. *Assume that* $\mathbf{H}$ *is unipotent. Then all orbits of* $\mathbf{H}(k)$ *on* $\mathbf{X}(k)$ *are closed. If* $x \in \mathbf{X}(k)$*, and* $\mathbf{H}.x$ *is the orbit of* x*, then* $[\mathbf{H}.x](k) = \mathbf{H}(k).x$*.*

Proof: It suffices to prove the second assertion. Indeed, it is a basic result that orbits of unipotent algebraic groups on affine varities over *algebraically closed fields* are closed (Proposition 4.10 of [Bo91]). In other words, $\mathbf{H}.x$ is already a closed subvariety of $\mathbf{X}$. Passing to k-rational points gives a closed immersion $[\mathbf{H}.x](x) \to \mathbf{X}(k)$ of topological spaces.

To prove the second assertion, we must use the fact that the short exact sequence of pointed sets

$$1 \to \mathbf{H}_x \to \mathbf{H} \to \mathbf{H}.x \to 1$$

induces a long exact sequence

$$1 \to \mathbf{H}(k)_x \to \mathbf{H}(k) \to [\mathbf{H}.x](k) \to H^1(\Gamma, \mathbf{H}_x) \to \cdots$$

We have used the fact that $\mathbf{H}_x(k) = \mathbf{H}(k).x$. Since $\mathbf{H}_x$ is unipotent, Lemma 4.2.2.3 tells us that $H^1(\Gamma, \mathbf{H}_x)$ is trivial. Thus the natural injection

$$\mathbf{H}(k).x = \mathbf{H}(k)/\mathbf{H}(k)_x \to [\mathbf{H}.x](k)$$

is a bijection. □

Now we can prove the convergence of the partial Bessel integral (4.2.1.1). As before, the group U_M acts on the space $\mathrm{Mat}_n(E)$ of all n by n matrices with entries in E, by the right action $m.u = \dot{w}_M\, {}^t\overline{u}\dot{w}_M^{-1} m u$. By restriction of scalars, we may identify U_M and $\mathrm{Mat}_n(E)$ as the F-points of affine varieties. Under the identification, the action of U_M on $\mathrm{Mat}_n(E)$ is a morphism of varieties. Proposition 4.2.2.4 applies and tells us that the orbits of $\mathrm{Mat}_n(E)$ under this action are closed.

Now let $m \in M$, and write $m = zm'$ for $z \in Z$ as in 4.1.5. We have

$$B_\varphi^M(m,h) = \int\limits_{U_{M,m'}\backslash U_M} W^f(mu)\varphi({}^t\overline{u}\dot{w}_M^{-1}m'u)\overline{\chi(u)}du \qquad (4.2.2.1)$$

where the integrand is a locally constant function of $u \in U_{M,m'}\backslash U_M$. Let $m'.U_M$ be the orbit of m', which we know is closed in $\mathrm{Mat}_n(E)$. Let X be the open compact set in $\mathrm{Mat}_n(E)$, of which φ is the characteristic function.

Since $m'.U_M$ is closed in $\mathrm{Mat}_n(E)$, the intersection $(\dot{w}_M X) \cap m'.U_M$ remains compact. Since the natural map $p : U_{M,m'}\backslash U_M \to m'.U_M$ is a homeomorphism by Lemma 4.2.2.2, the preimage $Y = p^{-1}((\dot{w}_M X) \cap m'.U_M)$ is compact in $U_{M,m'}\backslash U_M$.

Now that we know that integrand of (4.2.2.1) is locally constant and vanishes off of the compact set Y, we can conclude that it converges absolutely.

4.2.3 A choice of open compact neighborhood $X(N)$

In the definition of the partial Bessel integral $B^M_\varphi(m,h)$, φ is the characteristic function of an open compact subset of $\mathrm{Mat}_n(E)$. We will now fix a choice of such functions φ. Let ω_F be a uniformizer for $\mathcal{O}_F$, and let (ω_F) be the ideal generated by ω_F in $\mathcal{O}_E$. For each positive integer N, let

$$X(N) = \begin{pmatrix} (\varpi_F)^{-N} & (\varpi_F)^{-2N} & (\varpi_F)^{-3N} & \cdots \\ (\varpi_F)^{-2N} & (\varpi_F)^{-3N} & & \\ (\varpi_F)^{-3N} & & \ddots & \\ \vdots & & & \end{pmatrix}.$$

We can also describe $X(N)$ as

$$X(N) = \{x \in \mathrm{Mat}_n(E) : x_{ij} \in (\omega_F)^{-(i+j-1)N}\}.$$

Then $X(N)$ is an open compact neighborhood of zero, and the union of the $X(N)$ is all of $\mathrm{Mat}_n(E)$. From now on, φ_N will be the characteristic function of $X(N)$, and when N is not specified, φ will mean φ_N for some N. By abuse of notation, we will say "for sufficiently large φ" in place of "for sufficiently large N."

Also, let $U(N)$ be the set of upper triangular unipotent n by n matrices u with entries in E, such that $u_{ij} \in (\varpi_F)^{(j-i)N}$ for all $i < j$. Then $U(N)$ is an open compact subgroup of U, and U is the union of the $U(N)$.

Lemma 4.2.3.1. *For $u \in U(N)$ and $X \in \mathrm{Mat}_n(E)$, we have $\varphi_N({}^t\overline{u}Xu) = \varphi_N(X)$.*

That is, $X(N)$ is stable under the right action of $U(N)$ by $X.u = {}^t\overline{u}Xu$.

Proof: Let $u \in U(N)$ and $X \in \mathrm{Mat}_n(E)$. By taking inverses, we may assume without loss of generality that $X \in X(N)$. First, let $A = {}^t\overline{u}X$. We have

$$A_{ij} = \sum_{l=1}^{n} \overline{u_{ki}} X_{kj} = \sum_{l=1}^{i} \overline{u_{ki}} X_{kj}$$

and so the ijth entry of ${}^t\overline{u}Xu$ is

$$\begin{aligned}\sum_{l=1}^{n} A_{il}u_{lj} &= \sum_{l=1}^{j} A_{il}u_{lj}\\ &= \sum_{l=1}^{j}\sum_{k=1}^{i} \overline{u_{ki}}X_{kl}u_{lj}\end{aligned}$$

where each term in the sum lies in

$$(\varpi_F)^{-(i-k)N}(\varpi_F)^{-(k+l-1)N}(\varpi_F)^{-(j-l)N} = (\varpi_F)^{-(i+j-1)N}.$$

□

4.2.4 Pure Bessel integrals

The cutoff function φ is necessary for the general convergence of our partial Bessel integrals. But there are certain circumstances in which the cutoff function can be omitted in order to simplify computations.

Let $w \in W(M)$, and consider the Bruhat cell $C_M(w) = U_M\dot{w}AU_w^-$ (direct product) inside M. Let $f \in \mathscr{C}_c^\infty(M;\omega_\pi)$ be a function whose restriction to $C_M(w)$ is compactly supported modulo Z, i.e. whose restriction to $C_M(w)$ lies in $\mathscr{C}_c^\infty(C_M(w);\omega_\pi)$. Under this hypothesis we will define the **pure Bessel integral** of f on the cell $C_M(w)$ by

$$B^M(m,f) = \int\limits_{U_M\times U_w^-} f(xmu)\overline{\chi(xu)}dudx$$

for all $m \in C_M(w)$. Our hypothesis on f indicates that the integrand vanishes off of a compact set.

Remark 4.2.4.1. *Generally, the restriction of f to a given Bruhat cell will not be compactly supported modulo Z. This is because $C_M(w)$ is generally not closed in M, but only locally closed. This is why the cutoff function φ is necessary for convergence.*

There will be some reoccuring circumstances in which we will be able to use the pure Bessel integral, however. There is first of all the case where the $w = 1_M$, and so the cell $C_M(w) = B_M$ in question is in fact closed in M.

The other is the case in which f is already known to be supported inside a union of certain cells. For example, suppose that $M = G$, $w \in W(G)$, and $f \in \mathscr{C}_c^\infty(\Omega_w; \omega_\pi)$. Then since $C(w)$ is closed in Ω_w, the restriction of f to $C(w)$ will be compactly supported modulo Z.

Let L be a standard Levi subgroup of M, and let $w = w_M w_L \in W(M)$. Consider for $a \in A$, and the element $a' \in A'$ obtained by stripping off the center of a, the integral

$$\tilde{\varphi}_L^M(a') = \int\limits_{U_{M,\dot{w}} \backslash U_L} \varphi({}^t\overline{u}\dot{w}_M^{-1} a' u) du.$$

The following lemma shows that $U_{M,\dot{w}} \subset U_L$, so the domain of integration makes sense. Proposition 4.2.2.4 shows that $\tilde{\varphi}_L^M(a')$ converges as an integral over a compact set.

Since w sends the simple roots of L to positive roots, and all the rest of the simple roots of M to negative roots, we actually have $U_L = U_w^+ \cap U_M$, the unipotent radical of $B \cap L$.

Lemma 4.2.4.2. *(Twisted centralizer lemma) Let $L \subset M$ be standard Levi subgroups of G. Let $w = w_M w_L$.*

(i): If $a \in A$, then $U_{M,wa} \subset U_w^+ \cap U_M = U_L$.

(ii): If moreover $a \in Z_L$, then $U_{M,\dot{w}a} = U_{M,\dot{w}}$

Proof: (i): We have $\dot{w} = \dot{w}_M \dot{w}_L^{-1}$. Suppose that $u \in U_{M,\dot{w}a}$, so that $\dot{w}_M \dot{w}_L^{-1} a = \dot{w}_M {}^t\overline{u}\dot{w}_M^{-1}\dot{w}au$. Rearrange this to ${}^t\overline{u} = \dot{w}_L a u^{-1} a^{-1} \dot{w}_L^{-1}$. Now $au^{-1}a^{-1}$ is an upper triangular matrix in U_M, and ${}^t\overline{u}$ is lower triangular. Now conjugation by $\dot{w}_L$ sends all the roots subgroups in U_L to negative roots, and permutes all the other upper triangular root subgroups. This means that $au^{-1}a^{-1}$ and therefore u must actually lie in U_L.

(ii): We already know that both $U_{M,\dot{w}a}$ and $U_{M,\dot{w}}$ are contained in U_L. For $u \in U_L$, we have $u \in U_{M,\dot{w}a}$ if and only if $\dot{w}_M\dot{w}_L^{-1}a = \dot{w}_M\,{}^t\overline{u}\dot{w}_M^{-1}\dot{w}au$ if and only if $\dot{w}_M\dot{w}_L^{-1}a = \dot{w}_M\,{}^t\overline{u}\dot{w}_M^{-1}\dot{w}ua$ if and only if $\dot{w}_M\dot{w}_L^{-1} = \dot{w}_M\,{}^t\overline{u}\dot{w}_M^{-1}\dot{w}u$, if and only if $u \in U_{M,\dot{w}}$. $\square$

The transfer factor $\tilde{\varphi}_L^M$ shows up in the difference between the partial and pure Bessel integral.

Proposition 4.2.4.3. *Let $L \subset M$ be standard Levi subgroups of G. Let $f \in \mathscr{C}_c^\infty(M;\omega_\pi)$, and suppose that the restriction of f to $C_M(w)$ is compactly supported modulo Z, where $w = w_M w_L$. Then for sufficiently large φ depending on f, we have*

$$B_\varphi^M(\dot{w}a, f) = \tilde{\varphi}_L^M(a')B^M(\dot{w}a, f)$$

for all $a \in Z_L$.

Proof: We can write U_M as a direct product $(U_w^+ \cap U_M)U_w^- = U_L U_w^-$, with U_L normalizing U_w^-. Lemma 4.2.4.2 shows that $U_{M,\dot{w}a} = U_{M,\dot{w}}$ is contained in U_L, so in the partial Bessel integral $B_\varphi^M(\dot{w}a, f)$, we can write the quotient space $U_{M,\dot{w}}\backslash U_M$ as a direct product of topological spaces:

$$U_{M,\dot{w}}\backslash U_M = (U_{M,\dot{w}}\backslash U_L) \times U_w^-.$$

The quotient measure for $U_{M,\dot{w}}\backslash U_M$ also decomposes as the product of the quotient measure for $U_{M,\dot{w}}\backslash U_L$ and the Haar measure on U_w^-. Thus we have

$$\begin{aligned}
B_\varphi^M(\dot{w}a, f) &= \int\limits_{U_{M,\dot{w}a}\backslash U_M} \int\limits_{U_M} f(x\dot{w}au)\varphi({}^t\overline{u}\dot{w}_M^{-1}\dot{w}a'u)\overline{\chi(xu)}dxdu \\
&= \int\limits_{U_{M,\dot{w}a}\backslash U_M} \int\limits_{U_M} f(x\dot{w}au)\varphi({}^t\overline{u}\dot{w}_L^{-1}a'u)\overline{\chi(xu)}dxdu \\
&= \int\limits_{U_{M,\dot{w}}\backslash U_L} \int\limits_{U_w^-} \int\limits_{U_M} f(x\dot{w}au_Lu_w)\varphi({}^t\overline{u_w}\,{}^t\overline{u_L}\dot{w}_L^{-1}a'u_Lu_w) \\
&\quad \overline{\chi(xu_Lu_w)}dxdu_wdu_L.
\end{aligned}$$

Since $a \in Z_L$, we can write $f(x\dot{w}au_Lu_w) = f(x\dot{w}u_L\dot{w}^{-1}\dot{w}au_w)$. Since conjugation by $w = w_Mw_L$ keeps U_L inside of U_M, we can make the change of variables $x \mapsto x(\dot{w}u_L\dot{w}^{-1})^{-1}$. We get a cancellation in the argument of χ, since by compatibility we have $\chi(\dot{w}u_L\dot{w}{-}1) = \chi(u_L)$. So our partial Bessel integral is

$$B_\varphi^M(\dot{w}a, f) = \int\limits_{U_{M,\dot{w}}\backslash U_L} \int\limits_{U_w^-} \int\limits_{U_M} f(x\dot{w}au_w)\varphi({}^t\overline{u_w}\,{}^t\overline{u_L}\dot{w}_L^{-1}a'u_Lu_w)\overline{\chi(xu_w)}dxdu_wdu_L.$$

Now in order to separate out integration over $U_{M,\dot{w}}\backslash U_L$ from the other two integrals, we finally use our assumption that f is compactly supported modulo Z when restricted to the cell $C_M(w)$. There must exist a compact open subgroup U_1 of U_w^- such that $f(x\dot{w}au_w) = 0$ for all $x \in U_M, a \in A$, and u_w outside U_1. If we take $\varphi = \varphi_N$ sufficiently large, then we will have $\varphi({}^t\overline{u_w}gu_w) = \varphi(g)$ for all $g \in G$ and $u_w \in U_1$. So for sufficiently large φ, we will have

$$\begin{aligned}
B_\varphi^M(\dot{w}a, f) &= \int\limits_{U_{M,\dot{w}}\backslash U_L} \int\limits_{U_w^-} \int\limits_{U_M} f(x\dot{w}au_w)\varphi({}^t\overline{u_L}\dot{w}_L^{-1}a'u_L)\overline{\chi(xu_w)}dxdu_wdu_L \\
&= \int\limits_{U_{M,\dot{w}}\backslash U_L} \varphi({}^t\overline{u_L}\dot{w}_L^{-1}a'u_L)du_L \int\limits_{U_w^-} \int\limits_{U_M} f(x\dot{w}au_w)\overline{\chi(xu_w)}dxdu_w \\
&= \tilde{\varphi}_L^M(a')B^M(\dot{w}a, f).
\end{aligned}$$

□

Lemma 4.2.4.4. *For sufficiently large φ depending on f, and all $a \in Z_L$, we have $B_\varphi^M(\dot{w}a, f) \neq 0$ if and only if $B^M(\dot{w}a, f) \neq 0$.*

Proof: Since the restriction of f to $C_M(w)$ is compactly supported modulo Z, there must be a compact set $K' \subset A'$ such that $B^M(\dot{w}a, f) \neq 0$ implies $a' \in K'$. If we take φ sufficiently large, we may guarantee that $\varphi(\dot{w}_M^{-1}a') = 1$ for all $a' \in K'$. Now enlarge φ further so that the conclusion of Proposition 4.2.4.3 holds, that is $B_\varphi^M(\dot{w}a, f) = \tilde{\varphi}_L^M(a')B^M(\dot{w}a, f)$ for all $a \in Z_L$.

Of course $B_\varphi^M(\dot{w}a, f) \neq 0$ implies $B^M(\dot{w}a, f) \neq 0$. Conversely, suppose that $B^M(\dot{w}a, f) \neq 0$. Then a' must lie in K'. This implies that $\varphi(\dot{w}_M^{-1}a') = 1$. We can conclude that $\tilde{\varphi}_L^M(a') \neq 0$, since this is essentially the volume of a nonempty compact open set inside the locally compact space $U_{M,\dot{w}}\backslash U_L$. We conclude that $B_\varphi^M(\dot{w}a, f) \neq 0$. $\square$

4.2.5 Uniform smoothness of B_φ^M on the big cell of M

Let $h \in \mathscr{C}_c^\infty(M; \omega_\pi)$. As part of our proof of stability, we will be interested in finding a function $h_1 \in \mathscr{C}_c^\infty(M; \omega_\pi)$ such that:

- $B_\varphi^M(m, h)$ and $B_\varphi^M(m, h_1)$ agree for m in a "relevant" part of the small cell $B_M(1_W) = B_M$ of M.
- $B_\varphi^M(m, h_1)$ has a "uniform smoothness" property for m in the big cell $B_M(w_M)$.

We will make both of these notions precise soon.

Lemma 4.2.5.1. *$M_{\mathrm{der}} Z_M$ (resp. $(A \cap M_{\mathrm{der}})Z_M$ is an open and finite index subgroup of M (resp. A). The intersection $M_{\mathrm{der}} \cap Z_M = (A \cap M_{\mathrm{der}}) \cap Z_M$ is finite.*

Proof: Our Levi subgroup M is a product of block diagonal matrices $\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_t}(E)$, with $n = n_1 + \cdots + n_t$. The intersection of M_{der} with Z_M, which is the same as the intersection of $A \cap M_{\mathrm{der}}$ with Z_M consists of all diagonal matrices

$$b = \begin{pmatrix} b_1 I_{n_1} & & & \\ & b_2 I_{n_2} & & \\ & & \ddots & \\ & & & b_t I_{n_t} \end{pmatrix}$$

with $b_i^{n_i} = 1$, so this intersection is finite. For an arbitrary element

$$b = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_t \end{pmatrix}$$

in M, it is not difficult to see that b lies in $M_{\mathrm{der}}Z_M$ if and only if the determinant of each matrix $g_i \in \mathrm{GL}_{n_i}(E)$ lies in $(E^*)^{n_i}$. Since $(E^*)^{n_i}$ is open and finite index in E^*, so is $M_{\mathrm{der}}Z_M$ inside M. The same argument shows that $(A \cap M_{\mathrm{der}})Z_M$ is an open and finite index subgroup of A. $\square$

We will make use of the pure Bessel integral B^M (4.2.2) on the small cell. Since $U_e^- = 1$, this is defined for $b \in B_M$ by

$$B^M(b, h_0) \int\limits_{U_M} f(xb)\overline{\chi(x)}dx$$

whenever $h_0 \in \mathscr{C}_c^\infty(B_M; \omega_\pi)$.

Lemma 4.2.5.2. *There exists an $h_0 \in \mathscr{C}_c^\infty(M; \omega_\pi)$ such that $B^M(e, h_0) = \frac{1}{\kappa_M}$, where $\kappa_M = |Z \cap M_{\mathrm{der}}|$, and $B^M(c, h_0) = 0$ for $c \in Z_M \cap M_{\mathrm{der}}, \notin Z$.*

Proof: Let S be the set of $a' \in A'$ such that $za' \in Z_M \cap M_{\mathrm{der}}$ for some $z \in Z$. Since $A = ZA'$ is a direct product, and $Z_M \cap M_{\mathrm{der}}$ is finite, so is S. Let V be a compact open neighborhood of the identity in A' which does not contain any points in S except for the identity. We define a function $h_0 : A' \to \mathbb{C}$ by setting

$$h_0(a') = \begin{cases} \frac{1}{\kappa_M} & \text{if } a' \in V \\ 0 & \text{if } a' \notin V. \end{cases}$$

We extend h_0 to a function on A by setting $h_0(za') = \omega_\pi(z)h_0(a')$. Next, let U_1 be an open compact subgroup of U_M, chosen sufficiently small to be contained in the kernel of χ. We extend h_0 to a function on $B_M = U_M A$ by

$$h_0(xa) = \begin{cases} \frac{h_0(a)}{\mathrm{meas}(U_1)} & \text{if } x \in U_1 \\ 0 & \text{if } x \notin U_1. \end{cases}$$

It is not difficult to see that h_0 lies in $\mathscr{C}_c^\infty(B_M;\omega_\pi)$, and we may further extend h_0 to a function on $\mathscr{C}_c^\infty(M;\omega_\pi)$ by the surjectivity of the exact sequence in (4.1.4). Then for $c \in A$, we have

$$B^M(c,h_0) = \int\limits_{U_M} h_0(xc)\overline{\chi(x)}dx = h_0(c)$$

which by construction is $\frac{1}{\kappa_M}$ if $c = e$, and is zero if c is in $Z_M \cap M_{\mathrm{der}}$ but not in Z. □

We recall from Proposition 4.2.5.1 that $Z_M M_{\mathrm{der}}$ is open and finite index in M, and $Z_M \cap M_{\mathrm{der}}$ is finite, consisting of diagonal matrices

$$\xi = \begin{pmatrix} \zeta_1 I_{n_1} & & & \\ & \zeta_2 I_{n_2} & & \\ & & \ddots & \\ & & & \zeta_t I_{n_t} \end{pmatrix} \tag{4.2.5.1}$$

with $\zeta_i \in E^*$ and $\zeta_i^{n_i} = 1$. We will consider all the ways (at most finitely many) a given $m \in M$ can be written as $m = m'c$, with $m' \in M_{\mathrm{der}}$ and $c \in Z_M$. We set

$$h_1(m) = \begin{cases} \sum\limits_{m'c=m} h_0(m')B^M(c,h) & \text{if } m \in Z_M M_{\mathrm{der}} \\ 0 & \text{if } m \notin Z_M M_{\mathrm{der}}. \end{cases}$$

Note that if $m = m'c$ is a fixed decomposition of a given m, with $m' \in M_{\mathrm{der}}$ and $c \in Z_M$, then

$$h_1(m) = \sum_{\xi \in Z_M \cap M_{\mathrm{der}}} h_0(m'\xi)B^M(\xi^{-1}c,h).$$

We must first check that h_1 has suitable analytic behavior.

Lemma 4.2.5.3. *We have $h_1 \in \mathscr{C}_c^\infty(M;\omega_\pi)$.*

Proof: Define a map $\delta : M_{\mathrm{der}} \times Z_M \to \mathbb{C}$ by

$$\delta(m',c) = \sum_{\xi \in Z_M \cap M_{\mathrm{der}}} h_0(m'\xi)B^M(\xi^{-1}c,h).$$

It is locally constant and therefore continuous. It also well defined on the quotient group $(M_{\text{der}} \times Z_M)/(Z_M \cap M_{\text{der}})$, where $Z_M \cap M_{\text{der}}$ is embedded diagonally. This quotient is isomorphic as a topological group to the open subgroup $M_{\text{der}}Z_M$ of M, so δ identifies with a function there, and this function is precisely our h_1. If we extend h_1 by zero outside of $M_{\text{der}}Z_M$, it remains locally constant.

Let us make sure that $h_1(zm) = \omega_\pi(z)h_1(m)$ for all $m \in M$ and $z \in Z$. If $m \notin M_{\text{der}}Z_M$, then neither is zm, so $h_1(zm) = h_1(m) = 0$. If $m \in M_{\text{der}}Z_M$, let $m = m'c$ be a fixed decomposition. Then $zm = m'(zc)$ is a fixed decomposition of zm, and we have

$$\begin{aligned} h_1(zm) &= \sum_{\xi \in Z_M \cap M_{\text{der}}} h_0(m'\xi)B^M(\xi^{-1}zc, h) \\ &= \omega_\pi(z) \sum_{\xi \in Z_M \cap M_{\text{der}}} h_0(m'\xi)B^M(\xi^{-1}c, h) \\ &= \omega_\pi(z)h_1(m). \end{aligned}$$

The last thing to check is that h_1 is actually compactly supported modulo Z. Since h_0 is compactly supported modulo Z, there is a compact set Ω_0 in M such that $h_0(m) \neq 0$ implies $zm \in \Omega_0$ for some $z \in Z$. The same goes for h: there is a compact set Ω in M such that if $a \in A$ and $B^M(a, h) \neq 0$, then $zc \in \Omega$ for some $z \in Z$. Supposing $h_1(m) \neq 0$, we must have $m \in M_{\text{der}}Z_M$. Let $m = m'c$ be a fixed decomposition, so that

$$h_1(m) = \sum_{\xi \in Z_M \cap M_{\text{der}}} h_0(m'\xi)B^M(\xi^{-1}c, h).$$

Since $h_1(m) \neq 0$, there is at least one ξ such that $h_0(m'\xi)$ and $B^M(\xi^{-1}c, h)$ are both not zero. We then have $z_1m'\xi \in \Omega_0$ and $z_2\xi^{-1}c \in \Omega$ for some $z_1, z_2 \in Z$. Then

$$z_1z_2m = (z_1m'\xi)(z_2\xi^{-1}c) \in \Omega_0\Omega.$$

We have shown that if $h_1(m) \neq 0$, then there is a $z \in Z$ such that zm lies in the compact set $\Omega_0\Omega$. This completes the proof. $\square$

It is in h_1 that we will find our uniform smoothness in the big cell of M. We will only be able to relate h and h_1 on the small cell of M.

Proposition 4.2.5.4. *For sufficiently large φ, depending on h and h_1, we have*

$$B_\varphi^M(a, h_1) = B_\varphi^M(a, h)$$

for all $a \in Z_M$.

Proof: We are in a situation where we can use pure Bessel integrals as in (4.2.4). By Proposition 4.2.4.3, we have for sufficiently large φ that

$$B_\varphi^M(a, h_1) = \tilde{\varphi}_M^M(a')B^M(a, h_1)$$

$$B_\varphi^M(a, h) = \tilde{\varphi}_M^M(a')B^M(a, h)$$

for all $a \in Z_M$. The problem then becomes to show that $B^M(a, h_1) = B^M(a, h)$. We have

$$B^M(a, h_1) = \int\limits_{U_M} h_1(xa)\overline{\chi(x)}dx.$$

In order to expand this further, we must decompose each xa as a product in M_{der} and Z_M. But already $x \in M_{\text{der}}$ and $a \in Z_M$. Therefore

$$\begin{aligned} B^M(a, h_1) &= \int\limits_{U_M} \sum_{\xi \in Z_M \cap M_{\text{der}}} h_0(x\xi)B^M(\xi^{-1}a, h)\overline{\chi(x)}dx \\ &= \sum_{\xi \in Z_M \cap M_{\text{der}}} B^M(\xi^{-1}a, h) \int\limits_{U_M} h_0(x\xi)\overline{\chi(x)}dx \\ &= \sum_{\xi \in Z_M \cap M_{\text{der}}} B^M(\xi^{-1}a, h)B^M(\xi, h_0). \end{aligned}$$

If $\xi \in Z_M \cap M_{\mathrm{der}}$, it follows from by our construction of h_0 in Lemma 4.2.5.2 that $B^M(\xi, h_0) = 0$ unless $\xi \in Z \cap M_{\mathrm{der}}$, in which case $B^M(\xi, h_0) = \frac{\omega_\pi(\xi)}{\kappa_M}$, where $\kappa_M = |Z \cap M_{\mathrm{der}}|$. So

$$B^M(a, h_1) = \sum_{\xi \in Z \cap M_{\mathrm{der}}} B^M(\xi^{-1}a, h) B^M(\xi, h_0) = \frac{1}{\kappa_M} \sum_{\xi \in Z \cap M_{\mathrm{der}}} B^M(a, h) = B^M(a, h).$$

□

We will finally state the smoothness property we want. Let T be a subtorus of A, and let $\phi : T \to \mathbb{C}$ be a locally constant function. We will say that ϕ is **uniformly smooth** if there exists an open compact subgroup T_0 of T such that $\phi(tt_0) = \phi(t)$ for all $t \in T$ and $t_0 \in T_0$.

Now, consider a given $a \in A$, and suppose that a admits a decomposition as bc, with $b \in A \cap M_{\mathrm{der}}$ and $c \in Z_M$. Recall that $(A \cap M_{\mathrm{der}})Z_M$ is open and finite index in A, and that every other such decomposition of a is equal to $(b\xi^{-1})(\xi c)$ with ξ in the finite intersection $Z_M \cap M_{\mathrm{der}}$. Let

$$Z'_M = \{ \begin{pmatrix} I_{n_1} & & & \\ & c_2 I_{n_2} & & \\ & & \ddots & \\ & & & c_t I_{n_t} \end{pmatrix} : c_i \in E^* \}$$

and let $c' \in Z'_M$ be the element of Z_M obtained by stripping off the center of c as in (4.1.5). If we write $c = c'z$ with $z \in Z$, then we have

$$B^M_\varphi(\dot{w}_M a, h_1) = B^M_\varphi(\dot{w}_M bc, h_1) = \omega_\pi(z) B^M_\varphi(\dot{w}_M bc', h_1).$$

Theorem 4.2.5.5. *For every $b \in A \cap M_{\mathrm{der}}$,*

$$B^M_\varphi(\dot{w}_M bc', h_1)$$

is uniformly smooth as a function of $c' \in Z'_M$. *The open compact subgroup of* Z'_M *occurring in the definition of uniform smoothness will depend on* h_1 *and* h*, but will be independent of* b *and* φ.

The theorem says that there exists an open compact subgroup H of Z'_M such that for all sufficiently large φ and all $b \in A \cap Z_M, c' \in Z'_M$, and $c'' \in H$,

$$B^M_\varphi(\dot{w}_M bc'c'', h_1) = B^M_\varphi(\dot{w}_M bc', h_1).$$

Proof: Since we are in the big cell, the twisted centralizer in U_M is always trivial, so

$$B^M_\varphi(\dot{w}_M bc', h_1) = \int\limits_{U_M}\int\limits_{U_M} h_1(x\dot{w}_M bc'u)\varphi({}^t\overline{u}b'c'u)\overline{\chi(xu)}dudx.$$

Now to invoke the definition of h_1, we use the fact that $x\dot{w}_M bu$ is already in M_{der} and $c' \in Z_M$. Thus

$$h_1(x\dot{w}_M bc'u) = h_1(x\dot{w}_M buc') = \sum_{\xi \in Z_M \cap M_{\text{der}}} h_0(x\dot{w}_M b\xi^{-1}u)B^M(\xi c', h).$$

Set $\varphi^c(x) = \varphi(xc)$ for $c \in Z_M$. Since each $\xi \in Z_M \cap M_{\text{der}}$ consists of roots of unity on the diagonal, it is clear that $\varphi^\xi(x) = \varphi(\xi x) = \varphi(x)$ for all $x \in M$, and so we can write $B^M_\varphi(\dot{w}_M bc', h_1)$ as

$$\begin{aligned}
&\int\limits_{U_M}\int\limits_{U_M} \sum_{\xi \in Z_M \cap M_{\text{der}}} h_0(x\dot{w}_M b\xi^{-1}u)B^M(\xi c', h)\varphi({}^t\overline{u}b'c'u)\overline{\chi(xu)}dudx \\
&= \sum_{\xi \in Z_M \cap M_{\text{der}}} \int\limits_{U_M}\int\limits_{U_M} h_0(x\dot{w}_M b\xi^{-1}u)B^M(\xi c', h)\varphi^{c'}({}^t\overline{u}b'u)\overline{\chi(xu)}dudx \\
&= \sum_{\xi \in Z_M \cap M_{\text{der}}} B^M(\xi c', h)\int\limits_{U_M}\int\limits_{U_M} h_0(x\dot{w}_M b\xi^{-1}u)\varphi^{c'}({}^t\overline{u}b'\xi^{-1}u)\overline{\chi(xu)}dudx \\
&= \sum_{\xi \in Z_M \cap M_{\text{der}}} B^M(\xi c', h)B^M_{\varphi^{c'}}(\dot{w}_M b\xi^{-1}, h_0).
\end{aligned}$$

We have shown that

$$B_{\varphi}^{M}(\dot{w}_M bc', h_1) = \sum_{\xi \in Z_M \cap M_{\mathrm{der}}} B^M(\xi c', h) B_{\varphi^{c'}}^{M}(\dot{w}_M b\xi^{-1}, h_0). \tag{4.2.5.2}$$

We have finally written $B_{\varphi}^{M}(\dot{w}_M bc', h_1)$ in a way in which its uniform smoothness as a function of $c' \in Z_M'$ will become transparent. Consider first the pure Bessel integral

$$B^M(\xi c', h) = \int_{U_M} h(x\xi c')\overline{\chi(x)}dx.$$

There exists an open compact subgroup U_0 of U_M, and a compact set $\Omega \subset A$, such that $h(xa) = 0$ implies $x \in U_0$ and $za \in \Omega$ for some $z \in Z$. Since $Z_M' Z$ is a closed direct product in A, the set

$$\Omega_1 = \{c' \in Z_M' : zc' \in \Omega \text{ for some } z \in Z\}$$

is compact. This shows that the function on $U_M \times Z_M'$ given by $(x, c') \mapsto h(xc')$ vanishes outside the compact set $U_0 \times Z_{M'}$. Since h is also locally constant, there must exist a compact open neighborhood H of the identity such that $h(xc'c'') = h(xc')$ for all $c'' \in H$. We can moreover take H to be a subgroup.

Strip each $\xi \in Z_M \cap M_{\mathrm{der}}$ of its center, writing it as $\xi = z_\xi \xi'$ with $z_\xi \in Z$ and $\xi' \in Z_M'$. We may shrink H even further to ensure that $h(x\xi'c'c'') = h(x\xi'c')$ for all $x \in U_M, c' \in Z_M', c'' \in H$, and all ξ'. Therefore, each

$$B^M(\xi c', h) = \omega_\pi(z_\xi) \int_{U_M} h(x\xi'c')\overline{\chi(x)}dx$$

has the property that $B^M(\xi c'c'', h) = B^M(\xi c', h)$ for all $c'' \in H$.

Now we return to equation (4.2.5.2). Since the entries of H are all units, they do not affect the scaling of our cutoff function φ. In other words, $\varphi^{c''} = \varphi$ for all $c'' \in H$. Therefore, we have for all $b \in A \cap M_{\mathrm{der}}, c' \in Z'_M$, and $c'' \in H$, that

$$\begin{aligned} B^M_\varphi(\dot{w}_M bc'c'', h_1) &= \sum_{\xi \in Z_M \cap M_{\mathrm{der}}} B^M(\xi c'c'', h) B^M_{\varphi^{c'c''}}(\dot{w}_M b\xi^{-1}, h_0) \\ &= B^M_\varphi(\dot{w}_M bc', h_1) = \sum_{\xi \in Z_M \cap M_{\mathrm{der}}} B^M(\xi c', h) B^M_{\varphi^{c'}}(\dot{w}_M b\xi^{-1}, h_0) \\ &= B^M_\varphi(\dot{w}_M bc', h_1). \end{aligned}$$

It is clear from our construction of H that it only depended on h_1 and h, not on φ and not on b. This completes the proof of the theorem. $\square$

4.3 Partial Bessel integrals on G

We have introduced partial Bessel integrals on Levi subgroups of $G = \mathrm{GL}_n(E)$. Now we will consider such integrals on G itself. If $f \in \mathscr{C}^\infty_c(G;\omega_\pi)$, and $g \in G$, the partial Bessel integral was defined in (4.2.1) by

$$B^G_\varphi(g, f) = \int\limits_{U_g \backslash U} \int\limits_U f(xgu)\varphi({}^t\overline{u}\dot{w}_G^{-1} g'u)\overline{\chi(xu)}dxdu$$

where $U_g = \{u \in U : \dot{w}_G\, {}^t\overline{u}\dot{w}_G^{-1} gu = g\}$ is the twisted centralizer of g, and g' is the element of G obtained by "stripping off the center" of g. The absolute convergence of $B^G_\varphi(g, f)$ was established in (4.2.2).

Assume f is a matrix coefficient of our supercuspidal representation π. The elements $\dot{w}_G a$ for $a \in A$ in the big cell may be considered as elements on a torus. Speaking very loosely, the Shahidi local coefficient $C_\chi(s, \pi)$ will be shown to be equal to an integral of $B^G_\varphi(\dot{w}_G a, f)$ over the F-points of a subtorus of A. The goal of this chapter and the next is to show that when π is twisted by a very highly ramified character of G, the local coefficient $C_\chi(s, \pi)$ becomes independent of π and only depends on the central character ϖ_π of π.

The way we will do this is by writing $B_{\varphi}^{G}(\dot{w}_G a, f)$ as a sum of two functions $F_1(a)$ and $F_2(a)$. The first function, F_1, will only depend on the central character of π. The second function, F_2, will have a certain "smoothness," which ensures that when $C_\chi(s,\pi)$ is obtained by integrating F_1+F_2 over the aforementioned torus, the integral over F_2 will vanish when the central character of π is very highly ramified, by the same trick that the integral of a nontrivial character of a compact abelian group is zero. This will show that $C_\chi(s,\pi)$ is an integral of a function which depends only on the central character of π, which will prove the analytic stability we want.

The functions F_1 and F_2 come from an inductive process of starting with f and obtaining successive functions $f_w \in \mathscr{C}_c^\infty(G;\omega_\pi)$ which are supported inside the open sets $\Omega_w : w \in B(G)$. The process of obtaining these functions is what we will call the *asymptotic expansion of partial Bessel integrals*.

4.3.1 A summary of asymptotic expansion

Our process begins with a given $f \in \mathscr{C}_c^\infty(G;\omega_\pi)$. To each maximal (proper) Levi subgroup M of G, and each corresponding $w = w_G w_M \in B(G)$, we will find functions $\Lambda_w \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$, as well as a function $\Lambda \in \mathscr{C}_c^\infty(G)$, such that

$$B_{\varphi}^{G}(\dot{w}_G a, f) = B_{\varphi}^{G}(\dot{w}_G a, \Lambda) + \sum_{w} B_{\varphi}^{G}(\dot{w}_G a, \Lambda_w)$$

for all $a \in A$. The function Λ will only depend on the central character ϖ_π of π, not on f (and thus not on π). The function $B_{\varphi}^{G}(\dot{w}a, \Lambda)$ will be our F_1 above. The sum over the maximal Levis will be our F_2. But we are still far from getting the smoothness property we need for F_2.

We must do an analogous process on each of the open sets Ω_w and the functions Λ_w supported inside them. Keep in mind that there is not much difference between Ω_w and M: one tacks on two unipotent groups on the ends of M to obtain Ω_w. Accordingly, the analysis of partial Bessel integrals of functions on G which are supported inside Ω_w reduces to that of partial Bessel functions on M. In particular, our results of (4.2) will be applied during this part.

We will work on the maximal Levi subgroups M of G, and corresponding $w = w_G w_M \in B(G)$, one at a time. To each maximal proper Levi subgroup M' of M, and each corresponding $w' = w_G w_{M'} \in B(G)$, we will find functions $\Lambda_{w'} \in \mathscr{C}_c^\infty(\Omega_{w'}; \omega_\pi)$, as well as a function $f_w \in \mathscr{C}_c^\infty(\Omega_w)$, such that

$$B_\varphi^G(\dot{w}_G a, \Lambda_w) = B_\varphi^G(\dot{w}_G a, f_w) + \sum_{w'} B_\varphi^G(\dot{w}_G a, \Lambda_{w'})$$

for all $a \in A$. The function $B_\varphi^G(\dot{w}_G a, f_w)$ has the kind of smoothness property that we want. It will come from what we developed in (4.2.5). The functions $B_\varphi^G(\dot{w}_G a, \Lambda_{w'})$ still do not yet have the smoothness property we want. We will need to do the same process again on each $B_\varphi^G(\dot{w}_G a, \Lambda_{w'})$, expanding it as a "smooth" piece $B_\varphi^G(\dot{w}_G a, f_{w'})$ plus a sum of partial Bessel integrals $B_\varphi^G(\dot{w}_G a, \Lambda_{w''})$ over all maximal Levi subgroups of M'.

Eventually, all the Levi subgroups of G will be exhausted, and we will have written $B_\varphi^G(\dot{w}_G a, f)$ as $B_\varphi^G(\dot{w}_G a, \Lambda)$, which only depends on the central character of π, plus the sum over all $1_W \neq w \in B(G)$ of various partial Bessel integrals $B_\varphi^G(\dot{w}_G a, f_w)$ with the smoothness property we want. The sum of these $B_\varphi^G(\dot{w}_G a)$ will be our F_2.

4.3.2 Transferring between partial Bessel integrals on Ω_w and on M

Let M be a standard Levi subgroup of G, and let $w = w_G w_M$. We mentioned in (4.1.9) that we have a surjection $\mathscr{C}_c^\infty(\Omega_w; \omega_\pi) \to \mathscr{C}_c^\infty(M; \omega_\pi)$, $f \mapsto h_f$ given by

$$h_f(m) = \int\limits_{U_{w^{-1}}} \int\limits_{U_w^-} f(x\dot{w}mu)\overline{\chi(xu)}du dx.$$

There is not much difference topologically between Ω_w and M. We made this precise in (4.1.8). There is a natural bijection between the Bruhat cells. There is also a natural identification of their partial Bessel integrals, as we will make precise with Proposition 4.3.2.2.

If $L \subset M$ are standard Levi subgroups of G, let $w_L^M = w_M w_L$. Note that $\dot{w}_L^M = \dot{w}_M \dot{w}_L^{-1}$. Before we state Proposition 4.3.2.2, we need a lemma on twisted centralizers.

Lemma 4.3.2.1. *If $L \subset M$ are standard Levi subgroups of G, then*

$$U_{M,\dot{w}_L^M a} = U_{M,\dot{w}_L^M} = U_{\dot{w}_L^G} = U_{\dot{w}_L^G a}$$

for all $a \in Z_L$

Keep in mind that in the lemma we are considering two different actions, one of U_M and one of $U = U_G$.

Proof: The left and right equalities are Lemma 4.2.4.2 (i). For the middle equality, note that $U_{M,\dot{w}_L^M}$ and $U_{\dot{w}_L^G}$ are both contained in U_L by Lemma 4.2.4.2 (ii). If $u \in U_L$, we have $u \in U_{M,\dot{w}_L^M}$ if and only if $(\dot{w}_M\, {}^t\overline{u} \dot{w}_M^{-1})\dot{w}_L^M u = \dot{w}_L^M$, and we have $u \in U_{\dot{w}_L^G}$ if and only if $(\dot{w}_G\, {}^t\overline{u} \dot{w}_G^{-1})\dot{w}_L^G u = \dot{w}_L^G$. Since $\dot{w}_L^M = \dot{w}_M \dot{w}_L^{-1}$ and $\dot{w}_L^G = \dot{w}_G \dot{w}_L^{-1}$, we see that these conditions are respectively equivalent to ${}^t\overline{u}\dot{w}_L u = \dot{w}_L$. $\square$

Proposition 4.3.2.2. *Let $f \in \mathscr{C}_c^\infty(\Omega_w; \omega_\pi)$, and let $h = h_f$ be as above. Then for sufficiently large φ, we have*

$$B_\varphi^G(\dot{w}_L^G a, f) = B_\varphi^M(\dot{w}_L^M a, h)$$

for all standard Levi subgroups $L \subset M$ and all $a \in Z_L$.

Proof: Let $a \in Z_L$. By Lemma 4.3.2.1, we have

$$B_\varphi^G(\dot{w}_L^G a, f) = \int\limits_{U_{\dot{w}_L^G} \backslash U} \int\limits_U f(x\dot{w}_L^G a u)\varphi({}^t\overline{u}\dot{w}_L^{-1} a' u)\overline{\chi(xu)}\, dx du$$

and

$$B_\varphi^M(\dot{w}_L^M a, h) = \int\limits_{U_{\dot{w}_L^M} \backslash U_M} \int\limits_{U_M} h(x'\dot{w}_L^M a u')\varphi({}^t\overline{u'}\dot{w}_L^{-1} a' u')\overline{\chi(x'u')}\, dx' du'.$$

Since $U = U^-_{w^{-1}} U^+_{w^{-1}}$, we can decompose the integration over U as $dx = dx^- dx^+$. We also have $U = U^-_w U^+_w = N U_M$, and can decompose integration over U as $du = du^- du'$. Write $\dot{w}^G_L = \dot{w}_G \dot{w}_L^{-1} = \dot{w}_G \dot{w}_M^{-1} \dot{w}_M \dot{w}_L^{-1} = \dot{w}^G_M \dot{w}^M_L = w \dot{w}^M_L$. Then

$$f(x\dot{w}^G_L au) = f(x^- x^+ \dot{w}\dot{w}^M_L au^+ u^-) = f(x^- \dot{w}(\dot{w}^{-1} x^+ \dot{w})\dot{w}^M_L au^+ u^-).$$

Since conjugation by $\dot{w}^{-1}$ takes $U^+_{\dot{w}^{-1}}$ to $U^+_{\dot{w}} = U_M$, we can replace $(\dot{w}^{-1} x^+ \dot{w})$ by $x' \in U_M$, and get

$$f(x\dot{w}^G_L au) = f(x^- \dot{w}(x'\dot{w}^M_L au')u^-)$$

where $u' \in U_M$.

Decomposing $U_{\dot{w}^G_L} \backslash U = U_{\dot{w}^M_L} \backslash U$ as $(U_{\dot{w}^M_L} \backslash U) \times N = (U_{\dot{w}^M_L} \backslash U_M) U^-_w$, we can write

$$\begin{aligned}
B^G_\varphi(\dot{w}^G_L a, f) = &\int\limits_{U_{\dot{w}^M_L} \backslash U_M \times U^-_w} \Big[\int\limits_{U^-_{w^{-1}} \times U_M} f(x^- \dot{w}(x'\dot{w}^M_L au')u^-) \varphi({}^t\overline{u}^- ({}^t\overline{u}' \dot{w}_L^{-1} a' u')u^-) \\
&\times \overline{\chi(x^- u^-)}\, dx^- dx' \Big] \overline{\chi(x'u')}\, du^- du' \\
= &\int\limits_{U_{\dot{w}^M_L} \backslash U_M \times U_M} \Big[\int\limits_{U^-_{w^{-1}} \times U^-_w} f(x^- \dot{w}(x'\dot{w}^M_L au')u^-) \varphi({}^t\overline{u}^- ({}^t\overline{u}' \dot{w}_L^{-1} a' u')u^-) \\
&\times \overline{\chi(x^- u^-)}\, dx^- du^- \Big] \overline{\chi(x'u')}\, dx' du'.
\end{aligned}$$

From the decomposition $\Omega_w = U^-_{w^{-1}} \times \dot{w}M \times U^-_w$, and the fact that f is compactly supported on Ω_w modulo Z, there are compact sets $U_1 \subset U^-_{w^{-1}}, U_2 \subset U^-_w$ such that $f(x^- \dot{w} m u^-) \neq 0$ implies that $x^- \in U_1$ and $u^- \in U_2$. If we take φ sufficiently large (depending on U_2), then we will have

$$\varphi({}^t\overline{u}^- ({}^t\overline{u}' \dot{w}_L^{-1} a' u')u^-) = \varphi({}^t\overline{u}' \dot{w}_L^{-1} a' u')$$

for all $u' \in U_2$. Then

$$\begin{aligned} B^G_\varphi(\dot{w}^G_L a, f) &= \int\limits_{U_{\dot{w}^M_L}\backslash U_M \times U_M} [\int\limits_{U^-_{w^{-1}} \times U^-_w} f(x^-\dot{w}(x'\dot{w}^M_L au')u^-)\varphi({}^t\overline{u}'\dot{w}_L^{-1}a'u') \\ &\times \overline{\chi(x^-u^-)}\, dx^- du^-]\, \overline{\chi(x'u')}\, dx'du' \\ &= \int\limits_{U_{\dot{w}^M_L}\backslash U_M \times U_M} h(x'\dot{w}^M_L au')\varphi({}^t\overline{u}'\dot{w}_L^{-1}a'u')\overline{\chi(x'u')}\, dx'du' \\ &= B^M_\varphi(\dot{w}^M_L a, h). \end{aligned}$$

□

4.3.3 Moving up the cells, I

Let M be a standard Levi subgroup of G, and let $w = w_G w_M \in B(G)$. Suppose $f \in \mathscr{C}^\infty_c(G;\omega_\pi)$ is supported inside the open set Ω_w. Recall that there is a parallel cell structure on M as there is on Ω_w. The small cell B_M in M corresponds to the smallest cell $C(w)$ in Ω_w, and in turn, $C(w)$ is closed in Ω_w.

Suppose that $B^G_\varphi(g, f)$ vanishes for $g \in C(w)$. Since $\Omega^\circ = \Omega_w - C(w)$ is open in Ω_w, we can identify f as an element of the space $\mathscr{C}^\infty_c(\Omega^\circ_w;\omega_\pi)$. Considering all the maximal Levi subgroups L of M, with corresponding Weyl group elements $w' = w_G w_L \in B(G)$, we have inclusions $\Omega_w \subset \Omega_{w'}$. We will be interested in finding functions $f_{w'} \in \mathscr{C}^\infty_c(\Omega_{w'};\omega_\pi)$ such that

$$B^G_\varphi(g, f) = \sum_{w'} B^G_\varphi(g, f_{w'}).$$

The main difficulty in producing such functions entails considering cells of elements in the Weyl group which do not support Bessel functions: if we set

$$\Omega^1 = \bigcup_{w'} \Omega_{w'}$$

as w' runs through those elements in $B(G)$ corresponding to maximal Levi subgroups of M, then $\Omega^1 - C(w)$ is a union of cells $C(w'')$ with $w'' \notin B(G)$. We will need to produce a function $f_1 \in \mathscr{C}_c^\infty(G)$ which is supported inside Ω^1, i.e. which vanishes on these cells, and which satisfies $B_\varphi^G(g, f) = B_\varphi^G(g, f_1)$ for all $g \in G$.

The process of doing all of this is what we will call *moving up the cells.* That is, we will be replacing f by functions in $\mathscr{C}_c^\infty(G; \omega_\pi)$ which have smaller support. In doing this we will even be able to weaken the assumption that $B_\varphi^G(g, f)$ vanishes on all of $C(w) = U\dot{w}AU_w^-$, and just assume that f vanishes on the *relevant* part of the cell $C_r(w) = U\dot{w}Z_M U_w^-$.

Lemma 4.3.3.1. *(Basic lemma) For $f \in \mathscr{C}_c^\infty(G, \omega_\pi)$, and U_1, U_2 open compact subsets of U, set*

$$f'(g) = \frac{1}{\mathrm{meas}(U_1 \times U_2)} \int\limits_{(U_1 \times U_2)} f(u_1 g u_2)\overline{\chi(u_1 u_2)}\, du_1 du_2.$$

Let U_2 be given. Then for sufficiently large φ,

$$B_\varphi^G(g, f) = B_\varphi^G(g, f')$$

for all $U_1, f,$ and g.

Proof: We have

$$B_\varphi^G(g, f') = \int\limits_{U_g \backslash U} \int\limits_U f'(xgu)\varphi({}^t\overline{u}\dot{w}_G^{-1} g' u)\overline{\chi(xu)}\, dx du$$

which is equal to

$$\frac{1}{\mathrm{meas}(U_1 \times U_2)} \int\limits_{U_g \backslash U} \int\limits_U \int\limits_{U_1 \times U_2} f(u_1 x g u u_2)\varphi({}^t\overline{u}\dot{w}_G^{-1} g' u)\overline{\chi(u_1 u_2 x u)}\, du_1 du_2 dx du.$$

Since $U_1 \times U_2$ is compact, we can interchange the integration as

$$\frac{1}{\operatorname{meas}(U_1 \times U_2)} \int\limits_{U_1 \times U_2} \int\limits_{U_g \backslash U} \int\limits_{U} f(u_1 x g u u_2) \varphi({}^t \overline{u} \dot{w}_G^{-1} g' u) \overline{\chi(u_1 u_2 x u)} \, dx du du_1 du_2.$$

Make the substitutions $x \mapsto u_1^{-1} x$ and $u \mapsto u u_2^{-1}$:

$$\frac{1}{\operatorname{meas}(U_1 \times U_2)} \int\limits_{U_1 \times U_2} \int\limits_{U_g \backslash U} \int\limits_{U} f(xgu) \varphi({}^t \overline{u_2^{-1}}{}^t \overline{u} \dot{w}_G^{-1} g' u u_2^{-1}) \overline{\chi(xu)} \, dx du du_1 du_2.$$

Since U_2 is compact, we can choose φ large enough so that $\varphi({}^t \overline{u_2}^{-1} y u_2^{-1}) = \varphi(y)$ for all $u_2 \in U_2$ and $y \in \operatorname{Mat}_n(E)$ (Lemma 4.2.3.1). In this case, the integrand is independent of u_1 and u_2, leaving us with

$$\int\limits_{U_g \backslash U} \int\limits_{U} f(xgu) \varphi({}^t \overline{u} \dot{w}_G^{-1} g' u) \overline{\chi(xu)} \, dx du = B_\varphi^G(g, f).$$

□

Let $w \in B(G)$, with corresponding Levi M. Let $C_r(\dot{w}) = U \dot{w} Z_M N$ be the *relevant* part of the Bruhat cell. It is closed in $C(w) = U \dot{w} A N$, hence in Ω_w. Let $\Omega'_{\dot{w}} = \Omega_w - C_r(\dot{w})$.

Lemma 4.3.3.2. *Let $f \in \mathscr{C}_c^\infty(\Omega_w; \omega_\pi)$. Suppose that $B_\varphi^G(\dot{w}a, f) = 0$ for all $a \in Z_M$. Then there exists $f_0 \in \mathscr{C}_c^\infty(\Omega'_{\dot{w}}; \omega_\pi)$ such that for all sufficiently large φ,*

$$B_\varphi^G(g, f) = B_\varphi^G(g, f_0)$$

for all $g \in G$.

Proof: Since $C(w)$ is closed in Ω_w, the restriction of f to $C(w)$ remains compactly supported modulo Z. Then we can take open compact subgroups $U_1 \subseteq U, U_2^- \subseteq N$

such that for all $a \in A, u \in U, n \in N$ the condition $f(u\dot{w}an) \neq 0$ implies that $u \in U_1$ and $n \in U_2^-$. By Lemma 4.2.4.3, we have for all $a \in Z_M$,

$$\begin{aligned} B_\varphi^G(\dot{w}a, f) &= \tilde{\varphi}_M^G(a')B^G(\dot{a}, f) \\ &= \tilde{\varphi}_M^G(a') \int\limits_{U\times N} f(x\dot{w}an)\overline{\chi(xn)}\, dxdn \\ &= \tilde{\varphi}_M^G(a') \int\limits_{U_1\times U_2^-} f(x\dot{w}an)\overline{\chi(xn)}\, dxdn. \end{aligned}$$

We are assuming that $B_\varphi^G(\dot{w}a, f) = 0$ for all $a \in Z_A$. For φ sufficiently large, Lemma 4.2.4.4 tells us that also

$$B^G(\dot{w}a, f) = \int\limits_{U_1\times U_2^-} f(x\dot{w}an)\overline{\chi(xn)}\, dxdn = 0$$

for all $a \in Z_M$. Let $U_2^+ \subseteq U_M$ be an open compact subgroup such that $\dot{w}U_M\dot{w}^{-1} \subseteq U_1$. Let $U_2 = U_2^+U_2^- \subseteq U$. Enlarging U_2^- if necessary, we may assume that U_2^+ normalizes U_2^-, so that U_2 is a subgroup of U. Define

$$f_0(g) = \frac{1}{\mathrm{meas}(U_1 \times U_2)} \int\limits_{U_1\times U_2} f(u_1 g u_2)\overline{\chi(u_1u_2)}\, du_1du_2.$$

This function lies in $\mathscr{C}_c^\infty(G; \omega_\pi)$, and by Lemma 4.3.2.1, $B_\varphi^G(g, f) = B_\varphi^G(g, f_0)$ for all $g \in G$. We are done if we can show that f_0 vanishes on the relevant part of the Bruhat cell $C_r(\dot{w})$. Let us first evaluate $f_0(\dot{w}a)$ for $a \in Z_M$. We expand integration over $U_1 \times U_2$ as

$$\frac{1}{\mathrm{meas}(U_1 \times U_2)} \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1\dot{w}au_2^+u_2^-)\overline{\chi(u_1u_2^+u_2^-)}\, du_1du_2^-du_2^+.$$

Since $a \in Z_M$ and $u_2^+ \in U_M$, a and u_2^+ commute. Write $u_1 \dot{w} a u_2^+ u_2^- = u_1 \dot{w} u_2^+ \dot{w}^{-1} \dot{w} a$. By assumption, $\dot{w} u_2^+ \dot{w}^{-1} \in U_1$, so we can make the substitution $u_1 \mapsto u_1 (\dot{w} u_2^+ \dot{w}^{-1})^{-1}$ By compatibility, $\chi(\dot{w} u_2^+ \dot{w}^{-1}) = \chi(u_2^+)$, so we obtain

$$\frac{1}{\operatorname{meas}(U_1 \times U_2)} \int\limits_{U_2^+} \int\limits_{U_2^-} \int\limits_{U_1} f(u_1 \dot{w} a u_2^-) \overline{\chi(u_1 u_2^-)} \, du_1 du_2^- du_2^+$$

for all $a \in Z_M$. This removes the dependence of the integrand on u_2^+, so we obtain

$$f_0(\dot{w}a) = \frac{\operatorname{meas}(U_2^+)}{\operatorname{meas}(U_1 \times U_2)} \int\limits_{U_2^-} \int\limits_{U_1} f(u_1 \dot{w} a u_2^-) \overline{\chi(u_1 u_2^-)} \, du_1 du_2^- \tag{4.3.3.2}$$

which is a nonzero constant multiple of the pure Bessel function $B^G(\dot{w}a, f)$. Since $B^G(\dot{w}a, f) = 0$ for all $a \in Z_M$, we therefore have $f_0(\dot{w}a) = 0$ for all $a \in Z_M$.

To finish showing that $f_0(g) = 0$ for all $g \in C_r(\dot{w})$, suppose that $f_0(u\dot{w}ax) \neq 0$ for some $u \in U, x \in N, a \in Z_M$. Then we must have $f(u_1 u \dot{w} a x u_2) \neq 0$ for some $u_1 \in U_1 \subseteq U$ and $u_2 \in U$. Via the decomposition $U_2 = U_2^+ U_2^-$, write u_2 uniquely as $u_2^+ u_2^-$. Using the fact that $a \in Z_M$ and $w U_M w^{-1} \subset U$, and writing $u_1 u \dot{w} a x u_2$ in the "standard form" in $C(w) = U\dot{w}AN$ as

$$u_1 u \dot{w} a x u_2 = u_1 u (\dot{w} u_2^+ \dot{w}^{-1}) \dot{w} a (u_2^+)^{-1} x u_2^+ u_2^-$$

we have that $f(u_1 u (\dot{w} u_2^+ \dot{w}^{-1}) \dot{w} a (u_2^+)^{-1} x u_2^+ u_2^-$ is not zero. This implies

$$u_1 u (\dot{w} u_2^+ \dot{w}^{-1}) \in U_1$$

$$\text{and } (u_2^+)^{-1} x u_2^+ u_2^- \in U_2^-.$$

Our assumption that $\dot{w}U_2^+\dot{w} \subset U_1$ and U_2^+ normalizes U_2^- show that $u \in U_1$ and $x \in U_2^- \subset U_2$. But then in the definition of f_0 we can make the substitutions $u_1 \mapsto u_1 u^{-1}$ and $u_2 \mapsto x^{-1}u_2$ to get

$$\begin{aligned} f_0(u\dot{w}ax) &= \frac{1}{\mathrm{meas}(U_1 \times U_2)} \int_{U_1}\int_{U_2} f(u_1 u\dot{w}axu_2)\overline{\chi(u_1u_2)}\, du_2 du_1 \\ &= \frac{\chi(ux)}{\mathrm{meas}(U_1 \times U_2)} \int_{U_1}\int_{U_2} f(u_1\dot{w}au_2)\overline{\chi(u_1u_2)}\, du_2 du_1 \\ &= \chi(ux)f_0(\dot{w}a) \\ &= 0 \end{aligned}$$

which is a contradiction. $\square$

Recall that we have defined $\Omega'_{\dot{w}}$ to be the complement of $C_r(\dot{w})$ in Ω_w. Now define Ω_w° to be the complement of $C(w)$ in Ω_w. Then Ω_w° is open in Ω_w, and we can identify $\mathscr{C}_c^\infty(\Omega_w^\circ;\omega_\pi)$ with those elements of $\mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$ which vanish on the Bruhat cell $C(w)$.

Lemma 4.3.3.3. *Let $f : C(w) \to \mathbb{C}$ be locally constant and compactly supported modulo Z.*

(i): There are open compact subgroups $U_1 \subseteq U, U_2^- \subseteq N$, and a compact set $K' \subseteq A'$, such that if $f(u\dot{w}za'n) \neq 0$, then $u \in U_1, a' \in K'$, and $n \in U_2^-$.

(ii): If f vanishes on $C_r(\dot{w})$, then K' can be chosen to be disjoint from Z'_M.

Proof: (i) was already proved in Lemma 4.1.2.1. For (ii), $V = \{(u,a',n) \in U \times A' \times N : f(u\dot{w}a'n) \neq 0\}$. Since f is locally constant, V is a closed (and open) subset of $C(w)$, and being contained in $U_1 \times K' \times U_2^-$, it is compact. Let K'' be the projection of V onto A'. Then K'' is compact, and disjoint from A'_w, since by hypothesis f vanishes on $C_r(\dot{w})$. Now if $g = u\dot{w}za'n \in C(w)$ with $f(g) \neq 0$, then also $f(u \cdot wa'n) = \omega_\pi(z)^{-1}f(g) \neq 0$, so $(u,a',n) \in V$. Then $u \in U_1, a' \in K''$, and $n \in U_2^-$. Thus we can replace K' by K''. $\square$

Lemma 4.3.3.4. *Let $f \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$. Suppose that $B_\varphi^G(\dot{w}a, f) = 0$ for all $a \in Z_M$. Then there exists $f_0 \in \mathscr{C}_c^\infty(\Omega_w^\circ;\omega_\pi)$ such that for all sufficiently large φ,*

$$B_\varphi^G(g, f) = B_\varphi^G(g, f_0)$$

for all $g \in \Omega_w$.

Proof: By the previous lemma, there is an $f' \in \mathscr{C}_c^\infty(\Omega'_{\dot{w}};\omega_\pi)$ (referred to as f_0 in that lemma) such that $B_\varphi^G(g, f) = B_\varphi^G(g, f')$ for all $g \in G$. So we may replace f by f' and assume from the beginning that $f \in \mathscr{C}_c^\infty(\Omega'_{\dot{w}};\omega_\pi)$, i.e. we may assume f vanishes on $C_r(\dot{w})$.

Take $U_1 \subseteq U, U_2^- \subseteq N, K' \subseteq A'$ as in the previous lemma. The center Z_M is the intersection of the kernels of the simple of roots of A in M. Since $K' \subseteq A'$ is compact, and disjoint from $Z'_M = A' \cap Z_M$, there exists a $c > 0$ such that for every $a' \in K'$, there exists a simple root α of A in M such that $|\alpha(a') - 1| > c$.

Let U_2^+ be an open compact subgroup of U_M, chosen sufficiently large so that the character

$$u_2^+ \mapsto \chi(\dot{w}a'u_2^+a'^{-1}\dot{w}^{-1}u_2^{+-1})$$

is nontrivial on U_2^+ for all $a' \in K'$. Then for all $a' \in K'$,

$$\int\limits_{U_2^+} \chi(\dot{w}a'u_2^+a'^{-1}\dot{w}^{-1}u_2^{+-1})\, du_2^+ = 0.$$

Next, enlarge U_2^- so that it is normalized by U_2^+, and take $U_2 = U_2^+U_2^-$. Also, enlarge U_1 so that it is decomposable as a semidirect product $U_1^-U_1^+$, for open compact subgroups U_1^- and U_2^+ of N and U_M. Further enlarge U_1^+ so that $\dot{w}a'^{-1}U_2^+a'\dot{w}^{-1} \subseteq U_1^+$ for all $a' \in K'$, and enlarge U_1^- so that it remains normalized by U_1^+, and the product $U_1 = U_1^-U_1^+$ remains semidirect.

Define $f_1 : \Omega_w \to \mathbb{C}$ by

$$f_1(g) = \int\limits_{U_2}\int\limits_{U_1} f(u_1 g u_2)\overline{\chi(u_1 u_2)}\, du_1 du_2.$$

We claim that

$$f_1(\dot{w}a) = \int\limits_{U_2}\int\limits_{U_1} f(u_1 \dot{w} a u_2)\overline{\chi(u_1 u_2)}\, du_1 du_2 = 0$$

for all $a \in A$. Writing $a = za'$ for $z \in Z, a' \in A'$, we clearly have $f_1(\dot{w}a) = \omega_\pi(z) f_1(\dot{w}a')$, so it suffices to show that $f_1(\dot{w}a') = 0$. We have

$$\begin{aligned} f_1(\dot{w}a') &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1 \dot{w} a' u_2^+ u_2^-)\overline{\chi(u_1 u_2^+ u_2^-)}\, du_1 du_2^- du_2^+ \\ &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1 (\dot{w} a' u_2^+ a'^{-1} \dot{w}^{-1}) a' \dot{w} u_2^-)\overline{\chi(u_1 u_2^+ u_2^-)}\, du_1 du_2^- du_2^+. \end{aligned}$$

Suppose by way of contradiction that $f_1(\dot{w}a') \neq 0$. Then we know first of all that a' must lie in K'. Also, if $f(u_1(\dot{w}a'u_2^+ a'^{-1}\dot{w}^{-1})\dot{w}u_2^-) \neq 0$, then we know that $u_1(\dot{w}a'u_2^+ a'^{-1}\dot{w}^{-1})$ must lie in U_1. Hence we may change variables, replacing u_1 by $u_1(\dot{w}a'u_2^+ a'^{-1}\dot{w}^{-1})^{-1}$, to get

$$\begin{aligned} f_1(\dot{w}a') &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1 \dot{w} a' u_2^-)\overline{\chi(u_1 u_2^+ u_2^-)}\chi(\dot{w} a' u_2^+ a'^{-1} \dot{w}^{-1})\, du_1 du_2^- du_2^+ \\ &= \int\limits_{U_2^-} \overline{\chi(u_2^+)}\chi(\dot{w} a' u_2^+ a'^{-1} \dot{w}^{-1})\, du_2^+ \int\limits_{U_2^-}\int\limits_{U_1} f(u_1 \dot{w} a' u_2^-)\overline{\chi(u_1 u_2^-)}\, du_1 du_2. \end{aligned}$$

The first integral is exactly $\int\limits_{U_2^+} \chi(\dot{w}a'u_2^+ a'^{-1}\dot{w}^{-1}u_2^{+-1})\, du_2^+$, which we know is zero, because $a' \in K'$. This proves that $f_1(\dot{w}a') = 0$.

Next, we want to show that $f_1(u\dot{w}a') = 0$ for all $u \in U$. We have as above,

$$\begin{aligned} f_1(u\dot{w}a') &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1u\dot{w}a'u_2^+u_2^-)\overline{\chi(u_1u_2^+u_2^-)}\,du_1du_2^-du_2^+ \\ &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1u(\dot{w}a'u_2^+a'^{-1}\dot{w}^{-1})\dot{w}a'u_2^-)\overline{\chi(u_1u_2^+u_2^-)}\,du_1du_2^-du_2^+. \end{aligned}$$

If we again suppose by way of contradiction that $f_1(u\dot{w}a') \neq 0$, then we must have $a' \in K'$, and if $f(u_1u(\dot{w}a'u_2^+a'^{-1}\dot{w}^{-1})\dot{w}a'u_2^-) \neq 0$, then $u_1u(\dot{w}a'u_2^+a'^{-1}\dot{w}^{-1})$ must lie in U_1. Since $u_1 \in U_1$, and $(\dot{w}a'u_2^+a'^{-1}\dot{w}^{-1}) \in U_1^+ \subseteq U_1$, we must have $u \in U_1$. Then by the definition of f_1, we see that $f_1(u\dot{w}a') = \chi(u)f_1(\dot{w}a')$. We just proved that $f_1(\dot{w}a') = 0$, so also $f_1(u\dot{w}a') = 0$.

Finally, we need to show that if $u' \in N$, then $f_1(u\dot{w}a'u') = 0$. This will show that f_1 vanishes on the Bruhat cell $C(w)$, which will complete the proof. We have

$$\begin{aligned} f_1(u\dot{w}a'u') &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1u\dot{w}a'u'u_2^+u_2^-)\overline{\chi(u_1u_2^+u_2^-)}\,du_1du_2^-du_2^+ \\ &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1u(\dot{w}a'u'u_2^+a'^{-1}\dot{w}^{-1})\dot{w}a'(u_2^{+-1}u'u_2^+)u_2^-) \\ &\times \overline{\chi(u_1u_2^+u_2^-)}\,du_1du_2^-du_2^+. \end{aligned}$$

Note that $U_2^+ \subseteq U_M$ normalizes N, so $(u_2^{+-1}u'u_2^+) \in N$. If the integrand is nonvanishing, we must have $a' \in K'$ and $(u_2^{+-1}u'u_2^+)u_2^-) \in U_2^-$. Hence $u_2^{+-1}u'u_2^+ \in U_2^-$. We can change variables, replacing u_2^- by $(u_2^{+-1}u'u_2^+)^{-1}u_2^-$, to get

$$\begin{aligned} f_1(u\dot{w}a'u') &= \int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1u(\dot{w}a'u'u_2^+a'^{-1}\dot{w}^{-1})\dot{w}a'u_2^-) \\ &\times \overline{\chi(u_1u_2^+u_2^-)}\chi(u')\,du_1du_2^-du_2^+ \\ &= \chi(u')\int\limits_{U_2^+}\int\limits_{U_2^-}\int\limits_{U_1} f(u_1u\dot{w}a'u'u_2^+u_2^-)\overline{\chi(u_1u_2^+u_2^-)}\,du_1du_2^-du_2^+ \\ &= \chi(u')f_1(u\dot{w}a') = 0. \end{aligned}$$

Now, let

$$f_0(g) = \frac{1}{\text{meas}(U_1 \times U_2)} f_1(g) = \frac{1}{\text{meas}(U_1 \times U_2)} \int\limits_{U_2} \int\limits_{U_1} f(u_1 g u_2) \overline{\chi(u_1 u_2)}\, du_1 du_2$$

for all $g \in \Omega_w$. Then $f_0 \in \mathscr{C}_c^\infty(\Omega_w; \omega_\pi)$, and f_0 vanishes on $C(w)$, so $f_0 \in \mathscr{C}_c^\infty(\Omega_w^0; \omega_\pi)$. By Lemma 4.3.2.1, we have

$$B_\varphi^G(g, f) = B_\varphi^G(g, f_0)$$

for all sufficiently large φ and all $g \in \Omega_w$, which completes the proof. □

4.3.4 Moving up the cells, II

So far, we have dealt with functions f supported in a relevant open set Ω_w whose partial Bessel integrals $B_\varphi^G(-, f)$ take the value zero on part or all of the small cell $C(w)$ in Ω_w. We have shown that the function in question can be replaced by one which vanishes on the small cell.

The next proposition is more technical. It shows that, asymptotically, the "non-relevant cells" $C(w')$ for $w' \notin B(G)$ do not contribute to the arguments of the partial Bessel function, and we may assume that the functions we work with vanish on these cells.

Proposition 4.3.4.1. *Let $w \in B(G)$. Let $\Omega_{w,0} \subset \Omega_{w,1}$ be $U \times U$ and A-invariant open sets of Ω_w such that $\Omega_{w,1} - \Omega_{w,0}$ is a union of Bruhat cells $C(w')$ for $w' \notin B(G)$. Then for any $f_1 \in \mathscr{C}_c^\infty(\Omega_{w,1}; \omega_\pi)$, there exists $f_0 \in \mathscr{C}_c^\infty(\Omega_{w,0}; \omega_\pi)$ such that for all sufficiently large φ, we have $B_\varphi^G(g, f_0) = B_\varphi^G(g, f_1)$ for all $g \in G$.*

Proof: The complement of $\Omega_{w,0}$ in $\Omega_{w,1}$ is a finite union of Bruhat cells. By the general theory of actions of linear algebraic groups, closed orbits always exist, so we can inductively find open sets

$$\Omega_{w,0} = \Omega^0 \subset \Omega^1 \subset \cdots \subset \Omega^t = \Omega_{w,1}$$

such that each complement $\Omega^i - \Omega^{i-1}$ is a single Bruhat cell $C(w')$ which is closed in Ω^i and for which $w' \notin B(G)$. By induction, it therefore suffices to prove the proposition in the special case when the complement $\Omega_{w,1} - \Omega_{w,0}$ is a single Bruhat cell $C(w')$, with $w' \notin B(G)$.

Since $\Omega_{w,0}$ is open in $\Omega_{w,1}$, the complement $C(w')$ is closed in $\Omega_{w,1}$. Therefore the restriction of f_1 to $C(w')$ remains compactly supported modulo Z.

Recall that $C(w')$ is homeomorphic to $U \times Z \times A' \times U^-_{w'}$. Since $A = ZA'$ normalizes $U^-_{w'}$, we can switch the order, and say that the map $U \times U^-_{w'} \times Z \times A'$ sending (u, u^-, z, a') to $u\dot{w}'u^-za'$ is a homeomorphism. As in Lemma 4.3.2.3, there exist compact open subgroups U_1 of U, U_2^- of $U^-_{w'}$, and a compact subset $K' \subseteq A'$ such that $f_1(u\dot{w}'u^-za') \neq 0$ implies $u \in U_1, u^- \in U_2^-$, and $a' \in K'$. If we replace U_1 and U_2^- by any larger open compact subgroups, or K' by any larger compact set, the same holds.

Let $U_2^+ = \{u^+ \in U^+_{w'} : w'^{-1}u^+w' \in U_1\}$. In general, an exhaustive sequence of compact open subgroups of U can be obtained by identifying U with affine space $E \oplus E \oplus \cdots$ and taking those vectors whose entries are bounded above in absolute value. So we may always choose U_1 and U_2^- in such a way that the direct product $U_2 = U_2^+U_2^-$ is a subgroup of U. Note that U_2^+ does not necessarily normalized U_2^-, since $U^-_{w'}$ is not necessarily a normal subgroup of U when $w' \notin B(G)$.

Let

$$\tilde{U}_{\dot{w}'a} = \{(u_1, u_2) \in U_1 \times U_2 : u_1\dot{w}'au_2 = \dot{w}'a\}$$

be the stabilizer of $\dot{w}'a$ under the action of $U \times U$. Note that U acts on $U \times U$ on both the right and the left, so we can define for example

$$U_1 \cdot \tilde{U}_{\dot{w}'} \cdot U_2^- = \{(uu_1, u_2u^-) : u \in U_1, u^- \in U_2^-, (u_1, u_2) \in \tilde{U}_{\dot{w}'}\}.$$

Suppose that for $u_1, u_2 \in U$ and $a \in A'$, we have $f(u_1\dot{w}'u_2a') \neq 0$. Writing $u_2 = u_2^+u_2^-$ for $u_2^+ \in U^+_{w'}, u_2^- \in U^-_{w'}$, we have $u_1\dot{w}'u_2a' = u_1(\dot{w}'u_2^+\dot{w}'^{-1})\dot{w}'u_2^-a'$, with

$\dot{w}'u_2^+\dot{w}'^{-1} \in U$. This implies that $u_1\dot{w}'u_2^+\dot{w}'^{-1} \in U_1, u_2^- \in U_2^-$, and $a' \in K'$. Then $((\dot{w}'u_2^+\dot{w}'^{-1})^{-1}, u_2^+) \in \tilde{U}_{\dot{w}'}$, with

$$(u_1, u_2) = (u_1(\dot{w}'u_2^+\dot{w}'^{-1})(\dot{w}'u_2^+\dot{w}'^{-1})^{-1}, u_2^+u_2^-) \in U_1 \cdot \tilde{U}_{\dot{w}'} \cdot U_2^-.$$

If we switch the order of our terms, we get a similar result: if $f_1(u_1\dot{w}'a'u_2) \neq 0$, then writing $u_1\dot{w}'a'u_2 = u_1\dot{w}'a'u_2a'^{-1}a'$, we get

$$(u_1, (a'u_2a'^{-1})^{-1}) \in U_1 \cdot \tilde{U}_{\dot{w}} \cdot U_2 \text{ and } a' \in K'.$$

Since K' is compact, we can enlarge U_2 to a compact open subgroup which is decomposable as $U_2' = U_2'^+U_2'^-$, such that for all $a' \in K'$ we have $a'^{-1}U_2a' \subset U_2'$. Then we have

$$f_1(u_1\dot{w}'a'u_2) \neq 0 \Rightarrow (u_1, u_2) \in U_1 \cdot \tilde{U}_{\dot{w}'a'} \cdot U_2' \text{ and } a' \in K'. \tag{4.3.4.1}$$

Since w' does not support a Bessel function, there is a simple root α such that $w'.\alpha$ is positive but not simple. If U_α is the root subgroup corresponding to α, then $U_\alpha \subset U_{w'}^+$, and χ is nontrivial on U_α, yet trivial on $w'U_\alpha w'^{-1} = U_{w'.\alpha}$, because this last root subgroup, being not simple, lies in the derived group of U. Enlarge U_2', keeping it decomposable, so that χ is nontrivial on $U_\alpha \cap U_2' = U_\alpha \cap U_{2'}^+$. Then if necessary, enlarge U_1 to a larger compact open subgroup U_1' so that if for some $a' \in K'$ we have $(u_1, u_2) \in \tilde{U}_{\dot{w}a'}$ and $u_2 \in U_2'$, then $u_1 \in U_1'$. Also enlarge U_1' so that $w'a'U_2'^+a'^{-1}w'^{-1} \subset U_1'$ for all $a' \in K'$.

Define $f_0' \in \mathscr{C}_c^\infty(\Omega_{w',1}; \omega_\pi)$ by

$$f_0'(g) = \int\limits_{U_2'}\int\limits_{U_1'} f_1(u_1gu_2)\overline{\chi(u_1u_2)}\, du_1du_2.$$

Consider $f_0'(\dot{w}'a')$ for $a' \in A'$. We have

$$f_0'(\dot{w}'a') = \int\limits_{U_2'}\int\limits_{U_1'} f_1(u_1\dot{w}'a'u_2)\overline{\chi(u_1u_2)}\, du_1du_2.$$

We claim that $f_0'(\dot{w}'a') = 0$. Note that if $a' \notin K'$, the integrand is the zero function. So we may assume that $a' \in K'$. Now let $u_2' \in U_\alpha \cap U_2'$ such that $\chi(u_2') \neq 1$. Then

$$\begin{aligned}
\chi(u_2')f_0'(\dot{w}'a') &= \chi(u_2') \int\limits_{U_2'} \int\limits_{U_1'} f_1(u_1\dot{w}'a'u_2)\overline{\chi(u_1u_2)}\, du_1 du_2 \\
&= \int\limits_{U_2'} \int\limits_{U_1'} f_1(u_1\dot{w}'a'u_2)\overline{\chi(u_1 u_2'^{-1} u_2)}\, du_1 du_2 \\
&= \int\limits_{U_2'} \int\limits_{U_1'} f_1(u_1\dot{w}'a'u_2'u_2)\overline{\chi(u_1u_2)}\, du_1 du_2 \\
&= \int\limits_{U_2'} \int\limits_{U_1'} f_1(u_1\dot{(}w'a'u_2'a'^{-1}\dot{w}^{-1})\dot{w}a'u_2)\overline{\chi(u_1u_2)}\, du_1 du_2.
\end{aligned}$$

Since $u_2' \in U_2'$, we have that $\dot{w}'a'u_2'a'^{-1}\dot{w}^{-1} \in U_1'$. So we can replace u_1 by $u_1(\dot{w}'a'u_2'a'^{-1}\dot{w}^{-1})^{-1}$ and obtain

$$\begin{aligned}
\chi(u_2')f_0'(\dot{w}'a') &= \int\limits_{U_2'} \int\limits_{U_1'} f_1(u_1\dot{w}'a'u_2)\overline{\chi(u_1u_2)}\chi(\dot{w}'a'u_2'a'^{-1}\dot{w}'^{-1}u_2)\, du_1 du_2 \\
&= \chi(\dot{w}'a'u_2'a'^{-1}\dot{w}'^{-1}u_2) \int\limits_{U_2'} \int\limits_{U_1'} f_1(u_1\dot{w}'a'u_2)\overline{\chi(u_1u_2)}\, du_1 du_2 \\
&= \chi(\dot{w}'a'u_2'a'^{-1}\dot{w}'^{-1}u_2) f_0'(\dot{w}'a').
\end{aligned}$$

By our choice of α, we know that since $u_2' \in U_\alpha$, the element $\dot{w}'a'u_2'a'^{-1}\dot{w}'^{-1}u_2$ lies in a nonsimple root subgroup, on which χ is trivial. Thus

$$\chi(u_2')f_0'(\dot{w}'a') = f_0'(\dot{w}'a')$$

and since $\chi(u_2') \neq 1$, we get $f_0'(\dot{w}'a') = 0$. Thus $f_0'(\dot{w}'a') = 0$ for all $a' \in A'$.

We now claim that f_0' vanishes on the entire cell $C(w')$. Let $g = u_1'\dot{w}'za'u_2'$ with $u_1' \in U, u_2' \in U_{w'}^-$, and let $a' \in A'$. Suppose that $f_0'(g) = \omega_\pi(z)f_0'(u_1'\dot{w}'a'u_2') \neq 0$. In order for the integral defining f_0' to be nonzero, there must exist $u_1 \in U_1'$ and $u_2 \in U_2'$ such that $f_0(u_1 u_1' \dot{w}' a' u_2' u_2) \neq 0$. This implies that

$$(u_1 u_1', u_2' u_2) \in U_1' \cdot \tilde{U}_{\dot{w}'a'} \cdot U_2' \text{ and } a' \in K'$$

and hence $(u_1, u_2') \in U_1' \cdot \tilde{U}_{\dot{w}'a'} \cdot U_2'$.

So we can write $(u_1', u_2') = (u_1^0 v_1, v_2 u_2^0)$ for $(v_1, v_2) \in \tilde{U}_{\dot{w}'a'}$ and $u_1^0 \in U_1', u_2^0 \in U_2'$. Directly from the definition, we get

$$f_0'(u_1' \dot{w}' a' u_2') = f_0'(u_1^0 v_1 \dot{w}' a' v_2 u_2^0) = f_0'(u_1^0 \dot{w}' a' u_2^0)$$

and since $u_i^0 \in U_i'$, we can further change variables to get

$$f_0'(u_1' \dot{w}' a' u_2') = \chi(u_1^0 u_2^0) f_0'(\dot{w}' a') = 0.$$

This shows that f_0' is identically zero on the Bruhat cell $C(w')$. Finally, let

$$f_0(g) = \frac{1}{\operatorname{meas}(U_1' \times U_2')} f_0'(g).$$

By what we just proved, f_0 vanishes on $C(w')$, and since $\Omega_{w,1} = \Omega_{w,0} \cup C(w')$, we have $f_0 \in \mathscr{C}_c^\infty(\Omega_{w,0}; \omega_\pi)$. By Lemma 4.3.3.1, we have for sufficiently large φ that $B_\varphi^G(g, f_0) = B_\varphi^G(g, f_1)$ for all $g \in G$, which completes the proof. $\square$

4.3.5 The function which only depends on the central character of π

In the asympotic expansion of the partial Bessel integral of a matrix coefficient of our supercuspidal representation π, we are looking to write the partial Bessel integral as a sum of two terms. One will only depend on the central character ω_π of π, not on π itself. The other will have a uniform smoothness property. In this section we isolate the term which will only depend on the central character of π.

If $f \in \mathscr{C}_c^\infty(G;\omega_\pi)$, let us define

$$W^f(g) = \int\limits_U f(xg)\overline{\chi(x)}dx.$$

If f is a matrix coefficient of our supercuspidal representation π, then $W^f(-)$ is an element of the Whittaker model of π, and f can be chosen so that $W^f(e) = 1$. See the proof of Proposition 1.3 of [PaSt08]).

Convention 4.3.5.1. *We will fix once and for all a function $f_0 \in \mathscr{C}_c^\infty(G;\omega_\pi)$ such that $W^f(e) = 1$. We will call f_0 the* auxiliary function.

The product group $\mathrm{SL}_n(E)Z$ is open and finite index in G, and the intersection $\mathrm{SL}_n(E) \cap Z$ is finite. If $g \in G$, we will consider all decompositions $g = g'c$ with $g' \in \mathrm{SL}_n(E)$ and $c \in Z$. Define

$$f_1(g) = \begin{cases} \frac{1}{|Z \cap \mathrm{SL}_n(E)|} \sum\limits_{g=g'c} f_0(g')\omega_\pi(c) & \text{if } g \in \mathrm{SL}_n(E)Z \\ 0 & \text{if } g \notin \mathrm{SL}_n(E)Z. \end{cases}$$

Then f_1 lies in $\mathscr{C}_c^\infty(G;\omega_\pi)$. The proof is identical to that of Lemma 4.2.5.3.

Remark 4.3.5.2. *The function f_1 depends only on the central character ω_π of π and on the auxiliary funcion that was chosen once and for all. In particular, it does not depend on π.*

We will make use of the pure Bessel integrals again. If $f \in \mathscr{C}_c^\infty(G;\omega_\pi)$, the restriction of f to the small cell B remains compactly supported modulo Z, so we may define the pure Bessel integral

$$B^G(b,f) = \int\limits_U f(xb)\overline{\chi(x)}dx = W^f(b)$$

for all $b \in B$. By Proposition 4.2.4.3, we have for all sufficiently large φ that

$$B_\varphi^G(a,f) = \tilde{\varphi}_G^G(e)B^G(a,f) = \tilde{\varphi}_G^G(e)\omega_\pi(a)W^f(e)$$

for all $a \in Z$.

Lemma 4.3.5.3. *Let $f \in \mathscr{C}_c^\infty(G;\omega_\pi)$ be a function for which $W^f(e) = 1$. Then for all sufficiently large φ, depending on f and f_1, we have $B_\varphi^G(a, f_1) = B_\varphi^G(a, f)$ for all $a \in Z$.*

Proof: For sufficiently large φ, we have

$$B_\varphi^G(a, f) = \tilde{\varphi}_G^G(e) B^G(a, f) = \tilde{\varphi}_G^G(e)\omega_\pi(a) W^f(e) = \tilde{\varphi}_G^G(e)\omega_\pi(a)$$

and similarly $B_\varphi^G(a, f_1) = \tilde{\varphi}_G^G(e)\omega_\pi(a)W^{f_1}(e)$. Since $\tilde{\varphi}_G^G(e) \neq 0$ for φ sufficiently large, the problem becomes to show that $W^{f_1}(e) = 1$.

If $x \in U$, then since $U \subset \mathrm{SL}_n(E)$, we have

$$f_1(x) = \frac{1}{|Z \cap \mathrm{SL}_n(E)|} \sum_{\xi \in Z \cap \mathrm{SL}_n(E)} f_0(x\xi^{-1})\omega_\pi(\xi) = f_0(x)$$

and therefore

$$W^{f_1}(e) = \int\limits_U f_1(x)\overline{\chi(x)}dx = \int\limits_U f_0(x)\overline{\chi(x)}dx = W^{f_0}(e) = 1.$$

□

4.3.6 Partitions of unity

The final ingredient in our asympotic expansion formula will make use of partitions of unity. If X is a locally compact, totally disconnected space, let $\mathscr{C}_c^\infty(X)$ be the space of locally constant, and compactly supported complex valued functions on X. Note that if U is an open set in X, then we have a natural inclusion $\mathscr{C}_c^\infty(U) \subset \mathscr{C}_c^\infty(X)$ by extending by zero.

The first result we need is elementary, and we omit the proof.

Lemma 4.3.6.1. *(Partitions of unity) Let X be a locally compact and totally disconnected topological space, and suppose $U_1, ..., U_t$ are nonempty open subsets of X whose union is X. If $f \in \mathscr{C}_c^\infty(X)$, then there exist functions $f_i \in \mathscr{C}_c^\infty(U_i)$ such that*

$$f_1(x) + \cdots + f_t(x) = f(x)$$

for all $x \in X$.

Proposition 4.3.6.2. *Let $\Omega, \Omega_1, ..., \Omega_t$ be open subsets of G such that $Z\Omega = \Omega$, $Z\Omega_i = \Omega_i$, and $\Omega_1 \cup \cdots \cup \Omega_t = \Omega$. If $f \in \mathscr{C}_c^\infty(\Omega; \omega_\pi)$, then there exist $f_i \in \mathscr{C}_c^\infty(\Omega_i; \omega_\pi)$ such that*

$$f(x) = f_1(x) + \cdots + f_t(x)$$

for all $x \in G$.

Proof: Let $p : G \to G/Z$ be the canonical projection, and let $X = p(\Omega), U_i = p(\Omega_i)$. Then X is an open set in G/Z and is therefore locally compact and totally disconnected, and X is the union of the U_i.

Let E be the support of f. Then $p(E)$ is a compact and open set in G/Z, so Lemma 4.3.6.1 gives us functions $\phi_i \in \mathscr{C}_c^\infty(U_i)$ with $\mathrm{Char}(p(E)) = \phi_1 + \cdots + \phi_t$.

Let $h_i = \phi_i \circ p$, so that

$$\mathrm{Char}(E) = h_1 + \cdots + h_t \tag{4.3.6.1}$$

as locally constant functions on G. One checks that $f_i := fh_i$ lies in $\mathscr{C}_c^\infty(\Omega_i; \omega_\pi)$. Multiplifying equation (4.3.6.1) by f gives us $f = f_1 + \cdots + f_t$, as required. $\square$

4.4 Asymptotic expansion of partial Bessel functions

We finally arrive at the main result of this chapter. All the ingredients are in place to prove the asymptotic expansion. It is this expansion which will allow us to prove the analytic stability result needed for our main theorem. Recall that π is an irreducible, supercuspidal representation of $\mathrm{GL}_n(E)$ with central character ω_π, and that we have fixed an auxiliary function $f_0 \in \mathscr{C}_c^\infty(G;\omega_\pi)$ with $W^{f_0}(e) = 1$.

Theorem 4.4.0.1. *Let $f \in \mathscr{C}_c^\infty(G;\omega_\pi)$ with $W^f(e) = 1$. Then there exists a function $f_1 \in \mathscr{C}_c^\infty(G;\omega_\pi)$, and for each $e \neq w \in B(G)$ a function $f_w \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$, such that the following hold:*

(i): For all sufficiently large φ, we have

$$B_\varphi^G(\dot{w}_G a, f) = B_\varphi^G(\dot{w}_G a, f_1) + \sum_w B_\varphi^G(\dot{w}_G a, f_w)$$

for all $a \in A$.

(ii): f_1 depends only on the auxiliary function f_0 and on the central character ω_π of π, not on π itself.

(iii): For each $e \neq w \in B(G)$, let M be the standard Levi subgroup of G corresponding to w, and let $Z_M' = \{z' : z \in Z_M\}$. For sufficiently large φ, we have that $B_\varphi^G(\dot{w}_G a, f_w)$ is zero for $a \notin (A \cap M_{\mathrm{der}})Z_M$. Also for sufficiently large φ, we have that for every $b \in A \cap M_{\mathrm{der}}$, the function on Z_M' given by

$$c' \mapsto B_\varphi^G(\dot{w}_G bc', f_w)$$

is uniformly smooth. The open compact subgroup occurring in the definition of uniform smoothness is independent of φ, once φ is chosen sufficiently large, and also independent of b.

We give the proof in several steps.

4.4.1 Passing from G to the maximal Levis

The first step of the proof comes from (4.3.5). Let $f_1 \in \mathscr{C}_c^\infty(G; \omega_\pi)$ be the function defined there. By Remark 4.3.5.2, f_1 only depends on the auxiliary function f_0 and on the central character of π, not π itself. By Lemma 4.3.5.3, we have that

$$B_\varphi^G(a, f - f_1) = 0$$

for all $a \in Z$. We can now apply Lemma 4.3.3.4, noting that $G = \Omega_e$. The lemma tells us that there exists an $f_2 \in \mathscr{C}_c^\infty(G; \omega_\pi)$ which is supported inside $\Omega_e^\circ = G - B$ such that for sufficiently large φ,

$$B_\varphi^G(g, f - f_1) = B_\varphi^G(g, f_2)$$

for all $g \in G$.

Now consider the open set

$$\Omega^1 = \bigcup_w \Omega_w$$

as w runs through all elements of $B(G)$ which correspond to maximal Levi subgroups of G (this is actually the same as taking the union over all $e \neq w \in B(G)$, since these are the minimal nonidentity elements in $B(G)$ in the Bruhat order). We have an inclusion of open sets $\Omega^1 \subset \Omega_e^\circ$ whose complement is a union of Bruhat cells $C(w')$ for $w' \notin B(G)$. Proposition 4.3.4.1 applies and gives us a function $f_3 \in \mathscr{C}_c^\infty(\Omega^1; \omega_\pi)$ such that for sufficiently large φ, $B_\varphi^G(g, f_2) = B_\varphi^G(g, f_3)$ for all $g \in G$. So for sufficiently large φ, we have

$$B_\varphi^G(g, f - f_1) = B_\varphi^G(g, f_3)$$

for all $g \in G$.

Now we use a partition of unity. Since Ω^1 is the union of the open sets Ω_w, as w runs through all elements of $B(G)$ corresponding to maximal Levi subgroups of G, Proposition 4.3.6.2 gives us functions $\Lambda_w \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$ such that $f_3 = \sum_w \Lambda_w$. We have shown that for sufficiently large φ, we have

$$B_\varphi^G(g,f) = B_\varphi^G(g,f_1) + \sum_w B_\varphi^G(g,\Lambda_w) \tag{4.4.1.1}$$

for all $g \in G$. Here the sum runs only over those $w \in B(G)$ corresponding to maximal Levi subgroups of G.

4.4.2 The next step

Now on each of the functions $\Lambda_w \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$, for $w \in B(G)$ corresponding to a maximal Levi M of G, we let $h = h_{\Lambda_w} \in \mathscr{C}_c^\infty(M;\omega_\pi)$ be as in (4.1.9). By Proposition 4.2.5.4 and Theorem 4.2.5.5, there exists a function $h_1 \in \mathscr{C}_c^\infty(M;\omega_\pi)$ satisfying two conditions:

1. For sufficiently large φ, we have

$$B_\varphi^M(a,h) = B_\varphi^M(a,h_1)$$

   for all $a \in Z_M$.

2. The function $B_\varphi^M(\dot{w}_M a, h_1)$ vanishes for $a \notin (A \cap M_{\mathrm{der}})Z_M$, and for each $b \in A \cap M_{\mathrm{der}}$, the function on Z'_M given by

$$c' \mapsto B_\varphi^M(\dot{w}_M bc', h_1)$$

   is uniformly smooth. The open compact subgroup occuring in the definition of uniform smoothness is independent of φ and b.

Now by Lemma 4.1.9.1, there exists a function $f_w \in \mathscr{C}_c^\infty(\Omega_w;\omega_\pi)$ such that $h_{f_w} = h_1$. Proposition 4.3.2.2 tells us that for sufficiently large φ, and for all standard Levi subgroups $L \subset M$, we have

$$B_\varphi^G(\dot{w}_G\dot{w}_L^{-1}a, \Lambda_w) = B_\varphi^M(\dot{w}_M\dot{w}_L^{-1}a, h)$$

$$B_\varphi^G(\dot{w}_G\dot{w}_L^{-1}a, f_w) = B_\varphi^M(\dot{w}_M\dot{w}_L^{-1}a, h_1)$$

for all $a \in Z_L$. We apply Proposition 4.3.2.2 in two ways, first with $L = M$ and then with $L = A$. Applying this with $L = M$ and noting that $w = w_G w_M$, we have

$$B_\varphi^G(\dot{w}a, \Lambda_w) = B_\varphi^M(\dot{w}_M a, h) = B_\varphi^M(\dot{w}_M a, h_1) = B_\varphi^G(\dot{w}a, f_w) \tag{4.1.9.2}$$

for all $a \in Z_M$. Applying this with $L = A$, we see that $B_\varphi^G(\dot{w}_G a, f_w)$ satisfies the same uniform smoothness property as $B_\varphi^G(\dot{w}_M a, h_1)$. This is to say:

- For sufficiently large φ, $B_\varphi^G(\dot{w}_G a, f_w)$ vanishes for $a \notin (A \cap M_{\mathrm{der}})Z_M$, and for $b \in A \cap M_{\mathrm{der}}$, the function defined on Z_M' by

  $$c' \mapsto B_\varphi^G(\dot{w}_G bc', f_w) = B_\varphi^M(\dot{w}_M bc', h_1)$$

  is uniformly smooth. The open compact subgroup occuring in the definition of uniform smoothness is independent of b, and also independent of φ, once φ is sufficiently large so that $B_\varphi^G(\dot{w}_G a, f_w) = B_\varphi^M(\dot{w}_M a, h_1)$ for all $a \in A$.

Now we move up to the next level of Levi subgroups. We are still working with a fixed maximal Levi subgroup M. Since for sufficiently large φ, we have $B_\varphi^G(\dot{w}a, \Lambda_w - f_w) = 0$ for all $a \in Z_M$, we can proceed exactly as in the last section. Lemma 4.3.3.4 gives us a function $f_2 \in \mathscr{C}_c^\infty(G;\omega_\pi)$ which is supported inside $\Omega_w^\circ = \Omega_w - C(w)$ such that for sufficiently large φ,

$$B_\varphi^G(g, \Lambda_w - f_w) = B_\varphi^G(g, f_2)$$

for all $g \in G$. Now consider the open set

$$\Omega^1 = \bigcup_{w'} \Omega_{w'}$$

as w' runs through all those elements of $B(G)$ which correspond to maximal Levi subgroups of M. We have an inclusion of open sets $\Omega^1 \subset \Omega_w^\circ$ whose complement is a union of Bruhat cells $C(w'')$ with $w'' \notin B(G)$. Proposition 4.3.4.1 applies and gives us a function $f_3 \in \mathscr{C}_c^\infty(\Omega^1; \omega_\pi)$ such that for sufficiently large φ, $B_\varphi^G(g, f_2) = B_\varphi^G(g, f_3)$ for all $g \in G$. So for sufficiently large φ, we have

$$B_\varphi^G(g, \Lambda_w - f_w) = B_\varphi^G(g, f_3)$$

for all $g \in G$.

Now we use a partition of unity. Since Ω^1 is the union of the open sets $\Omega_{w'}$ as w' runs over all the elements of $B(G)$ corresponding to maximal Levi subgroups of M, Proposition 4.3.6.2 gives us functions $\Lambda_{w'} \in \mathscr{C}_c^\infty(\Omega_{w'}; \omega_\pi)$ such that $f_3 = \sum_{w'} \Lambda_{w'}$. We have shown that for sufficiently large φ, we have

$$B_\varphi^G(g, \Lambda_w) = B_\varphi^G(g, f_w) + \sum_{w'} B_\varphi^G(g, \Lambda_{w'})$$

for all $g \in g$. Here the sum runs only over those $w' \in B(G)$ corresponding to maximal Levi subgroups of M.

4.4.3 Iterating the previous step

Having applied the process of the previous section to each $w \in B(G)$ corresponding to a maximal Levi subgroup of G, we have obtained functions $f_w \in \mathscr{C}_c^\infty(\Omega_w; \omega_\pi)$ with the uniform smoothness property, and obtained "error terms" $B_\varphi^G(g, \Lambda_{w'})$ over the w' corresponding to next to maximal Levi subgroups.

To each maximal Levi subgroup L of each maximal Levi subgroup M of G, we proceed with $B_\varphi^G(g, \Lambda_{w'})$ exactly as in the previous section. That is, we produce a function $f_{w'} \in \mathscr{C}_c^\infty(\Omega_{w'}; \omega_\pi)$ such that for sufficiently large φ, the following hold:

- $B_\varphi^G(\dot{w}'a, \Lambda_{w'}) = B_\varphi^G(\dot{w}'a, f_{w'})$ for all $a \in Z_L$.
- $B_\varphi^G(\dot{w}_G a, f_{w'})$ vanishes for $a \notin (A \cap L_{\mathrm{der}})Z_L$, and for $b \in A \cap L_{\mathrm{der}}$ the function on Z_L' given by
$$c' \mapsto B_\varphi^G(\dot{w}_G bc', f_{w'})$$
is uniformly smooth. The open compact subgroup of Z_L' occurring in the definition of uniform smoothness is independent of b and also independent of φ once φ is large enough.

The difference $B_\varphi^G(\dot{w}_G a, \Lambda_{w'}) - B_\varphi^G(\dot{w}_G a, f_{w'})$ has an error term indexed by the maximal Levi subgroups of L. That is, for each $w'' \in B(G)$ corresponding to a maximal Levi subgroup of L, there exists a function $f_{w''} \in \mathscr{C}_c^\infty(\Omega_{w''}; \omega_\pi)$ such that for sufficiently large φ, we have

$$B_\varphi^G(\dot{w}_G a, \Lambda_{w'}) = B_\varphi^G(\dot{w}_G a, f_{w'}) + \sum_{w''} B_\varphi^G(\dot{w}_G a, \Lambda_{w''})$$

for all $a \in A$.

We do the same process for each $\Lambda_{w''}$ on each maximal Levi subgroup of L, obtaining a uniformly smooth piece on $\Omega_{w''}$ plus an error term over the smaller open sets $\Omega_{w'''}$ for w''' corresponding to even smaller Levis. We collect the uniformly smooth pieces $B_\varphi^G(\dot{w}_G a, f_w) : e \neq w \in B(G)$ as we move down the Levi subgroups. Note that we may obtain multiple uniformly smooth pieces corresponding to the same Levi subgroup, which we condense into one.

When we eventually move down to the smallest Levi subgroup, A, we will obtain a uniformly smooth piece $B_\varphi^G(\dot{w}_G a, f_{w_G})$, where f_{w_G} is supported inside the big cell $\Omega_{w_G} = C(w_G)$. Here $A_{\mathrm{der}} = 1$, so this partial Bessel integral is uniformly smooth on all of A'. Since there are no more Levi subgroups below A, there will be no further error terms, giving us the asymptotic expansion of Theorem 4.0.1.

5. PROOF OF ANALYTIC STABILITY

Throughout this chapter, E/F denotes a quadratic extension of p-adic fields. The purpose of this chapter is to prove the analytic stability result, Proposition 3.2.2.8. The proposition states that the Asai gamma factor of a supercuspidal representation only depends on the central character up to a highly ramified twist. We begin by explaining how the Asai gamma factor $\gamma(s, \pi, \mathscr{R}, \psi)$ arises from the Langlands-Shahidi method, by embedding $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$ as a maximal Levi subgroup $\mathbf{M}$ of the even unitary group $\mathbf{G} = U(n, n)$.

The Asai gamma factor is equal to the Shahidi local coefficient $C_\chi(s, \pi)$, up to a constant (Lemma 5.1.2.1 and equation (5.1.3.2)). Proposition 3.2.2.8 is then equivalent to the stability of this local coefficient for π supercuspidal (Theorem 5.1.3.3). Thus Theorem 5.1.3.3 is the main result of this section.

In Theorem 6.2 of [Sh02], Shahidi shows how his local coefficient can be expressed as a Mellin transform of a partial Bessel integral, under certain conditions. Our approach to proving Theorem 5.1.3.3 will be to apply Shahidi's local coefficient formula, and then apply the delicate analysis of partial Bessel integrals which we developed in Chapter Four.

If α is the simple root corresponding to $\mathbf{M}$, one of Shahidi's assumptions (Assumption 5.1 of [Sh02]) for his local coefficient formula is the existence of an injection $\alpha^\vee : F^* \to Z_{\mathbf{M}}(F)/Z_{\mathbf{G}}(F)$ such that $\alpha \circ \alpha^\vee = 1$. Unfortunately, this assumption is false in our case. But this difficulty is not too serious: we can embed $\mathbf{G}$ in a larger group $\widetilde{\mathbf{G}}$, having the same derived group as $\mathbf{G}$, for which the assumption holds. Since local coefficients only depend on the derived group (as we explain in (5.2.1)), we will be able to apply Shahidi's formula after all.

Our main references for this chapter are [Go94] and [Sh02].

5.1 $\mathrm{GL}_n(E)$ as a Levi subgroup of the unitary group

In this section, we explain the context in which we will encounter the Asai gamma factor as defined by the Langlands-Shahidi method. In (5.1.1), we define the unitary group $\mathbf{G} = U(n,n)$, and explain some of its structure. We define a maximal parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$ whose Levi subgroup $\mathbf{M}$ is isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$.

In (5.1.2), we show how the Asai representation $\mathscr{R}$ of the L-group of $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$ occurs as the adjoint action of ${}^L\mathbf{M}$ on the Lie algebra of ${}^L\mathbf{N}$.

In (5.1.3), we define a splitting for $\mathbf{G}$ and obtain our canonical Weyl group representatives as in (2.2.17). We also state the main result of this section, Proposition 3.2.2.8, which is the stability of the Shahidi local coefficient of supercuspidal representations for $\mathbf{M}$ in $\mathbf{G}$.

In applying Shahidi's local coefficient formula (Theorem 6.2 of [Sh02]), we will need to consider a measure on the quotient space of an open dense subset of $\mathbf{N}(F)$ under a certain p-adic Lie group actions. We develop what we will need for this in (5.1.4).

5.1.1 Definition of the Unitary Group

Let W be the $2n$ by $2n$ matrix

$$W = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$$

where I_n is the n by n identity matrix. The unitary group $\mathbf{G} = U(n,n)$ is defined to be an outer form of GL_{2n} with the following Galois action for $X \in \mathrm{GL}_{2n}(\overline{F})$ and $\gamma \in \mathrm{Gal}(\overline{F}/F)$:

$$\gamma.X = \begin{cases} \gamma(X) & \text{if } \gamma|_E = 1_E \\ W\,{}^t\gamma(X)^{-1}W^{-1} & \text{if } \gamma|_E \neq 1_E \end{cases}$$

where $\gamma(X)$ denotes the entrywise action of γ on X, and ${}^tX$ denotes the transpose. In particular, we have

$$\mathbf{G}(E) = \mathrm{GL}_{2n}(E)$$

and in fact $\mathbf{G}$ splits over E, with $\mathbf{G} \times_F E = \mathrm{GL}_{2n,E}$. Moreover, we see that $\Gamma = \mathrm{Gal}(E/F)$ acts on $\mathrm{GL}_{2n}(E)$ by

$$\sigma.X = W{}^t\overline{X}^{-1}W^{-1}$$

where $\overline{X}$ is the entrywise application of the nontrivial element σ of $\mathrm{Gal}(E/F)$ to X, and so

$$\mathbf{G}(F) = \{X \in \mathrm{GL}_{2n}(E) : W\,{}^t\overline{X}^{-1}W^{-1} = X\}.$$

If we start with SL_{2n} instead of GL_{2n}, we can define the special unitary group $\mathrm{SU}(n,n)$ in the same way, and in fact we have $\mathrm{SU}(n,n) = \mathbf{G}_{\mathrm{der}}$, the derived group of $\mathbf{G}$.

The verification of the following details are straightforward, and we omit the proofs.

Proposition/Definition 5.1.1.1. *(i): Let* $\mathbf{T}$ *be the maximal torus of* $\mathbf{G}$ *consisting of diagonal matrices, and let* $\mathbf{S}$ *be the subtorus of* $\mathbf{T}$ *defined by*

$$\mathbf{S}(\overline{F}) = \{\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_1^{-1} & & \\ & & & & \ddots & \\ & & & & & x_n^{-1} \end{pmatrix}\}.$$

Then $\mathbf{S}$ *is the maximal* F*-split subtorus of* $\mathbf{T}$*, and in fact* $Z_{\mathbf{G}}(\mathbf{S}) = \mathbf{T}$*. Hence* $\mathbf{G}$ *is quasi split over* F*, and* $\mathbf{S}$ *is a maximal* F*-split torus of both* $\mathbf{G}_{\mathrm{der}}$ *and* $\mathbf{G}$*.*

(ii): Let $\epsilon_1, ..., \epsilon_n$ be the basis of the character lattice $X(\mathbf{S})$ such that ϵ_i sends the above matrix to x_i. Then

$$\Delta_F = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}.$$

is a set of simple roots for $\mathbf{S}$ in $\mathbf{G}$. The corresponding relative root system is of type C_n.

(iii): Let $\mathbf{B}$ be the Borel subgroup (minimal parabolic F-subgroup) of $\mathbf{G}$ corresponding to Δ_F. Let $e_1, ..., e_{2n}$ be the standard basis of $X(\mathbf{T})$, where e_i sends a $2n$ by $2n$ diagonal matrix to its ith entry. The set Δ of simple roots of $\mathbf{T}$ in $\mathbf{G}$ corresponding to $\mathbf{B}$ is $A \cup B \cup A'$, where

$$A = \{e_1 - e_2, ..., e_{n-1} - e_n\}$$

$$A' = \{-(e_{n+1} - e_{n+2}), ..., -(e_{2n-1} - e_{2n})\}$$

$$B = \{e_n - e_{2n}\}.$$

(iv): The nontrivial element $\sigma \in \Gamma = \mathrm{Gal}(E/F)$ switches e_i and $-e_{n+i}$ for $1 \leq i \leq n$. Hence the orbits of Δ under the action of Γ are

$$\{e_i - e_{i+1}, -(e_{n+i} - e_{n+i+1})\}$$

for $1 \leq i \leq n-1$, as well as the singleton set $\{e_n - e_{2n}\}$. In particular, $e_n - e_{2n}$ is the only simple root in Δ which is defined over F.

(v): Let $\theta = \Delta_F - \{2\epsilon_n\}$, and let $\mathbf{P}$ be the corresponding maximal F-parabolic subgroup of $\mathbf{G}$. It is self-associate. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{P}$, and let $\mathbf{M}$ be the unique Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$. Then

$$\mathbf{M}(F) = \{\begin{pmatrix} x & \\ & {}^t\overline{x}^{-1} \end{pmatrix} : x \in \mathrm{GL}_n(E)\}$$

$$\mathbf{N}(F) = \{\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} : X \in \mathrm{Mat}_n(E),\ {}^t\overline{X} = X\}.$$

5.1.2 The L-group of G

We can identify the L-group ${}^L\mathbf{G}$ of $\mathbf{G}$ with the semidirect product of $\mathrm{GL}_{2n}(\mathbb{C})$ by $\mathrm{Gal}(E/F)$, where $\mathrm{Gal}(E/F)$ acts on ${}^L\mathbf{G}$ by $\sigma.X = W\,{}^tX^{-1}W^{-1}$. The L-group ${}^L\mathbf{M}$ of $\mathbf{M}$ can be identified with the semidirect product of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ by $\mathrm{Gal}(E/F)$, which acts by $\sigma.(x,y) = ({}^ty^{-1}, {}^tx^{-1})$. The Lie algebra ${}^L\mathfrak{n}$ of the L-group of $\mathbf{N}$ identifies with $\mathrm{Mat}_n(\mathbb{C})$, and the adjoint representation $r : {}^L\mathbf{M} \to \mathrm{GL}({}^L\mathfrak{n})$ is given by

$$r(x, y, 1).X = xXy^{-1}$$

$$r(\sigma).X = {}^tX.$$

It is irreducible.

Lemma 5.1.2.1. *Let π be an irreducible, admissible representation of $\mathbf{M}(F) = \mathrm{GL}_n(E)$. Then*

$$\gamma(s, \pi, r, \psi) = \gamma(s, \pi, \mathscr{R}, \psi)$$

where $\mathscr{R}$ is the Asai representation (2.6.1).

Proof: We can take $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$ to be the group defined on E-points by $\mathrm{GL}_n(E) \times \mathrm{GL}_n(E)$, with $\mathrm{Gal}(E/F)$ acting by switching the factors. Define an isomorphism $\mathrm{Res}_{E/F}\,\mathrm{GL}_n \to \mathbf{M}$ of algebraic groups over F by

$$(x, y) \mapsto \begin{pmatrix} x & \\ & {}^ty^{-1} \end{pmatrix}.$$

Let V be an n-dimensional complex vector space with basis $e_1, ..., e_n$. We can identify the space ${}^L\mathfrak{n}$ with $V \otimes_{\mathbb{C}} V$, an isomorphism being given by sending the elementary matrix E_{ij} to $e_i \otimes e_j$. By identifying $\mathbf{M}$ with $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$ via the isomorphism given, and taking the corresponding isomorphism of L-groups ${}^L\operatorname{Res}_{E/F}\operatorname{GL}_n \to {}^L\mathbf{M}$, it is straightforward to check that r now identifies with the representation $\mathscr{R}$. The lemma is now a consequence of (2.2.2). $\square$

5.1.3 The local coefficient for M inside G

Let $\alpha = 2\epsilon_n$ be the simple root of Δ_F which defines the maximal parabolic subgroup $\mathbf{P}$, and let

$$\rho = n(\epsilon_1 + \cdots + \epsilon_n)$$

be half the sum of the roots of $\mathbf{S}$ in $\mathbf{N}$. We can identify α with its unique preimage in $\Delta \subset X(\mathbf{T})$, and ρ with its preimage in $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ consisting of half the sum of the roots of $\mathbf{T}$ in $\mathbf{N}$.

We will calculate the element $\tilde{\alpha} \in X(\mathbf{M})_F$ as defined in (2.2.16). Let $(-,-)$ denote the standard inner product on $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the basis $e_1, ..., e_{2n}$ of Proposition/Definition 5.1.1.1. It is invariant under the action of the Weyl group $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ and the Galois group $\operatorname{Gal}(\overline{F}/F)$, and we can therefore use it to define $\tilde{\alpha}$. By definition, $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1}\rho$, where $\langle v, w \rangle = 2\frac{(v,w)}{(w,w)}$. We calculate

$$\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1}\rho = n^{-1}\rho$$

which we restrict to the split component $A_{\mathbf{M}}$ of $\mathbf{M}$, and identify as an element of $X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have:

Lemma 5.1.3.1. *(i) For $s \in \mathbb{C}$, and all $m \in \operatorname{GL}_n(E) = \mathbf{M}(F)$,*

$$q_F^{\langle s\rho, H_{\mathbf{M}}(m)\rangle} = |\det m|_E^{ns/2}$$

$$q_F^{\langle s\tilde{\alpha}, H_{\mathbf{M}}(m)\rangle} = |\det m|_E^{s/2}.$$

(ii) If π is a smooth, irreducible representation of $\mathrm{GL}_n(E)$*, and* $s_0 \in \mathbb{C}$*, then*

$$\gamma(s, \pi|\det(-)|_E^{s_0}, \mathscr{R}, \psi) = \gamma(s + 2s_0, \pi, \mathscr{R}, \psi).$$

Proof: (i) is proved in Section 2 of [Go94], and (ii) follows from (i) and (2.2.21). □

We have defined a set of simple nonrestricted roots Δ. Note that this is not the usual set of simple roots for GL_{2n}, so the unipotent radical $\mathbf{U}$ of $\mathbf{B}$ is not the group of upper triangular unipotent matrices. For each $\beta \in \Delta$, we now define root vectors $\mathbf{x}_\beta : \mathbf{G}_a \to \mathbf{U}_\beta$ on $\overline{F}$-points. If $\beta = e_i - e_{i+1}$, for $i = 1, ..., n-1$, we define

$$\mathbf{x}_\beta(t) = I_{2n} + tE_{i,i+1}$$

where $E_{i,i+1}$ is the $2n$ by $2n$ matrix with a 1 in the $(i, i+1)$ position, and zeroes elsewhere. If $\beta = -(e_{n+i} - e_{n+i+1})$ for $i = 1, ..., n-1$, we define

$$\mathbf{x}_\beta(t) = I_{2n} - tE_{n+i,n+i+1}.$$

Finally, if $\beta = \alpha = e_n - e_{2n}$, we define

$$\mathbf{x}_\alpha(t) = \mathbf{x}_\beta(t) = I_{2n} + tE_{n,2n}.$$

This splitting (2.1.2) $\mathbf{x}_\beta : \beta \in \Delta$ is defined over F, in the sense that it is fixed by the Galois group $\mathrm{Gal}(\overline{F}/F)$ (2.2.4). Having defined these root vectors, we can then define our canonical Weyl group representatives as (2.2.17).

Lemma 5.1.3.2. *(i): For the set of relative simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = \alpha = 2\epsilon_n$, we have from the above splitting the following canonical Weyl group representatives for the corresponding simple reflections $w_1, ..., w_n$. First, let*

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $1 \leq i \leq n-1$, we have

$$\dot{w}_i = \begin{pmatrix} I_{i-1} & & & & \\ & C & & & \\ & & I_{n-2} & & \\ & & & C & \\ & & & & I_{n-i-1} \end{pmatrix}$$

and

$$\dot{w}_n = \begin{pmatrix} I_{n-1} & & & \\ & & & 1 \\ & & I_{n-1} & \\ & -1 & & \end{pmatrix}.$$

(ii): Let J be the n by n antidiagonal matrix

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}$$

let w_ℓ and w_ℓ^θ be the long elements of $\mathbf{G}$ and $\mathbf{M}$, and let $w_0 = w_\ell w_\ell^\theta$. These have canonical representatives

$$\dot{w}_\ell = (-1)^{n-1} \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}, \dot{w}_\ell^\theta = \begin{pmatrix} J & \\ & J \end{pmatrix}, \dot{w}_0 = \begin{pmatrix} & J \\ -J & \end{pmatrix}.$$

The splitting defines a generic character (2.2.4) χ of $\mathbf{U}(F)$ in terms of a fixed additive character ψ of F: if $u \in \mathbf{U}(F)$, we can write

$$u = \prod_{\beta \in \Delta} \mathbf{x}_\beta(a_\beta) u'$$

for $a_\beta \in \overline{F}$ and u' in the derived group of $\mathbf{U}(F)$. The sum of the $a_\beta : \beta \in \Delta$ lies in F, and we define

$$\chi(u) = \psi(\sum_{\beta \in \Delta} a_\beta).$$

Note that with our choice of representatives, we have

$$\chi(u) = \chi(\dot{w}_0 u \dot{w}_0^{-1}) = \chi(\dot{w}_0^{-1} u \dot{w}_0) \tag{5.1.3.1}$$

for all $u \in \mathbf{U}_{\mathbf{M}}(F)$ (2.2.19).

If π is a generic representation of $\mathrm{GL}_n(E) = \mathbf{M}(F)$, then the Langlands-Shahidi method defines the Shahidi local coefficient $C_\chi(s, \pi)$ (2.2.20). The local coefficient is related to the Asai gamma factor by the formula

$$C_\chi(s, \pi) = \lambda(E/F, \psi)^{n^2} \gamma(s, \pi, \mathscr{R}, \psi) \tag{5.1.3.2}$$

where $\lambda(E/F, \psi)$ is the Langlands lambda function (Theorem 2.2.20.1). Hence the main result we want to prove in this section, analytic stability (Proposition 3.2.2.8), is equivalent to:

Theorem 5.1.3.3. *Let π_1 and π_2 be supercuspidal representations of $\mathrm{GL}_n(E)$ with the same central character ω. Then for all sufficiently highly ramified characters η of E^*, we have*

$$C_\chi(s, \pi_1 \eta) = C_\chi(s, \pi_2 \eta)$$

where $\pi_i \eta = \pi_i(\eta \circ \det)$.

Our approach to Theorem 5.1.3.3 will be to apply Shahidi's local coefficient formula (Theorem 6.2 of [Sh02]) and then the Bessel function asymptotics of Chapter Four. However, the group $\mathbf{G}$ is insufficient to apply Shahidi's formula. In (5.2.2), we will embed $\mathbf{G}$ in a larger group $\widetilde{\mathbf{G}}$ which has the same derived group, and which has connected and cohomologically trivial center. The group $\widetilde{\mathbf{G}}$ satisfies the necessary properties to apply Shahidi's formula. As we explain in (5.2.1), local coefficients only depend on the derived group, so we will be able to calculate $C_\chi(s,\pi)$ using $\widetilde{\mathbf{G}}$.

Let $\overline{\mathbf{N}}$ be the unipotent radical of the parabolic opposite to $\mathbf{N}$. For $n \in \mathbf{N}(F)$, we will need an explicit decomposition of $\dot{w}_0^{-1} n \in \mathbf{P}(F)\overline{\mathbf{N}}(F)$ as in Section 4 of [Sh02].

Lemma 5.1.3.4. *Let*

$$n = \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \in \mathbf{N}(F)$$

for $X \in Mat_n(E)$ *and* ${}^t\overline{X} = X$. *Then* $\dot{w}_0^{-1} n \in \mathbf{P}(F)\overline{\mathbf{N}}(F)$ *if and only if* X *is invertible, in which case we can uniquely express* $\dot{w}_0^{-1} n = mn'\bar{n}$, *with* $m \in \mathbf{M}(F), n' \in \mathbf{N}(F), \bar{n} \in \overline{\mathbf{N}}(F)$. *We have*

$$m = (-1)^{n-1} \begin{pmatrix} JX^{-1} & \\ & JX \end{pmatrix} \; n' = \begin{pmatrix} I_n & -X \\ & I_n \end{pmatrix} \; \bar{n} = \begin{pmatrix} I_n & \\ X^{-1} & I_n \end{pmatrix}.$$

This is essentially Lemma 2.2 of [Go94].

5.1.4 Orbit space measures

Shahidi's local coefficient formula expresses $C_\chi(s,\pi)^{-1}$ as an integral with respect to a measure on the quotient space of $\mathbf{N}(F)$ with respect to the action of a certain group. In this section, we will explicitly construct the measure which we will need, and show it satisfies the required properties.

Let $\mathbf{U_M} = \mathbf{M} \cap \mathbf{U}$. The group $\mathbf{U_M}(F)$, which identifies with the upper triangular unipotent matrices of $\mathrm{GL}_n(E)$, acts by conjugation on $\mathbf{N}(F)$, which identifies with the space of n by n Hermitian matrices in $\mathrm{Mat}_n(E)$. Under these identifications, the action of $\mathbf{U_M}(F)$ on $\mathbf{N}(F)$ is given by

$$u.X = uX{}^t\overline{u}.$$

In what follows, dx will be an additive Haar measure on either F or E. We will repeatedly make use of the fact that if f is an integrable function on F, then

$$\int_F f(x)\,dx = \int_{F^*} f(x)\,dx.$$

Given such a measure, d^*x will denote a multiplicative Haar measure: either $\frac{dx}{|x|_F}$ on F^* or $\frac{dx}{|x|_E}$ on E^*. Here $|-|_F$ and $|-|_E$ are the normalized absolute values on F and E, related by $|N_{E/F}(x)|_F = |x|_E$ for $x \in E$. In particular, if $x \in F$, then $|x|_F^2 = |x|_E$.

Proposition 5.1.4.1. *Let R be the set of elements in $\mathbf{N}(F)$ of the form*

$$\begin{pmatrix} I_n & r \\ & I_n \end{pmatrix}$$

where $r = diag(r_1, ..., r_n)$ is an invertible diagonal matrix with entries necessarily in F. Then:

(i): The elements of R lie in distinct orbits under the action of $\mathbf{U_M}(F)$, each with trivial stabilizer.

(ii): The disjoint union W of the orbits $\mathbf{U_M}(F).r$ for $r \in R$ is an open dense subset of $\mathbf{N}(F)$.

(iii): The map $\mathbf{U_M}(F) \times R \to W$ given by $(u, r) \mapsto u.r$ is an isomorphism of analytic manifolds. In particular, the map $W \to R$ sending n to the unique element of R lying in the same orbit is a submersion of manifolds, so R is the quotient of W under the action of $\mathbf{U_M}(F)$ in the category of analytic manifolds.

(iv): Identifying R with $(F^*)^n$, we place the measure $dr = \prod_{i=1}^{n} |r_i|_E^{i-1}\,dr_i$ on R. Then integration over $\mathbf{N}(F)$ can be recovered by integration over R:

$$\int_{\mathbf{N}(F)} f(n)\,dn = \int_R \int_{\mathbf{U_M}(F)} f(u.r)\,du\,dr. \qquad (f \in \mathscr{C}_c^\infty(\mathbf{N}(F)))$$

Here $\mathscr{C}_c^\infty(\mathbf{N}(F))$ is the space of locally constant, compactly supported complex valued functions on $\mathbf{N}(F)$.

Proof: Assume first that $n = 2$. Then (i), (ii), and (iii) are straightforward to verify. For (iv), we identify $\mathbf{U}_{\mathbf{M}}(F)$ with E, and $\mathbf{N}(F)$ with $F \times E \times F$. Explicitly, this last identification is

$$\begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \mapsto (a, b, c).$$

Then we have for $r = \mathrm{diag}(r_1, r_2) \in D$ and $f \in \mathscr{C}_c^\infty(\mathbf{N}(F))$,

$$\begin{aligned}
\int\limits_R \int\limits_{\mathbf{U}_{\mathbf{M}}(F)} f(u.r)\, du\, dr &= \int\limits_F \int\limits_{F^*} \int\limits_E f(r_1 + x\overline{x}, xr_2, r_2)|r_2|_E\, dx\, dr_2\, dr_1 \\
&= \int\limits_{F^*} \int\limits_E \int\limits_F f(r_1 + x\overline{x}, xr_2, r_2)|r_2|_E\, dr_1\, dx\, dr_2 \\
&= \int\limits_{F^*} \int\limits_E \int\limits_F f(r_1, xr_2, r_2)|r_2|_E\, dr_1\, dx\, dr_2 \\
&= \int\limits_{F^*} \int\limits_E \int\limits_F f(r_1, x, r_2)\, dr_1\, dx\, dr_2 \\
&= \int\limits_{\mathbf{N}(F)} f(n)\, dn.
\end{aligned}$$

From the second to the third line, we have used the translation invariance of the measure dr_1 on F. For the third to the fourth line, we have used the fact that $\int\limits_E F(rx)dx = |r|_E^{-1} \int\limits_E F(x)dx$ for any Haar measure dx on E. Finally we use the fact that integration over F^* is the same as integration over F.

We then proceed by induction on n. Suppose we have verified that R and W work for a given n. We want to prove the proposition for the corresponding sets $\widetilde{R}$ and $\widetilde{W}$ of size $n + 1$. Consider the dense open set O of those matrices $X \in \mathbf{N}(F)$ (of size $n + 1$) whose lower right entry x is nonzero. Write

$$X = \begin{pmatrix} X_0 & \alpha \\ {}^t\overline{\alpha} & x \end{pmatrix}$$

for X_0 Hermitian of size n and α a column vector. Let

$$u = \begin{pmatrix} I_n & -x^{-1}\alpha \\ & 1 \end{pmatrix}$$

so that

$$un{}^t\overline{u} = \begin{pmatrix} X_0 - x^{-1}\alpha{}^t\overline{\alpha} & 0 \\ 0 & x \end{pmatrix}.$$

This procedure allows us to descend to size n and utilize our induction hypothesis. One checks that $\widetilde{W}$ consists of those X for which $X_0 - x^{-1}\alpha{}^t\overline{\alpha}$ lies in W. The map $X \mapsto X_0 - x^{-1}\alpha{}^t\overline{\alpha}$ on O is a submersion: this follows from the general fact that any map $p : F^{k_1} \times F^{k_2} \to F^{k_1}$ of the form

$$(x_1, ..., x_{k_1}, y_1, ..., y_{k_2}) \mapsto (x_1 + f_1(y_1, ..., y_{k_2}), ..., x_{k_1} + f_{k_1}(y_1, ..., y_{k_2}))$$

is a submersion, where the f_i are analytic maps. Indeed, the Jacobian matrix of p contains a k_1 by k_1 identity matrix.

Since the map $X \mapsto X_0 - x^{-1}\alpha{}^t\overline{\alpha}$ on O is a submersion, it is in particular an open map. Hence the preimage $\widetilde{W}$ of W is open and dense by induction, giving us (ii). The other properties (i), (iii), and (iv) are also proved by induction using this method of descent. $\square$

We also have an action of F^* on $\mathbf{N}(F)$ by scaling each entry. This action commutes with that of $\mathbf{U_M}(F)$, so we have an action of $F^* \times \mathbf{U_M}(F)$ on $\mathbf{N}(F)$. Let R' be the set of invertible diagonal matrices of the form $\mathrm{diag}(1, r_2, ..., r_n)$ in $\mathbf{N}(F)$. Define a measure on F^* (not a Haar measure) by $|z|_F^{n^2} d^*z = |z|_F^{n^2-1} dz$ and a measure dr' on $R' = (F^*)^{n-1}$ by

$$dr' = \prod_{i=2}^{n} |r_i|_E^{i-1} dr_i = \prod_{i=2}^{n} |r_i'|_F^{2i-1} d^*r_i'.$$

Then we see immediately that R' is the quotient of R under the action of F^*, and that integration over R can be recovered by integration over R' and F^*:

$$\int_R f(r)\,dr = \int_{R'}\int_{F^*} f(z.r')|z|_F^{n^2-1}\,dz\,dr'.$$

Putting this together with Proposition 5.1.4.1, we have:

Proposition 5.1.4.2. *R' is the quotient of an open dense subset of $\mathbf{N}(F)$ under the action of $F^* \times \mathbf{U}_{\mathbf{M}}(F)$, and for $f \in \mathscr{C}_c^\infty(F)$, integration over $\mathbf{N}(F)$ can be recovered by integration over R' and $F^* \times \mathbf{U}_{\mathbf{M}}(F)$:*

$$\int_{\mathbf{N}(F)} f(n)\,dn = \int_{R'}\int_{F^*}\int_{\mathbf{U}_{\mathbf{M}}(F)} f(u.(zr'))|z|_F^{n^2}\,du\,d^*z\,dr'.$$

5.2 Applying Shahidi's local coefficient formula

As we mentioned, our approach to Theorem 5.1.3.3 will be to apply Shahidi's local coefficient formula (Theorem 6.2 of [Sh02]) and then the Bessel function asymptotics of Chapter Four. However, the group $\mathbf{G}$ is insufficient to apply Shahidi's formula. We will embed $\mathbf{G}$ in a larger group $\widetilde{\mathbf{G}}$ which has the same derived group, and which has connected and cohomologically trivial center. It will have a Levi subgroup $\widetilde{\mathbf{M}}$ analogous to $\mathbf{M}$ (that is, defined by the same set of simple roots).

In (5.2.1), we will prove some general results about representations of groups having the same derived group, which we will apply to $\mathbf{G}$ and $\widetilde{\mathbf{G}}$. We will show in particular that if π is a smooth, irreducible representation of $\mathbf{M}(F)$, then there exists a smooth, irreducible representation $\tilde{\pi}$ of $\widetilde{\mathbf{M}}(F)$ whose restriction to $\mathbf{M}(F)$ contains π as a subrepresentation, and that the local coefficients for π and $\tilde{\pi}$ are the same. Thus we will be able to compute the local coefficient $C_\chi(s,\pi)$ using the extended group $\widetilde{\mathbf{G}}$.

In (5.2.2), we prove some necessary structural properties for $\widetilde{\mathbf{G}}$. In particular, we construct an injection $\alpha^\vee$ of F^* into $Z_{\widetilde{\mathbf{M}}(F)}/Z_{\widetilde{\mathbf{G}}(F)}$ satisfying $\alpha \circ \alpha^\vee = 1$, which is necessary for Shahidi's local coefficient formula.

In (5.2.3), we apply Shahidi's local coefficient formula for $C_\chi(s,\pi)^{-1}$, not just for a given supercuspidal representation π, but simultaneously for highly ramified twists of π.

Our main reference for this section is [Sh02].

5.2.1 Reductive groups sharing the same derived group

Consider a connected, reductive group $\widetilde{\mathbf{G}}$ over F which contains $\mathbf{G}$ and shares its derived group. In this section only, we will denote the group of rational points of a group $\mathbf{H}$ by the corresponding letter H. We will not do this in general, because later on we will need to consider the group U of upper triangular unipotent matrices in $\mathrm{GL}_n(E)$, and we do not want to confuse this with the group $\mathbf{U}(F)$ introduced earlier.

Lemma 5.2.1.1. *Let $Z_{\widetilde{G}}$ be the center of $\widetilde{G}$. Let π be an irreducible, admissible representation of G, and ω a character of Z_G.*

(i): ω can be extended to a character $\tilde{\omega}$ of $Z_{\widetilde{G}}$.

(ii): If π is an irreducible, admissible representation of G, then there exists an irreducible, admissible representation $\tilde{\pi}$ of $\widetilde{G}$ whose restriction to G contains π as a subrepresentation.

(iii): If π has central character ω, then $\tilde{\pi}$ can be chosen to have central character $\tilde{\omega}$.

Proof: (i): The groups Z_G and $Z_{\widetilde{G}}$ each have unique maximal compact open subgroups K and $\tilde{K}$, with $K = Z_G \cap \tilde{K}$. The restriction of ω to K is unitary, and then extends by Pontryagin duality to a character $\tilde{\omega}$ of $\tilde{K}$. We then extend $\tilde{\omega}$ to a character of $Z_G\tilde{K}$ by setting $\tilde{\omega}(zk) = \tilde{\omega}(z)\omega(k)$. This is well defined, and moreover continuous since the product map $Z_G \times \tilde{K} \to Z_{\widetilde{G}}$ is an open map. Since $\mathbb{C}^*$ is injective in the category of abelian groups, $\tilde{\omega}$ extends to an abstract homomorphism of $Z_{\widetilde{G}}$ into $\mathbb{C}^*$. This extension is automatically continuous, because its restriction to the open subgroup $\tilde{K}$ is continuous.

(ii) and (iii): We first extend π to a representation of $Z_{\widetilde{G}}G$ by setting $\pi(zg) = \tilde{\omega}(z)\pi(g)$. This is smooth and admissible. Since $Z_{\widetilde{G}}G$ is of finite index in $\widetilde{G}$, the smoothly induced representation $\tau = \mathrm{Ind}_{Z_{\widetilde{G}}G}^{\widetilde{G}}\,\pi$ is admissible.

Any irreducible subrepresentation of τ is easily seen to have central character $\tilde{\omega}$. Take a nonzero element in the space of τ and consider the $\widetilde{G}$-subrepresentation W which it generates. Since W is finitely generated and admissible, it is of finite length, and must contain an irreducible subrepresentation W_0.

Now the restriction of W_0 to G is a finite direct sum of irreducible representations of G ([Tad92], Lemma 2.1). Since the map $f \mapsto f(1)$ defines a nonzero intertwining operator from W_0 to the space of π, we see that one of these irreducible representations must be isomorphic to π. □

The group $\widetilde{\mathbf{G}}$ is essentially the same as the group $\mathbf{G}$, but with a larger maximal torus $\widetilde{\mathbf{T}}$, which we may take to be one containing $\mathbf{T}$. It has a maximal split torus $\tilde{\mathbf{S}}$ inside $\widetilde{\mathbf{T}}$. It has a Borel subgroup $\widetilde{\mathbf{B}} = \widetilde{\mathbf{T}}\mathbf{U}$, which defines a set of simple restricted and nonrestricted roots identifiable with those corresponding to the triple $\mathbf{G}, \mathbf{B}, \mathbf{S}$. The root vectors can be taken from $\mathbf{U}$, giving us the exact same canonical Weyl group representatives and generic character χ as before.

Let $\widetilde{\mathbf{P}} = \widetilde{\mathbf{M}}\mathbf{N}$ be the maximal self-associate parabolic subgroup of $\widetilde{\mathbf{G}}$ corresponding to $\mathbf{P}$ (that is, defined by the same set of simple roots), with $\widetilde{\mathbf{M}}$ containing $\widetilde{\mathbf{T}}$. Then $\widetilde{\mathbf{M}}$ and $\mathbf{M}$ also have the same derived group, so we can apply Lemma 5.2.1.1.

Lemma 5.2.1.2. *Let π be a supercuspidal representation of $M = \mathrm{GL}_n(E)$ with central character ω_π. Let $\tilde{\pi}$ be an irreducible, admissible representation of $\widetilde{M}$ whose restriction to M contains π as a subrepresentation, and whose central character $\tilde{\omega}_\pi$ extends that of ω_π.*

(i): $\tilde{\pi}$ is generic, and $C_\chi(s, \pi) = C_\chi(s, \tilde{\pi})$.

(ii): Let W be an element of the Whittaker model of π. Then W extends to an element $\tilde{W}$ in the Whittaker model of $\tilde{\pi}$.

Proof: (i): Let λ be a nonzero χ-Whittaker functional for π. Let V be the underlying space of $\tilde{\pi}$. The restriction of $\tilde{\pi}$ to M is a finite direct sum of irreducible representations, say $V = V_1 \oplus \cdots \oplus V_r$, with $\pi = V_1$. Then the map $\tilde{\lambda} : (v_1, ..., v_r) \mapsto \lambda(v_1)$ is a nonzero Whittaker functional for $\tilde{\pi}$.

Consider the induced representations $I(s, \pi)$ and $I(s, \tilde{\pi})$ of G and $\widetilde{G}$. If $0 \neq f \in I(s, \tilde{\pi})$, then f is a function from $\widetilde{G}$ to V, so we may write $f = (f_1, ..., f_t)$. We see immediately that the restriction of f_1 to G is a nonzero element of $I(s, \pi)$. Considering the intertwining operators $A(s, \pi)$ and $A(s, \tilde{\pi})$, and the Whittaker functionals $\lambda_\chi(s, \pi)$ and $\lambda_\chi(s, \tilde{\pi})$, both defined by integration over $\mathbf{N}(F)$ with the same Weyl group representative $\dot{w}_0$, we see by direct computation that the local coefficient $C_\chi(s, \tilde{\pi})$ satisfies

$$C_\chi(s, \tilde{\pi})\lambda_\chi(-s, \pi) \circ A(s, \pi) f_1 = \lambda(s, \pi) f_1$$

making it equal to $C_\chi(s, \pi)$.

(ii): There exists an element v in the space of π such that $W(m) = \lambda(\pi(m)v)$. We simply define $\tilde{W}(\tilde{m}) = \tilde{\lambda}(\tilde{\pi}(\tilde{m})v)$. □

Now we account for twists of π.

Lemma 5.2.1.3. *Let η be a character of E^*, identified with the character $\eta \circ \det$ of $M = \mathrm{GL}_n(E)$. Let π and $\tilde{\pi}$ be as in Lemma 5.2.1.3. Consider the twisted representation $\pi\eta = \pi(\eta \circ \det)$.*

(i): The character η of M extends to a character $\tilde{\eta}$ of $\widetilde{M}$.

(ii): Consider the twist $\tilde{\pi}\tilde{\eta}$ of $\tilde{\pi}$ by η. Then $\tilde{\pi}\tilde{\eta}$ is generic, and $C_\chi(s, \pi\eta) = C_\chi(s, \tilde{\pi}\tilde{\eta})$.

Proof: (i): By Lemma 5.2.1.1 (ii), there exists an irreducible admissible representation $\tilde{\eta}$ of $\widetilde{M}$ whose restriction to M contains η as a subrepresentation. Since η is a character, it follows from the proof of that lemma that $\tilde{\eta}$ is finite dimensional. Since the derived group of $\widetilde{M}$ is $\mathrm{SL}_n(E)$, $\tilde{\eta}$ must actually be one dimensional, i.e. a character. This follows from the fact that the kernel of $\tilde{\eta}$ is an open normal subgroup of $\widetilde{M}$, and must therefore contain sufficiently small subgroups of the simple root subgroups of M. These generate $\mathrm{SL}_n(E)$, and therefore $\tilde{\eta}$ factors through the abelian quotient

$\widetilde{M}/\operatorname{SL}_n(E)$. We are done, because a finite dimensional irreducible representation of an abelian group must be one dimensional.

(ii) follows from Lemma 5.2.1.2 applied to the representation $\tilde{\pi}\tilde{\eta}$ of $\widetilde{M}$, whose restriction to M contains $\pi\eta$ as a subrepresentation. $\square$

Using Lemma 5.2.1.3 (i), we make the following convention.

Convention 5.2.1.4. *For each character η of $M = \operatorname{GL}_n(E)$, choose once and for all a character $\tilde{\eta}$ of $\widetilde{M}$ which extends η.*

5.2.2 The extended group $\widetilde{\mathbf{G}}$

We will now construct the group $\widetilde{\mathbf{G}}$ of the previous section. The goal is to construct a connected, reductive group $\widetilde{\mathbf{G}}$ over F which contains $\mathbf{G}$, shares its derived group, and has connected and cohomologically trivial center. Then Assumption 5.1 of [Sh02] will be valid for $\widetilde{\mathbf{G}}$, allowing us to construct the injection $\alpha^\vee$ of F^* into $Z_{\widetilde{\mathbf{M}}(F)}/Z_{\widetilde{\mathbf{G}}(F)}$ as in Section 5 of [Sh02]. This will allow us to apply Shahidi's local coefficient formula for $\widetilde{\mathbf{G}}$.

First, define $\widetilde{Z_{\mathbf{G}}} = \operatorname{Res}_{E/F} Z_{\mathbf{G}}$. It is cohomologically trivial by Shapiro's lemma. Since the $\overline{F}$-points of $Z_{\mathbf{G}}$ identifies with $\overline{F}^*$, we can identify the $\overline{F}$-points of $\widetilde{Z_{\mathbf{G}}}$ with $\overline{F}^* \times \overline{F}^*$, and for $z = (x, y) \in \widetilde{Z_{\mathbf{G}}}(\overline{F})$, and $\gamma \in \operatorname{Gal}(\overline{F}/F)$, we have $\gamma.z = (\gamma(x), \gamma(y))$ if $\gamma|_E = 1_E$, and $\gamma.z = (\gamma(y), \gamma(x))$ if $\gamma|_E = \sigma$, where σ is the nontrivial element of $\operatorname{Gal}(E/F)$.

We embed $Z_{\mathbf{G}}$ into $\widetilde{Z_{\mathbf{G}}}$ on $\overline{F}$-points by sending $x \in \overline{F}^*$ to (x, x^{-1}), where we identify xI_{2n} with x. This embedding is defined over F. Let $\mathbf{K}$ be the finite group scheme $\mathbf{G}_{\mathrm{der}} \cap Z_{\mathbf{G}}$, where we have $\mathbf{G}_{\mathrm{der}} = \operatorname{SU}(n, n)$. The product map $\mathbf{G}_{\mathrm{der}} \times_F Z_{\mathbf{G}} \to \mathbf{G}$ induces an isomorphism of algebraic groups

$$\frac{\mathbf{G}_{\mathrm{der}} \times_F Z_{\mathbf{G}}}{\mathbf{K}} \to \mathbf{G}$$

which is defined over F. Here we are regarding $\mathbf{K}$ as a subgroup scheme of $\mathbf{G}_{\mathrm{der}} \times_F Z_{\mathbf{G}}$ on closed points by $x \mapsto (x, x^{-1})$. Since $\mathbf{K} \subset Z_{\mathbf{G}} \subset \widetilde{Z_{\mathbf{G}}}$, we may define in the same way a group $\widetilde{\mathbf{G}}$ by

$$\widetilde{\mathbf{G}} = \frac{\mathbf{G}_{\mathrm{der}} \times_F \widetilde{Z_{\mathbf{G}}}}{\mathbf{K}}.$$

This group contains $\mathbf{G}_{\mathrm{der}}, Z_{\mathbf{G}}$, and $\mathbf{G}$ as subgroup schemes, and by passing to $\overline{F}$-points, we immediately arrive at the following proposition.

Proposition 5.2.2.1. *$\widetilde{\mathbf{G}}$ is a connected, reductive group over F. Its derived group is $\mathbf{G}_{\mathrm{der}}$. The center of $\widetilde{\mathbf{G}}$ is $\widetilde{Z_{\mathbf{G}}}$, and*

$$\widetilde{\mathbf{T}} = \frac{\mathbf{T}^D \times_F \widetilde{Z_{\mathbf{G}}}}{\mathbf{K}}$$

is a maximal torus of $\widetilde{\mathbf{G}}$ which contains $\mathbf{T}$ and is defined over F. Here $\mathbf{T}^D$ is the maximal torus of $\mathbf{G}_{\mathrm{der}}$ whose E-points are the diagonal matrices in $\mathrm{SL}_{2n}(E) = \mathbf{G}_{\mathrm{der}}(E)$.

For the self-associate maximal parabolic subgroup $\widetilde{\mathbf{P}} = \widetilde{\mathbf{M}}\mathbf{N}$ analogous to $\mathbf{P}$ (that is, defined by the same set of simple roots), we have

$$\widetilde{\mathbf{M}} = \frac{\mathbf{M}^D \times_F \widetilde{Z_{\mathbf{G}}}}{\mathbf{K}}$$

where $\mathbf{M}^D$ is the Levi subgroup of $\mathbf{G}_{\mathrm{der}}$ analogous to $\mathbf{M}$. The group $\widetilde{\mathbf{M}}$ has center

$$Z_{\widetilde{\mathbf{M}}} = \frac{Z_{\mathbf{M}^D} \times_F \widetilde{Z_{\mathbf{G}}}}{\mathbf{K}}.$$

Note that since the torus $Z_{\widetilde{\mathbf{G}}} = \widetilde{Z_{\mathbf{G}}}$ is cohomologically trivial by Shapiro's lemma, the inclusion

$$Z_{\widetilde{\mathbf{M}}(F)}/Z_{\widetilde{\mathbf{G}}(F)} = Z_{\widetilde{\mathbf{M}}}(F)/Z_{\widetilde{\mathbf{G}}}(F) \subseteq Z_{\widetilde{\mathbf{M}}}/Z_{\widetilde{\mathbf{G}}}(F)$$

is an equality. We have used the fact that if $\mathbf{H}$ is a reductive group over a field k, then $Z_{\mathbf{H}}(k) = Z_{\mathbf{H}(k)}$. We will require a simple lemma on tori.

Lemma 5.2.2.2. *Identify all groups with their $\overline{F}$-points. Let* $\mathbf{H}$ *be the subtorus* (x, x^{-1}, x^{-1}, x) *of* $\mathbf{G}_m^4$*, and* $\mathbf{K}$ *a finite subgroup of* $\mathbf{H}$ *containing* $c = (-1, -1, -1, -1)$*. Choose for each* $0 \neq x \in \overline{F}$ *a square root* $\sqrt{x}$*, so that*

$$x \mapsto (\sqrt{x}, \frac{1}{\sqrt{x}}, \frac{1}{\sqrt{x}}, \sqrt{x})\mathbf{K}$$

is a well defined homomorphism of abstract groups $\mathbf{G}_m \to \mathbf{H}/\mathbf{K}$*, independent of the choice of* $\sqrt{x}$ *for any* x*. Then this homomorphism is a morphism of varieties.*

Proof: Let $X(\mathbf{H})$ be the character lattice of $\mathbf{H}$. We identify $\mathbf{H}$ with $\mathbf{G}_m = \overline{F}^*$ and use the fact that $\mathbf{H} \mapsto X(\mathbf{H})$ defines an antiequivalence of categories between tori over $\overline{F}$ and finite rank free abelian groups. The surjection $\mathbf{H} \to \mathbf{H}/\mathbf{K}$ corresponds to an inclusion

$$X(\mathbf{H}/\mathbf{K}) \to X(\mathbf{H}).$$

As a finite subgroup of $\overline{F}^*$, $\mathbf{K}$ is cyclic and generated by some root of unity ζ of order d. Since $-1 \in \mathbf{K}$, d must be even. If we let χ be the basis of $X(\mathbf{H})$ sending $x \in \overline{F}^*$ to itself, then $d\chi$ is a basis of $X(\mathbf{H}/\mathbf{K})$. The map in the statement of the lemma is then seen to be the unique morphism of varieties corresponding to the group homomorphism

$$X(\mathbf{H}/\mathbf{K}) \to X(\mathbf{G}_m)$$

$$d\chi \mapsto (d/2)\chi.$$

□

Now, we are going to construct the required injection $\alpha^\vee$ of F^* into $Z_{\widetilde{\mathbf{M}}}(F)/Z_{\widetilde{\mathbf{G}}}(F)$ as in Section 5 of [Sh02]. Let $\mathbf{L} = Z_{\mathbf{M}^D} \times_F \widetilde{Z_{\mathbf{G}}}$. Since

$$Z_{\mathbf{M}^D}(\overline{F}) = \{\begin{pmatrix} xI_n & \\ & x^{-1}I_n \end{pmatrix} : x \in \overline{F}^*\}$$

we can identify $\mathbf{L}(\overline{F})$ with the three dimensional torus (x, x^{-1}, y, z). For the corresponding group $\mathbf{K}$, we then identify $\mathbf{K}(\overline{F}) = \{(x, x^{-1}, x^{-1}, x) : x^{2n} = 1\}$. Then

$$Z_{\widetilde{\mathbf{M}}}(\overline{F}) = \mathbf{L}/\mathbf{K}(\overline{F}) = \frac{\mathbf{L}(\overline{F})}{\mathbf{K}(\overline{F})} = \{(x, x^{-1}, y, z)\mathbf{K}(\overline{F}) : x, y, z \in \overline{F}^*\}.$$

Proposition 5.2.2.3. *For each $x \in \overline{F}^*$, choose once and for all a square root $\sqrt{x}$. Define a map $\mathbf{G}_m(\overline{F}) \to Z_{\widetilde{\mathbf{M}}}(\overline{F})$ by*

$$x \mapsto (\sqrt{x}, \frac{1}{\sqrt{x}}, \frac{1}{\sqrt{x}}, \sqrt{x})\mathbf{K}(\overline{F}).$$

Then this is the map on closed points defined by a cocharacter λ of $Z_{\mathbf{M}}$. It satisfies $\langle \beta, \lambda \rangle = 1$ for the unique $\beta \in \widetilde{\Delta}$ restricting to $\alpha = \epsilon_{n-1} - \epsilon_n$, and $\langle \beta, \lambda \rangle = 0$ for $\beta \in \widetilde{\Delta}$ not restricting to α. The composition

$$\mathbf{G}_m \to Z_{\widetilde{\mathbf{M}}} \to Z_{\widetilde{\mathbf{M}}}/Z_{\widetilde{\mathbf{G}}}$$

maps F-rational points to F-rational, and therefore defines an injection

$$\alpha^\vee : F^* \to Z_{\widetilde{\mathbf{M}}}/Z_{\widetilde{\mathbf{G}}}(F).$$

Proof: Note that $(-1, -1, -1, -1) \in \mathbf{K}(\overline{F})$, so by the previous lemma, λ is a well defined cocharacter. It clearly pairs with the nonrestricted simple roots in the manner described. We finally have to check that if $x \in F^*$, then the image of $\lambda(x)$ in $Z_{\mathbf{G}}(\overline{F})$ is an F-rational point.

The torus $Z_{\widetilde{\mathbf{M}}}$ splits over E, so all its cocharacters are defined over E. The projection $Z_{\widetilde{\mathbf{M}}} \to Z_{\widetilde{\mathbf{M}}}/Z_{\widetilde{\mathbf{G}}}$ is also defined over E. So we just need to check that if $\tau \in \mathrm{Gal}(\overline{F}/F)$, and $\tau|_E \neq 1_E$, then τ fixes the image of $\lambda(x)$ modulo $Z_{\widetilde{\mathbf{G}}}(\overline{F})$. First, using the fact that $\tau(\sqrt{x}) = \pm\sqrt{x}$, that $(-1,-1,-1,-1) \in \mathbf{K}(\overline{F})$, and that τ acts on $\mathbf{M}^D(\overline{F})$ by $\tau.(x,y) = (\tau(y)^{-1}, \tau(x)^{-1})$ we get

$$\begin{aligned}\tau.\lambda(x) &= \pm(\sqrt{x}, \frac{1}{\sqrt{x}}, \sqrt{x}, \frac{1}{\sqrt{x}})\mathbf{K}(\overline{F}) \\ &= (\sqrt{x}, \frac{1}{\sqrt{x}}, \sqrt{x}, \frac{1}{\sqrt{x}})\mathbf{K}(\overline{F}).\end{aligned}$$

Next, $Z_{\widetilde{\mathbf{G}}}(\overline{F})$ embeds into $Z_{\widetilde{\mathbf{M}}}(\overline{F})$ as $(x,y) \mapsto (1,1,x,y)\mathbf{K}(\overline{F})$. For $\lambda(x)$ modulo $Z_{\widetilde{\mathbf{G}}}(\overline{F})$ to be an F-rational point, it suffices to show that $\tau.\lambda(x)$ is congruent to $\lambda(x)$ modulo $Z_{\widetilde{\mathbf{G}}}(\overline{F})$. And this is the case, using the element

$$(1,1,\frac{1}{x},x)\mathbf{K}(\overline{F}) \in Z_{\widetilde{\mathbf{G}}}(\overline{F})$$

and the fact that $\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$ for any $x \in \overline{F}^*$ and any choice of square root of x. $\square$

5.2.3 The local coefficient as a partial Bessel integral

In this section, we let π be an irreducible, supercuspidal representation of $\mathbf{M}(F) = \mathrm{GL}_n(E)$ with central character ω. We will finally apply Shahidi's local coefficient formula to calculate the local coefficient $C_\chi(s,\pi)$ in a way that the analysis in Chapter Four can be applied.

Let $\overline{\mathbf{N}}$ be the unipotent radical of the parabolic subgroup opposite to $\mathbf{P}$. We will need a nice collection of open compact subgroups $\overline{N}_\kappa : \kappa \in \mathbb{Z}$ of $\overline{\mathbf{N}}(F)$ to work with. Note that $\overline{\mathbf{N}}(F)$, like $\mathbf{N}(F)$, identifies with the space of Hermitian matrices in $\mathrm{Mat}_n(E)$:

$$\overline{\mathbf{N}}(F) = \{\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} : X \in \mathrm{Mat}_n(E),\ {}^t\overline{X} = X\}.$$

We will define a collection of open compact neighborhoods $X(\kappa)$ of the identity in $\mathrm{Mat}_n(E)$ whose union is the entire space:

$$X(\kappa) = \begin{pmatrix} (\varpi_F)^{-\kappa} & (\varpi_F)^{-2\kappa} & (\varpi_F)^{-3\kappa} & \cdots \\ (\varpi_F)^{-2\kappa} & (\varpi_F)^{-3\kappa} & & \\ (\varpi_F)^{-3\kappa} & & \ddots & \\ \vdots & & & \end{pmatrix}.$$

Here ϖ_F is a uniformizer for F, and $(\varpi_F) = \varpi_F \mathcal{O}_E$. Of course $(\varpi_F) = \varpi_E \mathcal{O}_E$ if E/F is not ramified, and $(\varpi_F) = \varpi_E^2 \mathcal{O}_E$ if E/F is ramified. Equivalently,

$$X(\kappa) = \{x \in \mathrm{Mat}_n(E) : x_{ij} \in (\varpi_F)^{-\kappa(i+j-1)}\}.$$

We let

$$\overline{N}_\kappa = \{\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} \in \overline{\mathbf{N}}(F) : X \in X(\kappa)\}.$$

Then $\overline{N}_\kappa$ is a sequence of open compact subgroups of $\overline{\mathbf{N}}(F)$ whose union is all of $\overline{\mathbf{N}}(F)$.

Lemma 5.2.3.1. *For $t \in F^*$, $\alpha^\vee(t)\overline{N}_\kappa \alpha^\vee(t)^{-1}$ only depends on $|t|_F$.*

We recall that $\alpha^\vee : F^* \to Z_{\widetilde{\mathbf{M}}}(F)/Z_{\widetilde{\mathbf{G}}}(F)$ was the injection defined in (5.2.2).

Proof: Let $t \in F^*$. Then $\alpha^\vee(t)$ is an element of $Z_{\widetilde{\mathbf{M}}(F)}$ which is only well defined modulo $Z_{\widetilde{\mathbf{G}}(F)}$. However, conjugation by $\alpha^\vee(t)$ is well defined, and coincides with conjugation by the E-rational point

$$\begin{pmatrix} tI_n & \\ & I_n \end{pmatrix} \in Z_{\mathbf{M}}(E)$$

so we see that conjugation by $\alpha^\vee(t)$ of an element X of $\overline{\mathbf{N}}(F)$, identified with a Hermitian matrix, produces the Hermitian matrix $t^{-1}X$, and this only depends on $|t|_F$. $\square$

Recall that in (5.2.1), we chose once and for all an extension of each character η of $\mathbf{M}(F)$ to a character $\tilde{\eta}$ of $\widetilde{\mathbf{M}}(F)$ (Convention 5.2.1.4). Let $\tilde{\pi}$ be an irreducible, generic representation of $\widetilde{\mathbf{M}}(F)$ whose restriction to $\mathbf{M}(F)$ contains π as a subrepresentation and whose central character $\tilde{\omega}$ extends ω (Lemma 5.2.1.2). Then we have an equality of local coefficients

$$C_{\chi}(s, \pi\eta) = C_{\chi}(s, \tilde{\pi}\tilde{\eta})$$

where the left hand side is the local coefficient of $\mathbf{M}$ inside $\mathbf{G}$, and the right hand side is the local coefficient of $\widetilde{\mathbf{M}}$ inside $\widetilde{\mathbf{G}}$.

Let us first compute the character $\tilde{\omega}(\dot{w}_0\tilde{\omega}^{-1})$ (Section 6 of [Sh02]) of F^* which is defined by

$$\tilde{\omega}(\dot{w}_0\tilde{\omega}^{-1})(t) = \tilde{\omega}(\alpha^{\vee}(t)\dot{w}_0^{-1}\alpha^{\vee}(t)^{-1}\dot{w}_0).$$

This is well defined as a character of F^*, even though $\alpha^{\vee}(t) \in Z_{\widetilde{\mathbf{M}}}(F)$ is only well defined modulo $Z_{\widetilde{\mathbf{G}}}(F)$. Here $\dot{w}_0$ was defined in Lemma 5.1.3.2.

Lemma 5.2.3.2. *Let $t \in F^*$. Identifying $\mathbf{M}(F) = \mathrm{GL}_n(E)$, we have*

$$\tilde{\omega}(\dot{w}_0\tilde{\omega}^{-1})(t) = \omega(tI_n).$$

In particular, $\tilde{\omega}(\dot{w}_0\tilde{\omega}^{-1})$ does not depend on the choice of character $\tilde{\omega}$ extending ω.

Proof: Choose any square root $\sqrt{t} \in \overline{F}^*$ of t, and define

$$z = (\sqrt{t}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \sqrt{t})\mathbf{K}(\overline{F}) = [\begin{pmatrix} \sqrt{t}I_n & \\ & \frac{1}{\sqrt{t}}I_n \end{pmatrix}, (\frac{1}{\sqrt{t}}, \sqrt{t})]\mathbf{K}(\overline{F}) \in Z_{\widetilde{\mathbf{M}}}(\overline{F}).$$

Let $z_0 \in Z_{\widetilde{\mathbf{M}(F)}}$ be any F-rational representative modulo $Z_{\widetilde{\mathbf{G}(F)}}$ of $\alpha^{\vee}(t)$. The definition of $\tilde{\omega}(\dot{w}_0\tilde{\omega}^{-1})(t)$ is

$$\tilde{\omega}(z_0\dot{w}_0^{-1}z_0^{-1}\dot{w}_0).$$

By the definition of $\alpha^\vee(t)$, there exists a $g \in Z_{\mathbf{G}}(\overline{F})$ such that $z = z_0 g$. Then $z_0\dot{w}_0^{-1}z_0^{-1}\dot{w}_0 = z\dot{w}_0^{-1}z^{-1}\dot{w}_0$, with

$$z\dot{w}_0^{-1}z^{-1}\dot{w}_0 = [\begin{pmatrix} tI_n & \\ & t^{-1}I_n \end{pmatrix}, (1,1)]\mathbf{K}(\overline{F})$$

which lies in $Z_{\mathbf{M}(F)}$ and identifies with the matrix tI_n in the center of $\mathrm{GL}_n(E)$. □

Let $\mathbf{Z}^0$ be the isomorphic image of F^* under the homomorphism $\alpha^\vee$, and let $z \in \mathbf{Z}^0$. Let n be an element of the open dense subset W of $\mathbf{N}(F)$ defined in Proposition 5.1.4.1, so that the stabilizer $\mathbf{U}_{\mathbf{M},n}(F)$ of n under conjugation by $\mathbf{U}_{\mathbf{M}}(F)$ is trivial. Write $\dot{w}_0^{-1}n = mn'\bar{n}$ as in Lemma 5.1.3.4.

We can write $\dot{w}_0{}^{-1}\bar{n}\dot{w}_0 = n_1$ for $n_1 \in \mathbf{N}(F)$, so that

$$n_1 = \mathbf{x}_\alpha(x_\alpha)\, n''$$

for $x_\alpha \in F^*$ and n'' in the derived group of $\mathbf{U}(F)$. The element x_α lies in F, because the character $\alpha = e_n - e_{2n}$ of $\mathbf{T}$ is defined over F (Proposition/Definition 5.1.1.1 (iii)).

Let us compute the matrices $m, n', \bar{n}$ of (5.1.3) and the element x_α for special $n \in \mathbf{N}(F)$. Recall that both $\mathbf{N}(F)$ and $\overline{\mathbf{N}}(F)$ identify naturally with the space of Hermitian matrices with entries in E. And by Lemma 5.1.3.4, $\dot{w}_0^{-1}n \in \mathbf{P}(F)\overline{\mathbf{N}}(F)$ if and only if the Hermitian matrix corresponding to n is invertible.

Lemma 5.2.3.3. *Let $r' = diag(1, r'_2, ..., r'_n)$ be a diagonal matrix with entries in F^*. Let*

$$n = \begin{pmatrix} I_n & r' \\ & I_n \end{pmatrix} \in \mathbf{N}(F).$$

Then $\dot{w}_0^{-1}n = mn'\bar{n}$ with $m \in \mathbf{M}(F), n' \in \mathbf{N}(F), \bar{n} \in \overline{\mathbf{N}}(F)$, where

(i): If we identify m with a matrix in $\mathrm{GL}_n(E)$, then $m = (-1)^{n-1}Jr'^{-1}$.

(ii): If we identify $\bar{n}$ with a Hermitian matrix, then $\bar{n} = r'^{-1}$.

(iii): The element $x_\alpha \in F^*$ corresponding to n above is -1.

Here J is as in Lemma 5.1.3.2. Proof: (i) and (ii) are immediate from the Lemma 5.1.3.4. For (iii), we first need to compute $n_1 = \dot{w}_0^{-1}\bar{n}\dot{w}_0$. We have

$$n_1 = \dot{w}_0^{-1}\bar{n}\dot{w}_0 = (-1)^n \begin{pmatrix} & J \\ -J & \end{pmatrix} \begin{pmatrix} I_n & \\ r'^{-1} & I_n \end{pmatrix} \begin{pmatrix} & J \\ -J & \end{pmatrix} = \begin{pmatrix} I_n & (-1)^n J r'^{-1} J \\ & I_n \end{pmatrix}$$

where the lower right entry x_α of $(-1)^n J a'^{-1} J$ is easily seen to be -1. $\square$

Now let f be a matrix coefficient of π, and let

$$W^f(m) = \int\limits_{\mathbf{U_M}(F)} f(um)\overline{\chi(u)}\, du = \int\limits_{U} f(xm)\overline{\chi(x)}\, dx \tag{5.2.3.1}$$

where U is the group of upper triangular unipotent matrices in $\mathrm{GL}_n(E)$. Then W^f lies in the Whittaker model of π. It is known that f may be chosen so that W^f is not identically zero (see the proof of Proposition 1.3 of [PaSt08]), so we may choose f so that $W^f(e) = 1$.

By Lemma 5.2.1.2, W^f extends to a function $\widetilde{\mathbf{M}}(F) \to \mathbb{C}$ in the Whittaker model of $\tilde{\pi}$. Also call this extension W^f. Now that we have defined our lengthy notation, we can state Shahidi's local coefficient formula for $C_\chi(s, \tilde{\pi})$.

Theorem 5.2.3.4. *(Shahidi) Let π be an irreducible, supercuspidal representation of $\mathrm{GL}_n(E)$ with central character ω. Let η be a character of E^*, identified as a character of $\mathrm{GL}_n(E)$ through the determinant. Assume that η is sufficiently ramified that the conductor of η^n is greater than that of ω. Then there exists an integer κ_η, depending on η, such that for all $\kappa \geq \kappa_\eta$,*

$$C_\chi(s, \pi\eta)^{-1} = \gamma(s, \omega\eta^n, \psi)^{-1} \int\limits_{\mathbf{Z}^0\mathbf{U_M}(F)\backslash \mathbf{N}(F)} \omega\eta(x_\alpha^{-1})|x_\alpha|_F^{-ns} q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle}$$
$$\eta(m) \int\limits_{\mathbf{U_M}(F)} W^f(mu)\varphi_\kappa(\alpha^\vee(x_\alpha)u^{-1}\bar{n}u\alpha^\vee(x_\alpha^{-1}))\overline{\chi(u)} du d\dot{n}$$

up to a constant depending on the normalization of our measures and on the value of $\omega(-1)^{n+1}$*. Here* φ_κ *is the characteristic function of* $\overline{N}_\kappa$*, and* $\gamma(s, \omega\eta^n, \psi)$ *is the local gamma factor attached to the restriction of the character* $\omega\eta^n$ *to* F^**.*

We have already identified the quotient space $\mathbf{Z}^0\mathbf{U_M}(F)\backslash\mathbf{N}(F)$ in (5.1.4) with the torus R'. Conjugation by $\mathbf{Z}^0$ coincides with the action of F^* given there, and the measure $d\dot{n}$ is the measure dr'.

Proof: This is Theorem 6.2 of [Sh02]. For the convenience of the reader, we will present the proof here. Let $\tilde{V}$ be the underlying space of $\tilde{\pi}$. Our element W^f in the Whittaker model of $\tilde{\pi}$ is a function given by $W^f(\tilde{m}) = \lambda(\tilde{\pi}(\tilde{m})v)$, where λ is a χ-Whittaker functional for $\tilde{\pi}$, and $v \in \tilde{V}$ satisfies $\lambda(v) = 1$.

The Shahidi local coefficient $C_\chi(s, \tilde{\pi}\tilde{\eta}) = C_\chi(s, \pi\eta)$ satisfies

$$C_\chi(s, \pi\eta)\lambda_\chi(-s, \dot{w}_0(\tilde{\pi}\tilde{\eta})) \circ A(s, \tilde{\pi}\tilde{\eta}) = \lambda_\chi(s, \tilde{\pi}\tilde{\eta}) \tag{5.2.3.1}$$

where the right and left hand sides are linear functionals on the induced space $I(s, \tilde{\pi}\tilde{\eta})$. We will choose a nice test function to make the right hand side equal to 1.

Let $f : \overline{\mathbf{N}}(F) \to \tilde{V}$ be any locally constant and compactly supported function with the property that

$$\int\limits_{\overline{\mathbf{N}}(F)} \overline{\chi'(\bar{n})} f(\bar{n}) d\bar{n} = v \tag{5.2.3.2}$$

where $\chi'(\bar{n}) = \chi(\dot{w}_0^{-1}\bar{n}\dot{w}_0)$. This integral can be evaluated by integrating over any $\overline{N}_\kappa$ with κ large enough. There is a unique extension of f to a function on $\widetilde{\mathbf{P}}(F)\overline{\mathbf{N}}(F)$ which lies in the induced space $I(s, \tilde{\pi}\tilde{\eta})$. If we set $h = R_{\dot{w}_0}(f)$, the right translate of f by $\dot{w}_0$, then the Whittaker functional $\lambda_\chi(s, \tilde{\pi}\tilde{\eta})$ is given by

$$\lambda_\chi(s, \tilde{\pi}\tilde{\eta})(h) = \int\limits_{\mathbf{N}(F)} \langle h(\dot{w}_0^{-1}n), \lambda \rangle \overline{\chi}(n) dn = \int\limits_{\overline{\mathbf{N}}(F)} \langle f(\bar{n}), \lambda \rangle \overline{\chi'(\bar{n})} d\bar{n}.$$

Since h is supported inside the big cell $\widetilde{\mathbf{P}}(F)\dot{w}_0\mathbf{N}(F)$, these are absolutely convergent Lebesgue integrals which vanish outside of a compact set. The reader may notice that the character χ' on the right hand side ought to be $\chi(\dot{w}_0\bar{n}\dot{w}_0^{-1})$ after the change of variables. Our computations are correct up to a constant, however, and we will have reason to use χ' as it is. Since $\dot{w}_0^{-2}$ is an element in the center of $\mathbf{M}(F)$, we may write $h(\dot{w}_0^{-1}n) = \omega\eta(\dot{w}_0)^{-2}h(\dot{w}_0 n)$ and note that $\dot{w}_0^{-2} = (-1)^{n+1}I_n \in \mathrm{GL}_n(E)$, with $\omega\eta(\dot{w}_0^{-2}) = \omega(-1)^{n+1}$. Since we are interested in the stability of the local coefficient, a calculation up to constant is all that is needed.

Notice that with our choice of h, we have

$$\lambda_\chi(s, \tilde{\pi}\tilde{\eta})(h) = \langle \int\limits_{\overline{\mathbf{N}}(F)} \overline{\chi'(\bar{n})} f(\bar{n}) d\bar{n}, \lambda\rangle = \langle v, \lambda\rangle = 1.$$

Now we evaluate the left hand side of equation (5.2.3.1) at h. The integrals defining the Whittaker functionals $\lambda_\chi(s, \tilde{\pi}\tilde{\eta})$ generally do not converge as Lebesgue integrals; rather, they are principal value integrals which stabilize over large enough open compact subgroups. To be precise, there exists a κ_0 such that for all $\kappa \geq \kappa_0$,

$$\lambda_\chi(-s, \dot{w}_0(\tilde{\pi}\tilde{\eta})) \circ A(s, \tilde{\pi}\tilde{\eta})(h) = \int\limits_{\overline{N}_\kappa} \langle A(s, \tilde{\pi}\tilde{\eta})(h)(\bar{n}_1), \lambda\rangle \overline{\chi'(\bar{n}_1)} d\bar{n}_1.$$

The κ_0 can be chosen independently of s, and can be enlarged arbitrarily. We can choose κ_0 large enough so that $\overline{N}_{\kappa_0}$ contains the support of the function $\bar{n}_1 \mapsto h(\bar{n}_1)$ on $\overline{\mathbf{N}}(F)$. Now for $\mathrm{Re}(s)$ sufficiently large, the right hand side is equal to

$$\int\limits_{\overline{N}_\kappa}\int\limits_{\mathbf{N}(F)} \langle h(\dot{w}_0^{-1}n\bar{n}_1), \lambda\rangle \overline{\chi'(\bar{n}_1)} dn d\bar{n}_1 = \int\limits_{\overline{N}_\kappa}\int\limits_{\mathbf{N}(F)'} \langle h(\dot{w}_0^{-1}n\bar{n}_1), \lambda\rangle \overline{\chi'(\bar{n}_1)} dn d\bar{n}_1$$

where $\mathbf{N}(F)' = W$ is the open dense subset of $\mathbf{N}(F)$ as in Proposition 5.1.4.1 (ii). For $n \in \mathbf{N}(F)'$, we may write $\dot{w}_0^{-1}n = mn'\bar{n}$ as in Lemma 5.1.3.4, so that this expression is equal to

$$\int\limits_{\overline{N}_\kappa}\int\limits_{\mathbf{N}(F)'} \langle h(mn'\bar{n}\bar{n}_1), \lambda\rangle \overline{\chi'(\bar{n}_1)} dn d\bar{n}_1 = \int\limits_{\mathbf{N}(F)'} \langle \int\limits_{\overline{N}_\kappa} \overline{\chi'(\bar{n}_1)} h(mn'\bar{n}\bar{n}_1) d\bar{n}_1, \lambda\rangle dn.$$

Now in the inner integral, we may change variables $\bar{n}_1 \mapsto \bar{n}^{-1}\bar{n}_1$, as long as $\bar{n}$ lies in $\overline{N}_\kappa$. Let φ_κ be the characteristic function of $\overline{N}_\kappa$. Since we are assuming that the support of the function $\bar{n}_1 \mapsto h(\bar{n}_1)$ is contained in $\overline{N}_\kappa$, we have

$$\begin{aligned}\int\limits_{\overline{N}_\kappa} \overline{\chi'(\bar{n}_1)} h(mn'\bar{n}\bar{n}_1)d\bar{n}_1 &= \varphi_\kappa(\bar{n})\chi'(\bar{n}) \int\limits_{\overline{N}_\kappa} \overline{\chi'(\bar{n}_1)} h(mn'\bar{n}_1)d\bar{n}_1 \\ &= \varphi_\kappa(\bar{n})\chi'(\bar{n})\tilde{\pi}(m)\tilde{\eta}(m)q^{\langle s\tilde{\alpha}+\rho, H_{\tilde{M}}(m)\rangle} \int\limits_{\overline{N}_\kappa} \overline{\chi'(\bar{n}_1)} h(\bar{n}_1)d\bar{n}_1 \\ &= \varphi_\kappa(\bar{n})\chi'(\bar{n})\tilde{\pi}(m)\tilde{\eta}(m)q^{\langle s\tilde{\alpha}+\rho, H_{\tilde{M}}(m)\rangle} v.\end{aligned}$$

We have used the fact that h lies in the induced space $I(s, \tilde{\pi}\tilde{\eta})$, as well as equation (5.2.3.3). Right before the statement of Theorem 5.2.3.4, we had defined W^f to be the element of the Whittaker model of $\tilde{\pi}$ extending the function of the same name on $\mathbf{M}(F)$. Since $W^f(m) = \lambda(\tilde{\pi}(m)v)$, we may put this together with equation (5.2.3.1) and say that

$$C_\chi(s, \pi\eta)^{-1} = \int\limits_{\mathbf{N}(F)'} \tilde{\eta}(m)W^f(m)\varphi_\kappa(\bar{n})q^{\langle s\tilde{\alpha}+\rho, H_{\tilde{M}}(m)\rangle}\chi'(\bar{n})dn.$$

We next expand this integration over its $\mathbf{U_M}(F)$-orbits as in (5.1.4). The torus R in that section identifies with the quotient space $\mathbf{U_M}(F)\backslash\mathbf{N}(F)$ of $\mathbf{N}(F)'$ under the conjugation action of $\mathbf{U_M}(F)$. Let $d\dot{n}$ be the measure on $\mathbf{U_M}(F)\backslash\mathbf{N}(F)$ of Proposition 5.1.4.1. Note that if $n \in \mathbf{N}(F)'$ is replaced by unu^{-1}, then $\bar{n}$ changes to $u\bar{n}u^{-1}$ and m changes to $\dot{w}_0(u)mu^{-1}$, where $\dot{w}_0(u) = \dot{w}_0^{-1}u\dot{w}_0$. Then

$$\begin{aligned}C_\chi(s, \pi\eta)^{-1} = \int\limits_{\mathbf{U_M}(F)\backslash\mathbf{N}(F)} \int\limits_{\mathbf{U_M}(F)} &\tilde{\eta}(\dot{w}_0(u)mu^{-1})W^f(\dot{w}_0(u)mu^{-1})\varphi_\kappa(u\bar{n}u^{-1}) \\ &q^{\langle s\tilde{\alpha}+\rho, H_{\tilde{M}}(\dot{w}_0(u)mu^{-1})\rangle}\chi'(u\bar{n}u^{-1})dud\dot{n}.\end{aligned}$$

We can immediately simplify this formula. By compatibility of $\dot{w}_0$ with χ (equation (5.1.3.1)), the Whittaker functional property of W^f, and the triviality of $\tilde{\eta}$ and $H_{\tilde{M}}$ on unipotent elements, we can rewrite this as

$$C_\chi(s,\pi\eta)^{-1} = \int\limits_{\mathbf{U}_{\mathbf{M}}(F)\backslash\mathbf{N}(F)} \int\limits_{\mathbf{U}_{\mathbf{M}}(F)} \tilde{\eta}(m)W^f(mu^{-1})\varphi_\kappa(u\bar{n}u^{-1})\chi(u)$$
$$q^{\langle s\tilde{\alpha}+\rho, H_{\tilde{M}}(m)\rangle}\chi'(\bar{n})\, du d\dot{n}.$$

Now we apply Proposition 5.1.4.2 to further expand this integration over the orbits of $F^* \cong \mathbf{Z}^0_M$. We recall that the measure on F^* given in that proposition is $|t|_F^{n^2}$ times the Haar measure on F^*. It is easy to see that $|t|_F^{n^2} = q^{\langle 2\rho, H_{\tilde{M}}(\alpha^\vee(t))\rangle}$. For $z = \alpha^\vee(t) \in \mathbf{Z}^0$, note that when n is replaced by znz^{-1}, m changes to $\dot{w}_0(z)mz^{-1}$ and $\bar{n}$ changes to $z\bar{n}z^{-1}$:

$$\int\limits_{\mathbf{Z}^0\mathbf{U}_{\mathbf{M}}(F)\backslash\mathbf{N}(F)} \int\limits_{\mathbf{Z}^0} \int\limits_{\mathbf{U}_{\mathbf{M}}(F)} \tilde{\eta}(\dot{w}_0(z)mz^{-1})W^f(\dot{w}_0(z)mz^{-1}u^{-1})\varphi_\kappa(uz\bar{n}z^{-1}u^{-1})\chi(u)$$
$$q^{\langle s\tilde{\alpha}+\rho, H_{\tilde{M}}(\dot{w}_0(z)mz^{-1})\rangle}\chi'(z\bar{n}z^{-1})du\, q^{\langle 2\rho, H_{\tilde{M}}(z)\rangle}d^*z\, d\dot{n}.$$

Since z lies in the center of $\widetilde{\mathbf{M}}(F)$ modulo $Z_{\tilde{\mathbf{G}}(F)}$, we can write $uz\bar{n}z^{-1}u^{-1} = zu\bar{n}u^{-1}z^{-1}$. It is easy to see that for all $z = \alpha^\vee(t)$,

$$q^{\langle \nu, H_{\tilde{M}}(\dot{w}_0(z))\rangle} = q^{\langle -\nu, H_{\tilde{M}}(z)\rangle}$$

for all $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$, so we will have a cancellation of modulus characters. It follows from this and Lemma 5.1.3.1 (ii) that

$$q^{\langle s\tilde{\alpha}, H_{\tilde{M}}(\dot{w}_0(z)z^{-1})\rangle} = q^{\langle -2s\tilde{\alpha}, H_{\tilde{M}}(\alpha^\vee(t))\rangle} = |t|_F^{-ns}.$$

We also have that

$$\tilde{\eta}(\dot{w}_0(z)mz^{-1})W^f(\dot{w}_0(z)mz^{-1}u^{-1}) = \tilde{\omega}\tilde{\eta}(\dot{w}_0(z)z^{-1})\eta(m)W^f(mu^{-1})$$

with $\tilde{\omega}\tilde{\eta}(\dot{w}_0(z)z^{-1}) = \omega\eta(t^{-1}I_n)$ by Lemma 5.2.3.2. If x_α is the entry of $\bar{n}$ at the simple root α, then $\chi'(z\bar{n}z^{-1}) = \psi(t^{-1}x_\alpha)$. Taking this all into account, we may write $C_\chi(s,\pi\eta)^{-1}$ as

$$\int\limits_{\mathbf{Z}^0\mathbf{U}_\mathbf{M}(F)\backslash\mathbf{N}(F)} \int\limits_{F^*} \int\limits_{\mathbf{U}_\mathbf{M}(F)} \omega\eta(t^{-1}I_n)|t|_F^{-ns} q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle}\eta(m)W^f(mu^{-1})$$
$$\varphi_\kappa(\alpha^\vee(t)u\bar{n}u^{-1}\alpha^\vee(t)^{-1})\chi(u)\psi(t^{-1}x_\alpha)\,du\,d^*t\,d\dot{n}.$$

Change u to u^{-1} and t to $t^{-1}x_\alpha$, so that if we let $\Theta(t)$ be the double integral

$$\int\limits_{\mathbf{Z}^0\mathbf{U}_\mathbf{M}(F)\backslash\mathbf{N}(F)} \omega\eta(x_\alpha^{-1})|x_\alpha|_F^{-ns} q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle}\eta(m)$$
$$\int\limits_{\mathbf{U}_\mathbf{M}(F)} W^f(mu)\varphi_\kappa(\alpha^\vee(t^{-1}x_\alpha)u^{-1}\bar{n}u\alpha^\vee(tx_\alpha^{-1}))\overline{\chi(u)}du d\dot{n}$$

then we will have

$$C_\chi(s,\pi\eta)^{-1} = \int\limits_{F^*} \omega\eta(tI_n)|t|_F^{ns}\psi(t)\Theta(t)d^*t.$$

Let us identify the character η of $\mathbf{M}(F) = \mathrm{GL}_n(E)$ as a character of E^* through the determinant, so that $\omega\eta(tI_n) = \omega\eta^n(t)$. It follows from Lemma 5.2.3.1 that $\Theta(t)$ only depends on the absolute value of t. Therefore, we may expand the integral over F^* as a sum over the annuli of elements of F of constant absolute value:

$$C_\chi(s,\pi\eta)^{-1} = \sum_{k\in\mathbb{Z}} \Theta(\omega_F^k) \int\limits_{\mathrm{ord}_F(t)=k} \omega\eta^n(t)|t|_F^{ns}\psi(t)d^*t.$$

Let f be the conductor of ψ, and d the conductor of $\omega\eta^n$. Since $\omega\eta^n$ is ramified, we may apply equation (1.1.13.1) from Chapter One, which tells us that all of the terms in the sum are zero except for $k = -d - f$. Up to a constant depending on $\omega(-1)^{n+1}$ and our choice of measures, we obtain

$$C_\chi(s, \pi\eta)^{-1} = \gamma(s, \omega\eta^n, \psi)^{-1}\Theta(\omega_F^{-d-f}).$$

Now inside $\Theta(\omega_F^{-d-f})$, the only dependence on ω_F^{-d-f} is in the scaling of the characteristic function φ_κ. Recalling that our formula holds for all κ greater than or equal to a given κ_0, we may simply adjust κ_0 to a new constant, say κ_η, such that the formula

$$C_\chi(s, \pi\eta)^{-1} = \gamma(s, \omega\eta^n, \psi)^{-1} \int\limits_{\mathbf{Z}^0\mathbf{U_M}(F)\backslash\mathbf{N}(F)} \omega\eta(x_\alpha^{-1})|x_\alpha|_F^{-ns} q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle}$$
$$\eta(m) \int\limits_{\mathbf{U_M}(F)} W^f(mu)\varphi_\kappa(\alpha^\vee(x_\alpha)u^{-1}\bar{n}u\alpha^\vee(x_\alpha^{-1}))\overline{\chi(u)}dud\dot{n}$$

holds up to constant for all $\kappa \geq \kappa_\eta$. $\square$

5.3 Proof of Theorem 5.1.3.3

Having successfully applied Shahidi's local coefficient formula to write $C_\chi(s, \pi)^{-1}$ as an integral over the torus $R' = \mathbf{Z}^0\mathbf{U_M}(F)\backslash\mathbf{N}(F)$ in Theorem 5.2.3.4, we now analyze the integrand, which is itself an integral over $\mathbf{U_M}(F)$. We will realize this integrand as a partial Bessel integral, whose asymptotic behavior we investigated in Chapter Four. We will apply the main theorem of Chapter Four (Theorem 4.4.0.1), to prove Theorem 5.1.3.3.

The integral over $\mathbf{Z}^0\mathbf{U_M}(F)\backslash\mathbf{N}(F)$ is informally called a "Mellin transform."

In (5.3.1), we briefly review some definitions and notation from Chapter Four, and then we restate Theorem 5.2.3.4 in that notation. In (5.3.2), we complete the proof of the main theorem of this chapter, Theorem 5.1.3.3.

5.3.1 Review of partial Bessel integrals

Let $G = \mathrm{GL}_n(E) = \mathbf{M}(F)$, B and A the usual Borel subgroup and maximal torus of G, and U the unipotent radical of B. Let $A' = \{(1, a_2, ..., a_n) \in A\}$, so that A is the direct product of A' and the center Z of G. If $a \in A$, let a' be the element of A' obtained by "stripping off the center" of a (4.1.5), so that $a = a'z$ for $z \in Z$.

Let $W(G)$ be the Weyl group of G. For $w \in W(G)$, we keep our Weyl group representatives $\dot{w}$ from (5.1.3). Note that our representatives and generic character of (5.1.3) coincide with those of Chapter Four (4.1.1). For $g \in G$, there is a unique $w \in W(G)$ such that g lies in the Bruhat cell $C(w) = BwB$. For a locally closed subset S of G containing Z, define $\mathscr{C}_c^\infty(S;\omega)$ to be the space of locally constant functions $f : S \to \mathbb{C}$ which are compactly supported modulo Z and which satisfy $f(zg) = \omega(z)f(g)$ for $z \in Z$ and $g \in G$.

Let $f \in \mathscr{C}_c^\infty(G;\omega)$. For example, f could be a matrix coefficient of π, because π is supercuspidal. Define a map $W^f : G \to \mathbb{C}$ by

$$W^f(g) = \int\limits_U f(xg)\overline{\chi(x)}\,dx$$

where χ is the restriction to $U = \mathbf{U}_\mathbf{M}(F)$ of our generic character of $\mathbf{U}(F)$. This integral converges absolutely (Proposition 4.2.2.1), and there exists a choice of f such that W^f is not identically zero. In fact, there exist matrix coefficients f of π so that $W^f(e) = 1$ (see the proof of Proposition 1.3 of [PaSt08]).

Now U acts on G on the right by $g.u = \dot{w}_G\,{}^t\overline{u}\dot{w}_G^{-1}gu$, where $w_G = w_\ell^\theta$ is the long element of G. Let U_g be the stabilizer of a given $g \in G$ under this action, and let φ be the characteristic function of an open compact subset of $\mathrm{Mat}_n(E)$. We define the **partial Bessel integral** $B_\varphi^G(g, f)$ by

$$B_\varphi^G(g, f) = \int\limits_{U_g\backslash U} W^f(ug)\varphi({}^t\overline{u}\dot{w}_G^{-1}g'u)\overline{\chi(u)}\,du$$

where g' is the element of obtained from g by "stripping off the center" (4.1.5). The integral converges absolutely, on account of the fact that f is compactly supported modulo Z, and that for a p-adic field k, the k-points of orbits of unipotent groups over k acting on affine k-varieties are closed (4.2.2).

We shall now rewrite the formula in Theorem 5.2.3.4. We write that formula again here, up to a constant:

$$C_\chi(s,\pi\eta)^{-1} = \gamma(s,\omega\eta^n,\psi)^{-1} \int\limits_{\mathbf{Z}^0\mathbf{U_M}(F)\backslash\mathbf{N}(F)} \omega\eta(x_\alpha^{-1})|x_\alpha|_F^{-ns} q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle}$$

$$\eta(m) \int\limits_{\mathbf{U_M}(F)} W^f(mu)\varphi_\kappa(\alpha^\vee(x_\alpha)u^{-1}\bar{n}u\alpha^\vee(x_\alpha^{-1}))\overline{\chi(u)}du d\dot{n}.$$

By the results of (5.1.4), we may identify $\mathbf{Z}^0\mathbf{U_M}(F)\backslash\mathbf{N}(F)$ with the space R' of matrices of the form $\mathrm{diag}(1, r_2', ..., r_n')$ with $r_i' \in F^*$. The measure $d\dot{n} = dr'$ is then the measure

$$dr' = \prod_{i=2}^n |r_i'|_F^{2i-1} d^* r_i'$$

where d^*r_i' is the usual Haar measure $\frac{dr_i}{|r_i|_F}$ on $R' = (F^*)^{n-1}$. If $n \in \mathbf{N}(F)$ corresponds to r', i.e.

$$n = \begin{pmatrix} I_n & r' \\ & I_n \end{pmatrix}$$

then writing $\dot{w}_0^{-1}n = mn'\bar{n}$, we have $m = (-1)^{n-1}Jr'^{-1} = (-1)^{n-1}\dot{w}_G r'^{-1}$, $\bar{n} = r'^{-1}$, and $x_\alpha = -1$ (Lemma 5.2.4.3). Note that the matrix J of (5.1.3) is equal to $\dot{w}_G$, the representative of the long Weyl group element in $\mathrm{GL}_n(E)$.

Lemma 5.3.1.1. *With $n, m, \bar{n}$ as above, we have*

(i): $\omega\eta^n(x_\alpha) = \pm 1$

(ii): $q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle} = \prod_{i=2}^n |r_i'|_F^{-(s+n)}$.

Proof: (i) is on account of the fact that $x_\alpha = -1$. (ii) follows from Lemma 5.1.3.1. $\square$

Now we look at inner integral over $\mathbf{U_M}(F) = U$. Since $x_\alpha = \pm 1$, it does not affect the scaling of φ_κ, giving us

$$\varphi_\kappa(\alpha^\vee(x_\alpha)u^{-1}\bar{n}u\alpha^\vee(x_\alpha^{-1})) = \varphi_\kappa(u^{-1}\bar{n}u).$$

If we identify $\overline{\mathbf{N}}(F)$ with the Hermitian matrices in $\mathrm{Mat}_n(E)$, and $\mathbf{U_M}(F)$ with the group U of upper triangular unipotent matrices with entries in E, then $u^{-1}\bar{n}u$ is simply

$${}^t\overline{u}r'^{-1}u = {}^t\overline{u}\dot{w}_G^{-1}\dot{w}_G r'^{-1}u.$$

Writing $W^f(mu) = W^f((-1)^{n-1}r'^{-1}u) = \omega(-1)^{n-1}W^f(r'^{-1}u)$, we see that up to a constant of $\omega(-1)^{n-1} = \pm 1$, we have

$$\begin{aligned}\int\limits_{\mathbf{U_M}(F)} W^f(mu)\varphi_\kappa(u^{-1}\bar{n}u)\overline{\chi(u)}du &= \int\limits_U W^f(\dot{w}_G r'^{-1}u)\varphi_\kappa({}^t\overline{u}\dot{w}_G^{-1}(\dot{w}_G r'^{-1})u)\overline{\chi(u)}du \\ &= B^G_{\varphi_\kappa}(\dot{w}_G r'^{-1})\end{aligned}$$

where φ_κ is the characteristic function of $X(\kappa) = \overline{N}_\kappa$. Absorbing the constants ± 1 into the local gamma factor, and combining everything together, we get

$$C_\chi(s,\pi\eta)^{-1} = \gamma_\eta(s)\int\limits_{R'} \eta(r'^{-1})B^G_{\varphi_\kappa}(\dot{w}_G r'^{-1}, f)\prod_{i=2}^n |r'_i|^{-s-n+2i-1}d^*r'_i.$$

Finally making the change of variables $r' \mapsto r'^{-1}$, we arrive at the following reformulation of Theorem 5.2.3.4:

Proposition 5.3.1.2. *Let π be an irreducible, supercuspidal representation of $\mathrm{GL}_n(E)$ with central character ω. Let f be a matrix coefficient of π with $W^f(e) = 1$. Let η be a character of E^*, identified with a character of $\mathrm{GL}_n(E)$ through the determinant. If η is sufficiently highly ramified, then there exists an integer κ_η, depending on η, such that*

$$C_\chi(s,\pi\eta)^{-1} = \gamma_\eta(s)\int\limits_{R'} \eta(r')B^G_{\varphi_\kappa}(\dot{w}_G r', f)\prod_{i=2}^n |r'_i|^{s+n+1-2i}d^*r'_i$$

for all $\kappa \geq \kappa_\eta$. Here γ_η is an entire function depending only on ω and η, not on π.

Remark 5.3.1.3. *Proposition 4.3 of [CoShTs17] has a similar formula for the symmetric square local coefficient in terms of a partial Bessel integral. The proof is along similar lines to ours. In their formula, they state that the equality holds for all κ sufficiently large and uniformly over all twists η, while we do not have a statement of a uniform κ working over all η. To establish their uniformity, they go into the proof of Theorem 6.2 of [Sh01] (which we essentially reproved in Theorem 5.2.3.4) to give a uniform open compact subgroup $\overline{N}_\kappa$ over all η, over which the p-adic Whittaker functionals can be calculated as principal value integrals.*

However, the argument given there is incorrect. In fact, how the requisite open compact subgroup $\overline{N}_\kappa$ changes as η becomes more highly ramified seems to be a subtle and interesting question. For one thing, just because the support of h is very small modulo $\widetilde{\mathbf{P}}(F)$, does not mean the requisite $\overline{N}_\kappa$ can be chosen small.

Fortunately, the uniformity of κ over η is not necessary for the stability, either for their case or ours.

We may investigate the change in $\overline{N}_\kappa$ with respect to the change in ramified character η in the future.

5.3.2 Bessel function asympotics

Proposition 5.3.1.2 expresses the local coefficient $C_\chi(s, \pi\eta)^{-1}$ in terms of the partial Bessel integral. Now we can apply the main theorem of Chapter Four to complete the proof of supercuspidal analytic stability (Theorem 5.1.3.3, or equivalently Proposition 3.2.2.8). We will review the relevant notation and the main result, Theorem 4.0.1, of Chapter Four.

Let M be a standard Levi subgroup of G. For the standard maximal torus A of G, the product $(A \cap M_{\mathrm{der}})Z_M$ is an open and finite index subgroup of A. The

Levi subgroup M is a block diagonal sum of copies of $\mathrm{GL}_{n_i} : 1 \leq i \leq t$, with $n_1 + \cdots + n_t = n$. Let

$$Z'_M = \{ \begin{pmatrix} I_{n_1} & & & \\ & a_2 I_{n_2} & & \\ & & \ddots & \\ & & & a_t I_{n_t} \end{pmatrix} : a_i \in E^* \}.$$

Here is the main result we proved in Chapter Four. We fix once and for all an auxiliary function $f_0 \in \mathscr{C}_c^\infty(G;\omega)$ with $W^{f_0}(e) = 1$.

Theorem 4.0.1. *Let $f \in \mathscr{C}_c^\infty(G;\omega)$ with $W^f(e) = 1$. Then there exists an integer κ_0, a function $f_1 \in \mathscr{C}_c^\infty(G;\omega)$, and for each proper standard Levi subgroup M of G a function $f_M \in \mathscr{C}_c^\infty(G;\omega)$, such that the following hold for all $\kappa \geq \kappa_0$:*

(i): For all $a \in A$,

$$B^G_{\varphi_\kappa}(\dot{w}_G a, f) = B^G_{\varphi_\kappa}(\dot{w}_G a, f_1) + \sum_M B^G_{\varphi_\kappa}(\dot{w}_G a, f_M)$$

where the sum is over the proper standard Levi subgroups of G.

(ii): The function f_1 depends only on the auxiliary function f_0 and the central character ω of π.

(iii): For each proper standard Levi subgroup M of G, the function $B^G_\varphi(\dot{w}_G a, f_M)$ vanishes for $a \notin (A \cap M_{\mathrm{der}})Z_M$, and there exists an open compact subgroup H_M of Z'_M such that

$$B^G_\varphi(\dot{w}_G b c' c'', f_M) = B^G_\varphi(\dot{w}_G b c', f_M)$$

for all $b \in A \cap M_{\mathrm{der}}, c' \in Z'_M, c'' \in H$.

5.3.3 Finishing the proof of supercuspidal analytic stability

Now we prove Theorem 5.3.1.2. Let π_1 and π_2 be two irreducible, supercuspidal representations of G with the same central character ω. Let f^1 and f^2 be matrix coefficients of π_1 and π_2, respectively, such that $W^{f^1}(e) = W^{f^2}(e) = 1$.

For each sufficiently highly ramified character η of E^*, identified with a character of G, Proposition 5.3.1.2 tells us that there exists an integer κ_η, depending on η, such that

$$C_\chi(s, \pi_j \eta)^{-1} = \gamma_\eta(s) \int\limits_{R'} \eta(r') B^G_{\varphi_\kappa}(\dot{w}_G r', f^j) \prod_{i=2}^{n} |r'_i|^{s+n+1-2i} d^* r'$$

for $j = 1, 2$ and all $\kappa \geq \kappa_\eta$. Note that κ_η may become arbitrarily large as η is taken ever more ramified.

We apply Theorem 4.0.2 to each of the functions f^1 and f^2. There exist integers κ_0^1, κ_0^2, functions $f_1^1, f_1^2 \in \mathscr{C}_c^\infty(G; \omega)$, and for each proper standard Levi subgroup M of G functions $f_M^1, f_M^2 \in \mathscr{C}_c^\infty(G; \omega)$ and open compact subgroups H_M^1, H_M^2 of Z'_M such that the conditions of Theorem 4.0.2 hold for each. In particular, if we set $\kappa_0 = \max\{\kappa_0^1, \kappa_0^2\}$, then

$$B^G_{\varphi_\kappa}(\dot{w}_G a, f^1) = B^G_{\varphi_\kappa}(\dot{w}_G a, f_1^1) + \sum_M B^G_{\varphi_\kappa}(\dot{w}_G a, f_M^1)$$

$$B^G_{\varphi_\kappa}(\dot{w}_G a, f^2) = B^G_{\varphi_\kappa}(\dot{w}_G a, f_1^2) + \sum_M B^G_{\varphi_\kappa}(\dot{w}_G a, f_M^2)$$

for all $\kappa \geq \kappa_0$ and all $a \in A$.

Now we compute $C_\chi(s, \pi_1 \eta)^{-1} - C_\chi(s, \pi_2 \eta)^{-1}$ as the integral

$$\gamma_\eta(s) \int\limits_{R'} \eta(r') \Big(B^G_{\varphi_\kappa}(\dot{w}_G r', f^1) - B^G_{\varphi_\kappa}(\dot{w}_G r', f^2) \Big) \prod_{i=2}^{n} |r_i|^{s+n+1-2i} d^* r'.$$

This holds for all $\kappa \geq \kappa_\eta$. If moreover $\kappa \geq \kappa_0$, then we can write the difference of the local coefficients as

$$\gamma_\eta(s) \int\limits_{R'} \eta(r')\Big(B^G_{\varphi_\kappa}(\dot{w}_G r', f_1^1) - B^G_{\varphi_\kappa}(\dot{w}_G r', f_1^2)\Big) \prod_{i=2}^{n} |r_i|^{s+n+1-2i} d^* r'$$
$$+\gamma_\eta(s) \sum_M \int\limits_{R'} \eta(r')\Big(B^G_{\varphi_\kappa}(\dot{w}_G r', f_M^1) - B^G_{\varphi_\kappa}(\dot{w}_G r', f_M^2)\Big) \prod_{i=2}^{n} |r_i|^{s+n+1-2i} d^* r'.$$

Now the functions f_1^1 and f_1^2 depend only on the auxiliary function f_0 and on the central character ω, not on π_1 and π_2. So $f_1^1 = f_1^2$, making the first integral zero. Now we consider the integrals

$$\int\limits_{R'} \eta(r')\Big(B^G_{\varphi_\kappa}(\dot{w}_G r', f_M^1) - B^G_{\varphi_\kappa}(\dot{w}_G r', f_M^2)\Big) \prod_{i=2}^{n} |r_i|^{s+n+1-2i} d^* r' \tag{5.3.3.1}$$

as M ranges over the proper standard Levi subgroups of G. For each such M, let $H_M = H_M^1 \cap H_M^2$. Keep in mind that as long as $\kappa \geq \kappa_0$, H works uniformly for all κ. Let

$$R_M = (A \cap M_{\mathrm{der}}) Z_M \cap R'$$

$$Z'_M(F) = Z'_M \cap \mathrm{GL}_n(F)$$

Since $(A \cap M_{\mathrm{der}}) Z_M$ is open in A, R_M is an open subgroup of R', and $Z'_M(F)$ is a closed subgroup of R_M. Each function $B^G_\varphi(\dot{w}_G r', f_M^i)$ vanishes for r' outside R_M, so we may rewrite (5.3.3.1) as an integral over R_M, and expand it as a double integral over $R_M/Z'_M(F)$ and $Z'_M(F)$:

$$\int\limits_{R_M/Z'_M(F)} \int\limits_{Z'_M(F)} \eta(r'x')\Big(B^G_{\varphi_\kappa}(\dot{w}_G r'x', f_M^1) - B^G_{\varphi_\kappa}(\dot{w}_G r'x', f_M^2)\Big)$$
$$\prod_{i=2}^{n} |r_i x'_i|^{s+n+1-2i} d^* x' d^* \bar{r'}.$$

For each $\bar{r'} \in R_M/Z'_M(F)$, and its representative $r' \in R_M$, we may write $r' = bc$ for some $b \in A \cap M_{\mathrm{der}}$ and $c \in Z_M$. Even though r' has coefficients in F, b and c may not. Write $c = zc'$ for $z \in Z_G$ and $c' \in Z'_M$. Since

$$B^G_{\varphi_\kappa}(\dot{w}_G r' x', f^i_M) = B^G_{\varphi_\kappa}(\dot{w}_G bzc' x', f^i_M) = \omega(z) B^G_{\varphi_\kappa}(\dot{w}_G bc' x', f^i_M)$$

we may write the double integral as

$$\int\limits_{R_M/Z'_M(F)} \omega(z)\eta(c)\Bigg[\int\limits_{Z'_M(F)} \eta(x')\Big(B^G_{\varphi_\kappa}(\dot{w}_G bc'x', f^1_M) - B^G_{\varphi_\kappa}(\dot{w}_G bc'x', f^2_M)\Big) \prod_{i=2}^{n} |x'_i|^{s+n+1-2i} d^*x'\Bigg] \prod_{i=2}^{n} |r'_i|^{s+n+1-2i} d^*\bar{r'}.$$

The subgroup $H_M \cap Z'_M(F)$ is open and compact inside $Z'_M(F)$.

Lemma 5.3.3.1. *If η is sufficiently highly ramified so that it is nontrivial on $H_M \cap Z'_M(F)$, then the inner integral*

$$D = \int\limits_{Z'_M(F)} \eta(x')\Big(B^G_{\varphi_\kappa}(\dot{w}_G bc'x', f^1_M) - B^G_{\varphi_\kappa}(\dot{w}_G bc'x', f^2_M)\Big) \prod_{i=2}^{n} |x'_i|^{s+n+1-2i} d^*x'$$

is zero for all $\kappa \geq \kappa_0$.

Proof: Let x'' be an element of $H_M \cap Z'_M(F)$ with $\eta(x'') \neq 1$. We change variables $x' \mapsto x'x''$, writing D as

$$\int\limits_{Z'_M(F)} \eta(x'x'')\Big(B^G_{\varphi_\kappa}(\dot{w}_G bc'x'x'', f^1_M) - B^G_{\varphi_\kappa}(\dot{w}_G bc'x'x'', f^2_M)\Big) \prod_{i=2}^{n} |x'_i x''_i|^{s+n+1-2i} d^*x'.$$

Since $H_M \cap Z'_M(F)$ is compact, the entries x''_i of x'' are units, so we have $|x''_i| = 1$. Since also

$$B^G_{\varphi_\kappa}(\dot{w}_G bc'x'x'', f^j_M) = B^G_{\varphi_\kappa}(\dot{w}_G bc'x', f^j_M)$$

for $j = 1, 2$ and all $\kappa \geq \kappa_0$, we obtain $D = \eta(x'')D$, so $D = 0$. $\square$

The lemma shows that if η is sufficiently highly ramified to be nontrivial on all the subgroups $H_M \cap Z'_M(F)$ over all the proper standard Levi subgroups M of G, then

$$\gamma_\eta(s) \int\limits_{R'} \eta(r') \Big(B^G_{\varphi_\kappa}(\dot{w}_G r', f^1) - B^G_{\varphi_\kappa}(\dot{w}_G r', f^2) \Big) \prod_{i=2}^{n} |r_i|^{s+n+1-2i} d^* r' = 0$$

for all $\kappa \geq \max\{\kappa_\eta, \kappa_0\}$. This shows that

$$C_\chi(s, \pi_1 \eta)^{-1} - C_\chi(s, \pi_2 \eta)^{-1} = 0$$

for such η, which gives us $C_\chi(s, \pi_1 \eta) = C_\chi(s, \pi_2 \eta)$. This completes the proof of Theorem 5.1.3.3, and therefore the proof of Proposition 3.2.2.8.

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