SPECTRA OF COMPOSITION OPERATORS ON THE UNIT BALL IN TWO COMPLEX VARIABLES

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For my wonderful wife, Patty, and for my father.

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ABSTRACT

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Let ϕ be a self-map of $\mathbb{B}_2$, the unit ball in $\mathbb{C}^2$. We investigate the equation $C_\phi f = \lambda f$ where we define $C_\phi f := f \circ \phi$, with f a function in the Drury Arveson Space. After imposing conditions to keep C_ϕ bounded and well-behaved, we solve the equation $C_\phi f = \lambda f$ and determine the spectrum $\sigma(C_\phi)$ in the case where there is no interior fixed point and boundary fixed point without multiplicity. We then investigate the existence of one-parameter semigroups for such maps and discuss some generalizations.

1. INTRODUCTION

1.1 Background

Let ϕ be a self map of the unit ball $\mathbb{B}_N = \{z \in \mathbb{C}^N \mid |z_1|^2 + \cdots + |z_N|^2 < 1\}$ in $\mathbb{C}^N$, not an automorphism. We assume ϕ is not a constant function. In the case of the disk $\mathbb{D}$, work by Koenig in 1884, presuming $\phi(0) = 0$ and $\phi'(0) = \lambda$ where $0 < |\lambda| < 1$, demonstrated that the functional equation, known as Schroeder's equation,

$$f \circ \phi = \lambda f$$

has an essentially unique solution in the disk [22]. Of course, there is nothing special about the interior fixed point at 0. One can always conjugate by an automorphism of the disk to send any interior fixed point to the origin. Except for the identity, recall that Schwarz lemma tells us that there are no other fixed points in the disk. Assuming some additional hypotheses, Enoch [13] and Bridges [5] were able to extend this result to analytic self maps of $\mathbb{B}_N$ with interior fixed point.

Now, it may also be the case that our self map of the ball ϕ has no interior fixed points. If ϕ has no interior fixed points in the disk, then it is well known that there is a privileged fixed point a on the boundary such that iterates of ϕ converge to a on compact subsets of the disk. This privileged point is called the Denjoy-Wolff point. MacCluer demonstrated an analogue of this for $\mathbb{B}_N$ [24]. In particular, suppose ϕ is a holomorphic, fixed point free self-map of $\mathbb{B}_N$. Then there exists a unique point ζ on the boundary such that the iterates of ϕ converge uniformly to ζ on compact subsets of $\mathbb{B}_N$. We will also call this point the Denjoy-Wolff point. It also follows from [24] that

$$0 < d(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\phi(z)|^2}{1 - |z|^2} = \alpha \leq 1$$

where we call $d(\zeta)$ the *dilation coefficient* of ϕ for reasons to be explained below. This dilation coefficient will further partition the cases where we have an attractive boundary fixed point and will be the focus of our attention.

1.2 Geometric Function Theory in $\mathbb{B}_N$

In the disk, investigations of the function theoretic properties of C_ϕ make frequent use of the fact that an analytic function in the disk has nontangential limit at $\zeta \in \partial\mathbb{D}$ in the nontangential approach region given by

$$\Gamma(\zeta, \alpha) = \{z \in \mathbb{D} \mid |z - \zeta| < \alpha(1 - |z|)\}$$

where $\alpha > 1$.

In several variables, this is not sufficient. Instead, we must use a more restricted notion of convergence.

Definition 1 *Let $\zeta \in \partial\mathbb{B}_N$ and let $\Gamma(t)$ define a curve from $[0, 1)$ to $\mathbb{B}_N$ such that $\Gamma \to \zeta$ as $t \to 1$. Next, let $\gamma(t) = \langle \Gamma(t), \zeta\rangle\zeta$ be the orthogonal projection of Γ onto the complex line through 0 and ζ. A curve is called restricted if the following criteria are satisfied:*

$$\lim_{t\to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0 \quad \textit{and} \quad \frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq M$$

for some constant M.

We say that $f : \mathbb{B}_N \to \mathbb{C}^N$ has restricted limit L at ζ if $\lim_{z\to\zeta} f(z) = L$ along every restricted curve.

Intuitively, the second criteria is telling us that the projection $\gamma(t)$ resides in a nontangential approach region in the copy of the unit disk lying in the complex line through 0 and ζ.

We will also make use of the generalized Julia-Carathéodory theorem, the proof and full statement of which can be found in [27] as Theorem 8.5.6 or [10] as Theorem

2.81. For $\zeta \in \partial\mathbb{B}_N$, we let $\phi_\zeta(z) = \langle \phi(z), \zeta \rangle$, the coordinate of ϕ in the direction of ζ. We will also let $\phi'(z)$ represent the Jacobian matrix of ϕ. We state the truncated version with the relevant equivalences here for convenience.

Theorem 2 (Julia-Carathéodory theorem in $\mathbb{B}_N$) *Let ϕ be an analytic map from $\mathbb{B}_N$ into itself and let $\zeta \in \partial\mathbb{B}_N$. Then the following are equivalent:*

1. *$d(\zeta) = \liminf_{z\to\zeta}(1 - |\phi(z)|)/(1 - |z|) < \infty$ where the limit is taken as z approaches ζ unrestrictedly in $\mathbb{B}_N$.*
2. *The map ϕ has restricted limit η at ζ, where $|\eta| = 1$ and $D_\zeta\phi_\eta(z) = \langle \phi'(z)\zeta, \eta \rangle$ has finite restricted limit at ζ.*

Given these conditions, $\langle \phi'(z)\zeta, \eta \rangle$ has restricted limit $d(\zeta)$ at ζ.

Thus, by the Julia-Carathéodory theorem in $\mathbb{B}_N$, the complex directional derivative $D_\zeta\phi$ has a radial limit at ζ which is called the *dilation coefficient* of ϕ. It is called the dilation coefficient due to its connection to Julia's lemma in $\mathbb{B}_N$ and its geometric interpretation. We recall it here for convenience.

Lemma 3 (Julia's Lemma in $\mathbb{B}_N$) *Suppose ζ is in $\partial\mathbb{B}_N$ with $d(\zeta) < \infty$. Suppose $a_n \to \zeta$ satisfies*

$$\lim_{n\to\infty} \frac{1 - |\phi(a_n)|^2}{1 - |a_n|^2} = d(\zeta)$$

and $\lim_{n\to\infty} \phi(a_n) = \eta$ where η is in $\partial\mathbb{B}_N$. Then for every z in $\mathbb{B}_N$

$$\frac{|1 - \langle \phi(z), \eta \rangle|^2}{1 - |\phi(z)|^2} \leq d(\zeta)\frac{|1 - \langle z, \zeta \rangle|^2}{1 - |z|^2}.$$

If we let $E(k, \zeta) = \{z \in \mathbb{B}_N \mid |1 - \langle z, \zeta \rangle|^2 \leq k(1 - |z|^2\}$ denote the ellipsoid internally tangent to the unit sphere at ζ with center $\frac{1}{1+k}\zeta$, then ϕ maps the ellipsoid $E(k, \zeta)$ into the ellipsoid $E(d(\zeta)k, \eta)$ (see [10] Lemma 2.77).

Suppose ϕ is an analytic map from the ball $\mathbb{B}_N$ into itself with no interior fixed points. As we saw, MacCluer demonstrated in [24] that there exists a unique point

$\zeta \in \partial\mathbb{B}_N$, which we call the Denjoy-Wolff point of ϕ, with $\phi(\zeta) = \zeta$ such that the iterates of ϕ converge uniformly to ζ on compact subsets of $\mathbb{B}_N$. It also follows from [24] that

$$0 < d(\zeta) = \liminf_{z \to a} \frac{1 - |\phi(z)|^2}{1 - |z|^2} \leq 1$$

and thus by the Julia-Carathéodory theorem in $\mathbb{B}_N$, the complex directional derivative $D_a\phi$ has radial limit $d(\zeta)$ at ζ so that $D_\zeta \phi_\zeta(\zeta) = \langle \phi'(\zeta)\zeta, \zeta \rangle \leq 1$. Uniqueness follows from Julia's lemma in $\mathbb{B}_N$. Suppose ζ_1 and ζ_2 are distinct fixed points on the boundary of $\mathbb{B}_2$ such that $d(\zeta_1) \leq 1$ and $d(\zeta_2) \leq 1$. Set $E(k, \zeta) = \{z \in \mathbb{B}_2 \mid |1 - \langle z, \zeta \rangle|^2 \leq k(1 - |z|^2)\}$. Then we see geometrically that $E(k, \zeta_1)$ and $E(k, \zeta_2)$ are ellipsoids internally tangent to the unit sphere at ζ_1 and ζ_2, respectively. Recall that Julia's Lemma in $\mathbb{B}_N$ tells us that ϕ maps the ellipsoids $E(k, \zeta_1)$ and $E(k, \zeta_2)$ into the corresponding ellipsoids and $E(d(\zeta_1)k, \zeta_1)$ and $E(d(\zeta_2)k, \zeta_2)$, respectively. Choose k_1 and k_2 so that the closed ellipsoids $\overline{E(k_1, \zeta_1)}$ and $\overline{E(k_2, \zeta_2)}$ are tangent to each other at w in $\mathbb{B}_N$. Then $\phi(w)$ is in $\overline{E(k_1, \zeta_1)} \cup \overline{E(k_2, \zeta_2)} = \{w\}$, contradicting the hypothesis that ϕ does not have a fixed point in $\mathbb{B}_N$.

We state it here for reference.

Theorem 4 *If ϕ is an analytic map of the ball $\mathbb{B}_N$ into itself that has no fixed points in the ball, then there is a unique fixed point ζ (the Denjoy-Wolff point) of ϕ on the boundary with $d(\zeta) \leq 1$. If $\phi(b) = b$ with $|b| = 1$, not the Denjoy-Wolff point, then $d(b) > 1$.*

For the case when ζ is the Denjoy-Wolff point, by the generalized Julia-Carathéodory Theorem we have $d(\zeta) = \langle \phi'(\zeta)\zeta, \zeta \rangle = \alpha$. The dilation coefficient partitions our self maps of the ball into three classes, depending on the fixed point behavior of ϕ. We define these as follows.

Definition 5 *An analytic map ϕ from $\mathbb{B}_N$ into $\mathbb{B}_N$ is called*

- *elliptic if ϕ fixes an interior point of $\mathbb{B}_N$*

- *hyperbolic if ϕ has no fixed point in $\mathbb{B}_N$ and dilation coefficient $\alpha < 1$*
- *parabolic if ϕ has no fixed point in $\mathbb{B}_N$ and dilation coefficient $\alpha = 1$.*

1.3 A Uniform Approach to Solving Schroeder's Equation

In the disk, Cowen approached the problem of solving Schroeder's equation in a uniform way using linear fractional maps [9]. Under quite general conditions, it was shown that a nonconstant analytic map ϕ, not an automorphism, from the disk into the disk can be intertwined with a linear fractional map Φ and an analytic map σ such that

$$\Phi \circ \sigma = \sigma \circ \phi$$

where σ maps the disk into a domain Ω with Φ mapping Ω onto Ω. We have the following commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{\phi} & D \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \Omega & \xrightarrow{\Phi} & \Omega \end{array}$$

If Ω is the smallest set containing $\sigma(D)$ for which $\Phi(\Omega) = \Omega$, then the model parameters (σ, Ω, Φ) will be unique up to holomorphic equivalence. The classification depends on the behavior near the Denjoy-Wolff point a. The model results in the following four cases:

i. **(plane/dilation)** $\Omega = \mathbb{C}$, $\sigma(a) = 0$, and $\Phi(z) = sz$ where $0 < |s| < 1$.

ii. **(plane/translation)** $\Omega = \mathbb{C}$, $\sigma(a) = \infty$, and $\Phi(z) = z + 1$.

iii. **(half plane/dilation)** $\Omega = \{z \mid \Re z > 0\}$, $\sigma(a) = 0$, and $\Phi(z) = sz$ where $0 < s < 1$.

iv. **(plane/dilation)** $\Omega = \{z \mid \Im z > 0\}$, $\sigma(a) = \infty$, and $\Phi(z) = z \pm 1$.

We next turn our attention to Schroeder's equation $f \circ \phi = \lambda f$. We would like to interpret this as an eigenvalue equation. We start with the following definition.

Definition 6 *If $\mathcal{H}$ is a vector space of functions defined on a set X, given a function $f : X \to X$, we define the composition operator C_ϕ by*

$$C_\phi f = f \circ \phi.$$

Thus we may write Schroeder's equation as $C_\phi f = \lambda f$. We next ask ourselves over what vector space we are solving the equation. We must then determine when f is in the appropriate vector space. The vector spaces over which we would like to solve our functional equation are Hilbert function spaces. We recall that a Hilbert function space is a Hilbert space of complex-valued functions with pointwise vector operations along with the property that for each z in our set, the linear functional given by evaluation at z, $f \to f(z)$ is continuous. By the Riesz representation theorem, there exists a function k_z which we will call the kernel function, in the Hilbert space that induces the linear functional $f(z) = \langle f, k_z \rangle$. In such a case, we call the function k_z the reproducing kernel. A Hilbert function space is also known as a reproducing kernel Hilbert space (RKHS) or a Hilbert space of analytic functions.. The most celebrated example in $\mathbb{C}$ is the Hardy space $H^2(\mathbb{D})$.

Definition 7 *For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the set of functions analytic on the unit disk $\mathbb{D}$ for which*

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

For $1 \leq p < \infty$, $H^p(\mathbb{D})$ is a Banach space with norm $||f||_p$ given by the p^{th} root of this supremum. The Hardy space $H^\infty(\mathbb{D})$ is the set of analytic functions that are bounded in $\mathbb{D}$ with supremum norm $||f||_\infty$.

When $p = 2$, one can show, letting $f(z) = \sum_{j=0}^{\infty} a_j z^j$, that an equivalent definition is given by

$$H^2(\mathbb{D}) = \{f \mid \sum_{j=0}^{\infty} |a_j|^2 < \infty\}.$$

If we restrict f to reside in a Hilbert function space, then we may interpret Schroeder's equation as an eigenvalue equation over our Hilbert function space for a composition operator C_ϕ.

Since $H^2(\mathbb{D})$ is an infinite-dimensional vector space, we also generalize the notion of eigenvalues.

Definition 8 *Let X be a Hilbert space and T a bounded linear operator from X into X. The spectrum of T is defined to be*

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\}.$$

For finite-dimensional vector spaces, the spectrum coincides with the set of eigenvalues. For infinite-dimensional vector spaces, however, the set of eigenvalues is a subset of the spectrum. If our Hilbert space is $H^2(\mathbb{D})$, we do not need to be concerned about the boundedness of the composition operator C_ϕ since one can use the Littlewood subordination principle to demonstrate that C_ϕ is bounded for every analytic map $\phi : \mathbb{D} \to \mathbb{D}$ with

$$||C_\phi|| \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right)^{\frac{1}{2}}.$$

See Corollary 3.7 of [10] for details.

In our case, the strategy to find the spectrum is to start with the equation $F \circ \Phi = \lambda F$ and thus, letting $f = F \circ \sigma$, use the model results to determine the spectrum of the equation

$$f \circ \phi = F \circ \sigma \circ \phi = F \circ \Phi \circ \sigma = \lambda F \circ \sigma = \lambda f.$$

In the disk, it is natural to consider C_ϕ acting on the Hardy space $H^2(\mathbb{D})$. Our goal is to make partial steps toward generalizing the results in the disk to the unit ball $\mathbb{B}_2$ when our self map ϕ has no interior fixed point.

1.4 Linear Fractional Maps in $\mathbb{C}^N$

Before we attempt to generalize the results to higher dimensions, we must first say what it means to be a linear fractional map in $\mathbb{C}^N$ for $N > 1$. For $N = 1$, given a linear fractional map $f(z) = \frac{az+b}{cz+d}$, recall that one may define the associated matrix of f as follows:

$$m_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One can show that m_f acts as a linear transformation on complex projective coordinates (see [25] pg.156). We take the perspective that linear fractional maps in $\mathbb{C}_N$ should have associated matrices which act as linear transformations on complex projective coordinates. Recall the following construction. We associate the point $z = (z_1', z_2)$ where $z_1' \in \mathbb{C}^N$ and $z_2 \in \mathbb{C}$, $z \neq 0$, with the point $\frac{z_1'}{z_2} \in \mathbb{C}^N$. This associated space is known as the complex projective space $\mathbb{CP}^N$. We now consider a linear transformation in $\mathbb{C}^N$ which can be represented by a complex matrix as

$$\begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and C^* denotes the conjugate transpose of C. Denote the rows of A by a_i for $i = 1, ..., N$ and $B = \begin{pmatrix} b_1 & \cdots & b_N \end{pmatrix}^T$. For a point $\begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ in $\mathbb{CP}^N$, we have

$$\begin{pmatrix} w_1' \\ w_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \begin{pmatrix} z_1' \\ z_2 \end{pmatrix} = \begin{pmatrix} \langle a_1, \overline{z_1'} \rangle + b_1 z_2 \\ \vdots \\ \langle a_N, \overline{z_1'} \rangle + b_N z_2 \\ \langle z_1', C \rangle + D z_2 \end{pmatrix}.$$

where $\langle \cdot, \cdot \rangle$ represents the standard inner product.

Let $z \sim \begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ and $w \sim \begin{pmatrix} w_1' \\ w_2 \end{pmatrix}$. Then we can associate the above linear transformation in $\mathbb{CP}^N$ with the non-linear transformation in $\mathbb{C}^N$ given by

$$\begin{aligned} w = \frac{w_1'}{w_2} &= \left(\frac{\langle a_1, \overline{z_1'} \rangle + b_1 z_2}{\langle z_1', C \rangle + D z_2}, ..., \frac{\langle a_N, \overline{z_1'} \rangle + b_N z_2}{\langle z_1', C \rangle + D z_2} \right) \\ &= \left(\frac{\langle a_1, \overline{\frac{z_1'}{z_2}} \rangle + b_1}{\langle \frac{z_1'}{z_2}, C \rangle + D}, ..., \frac{\langle a_N, \overline{\frac{z_1'}{z_2}} \rangle + b_N}{\langle \frac{z_1'}{z_2}, C \rangle + D} \right) \\ &= \left(\frac{\langle a_1, \overline{z} \rangle + b_1}{\langle z, C \rangle + D}, ..., \frac{\langle a_N, \overline{z} \rangle + b_N}{\langle z, C \rangle + D} \right) \\ &= \frac{Az + B}{\langle z, C \rangle + D}. \end{aligned}$$

This is precisely the definition given by Cowen and MacCluer [10] and is the one we will adopt.

Definition 9 *We say ϕ is a linear fractional map in $\mathbb{C}^N$ if*

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

This class of maps has been studied in more generality by others [16], [26], [29], [31] and can be demonstrated to share many additional properties with linear fractional maps in the disk that justify their definition as linear fractional maps [11].

We define the associated matrix m_ϕ of the linear fractional map $\phi(z) = \frac{Az+B}{\langle z,C \rangle + D}$ to be given by

$$m_\phi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

which, as we saw, is a linear transformation on $\mathbb{CP}^N$. If $\phi(z) = w$ and the point $v \in \mathbb{C}^N$ is associated with z, then $m_\phi v$ is associated with the point w and vice versa. A routine calculation also shows that $m_{\phi_1 \circ \phi_2} = m_{\phi_1} m_{\phi_2}$ and $m_{\phi^{-1}} = (m_\phi)^{-1}$.

1.5 A Uniform Approach in Several Variables

In several variables, an analogue of the linear fractional models used in the disk will not be adequate. The proof of the existence of the linear fractional models in the disk cannot be generalized to $\mathbb{B}_N$ due to its critical use of the Riemann mapping theorem, which fails in $\mathbb{C}_N$ for $N > 1$. This is not the only issue in higher dimensions. As we will see, to gain any traction for the case when $N > 1$, further refinement is needed.

While it can be demonstrated that linear fractional maps induce bounded composition operators over the standard Hardy spaces (see [11], [20]), one can't guarantee much more without additional assumptions. Not only can it be demonstrated that the composition operator C_ϕ is not necessarily bounded, but one can construct unbounded composition operators induced by polynomials [7]! We can, however, classify the set of linear fractional maps in definition 9 from $\mathbb{B}_2$ into $\mathbb{B}_2$ in a similar fashion as in the disk [8]. We hope to expand on this by including analytic maps under given conditions. In this classification, both for linear fractional maps and for more general analytic maps, we exclude two types of maps that we consider degenerate. We exclude maps that are not invertible as maps of $\mathbb{C}^N$ onto itself and maps of the ball that do not have a Denjoy-Wolff point. The first type maps the ball into a lower dimensional affine set and in the latter case one can show that the map acts as a generalized rotation on an affine subset that has a non-trivial intersection with the ball.

We find that for the class of linear fractional maps in two complex variables, Cowen's model theory generalizes to admit seven cases to be considered [8]. The seven cases are determined by the behavior of the map ϕ near the Denjoy-Wolff point and its characteristic domain. As in the disk, one can show that this classification is invariant under conjugation by an automorphism. We find that there are three characteristic domains to be considered. These are the whole space $\mathbb{C}^2$, the half space

$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re z_1 > 0\}$, and the Siegel half space $\mathbb{H}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re z_1 > |z_2|^2\}$. We have the following seven cases.

I. The Denjoy-Wolff point is inside the ball and the characteristic domain is the whole space. This is the **whole space/dilation** case.

II. There are three distinct fixed points, the Denjoy-Wolff point is on the boundary and the characteristic domain is a half space. This is the **half space/dilation** case.

III. There are three distinct fixed points, the Denjoy-Wolff point is on the boundary and the characteristic domain is a Siegel half space. This is the **Siegel half space/dilation** case.

IV. There is one fixed point of multiplicity three on the boundary and the characteristic domain is the whole space. This is the **whole space/Heisenberg translation-translation** case.

V. There is one fixed point of multiplicity three on the boundary and the characteristic domain is a Siegel half space. This is the **Siegel half space/Heisenberg translation** case.

VI. There are two fixed points with the Denjoy-Wolff point of multiplicity two on the boundary and the characteristic domain is the whole space. This is the **whole space/translation** case.

VII. There are two fixed points with the Denjoy-Wolff point of multiplicity one on the boundary and the characteristic domain is the whole space. This is the **whole space/asymptotic translation** case.

Case I corresponds to the case when ϕ has an interior fixed point, that is, ϕ is an elliptic map. Cases II and III correspond to the case when ϕ is a hyperbolic map. The remaining cases correspond to the cases when ϕ is a parabolic map. We reproduce the results of [8] here for convenience.

Theorem 10 (The Model for Iteration of Linear Fractional Maps) *Let ϕ be a linear fractional map of $\mathbb{B}_2$ into itself, not an automorphism of the ball and not constant. We can intertwine ϕ with a model linear fractional map Φ with characteristic domain Ω, either the half space, Siegel half space, or the whole space, and an open map σ from $\mathbb{B}_2$ into Ω such that*

$$\sigma \circ \phi = \Phi \circ \sigma.$$

If Ω is the smallest set containing $\sigma(\mathbb{B}_2)$ for which $\Phi(\Omega) = \Omega$, then the model parameters (σ, Ω, Φ) will be unique up to holomorphic equivalence.

We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{B}_2 & \xrightarrow{\phi} & \mathbb{B}_2 \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \Omega & \xrightarrow{\Phi} & \Omega \end{array}$$

In addition, there exists a set V, known as the fundamental set, such that V is an open, connected, simply connected subset of $\mathbb{B}_2$ such that $\phi(V) \subset V$ and for every compact set K in $\mathbb{B}_2$, there is a positive integer n with $\phi_n(V) \subset V$ with ϕ and σ univalent on V and with $\sigma(V)$ a fundamental set for Φ on Ω.

If our map ϕ is an analytic map that is not a linear fractional map but otherwise has the same model for iteration as Theorem 10, we will say that ϕ satisfies the requirements of Theorem 10.

1.6 The Drury-Arveson Space

Recall that the equation $f \circ \phi = \lambda f$ was studied over the Hardy space $H^2(\mathbb{D})$. In $\mathbb{B}_N$, the correct generalization of the Hardy space $H^2(\mathbb{D})$ seems to be the Hilbert function space known as the Drury-Arveson space $H^2_d(\mathbb{B}_N)$ or just H^2_d when it is understood that we are working in the ball. See [30] for a recent survey on the Drury-Arveson space.

We began by introducing some basic notation. For $z = (z_1, ..., z_n) \in \mathbb{C}^n$, we let

$$z^\alpha = \prod_{i=1}^{n} z_i^{\alpha_i}$$

for every multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$. Likewise we write

$$\alpha! = \prod_{i=1}^{n} \alpha_i! \quad \text{and} \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Suppose $f(z)$ is analytic in $\mathbb{B}_N$ so that

$$f(z) = \left(\sum_\alpha a_\alpha^{(1)} z^\alpha, ..., \sum_\alpha a_\alpha^{(N)} z^\alpha \right)$$

We define the Drury-Arveson space H_d^2 to be the reproducing kernel Hilbert space (RKHS) on $\mathbb{B}_N$ with kernel

$$k(z, w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product. For two functions $f, g \in H_d^2$ if we have Taylor expansions given by

$$f(z) = \sum_\alpha c_\alpha z^\alpha \quad \text{and} \quad g(z) = \sum_\alpha d_\alpha z^\alpha$$

then we define their inner product to be

$$\langle f, g \rangle_{H_d^2} = \langle f, g \rangle = \sum_\alpha \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d_\alpha}.$$

An orthonormal basis is then given by $\{e_\alpha\}$ where

$$e_\alpha = \sqrt{\frac{|\alpha|!}{\alpha!}} z^\alpha.$$

An analytic function $f(z) = \sum_\alpha c_\alpha z^\alpha$ is in H_d^2 if

$$||f||_{\mathbb{H}_d^2}^2 = ||f||^2 = \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty.$$

For $w \in \mathbb{B}_N$, we have

$$k(z,w) = k_w(z) = \frac{1}{1-\langle z,w\rangle} = \sum_{n=0}^{\infty}\langle z,w\rangle^n = \sum_{n=0}^{\infty}\sum_{|\alpha|=n}\frac{|\alpha|!}{\alpha!}\overline{w}^{\alpha}z^{\alpha}$$

which is in H^2_d and

$$f(w) = \sum_{\alpha} c_\alpha w^\alpha = \sum_{\alpha} c_\alpha \frac{|\alpha|!}{\alpha!} w^\alpha \langle z^\alpha, z^\alpha\rangle = \langle f, k_w\rangle.$$

Our goal then, will be as follows:

Goal 1 *Given an analytic map $\phi : \mathbb{B}_2 \to \mathbb{B}_2$, not an automorphism, such that ϕ is in the Siegel half space/dilation case. Find all analytic functions $f : \mathbb{B}_2 \to \mathbb{C}$ in the Drury-Arveson space and all complex numbers λ that satisfy the functional equation $C_\phi f = \lambda f$.*

Much of what we do will be applicable to both the Siegel half space/dilation case and the half space/dilation case. We will make note when we must presume the former. When applicable we will follow arguments given by [10].

2. SOLVING SCHROEDER'S EQUATION IN THE SIEGEL HALF SPACE/DILATION AND HALF SPACE/DILATION CASE

2.1 Conjugation

Recall that in the case where ϕ has an interior fixed point, we may conjugate by an automorphism to relocate the interior fixed point to the origin so that we may assume $\phi(0) = 0$ without loss of generality. For the case where ϕ has no interior fixed point and Denjoy-Wolff point on the boundary, we standardize the problem so that we may presume $\phi(e_1) = e_1$ without loss of generality, where $e_1 = (1, 0)$ is the "east pole". We justify this below.

Lemma 11 *Suppose η is an automorphism of the ball and suppose $\psi = \eta^{-1} \circ \phi \circ \eta$. Then f is a solution to $f \circ \phi = \lambda f$ if and only if $g = f \circ \eta$ is a solution to $g \circ \psi = \lambda g$.*

Proof This follows immediately from the relation

$$g \circ \psi = f \circ \eta \circ \psi = f \circ \phi \circ \eta = \lambda f \circ \eta = \lambda g.$$

■

Thus, according to the above lemma, since we are concerned with maps that have no interior fixed point and a boundary fixed point, without loss of generality we may conjugate by a rotation to place the Denjoy-Wolff point at $e_1 = (1, 0)$, the "east pole" of the ball.

For ϕ in the Siegel half space/dilation model or half space/dilation model, we may further conjugate ϕ by the appropriate automorphisms [8] to attain

$$\sigma \circ \phi = \Phi \circ \sigma$$

where $\Phi(z) = (\alpha z_1, \beta z_2)$ with $\alpha \geq |\beta|^2$ and $0 < \alpha < 1$ where $\alpha = |\beta|^2$ corresponds to the Siegel half space/dilation model and $\alpha > |\beta|^2$ corresponds to the half space/dilation model.

As in the disk, our strategy will be to first find the solutions of the equation $F \circ \Phi = \lambda F$ and thus, letting $f = F \circ \sigma$, determine the solutions of the equation

$$f \circ \phi = F \circ \sigma \circ \phi = F \circ \Phi \circ \sigma = \lambda F \circ \sigma = \lambda f.$$

We first need the following theorem.

Lemma 12 *Suppose that ϕ, V, Φ, σ, and Ω satisfy requirements of Theorem 10. Suppose $\lambda \neq 0$ is a complex number. If F is analytic on Ω and $F \circ \Phi = \lambda F$, then $f \circ \phi = \lambda f$ where $f = F \circ \sigma$. Conversely, if f is analytic on $\mathbb{B}_2$ and $f \circ \phi = \lambda f$, there is a function F analytic on Ω so that $F \circ \Phi = \lambda F$ and $f = F \circ \sigma$.*

Proof If $F \circ \Phi = \lambda F$, then

$$f \circ \phi = F \circ \sigma \circ \phi = F \circ \Phi \circ \sigma = \lambda \circ F \circ \sigma = \lambda f.$$

Conversely, if $f \circ \phi = \lambda f$, then since σ is univalent on V, we may define $\tilde{F}$ on $\sigma(V)$ by letting $\tilde{F} = f \circ \sigma^{-1}$ so that $\tilde{F} \circ \Phi = \lambda \tilde{F}$ on $\sigma(V)$. Since $\sigma(V)$ is a fundamental set for Φ on Ω, we can define F on Ω by $F(w) = \lambda^{-k} \tilde{F}(\Phi_k(w))$ where k is an integer large enough so that $\Phi_k(w)$ is in $\sigma(V)$. If $k' = k + m$ is another such integer with $m > 0$, then

$$\begin{aligned} \lambda^{-k'} \tilde{F}(\Phi_{k'}(w)) &= \lambda^{-k-m} \tilde{F}(\Phi_{k+m}(w)) = \lambda^{-k-m} \tilde{F}(\Phi_m(\Phi_k(w))) \\ &= \lambda^{-k-m} \left(\lambda^m \tilde{F}(\Phi_k(w)) \right) = \lambda^{-k} \tilde{F}(\Phi_k(w)) \end{aligned}$$

so that F is well defined. We see that it satisfies $F \circ \Phi = \lambda F$ and $f = F \circ \sigma$.

■

2.2 The Solution to $F \circ \Phi = \lambda F$.

We begin by finding the solution to the equation $F \circ \Phi = \lambda F$ and then use Lemma 12 to transfer the solution to $f \circ \phi = \lambda f$.

Theorem 13 *Let $\Phi(z) = (\alpha z_1, \beta z_2)$ with $\alpha \geq |\beta|^2$ and $0 < \alpha < 1$ so that Φ corresponds to the Siegel half space model or half space model. If $\beta \notin \mathbb{R}$ or Φ is in the half space model, then $F \circ \Phi = \lambda F$ has a nonzero solution F analytic in $\mathbb{C}^2$ if and only if $\lambda = \alpha^k \beta^l$ for $k, l \in \mathbb{N} \cup \{0\}$ with $F(z) = z_1^k z_2^l$. If $\beta \in \mathbb{R}$ and Φ is in the Siegel half space model (so that $\alpha = \beta^2$), then $F \circ \Phi = \lambda F$ has a nonzero solution F analytic in $\mathbb{C}^2$ if and only if $\lambda = \alpha^k \beta^l$ for $k, l \in \mathbb{N} \cup \{0\}$ with $F(z) = z_1^k z_2^l$ or a polynomial of the form*

$$F_m(z) = \sum_{\{k,l | m = 2k+l\}} a_{kl} z_1^k z_2^l$$

where m is a fixed nonnegative integer and a_{kl} can equal 0 or 1 for each k, l.

Proof Let $F(z) = \sum_{k,l \geq 0} a_{kl} z_1^k z_2^l$ be an analytic function in $\mathbb{C}^2$. To avoid trivialities, we presume $F(z) \neq 0$. We then proceed to compare coefficients of the Taylor expansion. Our functional equation then demands

$$\sum_{k,l} a_{kl} (\alpha z_1)^k (\beta z_2)^l = F(\Phi(z)) = \lambda F(z) = \lambda \sum_{k,l} a_{kl} z_1^k z_2^l$$

for all z_1 and z_2 which implies that

$$\alpha^k \beta^l a_{kl} = \lambda a_{kl}$$

for all $k, l \geq 0$. Let $\beta = re^{i\theta}$ with $\theta \neq \pi p$ for p integer. Then, for $a_{kl}, a_{mn} \neq 0$ we have

$$\alpha^k \beta^l = \lambda = \alpha^m \beta^n \rightarrow \alpha^k r^l e^{i\theta l} = \alpha^m r^n e^{i\theta n}$$

which implies $l = n$, since α is real, from which it follows that $k = m$ and therefore $a_{kl} = a_{mn}$. Thus if β is not real, our solutions must be the monomials $z_1^k z_2^l$. In such a case, for each monomial $F(z) = a_{kl} z_1^k z_2^l$, $\lambda = \alpha^k \beta^l$ satisfies our equation so that

$$F \circ \Phi(z) = a_{kl}(\alpha z_1)^k(\beta z_2)^l = \alpha^k \beta^l a_{kl} z_1^k z_2^l = \lambda F(z).$$

In the case that β is real, if $\alpha > \beta^2$ then again for $a_{kl}, a_{mn} \neq 0$ we obtain $\alpha^k \beta^l = \alpha^m \beta^n$. Since $\alpha \neq \beta$ and $\alpha > \beta^2$ implies $\alpha > \beta^j$ for all positive integers $j \geq 2$ (so that $\alpha \neq \beta^j$ for any positive integer j), we have $\alpha^{k-m} = \beta^{n-l}$ only when $k = m$ and $n = l$ and thus $F(z)$ is a monomial.

If we have $\alpha = \beta^2$, we conclude

$$\alpha^k \beta^l = \beta^{2k+l} = \lambda.$$

As before, any monomial $F(z) = a_{kl} z_1^k z_2^l$ satisfies our equation with $\lambda = \alpha^k \beta^l$. For $a_{kl} = a_{mn} \neq 0$, we must have $\beta^{2k+l} = \lambda = \beta^{2m+n}$ which implies $2k + l = 2m + n$. Thus, polynomials of the form

$$F_m(z) = \sum_{\{k,l | m=2k+l\}} a_{kl} z_1^k z_2^l$$

where a_{kl} can equal 0 or 1 for each k, l, satisfy our requirement.

■

As the simplest explicit example of a polynomial $F_m(z)$, take $F_2(z) = z_1 + z_2^2$. Then, since $\alpha = \beta^2$, we have

$$F \circ \Phi(z) = (\alpha z_1) + (\beta z_2)^2 = \beta^2 z_1 + \beta^2 z_2^2 = \beta^2(z_1 + z_2^2) = \lambda F(z).$$

2.3 The Solution to $f \circ \phi = \lambda f$

This brings us to our first new theorem.

Theorem 14 *Suppose ϕ is an analytic map, not an automorphism, of the unit ball $\mathbb{B}_N$ into itself with Denjoy-Wolff point ζ in the Siegel half space/dilation or half space/dilation case with intertwining $\sigma \circ \phi = \Phi \circ \sigma$ and $\Phi(z) = (\alpha z_1, \beta z_2)$. Then*

$f \circ \phi = \lambda\phi$ has a non-zero solution if and only if $\lambda = \alpha^k\beta^l$ for nonnegative integers k,l. Additionally, if $\beta \notin \mathbb{R}$ or $\alpha > |\beta|^2$, then f is a non-zero solution of $f \circ \phi = \alpha^k\beta^l f$ for some nonnegative integers k,l if and only if $f(z) = c\sigma_1^k\sigma_2^l$ where c is a constant. If $\beta \in \mathbb{R}$ and $\alpha = \beta^2$, then, in addition to the above solutions, the polynomials $f_m(z) = c\sum_{\{k,l|m=2k+l\}} a_{kl}\sigma_1(z)^k\sigma_2(z)^l$ where m is a fixed nonnegative integer and a_{kl} can equal 0 or 1 for each k, l, are permitted.

Proof Apply Lemma 12 and Theorem 13. ■

3. SPECTRA OF COMPOSITION OPERATORS INDUCED BY MAPS IN THE SIEGEL HALF SPACE/DILATION CASE

3.1 Growth Estimates

While we have acquired the analytic solutions desired, we have not completely solved the problem as the function f is an eigenvector of $C_\phi f = \lambda f$ if and only if f is in the Hilbert function space on which the composition operator C_ϕ acts. To help determine when our functions are in our space, we first find a growth estimate.

Theorem 15 *Let ϕ be an analytic map from the unit ball $\mathbb{C}^2$ into itself with Denjoy-Wolff point η, $|\eta| = 1$ and the dilation coefficent $d(\eta)$ less than or equal to 1. Let b be another fixed point of ϕ and suppose that ϕ is analytic in a neighborhood of b. Likewise suppose there is a $\delta > 0$ for which $|\phi(z)| < 1$ whenever $|z - b| < \delta$ and $|z| \leq 1$. If σ is the model map of ϕ that takes the ball into the Siegel half-space with $\sigma \circ \phi = \Phi \circ \sigma$ and $\Phi(z_1, z_2) = (\alpha z_1, \beta z_2)$ with $0 < |\beta|^2 \leq \alpha < 1$, then for every $p > |3 \log |\beta| / \log D_b\phi_b(b)|$ where $D_b\phi_b(z) = \langle \phi'(z)b, b\rangle$, there is a constant M so that*

$$|\sigma(z)| < M|z - b|^{-p}$$

in a neighborhood of b.

Proof Let $r_0 = D_b\phi_b(b)$ and $r = e^{\left(\frac{3|\log(|\beta|)|}{p}\right)} < r_0$ with $|\beta|^2 \leq \alpha = d(\eta) \leq 1$. WLOG assume that δ is such that if $0 < |z - b| < \delta$ and $|z| \leq 1$ then $|D_b\phi_b(b) - D_b\phi_b(z)| < r_0 - r$ as well as $|\phi(z)| < 1$. Now let

$$K = \{w \mid |b - w| \geq \delta \text{ and } w = \phi(z) \text{ for some } z \text{ with } |z| \leq 1 \text{ and } |b - z| \leq \delta\}.$$

Then K is a compact subset of the ball such that for z in the ball with $|b-z|<\delta$, either $|b-\phi(z)|<\delta$ or $\phi(z)$ is in K. Thus, for z in the closed ball in a neighborhood of b, we have that $\phi_k(z)$ is in K for some positive integer k. Let $\delta'=\sup\{|b-w| \mid w\in K\}$.

By our intertwining assumption, for positive integer n such that $\phi_n(z)$ is in K, we have $\sigma\circ\phi_n=\Phi_n\circ\sigma$ where $\Phi_n(z_1,z_2)=(\alpha^n z_1,\beta^n z_2)$, so $\sigma=\Phi_n^{-1}\circ\sigma\circ\phi_n$ where $\Phi_n^{-1}(z_1,z_2)=(\alpha^{-n}z_1,\beta^{-n}z_2)$ is in $\Phi_n^{-1}\circ\sigma(K)$. Since $\sigma(K)$ is a compact subset of the Siegel half-space, we have $|\sigma(z)|\le M_1|\alpha\beta|^{-n}$ where $M_1=\max\{|w| \mid w\in\sigma(K)\}$. Our goal is then to make an estimate of the integer n for which $\phi_n(z)$ is in K.

Let $\phi(z)=(\varphi_1(z),\varphi_2(z))$ and $b=(b_1,b_2)$. Also let $q(t)=tb+(1-t)z$ for $0\le t\le 1$ be the line segment from z to b. If $|b-z|<\delta$ then we have

$$\begin{aligned}
|b-\phi(z)| &= |\phi(b)-\phi(z)|\,|b| \ge |\langle \phi(b)-\phi(z), b\rangle| = \left|\langle \phi(tb+(1-t)z)\Big|_0^1, b\rangle\right| \\
&= \left|\langle \int_0^1 \frac{d}{dt}\phi(tb+(1-t)z)dt, b\rangle\right| = \left|\int_0^1 \langle \phi'(tb+(1-t)z)(b-z)^T, b\rangle dt\right| \\
&= \left|\langle \phi'(b)b,b\rangle|b-z| - \int_0^1 \left(\langle\phi'(b)b,b\rangle - \langle\phi'(tb+(1-t)z)\frac{(b-z)^T}{|b-z|},b\rangle\right)|b-z|dt\right| \\
&\ge |b-z|\,|\langle\phi'(b)b,b\rangle - (r_0-r)| = r|b-z|.
\end{aligned}$$

Now, if $|b-\phi(z)|<\delta$ then by the same reasoning we obtain

$$|b-\phi(\phi(z))| \ge r|b-\phi(z)| \ge r^2|b-z|.$$

Let n be the least integer such that $|b-\phi_n(z)|\ge\delta$. Then n is the least integer for which $\phi_n(z)$ is in K. We have $\delta'\ge|b-\phi_n(z)|\ge r^n|b-z|$. We isolate n to obtain $n\le\frac{\log(\delta'|b-z|^{-1})}{\log r}$. Thus, since $|\beta|^2\le\alpha$ implies $|\alpha|^{-n}\le|\beta|^{-2n}$, we have that

$$\begin{aligned}
|\sigma(z)| &\le M_1|\alpha\beta|^{-n} \le M_1|\beta|^{-\frac{3\log(\delta'|b-z|^{-1})}{\log r}} \\
&= M_2|b-z|^{3\frac{\log\beta}{\log r}} = M_2|b-z|^{-p}.
\end{aligned}$$

■

3.2 The Spectra in the Siegel Half Space/Dilation Case

In this section we will determine the spectrum of a composition operator on the Drury-Arveson Space in the Siegel Half Space/Dilation Case. Thus ϕ will have Denjoy-Wolff point ζ on the unit ball with $\langle \phi'(\zeta)\zeta, \zeta\rangle = \alpha < 1$. While some of our results will hold in great generality, others need additional hypotheses. These include a smoothness hypothesis on the boundary and a restriction of ϕ to a subset of analytic maps of the ball known as the Schur-Agler class. We have the following definition.

Definition 16 *The Schur-Agler class S_n is the set of all holomorphic mappings $\phi : \mathbb{B}_n \to \mathbb{B}_n$ for which the Hermitian kernel*

$$k^{\phi}(z, w) = \frac{1 - \langle \phi(z), \phi(w)\rangle}{1 - \langle z, w\rangle}$$

is positive semidefinite.

In one variable, every self-map of the disk resides in the Schur-Agler class. This is not true in higher dimensions. Explicitly, every self-map of the ball $\mathbb{B}_N$ is not necessarily in the Schur-Agler class for $N > 1$. It turns out, however, that linear fractional maps as defined above are in this class [20]. In many ways, this class should be seen as the appropriate analogue of the unit ball of $H^{\infty}(\mathbb{D})$ in the multivariable setting. See [18] for further justification of this. The assumption that our map is in this class is invoked due to the fact that, for $N > 1$, the composition operator C_{ϕ} induced by the analytic map ϕ is not necessarily bounded. As noted, one may even construct polynomials that give rise to unbounded composition operators. For ϕ in the Schur-Agler class, however, the composition operator C_{ϕ} induces a bounded composition operator over the standard Hilbert function spaces [19]. This suggests that maps in the Schur-Agler class should enjoy function-theoretic privileges over more generic maps of the ball and is the motivation for our assumption.

We began by showing circular symmetry. We utilize a similarity argument which illustrates a connection between Toeplitz operators and composition operators. In

the disk, the set of multipliers of the Hardy space $H^2(\mathbb{D})$ is precisely $H^\infty(\mathbb{D})$ and thus for f to be a multiplier one only needs to show that $f \in H^\infty(\mathbb{D})$. In order to generalize to higher dimensions, however, we note that it is not sufficient to show that f is bounded, that is $f \in H^\infty(\mathbb{B}_n)$. This is because for $H^2_d(\mathbb{B}_n)$ for $n > 1$, the set of multipliers is a proper subset of $H^\infty(\mathbb{B}_n)$ (see [30] for an example). We instead appeal to a positivity argument. We first recall the following properties of positive semi-definite functions.

- Kernel functions are positive semi-definite. This can easily be seen by the fact that for a kernel function $k(z, w)$, complex numbers $a_i, ..., a_n$, and distinct points $\{z_1, ..., z_n\}$, we have

$$\sum_{i,j=1}^{n} \overline{a}_i a_j k(z_i, z_j) = \langle \sum_{j=1}^{n} a_j k_{z_j}, \sum_{i=1}^{n} a_i k_{z_i} \rangle = \left|\left| \sum_{j=1}^{n} a_j k_{z_j} \right|\right|^2 \geq 0.$$

- The Schur product, also known as the Hadamard product, of two positive semi-definite functions is positive semi-definite. Recall that for two matrices M and N, both of dimension $m \times n$, the Schur product $M \circ N$ is an $m \times n$ matrix given entry-wise by

$$(M \circ N)_{ij} = (M)_{ij}(N)_{ij}.$$

Note the following result applies to the Siegel half space/dilation and half space/dilation case.

Theorem 17 *Suppose ϕ is an analytic map of $\mathbb{B}_2$ into itself with intertwining linear fractional map given by $\Phi(z) = (\alpha z_1, \beta z_2)$ with $\alpha \geq |\beta|^2$. Then for $\theta \in \mathbb{R}$ the operator C_ϕ acting on H^2_d is similar to the operator $e^{i\theta} C_\phi$. In particular, if $\lambda \in \sigma(C_\phi)$ then so is $\lambda e^{i\theta}$ for all real θ.*

Proof Since ϕ may be intertwined with Φ, there is a map σ from the ball into the Siegel half space or half space (depending on which case we are in) such that $(\sigma \circ \phi)(z) = (\Phi \circ \sigma)(z)$ for all z in the ball. Let $F(w) = e^{i\frac{\theta}{2}\frac{\log w_1}{\log \alpha}} e^{i\frac{\theta}{2}\frac{\log w_2}{\log \beta}}$ for w in the

Siegel half space/half space where log denotes the principal branch of the logarithm. Then a simple calculation shows $F \circ \Phi(w) = e^{i\theta}F(w)$. Writing $f = F \circ \sigma$ we see that $f \circ \phi = F \circ \sigma \circ \phi = F \circ \Phi \circ \sigma = e^{i\theta}F \circ \sigma = e^{i\theta}f$. We next desire to show that T_f and $T_{\frac{1}{f}}$ are multipliers on H^2_d. We know by [1] that T_f is a multiplier of norm at most ρ if and only if

$$\rho^2 I - T_f^\star T_f \geq 0.$$

A simple calculation shows that for $\theta \geq 0$ and $\beta \in \mathbb{R}$, we have that

$$e^{\frac{\pi\theta}{4\log\alpha}} e^{\frac{\pi\theta}{2\log\beta}} < |f(z)| < e^{-\frac{\pi\theta}{4\log\alpha}} e^{-\frac{\pi\theta}{2\log\beta}}.$$

Likewise, for $\theta < 0$ and and $\beta \in \mathbb{R}$, we have

$$e^{-\frac{\pi\theta}{4\log\alpha}} e^{-\frac{\pi\theta}{2\log\beta}} < |f(z)| < e^{\frac{\pi\theta}{4\log\alpha}} e^{\frac{\pi\theta}{2\log\beta}}.$$

Since the linear span of kernel functions are dense in H^2_d, it is sufficient to check the above equation on linear combinations of kernel functions. Thus, we want to show that $(\rho^2 - f(z)\overline{f(w)})k(z,w)$ is positive semi-definite. If $\theta \geq 0$, let $\rho = e^{-\frac{\pi\theta}{4\log\alpha}} e^{-\frac{\pi\theta}{2\log\beta}}$ and if $\theta < 0$ let $\rho = e^{\frac{\pi\theta}{4\log\alpha}} e^{\frac{\pi\theta}{2\log\beta}}$. Then we have

$$\sum_{k,l=1}^{N} c_k\overline{c_l}(\rho^2 - f(z_k)\overline{f(z_l)}) = \rho^2 \left|\sum_{j=1}^{N} c_j\right|^2 - \sum_{k,l=1}^{N} c_k\overline{c_l} f(z_k)\overline{f(z_l)} \geq 0.$$

Since kernel functions are positive semi-definite and the Schur product of positive semi-definite functions are positive semi-definite, it follows that $(\rho^2 - f(z)\overline{f(w)})k(z,w)$ is positive semi-definite. The above argument applies likewise to $T_{\frac{1}{f}}$ as well and with a little more algebra this same reasoning works for $\beta \notin \mathbb{R}$. Thus for $h \in H^2_d$, we have

$$((T_f)^{-1}C_\phi T_f))(h) = (T_f)^{-1}((f \circ \phi)(h \circ \phi)) = e^{i\theta}(T_f)^{-1}T_f C_\phi h = e^{i\theta}(C_\phi h)$$

from which we conclude that C_ϕ is similar to $e^{i\theta}C_\phi$.

Since similar operators have the same spectrum, we apply the spectral mapping theorem to conclude that the spectrum of $e^{i\theta}C_\phi$ is $e^{i\theta}$ times the spectrum of C_ϕ. ■

We next introduce some terminology. The K and M defined below can be $\pm\infty$ in addition to integers.

Definition 18 *We say the sequence of points* $\{z_k\}_{k=K}^{M}$ *in* $\mathbb{B}_2$ *is an* **iteration sequence for** ϕ *if* $\phi(z_k) = z_{k+1}$ *for* $K < k < M$.

Given an iteration sequence $\{z_k\}_{k=-\infty}^{0}$ of distinct points for ϕ, not an elliptic automorphism (i.e. an automorphism of the ball that fixes an interior point) of the ball $\mathbb{B}_2$ onto itself, we have $\lim_{k\to-\infty}|z_k| = 1$. Otherwise suppose b were a limit point of this sequence and $|b| < 1$. If a is the Denjoy-Wolff point of ϕ and $b = a$, then the pseudohyperbolic distance from b to z_{k_1} is greater than the pseudohyperbolic distance from b to $z_{k_2} = \phi_n(z_{k_1})$ for $n = k_2 - k_1 > 0$ which contradicts the assumption that a subsequence $\{z_{k_j}\}$ of the iteration sequence converges to b with k_j tending to $-\infty$. See [10], section 2.6 for further discussion on pseudohyperbolic distances in $\mathbb{B}_N$. If $b \neq a$, then there is an $\epsilon > 0$ so that $D_\epsilon = \{z \mid |z - b| \leq \epsilon\}$ is contained in $\mathbb{B}_2$ and does not contain a. In this case, the iterates of D_ϵ converge to a and there is an n so that $\phi_n(D_\epsilon) \cap D_\epsilon = \emptyset$, which also contradicts the assumption that a subsequence $\{z_{k_j}\}$ of the iteration sequence converges to b with k_j tending to $-\infty$.

If ϕ is not an elliptic automorphism of the ball $\mathbb{B}_2$ onto itself and z_0 is a point of the ball, then $z_k = \phi_k(z_0)$ defines an iteration sequence of distinct points for $k \geq 0$ or else there is a least M so that $\phi_M(z_0) = a$, the Denjoy-Wolff point of ϕ in $\mathbb{B}_2$, and there is an iteration sequence of distinct points defined for $0 \leq k \leq M$. Moreover, either there is no point w of $\mathbb{B}_2$ with $\phi(w) = z_0$ or we can find z_{-1} so that $\phi(z_{-1}) = z_0$. So every point of the ball is in at least one iteration sequence $\{z_k\}_{k=K}^{M}$ of distinct points for which either $M = \infty$ and $\lim_{k\to\infty} = a$ or M is finite and $\phi_M(z_0) = a$ and either $K = -\infty$ and $\lim_{k\to-\infty}|z_k| = 1$ or K is finite and there is no point w in $\mathbb{B}_2$ with $\phi(w) = z_K$.

We will be particularly interested in the case $\{z_k\}_{k=-\infty}^{0}$ in which $\lim_{k\to-\infty} z_k = b$ where $b = (b_1, b_2)$ is a fixed point of ϕ on the boundary of the ball. If b is not the Denjoy-Wolff point, then we have seen by Theorem 4 that $\langle \phi'(b)b, b\rangle > 1$. As mentioned in the introduction, the case in which ϕ, an analytic map from $\mathbb{B}_N$ to $\mathbb{B}_N$, has interior fixed point ζ which by conjugation we may presume is 0, was solved in [5] and [13]. In addition to a few natural conditions on our map ϕ, it was determined that there could also be an arithmetic obstruction called resonance. In the case of $\mathbb{B}_2$, this occurs when the eigenvalues of $\phi'(0)$, given by λ_1 and λ_2, are such that either $\lambda_1 = \lambda_2^n$ or $\lambda_2 = \lambda_1^m$ for some non-negative integers $n, m > 1$. This obstruction only exists for $\mathbb{B}_N$ with $N > 1$.

If ϕ is analytic in a neighborhood of b, then there is $\epsilon > 0$ so that ϕ is univalent on $B_\epsilon = \{z \mid |z - b| \leq \epsilon\}$ and ϕ^{-1} maps B_ϵ into itself. If we fix $z_2 = b_2$, then $\phi^{-1}(z_1, b_2)$ maps the disk $D_\epsilon = \{z = (z_1, b_2) \mid |z - b| \leq \epsilon\}$ into itself. Since $\phi^{-1}(b) = b$, we have that b is an attractive interior fixed point of the disk D_ϵ and thus, by modifying the results of [9], we utilize the model theory of linear fractional maps in the disk, which do not have resonance obstructions, to construct an intertwining and obtain that there is an analytic map ψ so that $\psi(b) = 0$

$$\psi(\phi^{-1}(z)) = (\phi^{-1})'(b)\psi(z) = (\phi')^{-1}(b)\psi(z)$$

for z in D_ϵ. Since ϕ^{-1} is univalent near b, so is ψ. Multiplying ψ by appropriate constants of modulus 1 if necessary, without loss of generality we may assume that ψ maps the radial segment $\{rb \mid 1 - \epsilon < r < 1\}$ to a curve tangent to the vector $\langle 1, 0\rangle$ at 0. Now the conformality of ψ shows that if $\delta > 0$ is small enough that the interval $\{(r, 0) \mid 0 \leq r < \delta\}$ is in $\psi(D_\epsilon)$, the image of $\{(r, 0) \mid 0 \leq r < \delta\}$ under ψ^{-1} is a curve tangent to the radius $\{rb \mid 0 < r < 1\}$ in the ball. Now for w so that $w \in \{(r, 0) \mid 0 \leq r < \delta\}$, defining $z_{-k} = \psi^{-1}\left((\phi'^{-1}(b))^k w\right)$ for $k = 0, 1, ...$ gives an iteration sequence $\{z_j\}_{j=-\infty}^{0}$ so that z_j has restricted limit b as j approaches negative infinity. In particular, there are uncountably many such iteration sequences.

Theorem 19 *Let ϕ be an analytic map from the unit ball $\mathbb{B}_2$ into itself, not an automorphism of the ball, with corresponding composition operator C_ϕ acting on H^2_d. If $b \in \partial\mathbb{B}_2$ is a fixed point of ϕ with $\langle \phi'(b)b, b\rangle > 1$ and ϕ is analytic in a neighborhood of b, then for each $\rho < \langle \phi'(b)b, b\rangle^{-\frac{1}{2}}$, the circle of radius ρ centered at the origin intersects the spectrum of C_ϕ.*

Proof Suppose that ϕ satisfies the above conditions. Then, as we saw above, the fact that ϕ is analytic near the boundary fixed point b implies that there are uncountably many iteration sequences $\{z_j\}_{j=-\infty}^{0}$ that have restricted limit b. For a given iteration sequence, let $k_j(z) = \frac{(1-|z_j|^2)^{\frac{1}{2}}}{1-\langle z, z_j\rangle}$ denote the normalized reproducing kernel for the point z_j.

Suppose $\rho < \rho_1 < \langle \phi'(b)b, b\rangle^{-\frac{1}{2}}$ with $|\lambda| = \rho$. Let

$$h_\lambda = \sum_{j=-\infty}^{-1} \lambda^{-j-1} \left(\frac{1-|z_0|^2}{1-|z_j|^2}\right)^{\frac{1}{2}} k_j.$$

By the Julia-Carathéodory Theorem in $\mathbb{B}_2$, we have

$$\lim_{j\to-\infty} \frac{1-|z_{j+1}|^2}{1-|z_j|^2} = \langle \phi'(b)b, b\rangle.$$

Thus there is a constant c so that

$$\begin{aligned} |\lambda|^{-j-1} \left(\frac{1-|z_0|^2}{1-|z_j|^2}\right)^{\frac{1}{2}} &= |\lambda|^{-j-1} \left(\prod_{l=j}^{-1} \frac{1-|z_{l+1}|^2}{1-|z_l|^2}\right)^{\frac{1}{2}} \\ &\leq c \left(\frac{|\lambda|}{\rho_1}\right)^{-j} = c\left(\frac{\rho}{\rho_1}\right)^{-j}. \end{aligned}$$

Since $||k_j|| = 1$, the series for h_λ converges absolutely which implies it converges in H^2_d. Also $(C^*_\phi - \lambda)h_\lambda$ is

$$\sum_{j=-\infty}^{-1} \lambda^{-j-1} \left(\frac{1-|z_0|^2}{1-|z_{j+1}|^2}\right)^{\frac{1}{2}} k_{j+1} - \sum_{j=-\infty}^{-1} \lambda^{-j} \left(\frac{1-|z_0|^2}{1-|z_j|^2}\right)^{\frac{1}{2}} k_j = k_0.$$

Moreover, if $\sigma(C_\phi)$ does not intersect the circle $|\lambda| = \rho$, then $\left(C^*_\phi - \rho e^{i\theta}\right)^{-1}$ exists for each real θ and we define Q by

$$Q = \frac{1}{2\pi}\int_0^{2\pi} \left(C_\phi^* - \rho e^{i\theta}\right)^{-1} d\theta.$$

Now

$$\begin{aligned} Qk_0 &= \frac{1}{2\pi}\int_0^{2\pi} \left(C_\phi^* - \rho e^{i\theta}\right)^{-1} k_0 d\theta = \frac{1}{2\pi}\int_0^{2\pi} h_{\rho e^{i\theta}} d\theta \\ &= \sum_{j=-\infty}^{-1} \frac{1}{2\pi}\int_0^{2\pi} \rho^{-j-1} e^{i(-j-1)\theta} k_j \left(\frac{1-|z_0|^2}{1-|z_j|^2}\right)^{\frac{1}{2}} d\theta \\ &= \left(\frac{1-|z_0|^2}{1-|z_{-1}|^2}\right)^{\frac{1}{2}} k_{-1}. \end{aligned}$$

since $\frac{1}{i(-j-1)} e^{i(-j-1)\theta}$ evaluated from 0 to 2π is zero for all values of j except $j = -1$. Thus, writing $K_w(z) = \frac{1}{1-\langle z,w\rangle}$, the kernel for evaluation at w, we have $QK_{z_0} = K_{z_{-1}}$, which means $C_\phi^* Q K_{z_0} = K_{z_0}$. Since Q is a rational function of C_ϕ^*, they commute and we have $QC_\phi^* K_{z_0} = K_{z_0}$ also.

For each such iteration sequence $\{z_k\}_{k=-\infty}^0$, the argument above showed that $C_\phi^* Q K_{z_0} = Q C_\phi^* K_{z_0} = K_{z_0}$. Since there are uncountably many such z_0, their kernel functions span H_d^2 and we get $C_\phi^* Q = Q C_\phi^* = I$. But we assumed that ϕ is not an automorphism, so we have a contradiction. Thus our assumption that $\sigma(C_\phi)$ does not intersect the circle $|\lambda| = \rho$ must be false.

■

Up to this point, our results apply to the half space/dilation and Siegel half space/dilation cases. We next demonstrate an annulus of eigenvalues that lies in our spectrum when we are in the Siegel half space/dilation case. In preparation of this, we first prove the following lemma.

Lemma 20 *The function*

$$G(z) = \left(\frac{1+z_1}{1-z_1}\right)^k \left(\frac{z_2}{1-z_1}\right)^l$$

belongs to H_d^2 *for* $l \in \mathbb{N} \cup \{0\}$ *and* $|2k+l| < 1$ *where* $\mathbb{N} = \{1, 2, 3, ...\}$.

Proof First we see that to have $G \in H^2_d$, G must be holomorphic in the ball so we must have $l \in \mathbb{N} \cup \{0\}$ due to the factor z_2^l. Thus l is a nonnegative integer. We proceed to determine the restrictions on k and l by cases.

- First suppose that $k \geq 0$. In this case $(1+z_1)^k$ will be a multiplier of H^2_d and thus we may ask when $(1-z_1)^{-k} z_2^l (1-z_1)^{-l} = z_2^l (1-z_1)^{-(k+l)}$ is in H^2_d. We use the generalized binomial expansion formula to obtain

$$\frac{z_2^l}{(1-z_1)^{k+l}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+k+l)}{\Gamma(n+1)\Gamma(k+l)} z_1^n z_2^l$$

  which resides in H^2_d when

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+k+l)^2}{\Gamma(n+1)^2\Gamma(k+l)^2} \frac{n!l!}{(n+l)!} < \infty.$$

  Using Stirling's approximation to determine the behavior as $n \to \infty$, we find

$$\begin{aligned} \frac{\Gamma(n+k+l)^2}{\Gamma(n+1)^2\Gamma(k+l)^2} \frac{n!l!}{(n+l)!} &\sim \frac{\frac{2\pi}{n+k+l}\left(\frac{n+k+l}{e}\right)^{2(n+k+l)} \sqrt{2\pi l}\left(\frac{l}{e}\right)^l}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \frac{2\pi}{k+l}\left(\frac{k+l}{e}\right)^{2(k+l)} \sqrt{2\pi(n+l)}\left(\frac{n+l}{e}\right)^{n+l}} \\ &\sim \frac{(n+k+l)^{2n+2(k+l)-1}}{n^{n+\frac{1}{2}}(n+l)^{n+l+\frac{1}{2}}} \sim n^{2k+l-2}. \end{aligned}$$

  where we see that our sum converges when $2-(2k+l) > 1$ or $2k+l = |2k+l| < 1$.

- Now suppose $k < 0$ and $k+l < 0$. Then $(1-z_1)^{-(k+l)}$ will be a multiplier of H^2_d and we consider when $z_2^l(1+z_1)^{-t}$ is in H^2_d where $t = -k > 0$. Using the above computation (replacing $1-z_1$ with $1+z_1$ makes no difference) by replacing $k+l$ with t, we find that we need

$$\begin{aligned} \frac{\Gamma(n+t)^2}{\Gamma(n+1)^2\Gamma(t)^2} \frac{n!l!}{(n+l)!} &\sim \frac{\frac{2\pi}{n+t}\left(\frac{n+t}{e}\right)^{2(n+t)} \sqrt{2\pi l}\left(\frac{l}{e}\right)^l}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \frac{2\pi}{t}\left(\frac{t}{e}\right)^{2t} \sqrt{2\pi(n+l)}\left(\frac{n+l}{e}\right)^{n+l}} \\ &\sim \frac{(n+t)^{2n+2t-1}}{n^{n+\frac{1}{2}}(n+l)^{n+l+\frac{1}{2}}} \sim n^{2t-l-2}. \end{aligned}$$

where our sum converges when $2-(2t-l) = 2+2k+l > 1$ which implies $2k+l > -1$. Since our conditions $k<0$ and $k+l<0$ imply $2k+l<1$, we conclude again that $|2k+l|<1$.

- Now suppose that $k<0$ and $k+l>0$. We further break up into two additional subcases. First suppose $2k+l<0$. Then

$$\left(\frac{1+z_1}{1-z_1}\right)^k\left(\frac{z_2}{1-z_1}\right)^l = \left(\frac{1+z_1}{1-z_1}\right)^k\left(\frac{z_2}{1-z_1}\right)^l\frac{(1-z_1)^k}{(1-z_1)^k} = \frac{z_2^l(1-z_1^2)^k}{(1-z_1)^{2k+l}}.$$

Since $2k+l<0$, $(1-z_1)^{-(2k+l)}$ is a multiplier. Let $t=-k$ so that $z_2^l(1-z_1^2)^{-t} = \sum_{n=0}^{\infty}\frac{\Gamma(n+t)}{\Gamma(n+1)\Gamma(t)}z_1^{2n}z_2^l$. Then we observe the behavior as $n\to\infty$ of the expression

$$\begin{aligned}\frac{\Gamma(n+t)^2}{\Gamma(n+1)\Gamma(t)}\frac{(2n)!l!}{(2n+l)!} &\sim \frac{(n+t)^{2(n+t)-1}}{n^{2n+1}}\frac{n^{\frac{1}{2}}2^{2n}n^{2n}}{(2n+l)^{\frac{1}{2}}(2n+l)^{2n+l}}\\ &\sim n^{2t-2-l}.\end{aligned}$$

Thus in order to converge, we must have $1<2-(2t-l)=2+2k+l$ which implies $2k+l>-1$ which again gives $|2k+l|<1$.

Finally, suppose $k<0$, $k+l>0$ and $2k+l>0$. Then we have

$$\left(\frac{1+z_1}{1-z_1}\right)^k\left(\frac{z_2}{1-z_1}\right)^l\frac{(1+z_1)^{k+l}}{(1+z_1)^{k+l}} = \frac{z_2^l(1+z_1)^{2k+l}}{(1-z_1^2)^{k+l}}.$$

Now we have that $(1+z_1)^{2k+l}$ is a multiplier and by the above calculation $z_2^l(1-z_1^2)^{-(k+l)}$ is in our space when $1<2-(2(k+l)-l)$ which implies $2k+l<1$ as desired.

This exhausts all of our cases.

■

We now proceed to determining our annulus of eigenvalues for $H_d^2(\mathbb{B}_2)$ for the Siegel half space/dilation case.

Lemma 21 *Suppose ϕ is an analytic map from $\mathbb{B}_2$ into $\mathbb{B}_2$ in the Siegel half-space model with model map $\Phi(z) = (\alpha z_1, \beta z_2)$ and Denjoy-Wolff point $(1,0)$. If*

$$\alpha^{\frac{1}{2}} < |\lambda| < \alpha^{-\frac{1}{2}}$$

then λ is an eigenvalue of C_ϕ on H^2_d.

Proof It is often useful to map problems in the ball to the Siegel half space, which one can show is biholomorphic to the ball in two variables via the map from $\mathbb{B}_2$ to $\mathbb{H}^2$ given by $\Psi(z) = (\Psi_1(z), \Psi_2(z)) = \left(\frac{1+z_1}{1-z_1}, \frac{z_2}{1-z_1}\right)$ with inverse $\Psi^{-1}(z) = \left(\frac{z_1-1}{z_1+1}, \frac{2z_2}{z_1+1}\right)$. Suppose ϕ is in the Siegel half space model so that $\sigma \circ \phi = \Phi \circ \sigma$ for appropriate $\sigma(z) = (\sigma_1(z), \sigma_2(z))$ and $\Phi(z) = (\alpha z_1, \beta z_2)$ where σ maps the ball into the Siegel half space. Let

$$\gamma(z) = \Psi^{-1} \circ \sigma(z) = \left(\frac{\sigma_1(z) - 1}{\sigma_1(z) + 1}, \frac{2\sigma_2(z)}{\sigma_1(z) + 1}\right)$$

where $\gamma : \mathbb{B}_2 \to \mathbb{B}_2$. For $\lambda = \alpha^k \beta^l$ and $F(z) = z_1^k z_2^l$ we have $f(z) = F \circ \sigma(z) = (\sigma_1(z))^k (\sigma_2(z))^l$. Let $G = \Psi_1^k \Psi_2^l$ so that $f = \sigma_1^k \sigma_2^l = G \circ \gamma$. From the above lemma we know that $G(z) = \left(\frac{1+z_1}{1-z_1}\right)^k \left(\frac{z_2}{1-z_1}\right)^l$ is in H^2_d when $|2k + l| < 1$. Thus for $\lambda = \alpha^k \beta^l$ we have

$$|\lambda| = |\alpha^k \beta^l| = |\beta|^{2k+l} \Rightarrow \alpha^{\frac{1}{2}} < |\lambda| < \alpha^{-\frac{1}{2}}.$$

■

Notice that we used the fact that σ maps the ball into the Siegel half space. This is the step where we had to deviate from considering the half space/dilation case.

We next turn our attention toward the spectral radius of C_ϕ.

Lemma 22 *If ϕ is in the Siegel half space/dilation or half space/dilation model and in the Schur-Agler class, then $||C_\phi||$ is bounded on H^2_d and*

$$\left(\frac{1}{1 - |\phi(0)|^2}\right)^{\frac{1}{2}} \leq ||C_\phi|| \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right)^{\frac{1}{2}}.$$

Proof The lower bound is attained by noting that $k(z,0) = 1$ and

$$||C_\phi^*|| \geq ||C_\phi^* k(z,0)|| = ||k(z,\phi(0))|| = \left(\frac{1}{1-|\phi(0)|^2}\right)^{\frac{1}{2}}.$$

The upper bound is obtained by appealing to [19] for our result.

■

Theorem 23 *Let ϕ be in the Schur-Agler class and in the Siegel half space/dilation or half space/dilation case with linear fractional model $\Phi(z) = (\alpha z, \beta z)$. Then the spectral radius of ϕ acting on H_d^2 is $\alpha^{-\frac{1}{2}}$.*

Proof Conjugating by a rotation, we may presume that ϕ has Denjoy-Wolff point given by $(1,0)$. We will move the problem to the Siegel half-space $\mathbb{H}^2$ via the bi-holomorphic map $\Psi(z) = \left(\frac{1+z_1}{1-z_1}, \frac{z_2}{1-z_1}\right)$. Now suppose ϕ has an intertwining map $\sigma : \mathbb{B}_2 \to \mathbb{H}^2$ so that $\sigma \circ \phi = \Phi \circ \sigma$. By [19], the spectral radius is given over H_d^2 by

$$r(C_\phi) = \lim_{n\to\infty} (1 - |\phi_n(0)|)^{-\frac{1}{2n}}.$$

It suffices, then, to show that $\lim_{n\to\infty}(1-|\phi_n(0)|^2)^{\frac{1}{n}} = \alpha$. Recall that for any $z \in \mathbb{B}_2$, if $w = \Psi(z)$ we have the identity

$$1 - |z|^2 = \frac{4}{|w_1+1|^2}\left(\Re(w_1) - |w_2|^2\right).$$

By our intertwining hypothesis, ϕ is conjugate to $\Phi(z) = (\alpha z_1, \beta z_2)$ where $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$. We note that $\Psi(0,0) = (1,0)$ and $\Phi_n(1,0) = (\alpha^n, 0)$ with $0 < \alpha < 1$. We calculate

$$\lim_{n\to\infty}(1-|\phi_n(0)|^2)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{4}{|\alpha^n+1|^2}\left(\Re(\alpha^n) - |0|^2\right)\right)^{\frac{1}{n}} = \alpha$$

from which our result follows.

■

Before proceeding further, it will be instructive to say a few words on a special class of functions known as inner functions.

Definition 24 *Let $\mathbb{B}_N$ be the unit ball in $\mathbb{C}^N$. We say a nonconstant function f is an inner function in $\mathbb{B}_N$ if $f \in H^\infty(\mathbb{B}_N)$ with radial limit f^* such that $|f^*(\zeta)| = 1$ for almost all $\zeta \in \partial\mathbb{B}_N$, the boundary of the ball.*

For $N = 1$, where the ball is the unit disk, inner functions play an important role for factorization theorems, invariant subspaces, and numerous other applications (see [6] for a recent survey). The most well-known being Blaschke products, which are functions of the form

$$B(z) = e^{i\theta} z^m \prod_{n \geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

where $\theta \in \mathbb{R}$, m is a non-negative integer, and $\{a_1, a_2, ...\}$ is a (not necessarily finite) sequence with $0 < |a_i| < 1$ for all $1 \leq i \leq \infty$ satisfying the Blaschke condition given by

$$\sum_{n \geq 1} (1 - |a_n|) < \infty.$$

These are by no means the only inner functions in the disk. One may even construct inner functions that have no zeroes in the disk. Consider $e^{\frac{z+1}{z-1}}$ with $z \in \mathbb{D}$ as an example. In the disk, methods of determining the spectrum of the composition operator C_ϕ bifurcate into two cases, depending on whether or not ϕ is an inner function from the disk into itself. While our annulus of eigenvalues determined in Lemma 21 and spectral radius determined in Theorem 23 did not depend on this bifurcation, our final result will.

In the ball $\mathbb{B}_N$ for $N > 1$, things become more complicated. In order for such functions to exist, they would have to exhibit certain pathological features including extreme oscillatory behavior near the boundary (see [27], section 19.1). The existence of such functions for $N > 1$ was first demonstrated by Aleksandrov [2] and independently by Low [23] building on results by Hakim and Sibony [14]. Later the results were strengthened by Rudin [28], including the fact that this set is dense in the unit ball of $H^\infty(\mathbb{B}_N)$ in the compact-open topology. While in the disk, inner functions

are sufficiently well-behaved as to merit further study, we find them to be sufficiently pathological in $\mathbb{B}_2$ as to exclude them.

It can be shown that a self map of the disk, analytic in a neighborhood of the closed disk and not an inner function, must map, at most, a finite amount of points from the boundary into the boundary (Lemma 1.3, [21]). The generalization of this result is not immediate due to the critical use of the identity principle and accumulation points in the case of the disk. In particular, although the identity principle generalizes, zeroes of a holomorphic function are never isolated and thus requiring an accumulation point of zeroes puts no restriction on the function [4].

We thus make the hypothesis that our function ϕ, analytic in a neighborhood of the closed unit ball and not an inner function, is such that the set $\{z \in \partial\mathbb{B}_2 \mid |\phi(z)| = 1\}$ consists of at most a finite set of points. The generalization that seems to be likely is that the family of curves $\{\Gamma_i\}$ that lie on the boundary of $\mathbb{B}_2$ such that ϕ maps each Γ_i back into the boundary, must be a finite set.

Conjecture 25 *Let ϕ be a map of $\mathbb{B}_2$ into itself, analytic in a neighborhood of the closed unit ball that does not map the ball into a lower-dimensional affine set. The family of curves $\{\Gamma_i\}$ on the boundary of $\mathbb{B}_2$ such that ϕ maps Γ_i back into the boundary is a finite set.*

It doesn't appear that replacing our hypothesis with the above conjecture would interrupt the validity of our results. From our hypothesis we acquire the following theorem.

Theorem 26 *If ϕ is analytic in a neighborhood of the closed unit ball and does not map the ball into a lower-dimensional affine set, then there is a positive integer n such that $\{z \mid |\phi_n(z)| = 1\}$ is either empty or consists entirely of fixed points of ϕ_n.*

Proof We know by our hypothesis that the set $\{b \in \partial\mathbb{B}_2 \mid |\phi(b)| = 1\}$ is finite. Denote this set of points by $\{b_i\}_1^m$. Given b_i, if for some positive integer n, we have $|\phi_n(b_i)| < 1$, then we are done. Otherwise, $|\phi_j(b_i)| = 1$ for all positive integers

j. Either $\phi_{j+k}(b_i) = \phi_j(b_i)$ for some positive integers j and k or the iterates of $\phi_j(b_i)$ form an infinite set of elements where each element is on the boundary and of magnitude 1, contrary to our assumption. For each b_i, we thus have a j_i and k_i such that $\phi_{j_i+k_i}(b_i) = \phi_{j_i}(b_i)$. Taking n to be the appropriate multiple of the k_i's gives our result. ■

Theorem 27 *Suppose ϕ maps the unit ball $\mathbb{B}_2$ into itself and is analytic in a neighborhood of the closed unit ball. Suppose ϕ has Denjoy-Wolff point a on the boundary of the ball with $\alpha = \langle \phi'(a)a, a\rangle < 1$. Suppose also that $\{z_j \mid |\phi(z_j)| = 1\} = \{a, b_1, b_2, ..., b_k\}$, where $\phi(b_j) = b_j$ for $j = 1, ..., k$. If*

$$\max\{\langle \phi'(b_j)b_j, b_j\rangle^{-\frac{1}{2}} \mid j = 1, 2, ..., k\} < |\lambda| < \langle \phi'(a)a, a\rangle^{-\frac{1}{2}}$$

then λ is an eigenvalue of C_ϕ on $H^2_d(\mathbb{B}_2)$ of infinite multiplicity.

Proof We have already seen the results for λ with $\alpha^{\frac{1}{2}} < |\lambda| < \alpha^{-\frac{1}{2}}$. Let $r_0 = \min\{\langle \phi'(b_j)b_j, b_j\rangle \mid j = 1, 2, ..., k\}$ and let $\alpha = \langle \phi'(a)a, a\rangle$. Suppose λ and λ_0 are positive numbers satisfying $r_0^{-\frac{1}{2}} < \lambda_0 < \lambda < 1$. Circular symmetry will show that the conclusion will follow if we show λ is an eigenvalue. Since $|\phi(z)| < 1$ for all $z \in \overline{\mathbb{B}}_2$ except for the values a and b_j for $j = 1, ..., k$, we see that for every $\epsilon > 0$, the set given by

$$\{\phi(z) \mid |z| \leq 1 \text{ and } |z - b_j| \geq \epsilon \text{ for } j = 0, 1, 2, ..., k\}$$

is a compact subset of $\mathbb{B}_2$. Thus by the hypothesis on our model for iteration, there is a constant M_ϵ so that $|\sigma(z)| \leq M_\epsilon$ for $|z - b_j| \geq \epsilon$. Since $\langle \phi'(b_j)b_j, b_j\rangle^{-\frac{1}{2}} < \lambda_0 < 1$, we have

$$\frac{|\log \alpha|}{\log\langle \phi'(b_j)b_j, b_j\rangle} < \frac{|\log \alpha|}{-2\log \lambda_0}$$

By our growth estimate given by Theorem 15 there are M_j and ϵ_j so that

$$|\sigma(z)| \leq M_j|z - b_j|^{-\left|\frac{3\log \alpha}{2\log \lambda_0}\right|}$$

for $|z - b_j| < \epsilon_j$. Thus there is an M so that for $z \in \mathbb{B}_2$, we have

$$|\sigma(z)| \leq M \prod_{j=1}^{k} |z - b_j|^{-\left|\frac{3 \log \alpha}{2 \log \lambda_0}\right|}$$

and for $r_0^{-\frac{1}{2}} < \lambda_0 < \lambda < 1$. Let $x = \frac{\log \lambda}{3 \log \alpha}$. Note that for the values of λ under consideration, $x > 0$. This implies

$$\begin{aligned} |\sigma(z)^{2x}| &\leq M^{2x} \prod_{j=1}^{k} \left(|z - b_j|^{-\left|\frac{3 \log \alpha}{2 \log \lambda_0}\right|} \right)^{2\left|\frac{\log \lambda}{3 \log \alpha}\right|} \\ &= M^{2x} \prod_{j=1}^{k} |z - b_j|^{-\left|\frac{\log \lambda}{\log \lambda_0}\right|} \end{aligned}$$

Since $0 < \lambda_0 < \lambda < 1$ implies $\left|\frac{\log \lambda}{\log \lambda_0}\right| < 1$, the product is seen to be integrable along the boundary of the unit ball $\mathbb{B}_2$ and thus σ^x is in $H_d^2(\mathbb{B}_2)$. We conclude that each λ such that $r_0^{-\frac{1}{2}} < \lambda < \alpha^{-\frac{1}{2}}$, is an eigenvalue of C_ϕ on $H_d^2(\mathbb{B}_2)$. Applying circular symmetry, we conclude our results.

■

We now combine the sum of our work to establish the main result.

Theorem 28 *Let α be the dilation coefficient of an analytic map ϕ, from $\mathbb{B}_2$ into itself, that resides in the Schur-Agler class and is in the Siegel half space/dilation model where ϕ is not inner and no slice function of ϕ is inner. Then the spectrum of ϕ acting on H_d^2 is given by*

$$\sigma(C_\phi) = \{\lambda \mid |\lambda| \leq \alpha^{-\frac{1}{2}}\}.$$

Proof Since ϕ is not an inner function, we know by our hypothesis that the set $\{z \mid |\phi(z)| = 1\}$ is a finite set. By Theorem 26 there is a positive integer n such that the set $\{z \mid |\phi_n(z)| = 1\}$ consists of fixed points $\{a, b_1, ...b_k\}$, where a is the Denjoy-Wolff point, for some integer k. We know from Theorem 4 that a is the unique point

in which $\langle \phi'(a)a, a\rangle \leq 1$. Now suppose a is the Denjoy-Wolff point for ϕ. Then since $a^T a = 1$, we have

$$\langle \phi_n'(a)a, a\rangle = \langle (\phi'(a))^n a, a\rangle = a^T (\phi'(a))^n a$$
$$a^T\phi'(a)aa^T\phi'(a)a\cdots a^T\phi'(a)a = \left(a^T\phi'(a)a\right)^n \leq 1.$$

Thus a is the Denjoy-Wolff point for ϕ_n as well. By Theorem 19 we have that $\sigma(C_{\phi_n})$ intersects the circle of radius r for $0 < r < r_0^{-\frac{1}{2}}$ where $r_0 = \min\{\langle \phi_n'(b_j)b_j, b_j\rangle\}$. We also have by Theorem 27 that $\sigma(C_{\phi_n})$ includes the circle of radius r for $r_0^{-\frac{1}{2}} < r < \alpha^{-\frac{n}{2}}$. Since $C_{\phi_n} = C_\phi^n$, we have by the spectral mapping theorem that $\sigma(C_\phi)$ intersects the circle of radius r for $0 < r < \alpha^{-\frac{1}{2}}$. By Theorem 17 we know that $\sigma(C_\phi)$ includes the disk $\{\lambda \mid |\lambda| \leq \alpha^{-\frac{1}{2}}\}$. Since we are presuming that ϕ is in the Schur-Agler class, Theorem 23 implies that this circle is exactly the spectrum and our proof is complete. ■

3.3 An Explicit Example

Example 29 We next construct an example of a self map of the ball in $\mathbb{C}^2$ that satisfies the requirements of Theorem 28. Consider the linear fractional map given by

$$\phi(z) = \left(\frac{z_1 + 3}{4}, \frac{z_2}{2}\right).$$

It is clear that $\phi : \mathbb{B}_2 \to \mathbb{B}_2$ has no fixed points in the ball and for $\zeta = (1, 0)$ we have $\phi(\zeta) = \zeta$ as the Denjoy-Wolff point with $D_\zeta\phi_\zeta(\zeta) = \frac{1}{4}$. By [20], linear fractional maps are in the Schur-Agler class. To conform to our model we put $\sigma(z) = e_1 - z$ and $\Phi(z) = (\frac{1}{4}z_1, \frac{1}{2}z_2)$ which gives us

$$\sigma \circ \phi(z) = \left(1 - \frac{z_1 + 3}{4}, -\frac{z_2}{2}\right) = \left(\frac{1}{4}(1 - z_1), -\frac{1}{2}z_2\right) = \Phi \circ \sigma(z).$$

Our conditions are satisfied and thus the spectrum of C_ϕ is given by

$$\sigma(C_\phi) = \{\lambda \in \mathbb{C} \mid |\lambda| < \left(\frac{1}{4}\right)^{-\frac{1}{2}} = 2\}.$$

where $\lambda_1 = |\lambda_2|^2$ and we thus have a Siegel Half Space/Dilation case.

4. SEMIGROUPS IN SEVERAL COMPLEX VARIABLES

4.1 Background

Recall that a **one-parameter semigroup** for a monoid $(S, *)$ is a map $\phi : [0, \infty) \to S$, such that

i. $\phi(0) = I$.

ii. $\phi(s + t) = \phi(s) * \phi(t)$.

Let $B_N = \{(z_1, ..., z_N) \in \mathbb{C}^N \mid \sum_{i=1}^{N} |z_i|^2 < 1\}$ be the unit ball in $\mathbb{C}^N$ and

$T = \{\phi : B_N \to B_N \mid \phi$ is a nonconstant analytic map, not an automorphism$\}$.

For $\phi \in T$, it is clear that the set of iterates $\{\phi_n\}$ under composition for $n = 0, 1, 2, ...$, defines a discrete semigroup. We know that this discrete semigroup can be extended to a one-parameter semigroup in one complex variable [9]. The proof of this result depends on Cowen's model theory of linear fractional maps, as discussed in our introduction, in which we intertwine analytic functions with model linear fractional maps that fall into four categories classified according to their behavior near the Denjoy-Wolff point and what is known as the characteristic domain [9]. As we saw, we may extend the definition of the Denjoy-Wolff point to maps of more than one complex variable using results by MacCluer [24]. This allows us to analyze the model theory in higher dimensions. In all cases, we suppose the derivative of our map ϕ at the Denjoy-Wolff point is not zero. We will use the generalization of this model theory in two complex variables applied to linear fractional maps in order to define a continuous one-parameter semigroup on the set of linear fractional maps in two complex variables.

Of necessity, σ will be an invertible linear fractional map and we may write

$$\phi = \sigma^{-1} \circ \Phi \circ \sigma. \tag{4.1}$$

We can thus define a discrete semigroup for the set $\{\phi_n\}$ by

$$\phi_n = \sigma^{-1} \circ \Phi_n \circ \sigma \tag{4.2}$$

where Φ_n is our model linear fractional map.

4.2 Semigroups for Linear Fractional Maps in Two Complex Variables

It can be shown that the eigenvectors of the associated matrix m_ϕ correspond to fixed points of ϕ [8]. We assume our maps are invertible and thus we do not have to consider zero eigenvalues. We may use Jordan form of a matrix to factor m_ϕ to obtain $m_\phi = S\Lambda S^{-1}$ where the columns of S are (generalized) eigenvectors of m_ϕ and Λ is in Jordan form. Given a linear fractional map ϕ of $\mathbb{B}_N$ into itself and an automorphism ψ of $\mathbb{B}_N$, we see that

$$m_\psi m_\phi m_\psi^{-1} = m_\psi S\Lambda S^{-1} m_\psi^{-1} = (m_\psi S)\Lambda(m_\psi S)^{-1}.$$

Thus, not only are our maps are equivalent up to conjugation by an automorphism, but conjugation by an automorphism yields the same Jordan form matrix Λ.

We see that, for $n = 0, 1, 2, \ldots$, the iterates are given by $m_{\phi_n} = (m_\phi)^n = S\Lambda^n S^{-1}$. We then would like to extend this definition by finding an expression for Λ^n and replacing $n = 0, 1, 2, \ldots$, with $t \in [0, \infty)$. The form of Λ will depend on which of the seven cases we are in. Since cases II, III,VI, and VII can be written as direct sums of lower dimensional associated matrices, our new result will consist of defining a one parameter semigroup for cases IV and V. We proceed to compute the cases with Denjoy-Wolff point on the boundary.

In order to define a one-parameter semigroup, we must be sure that our map stays in our space for fractional iterates. This will follow from the fact that the half-space and the Siegel half-space are convex domains in $\mathbb{C}^2$. We see this as follows.

It is clear that, since in each case Λ is in Jordan form (taken so that the off-diagonal elements are ones on the subdiagonal above the diagonal), each of the model maps Λ are associated with a map of the form $Az + B$.

Now, given two vectors (u_1, u_2) and (w_1, w_2) in the half-space, we have for $t \in [0, 1]$

$$\Re(tu_1 + (1-t)w_1) = t\Re u_1 + (1-t)\Re w_1 > 0$$

and thus the half space is convex.

Given two vectors (u_1, u_2) and (w_1, w_2) in the Siegel half-space, we have for $t \in [0, 1]$

$$\begin{aligned}\Re(tu_1 + (1-t)w_1) = t\Re u_1 + (1-t)\Re w_1 &> t|u_2|^2 + (1-t)|w_1|^2 \\ &\geq t^2|u_2|^2 + (1-t)^2|w_1|^2 \geq |tu_2 + (1-t)w_2|^2\end{aligned}$$

and thus the Siegel half space is convex.

In cases II and III, we have three distinct fixed points and thus m_ϕ is diagonalizable. Suppose

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

For n, a nonnegative integer, we have $m_{\phi_n} = (m_\phi)^n = S\Lambda^n S^{-1}$ where

$$\Lambda^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}.$$

We may embed this in a continuous semigroup defined by $m_{\phi_t} = S\Lambda^t S^{-1}$ for all $t \geq 0$ where

$$\Lambda^t = \begin{pmatrix} \lambda_1^t & 0 & 0 \\ 0 & \lambda_2^t & 0 \\ 0 & 0 & \lambda_3^t \end{pmatrix}.$$

We then have for $s, t \geq 0$,

$$m_{\phi_t} m_{\phi_s} = S\Lambda^t S^{-1} S\Lambda^s S^{-1} = S\Lambda^{t+s} S^{-1} = m_{\phi_{t+s}}$$

from which it follows that $\phi_{t \circ s} = \phi_{t+s}$.

In cases VI and VII, we note that Λ is given by

$$\Lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \eta \end{pmatrix}$$

where λ may equal η depending on whether we are in case VI or VII. Hence we find

$$m_{\phi_n} = S\Lambda^n S^{-1} = S(A^n \oplus B^n)S^{-1}$$

with

$$A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \quad \text{and} \quad B^n = (\eta^n). \tag{4.3}$$

Hence we can embed this into a continuous semigroup defined for all $t \geq 0$ by

$$\Lambda = \begin{pmatrix} \lambda^t & t\lambda^{t-1} & 0 \\ 0 & \lambda^t & 0 \\ 0 & 0 & \eta^t \end{pmatrix}.$$

We check this satisfies the one-parameter semigroup properties by noting

$$\Lambda^t \Lambda^s = \begin{pmatrix} \lambda^t & t\lambda^{t-1} & 0 \\ 0 & \lambda^t & 0 \\ 0 & 0 & \eta^t \end{pmatrix} \begin{pmatrix} \lambda^s & s\lambda^{s-1} & 0 \\ 0 & \lambda^s & 0 \\ 0 & 0 & \eta^s \end{pmatrix} = \begin{pmatrix} \lambda^{t+s} & (t+s)\lambda^{t+s-1} & 0 \\ 0 & \lambda^{t+s} & 0 \\ 0 & 0 & \eta^{t+s} \end{pmatrix} = \Lambda^{t+s}$$

and thus

$$m_{\phi_t} m_{\phi_s} = S\Lambda^t S^{-1} S\Lambda^s S^{-1} = S\Lambda^{t+s} S^{-1} = m_{\phi_{t+s}}$$

from which it follows that $\phi_{tos} = \phi_{t+s}$.

In cases IV and V, we have one fixed point of multiplicity three. In these cases, Λ has the form

$$\Lambda = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}.$$

We may assume that the diagonal elements are all 1 as, for $\lambda = \frac{1}{\alpha}$, we have

$$\Lambda = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$$

and any multiple of an associated vector in $\mathbb{C}^{N+1}$ is associated with the same vector in $\mathbb{C}^N$. Thus, without loss of generality,

$$\Lambda = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}.$$

It is a straightforward result to show that for n a nonnegative integer we have $m_{\phi_n} = (m_\phi)^n = S\Lambda^n S^{-1}$ where

$$\Lambda^n = \begin{pmatrix} 1 & \lambda n & \frac{\lambda^2 n(n-1)}{2} \\ 0 & 1 & \lambda n \\ 0 & 0 & 1 \end{pmatrix}.$$

We may embed this in a continuous semigroup defined by $m_{\phi_t} = S\Lambda^t S^{-1}$ for all $t \geq 0$ where

$$\Lambda^t = \begin{pmatrix} 1 & \lambda t & \frac{\lambda^2 t(t-1)}{2} \\ 0 & 1 & \lambda t \\ 0 & 0 & 1 \end{pmatrix}.$$

We check this satisfies the one-parameter semigroup properties by noting

$$\Lambda^t \Lambda^s = \begin{pmatrix} 1 & \lambda t & \frac{\lambda^2 t(t-1)}{2} \\ 0 & 1 & \lambda t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda s & \frac{\lambda^2 s(s-1)}{2} \\ 0 & 1 & \lambda s \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda(t+s) & \frac{\lambda^2 (t+s)(t+s-1)}{2} \\ 0 & 1 & \lambda(t+s) \\ 0 & 0 & 1 \end{pmatrix} = \Lambda^{t+s}$$

and thus

$$m_{\phi_t} m_{\phi_s} = S\Lambda^t S^{-1} S\Lambda^s S^{-1} = S\Lambda^{t+s} S^{-1} = m_{\phi_{t+s}}$$

from which it follows that $\phi_{tos} = \phi_{t+s}$.

Example 30 Let ϕ be the linear fractional map from $\mathbb{B}_2$ into $\mathbb{B}_2$ given by

$$\phi(z) = \left(\frac{z_1 + 2z_2 + 1}{-z_1 + 2z_2 + 3}, \frac{-2z_1 + 2z_2 + 2}{-z_1 + 2z_2 + 3} \right).$$

It can be shown that this map corresponds to case V. We have

$$\phi(z) = \frac{Az + B}{C^* z + D}$$

where $A = \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $C = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $D = 3$.

Then

$$m_\phi = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 2 & 2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

so $\lambda = \frac{1}{2}$ and we have

$$m_{\phi_t} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{2} & \frac{t(t-1)}{8} \\ 0 & 1 & \frac{t}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2-t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & \frac{t^2+2}{2} \end{pmatrix}$$

which implies

$$\phi_t(z_1, z_2) = \left(\frac{(2-t^2)z_1 + 2tz_2 + t^2}{-t^2 z_1 + 2tz_2 + t^2 + 2}, \frac{-2tz_1 + 2z_2 + 2t}{-t^2 z_1 + 2tz_2 + t^2 + 2} \right).$$

It is a straightforward calculation to see that $\phi_0 = I$ and $\phi_1 = \phi$.

4.3 Classification of Linear Fractional Maps in the Siegel Half Space/Dilation and Half Space/Dilation Cases

We see in [8], under some very general conditions, a complete classification of linear fractional maps in $\mathbb{C}^2$. We would like to know, given a linear fractional map ϕ from the ball into the ball, which of the seven cases we are in. We first note that, for a map ϕ, we have that ϕ is in case I if and only if ϕ has an interior fixed point of the ball. This follows from the fact that if ϕ has a interior fixed point in the ball, then Φ has an interior fixed point in $\sigma(B_2)$ and case I is the only case where Φ has an interior fixed point. We next note that cases II and III correspond to maps with three distinct fixed points. Following the reasoning used in [4], we know that for a map ϕ with three distinct fixed points, the associated matrix m_ϕ is diagonalizable. If we use Jordan form to write $m_\phi = S\Lambda S^{-1}$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

then we have

$$m_\phi^n = S \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix} S^{-1}.$$

As noted in [8], since Φ is an automorphism of Ω and $\sigma(B_2) \subset \Omega$, we get $\Phi^{-1}(\sigma(B_2)) \subset \Phi^{-1}(\Omega) = \Omega$ and, in general, $\Phi^{-n}(\sigma(B_2)) \subset \Omega$ for every positive integer n. Thus, since Ω is to be the smallest domain satisfying condition (1.2), we have $\Omega = \cup_{n=1}^{\infty} \Phi^{-n}(\sigma(B_2))$.

Given a linear fractional map ϕ with three distinct fixed points, we determine ϕ_n using the techniques above and then calculate

$$|\phi_n(z_1)|^2 + |\phi_n(z_2)|^2.$$

We want this expression to be inside the ball for sufficiently large n. For convenience we can take $\Phi = \phi$ and $\sigma(z) = z$ and ask what space eventually maps the above expression into the ball. Since the half space and Siegel half-space are holomorphically distinct in $\mathbb{C}^2$, once we determine the above space, we will be able to standardize it to a half space or Siegel half space, classifying the map.

4.4 Semigroups for Analytic Maps in Two Complex Variables

In one complex variable, it is known [9] that under very general conditions, an analytic map ϕ from the disk into the disk can be intertwined with a linear fractional map $\Phi : \Omega \to \Omega$ and an analytic map σ such that

$$\sigma \circ \phi = \Phi \circ \sigma \tag{4.4}$$

where Ω is the characteristic domain. The proof of this relies on the Riemann Mapping Theorem which does not hold for $\mathbb{C}^N$ if $N > 1$. However, since we know that our model maps in $\mathbb{C}^2$ stay in our appropriate space for fractional iterates, we can define a one-parameter semigroup for a map ϕ by explicitly constructing σ and Φ.

Example 31 In this example, we will construct a map for the Siegel half space/Heisenberg translation case (case V). We will follow the below commutative diagram where $\Psi(z) = \frac{z+1}{-z_1+1}$ is the Cayley map from $\mathbb{B}_N$ to $\mathbb{H}^2$ with $\Psi^{-1}(z) = \left(\frac{z_1-1}{z_1+1}, \frac{2z_2}{z_1+1}\right)$ and $\sigma = \omega \circ \Psi$.

$$\begin{array}{ccc} \mathbb{B}_2 & \xrightarrow{\phi} & \mathbb{B}_2 \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathbb{H}^2 & \xrightarrow{\Phi} & \mathbb{H}^2 \end{array}$$

We will choose $\omega(z) = (\sqrt{2z_1}, \sqrt{z_2})$ where we take the principal branch of the square root. One can show $\omega : \mathbb{H}^2 \to \mathbb{H}^2$. Then $\omega^{-1}(z) = (\frac{1}{2}z_1^2, z_2^2)$ with

$$\sigma(z) = \left(\sqrt{\frac{2(z_1+1)}{-z_1+1}}, \sqrt{\frac{z_2}{-z_1+1}}\right) \quad \text{and} \quad \sigma^{-1}(z) = \left(\frac{z_1^2-2}{z_1^2+2}, \frac{4z_2^2}{z_1^2+2}\right). \tag{4.5}$$

Next, we want to choose Φ to correspond to case V. By [8], we know that for case V, ϕ is equivalent to a Heisenberg translation whose associated matrix has one Jordan block. Recall that a Heisenberg translation in $\mathbb{C}^2$ is a linear fractional map of the form $h_b(z) = Az + b$ where $A = \begin{pmatrix} 1 & 2\overline{b_2} \\ 0 & 1 \end{pmatrix}$ and $b = (b_1, b_2)^T$. Thus we choose our map Φ to be the Heisenberg translation given by

$$\Phi(z) = \left(z_1 + \frac{1}{2}z_2 + \frac{1}{2}, z_2 + \frac{1}{4}\right).$$

We then define $\phi = \sigma^{-1} \circ \Phi \circ \sigma$. A calculation shows $\phi(z) = (\phi_1(z), \phi_2(z))$ where

$$\phi_1(z) = \frac{15z_1 + z_2 + 1 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}{-z_1 + z_2 + 17 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}$$

and

$$\phi_2(z) = \frac{16z_2 - z_1 + 1 + 8\sqrt{z_2(1-z_1)}}{-z_1 + z_2 + 17 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}.$$

To define a one-parameter semigroup, we note that

$$m_\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -8 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}$$

and thus

$$m_{\Phi_t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -8 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t(t-1)}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{8} \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{2} & \frac{t(t+7)}{16} \\ 0 & 1 & \frac{t}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

which gives

$$\Phi_t(z) = \left(z_1 + \frac{t}{2} z_2 + \frac{t(t+7)}{16}, z_2 + \frac{t}{4} \right).$$

Hence we define $\phi_t = \sigma^{-1} \circ \Phi_t \circ \sigma$ to be our one-parameter semigroup of ϕ. Define the following:

$$\begin{aligned} A :=& 1024 z_1 + 64 t^2 z_2 + t^2 (t+7)^2 (1 - z_1) + 256 t \sqrt{2 z_2 (z_1 + 1)} \\ & + 32 t (t+7) \sqrt{2(1 - z_1^2)} + 16 t^2 (t+7) \sqrt{z_2 (1 - z_1)} \\ B :=& 1024 + 64 t^2 z_2 + t^2 (t+7)^2 (1 - z_1) + 256 t \sqrt{2 z_2 (z_1 + 1)} \\ & + 32 t (t+7) \sqrt{2(1 - z_1^2)} + 16 t^2 (t+7) \sqrt{z_2 (1 - z_1)} \\ C :=& 64 t^2 (1 - z_1) + 1024 z_2 + 512 t \sqrt{z_2 (1 - z_1)} \\ D :=& 1024 + 64 t^2 z_2 + t^2 (t+7)^2 (1 - z_1) + 256 t \sqrt{2 z_2 (z_1 + 1)} \\ & + 32 t (t+7) \sqrt{2(1 - z_1^2)} + 16 t^2 (t+7) \sqrt{z_2 (1 - z_1)}. \end{aligned}$$

A calculation shows $\phi_t(z) = (\phi_{1_t}(z), \phi_{2_t}(z))$ where $\phi_{1_t}(z)$ and $\phi_{2_t}(z)$ are given by

$$\phi_{1_t}(z) = \frac{A}{B}$$

and

$$\phi_{2_t}(z) = \frac{C}{D}$$

It is a straightforward calculation to see that $\phi_0 = I$ and $\phi_1 = \phi$.

5. FUTURE WORK

5.1 Spectra of Maps in Different Model Cases

Recall that our model theory admits seven cases to be considered for analytic maps of $\mathbb{B}_2$ into itself. While case I (interior fixed point) and case III (Siegel half space/dilation) are solved, we have made great progress on case II (half space/dilation) as well. We may ask ourselves what the spectra of maps in the other four cases (attractive boundary fixed point with multiplicity) as well. Taken over the Hardy space in $\mathbb{B}_2$, work by Bayart [3] along with results by Jiang and Chen [17] solve the problem for C_ϕ when ϕ is a linear fractional map. It remains to show, however, how this generalizes to analytic maps that have an intertwining.

5.2 Generalized Hardy Spaces

In the disk, the Hardy space is a specific realization of a broader class of Hilbert function spaces known as weighted Hardy spaces. These are the Hilbert spaces whose vectors are functions that are analytic in the unit disk with monomials $\{1, z, z^2, ...\}$ representing a complete orthogonal set of non-zero vectors for the Hilbert space. These include well-studied spaces such as the Bergman space and the Dirichlet space. In several variables, the Drury-Arveson space can likewise be studied as a specific realization of a broader class of Hilbert function spaces. We define these below.

Definition 32 *Let m, β be positive integers. The space $H^2_{m,\beta}(\mathbb{B}_m)$ is the Hilbert function space with reproducing kernel*

$$\frac{1}{(1-\langle z, w\rangle)^\beta}.$$

The space $H^2_{2,\beta}$ can be identified with the space of holomorphic functions $f : \mathbb{B}_2 \to \mathbb{C}$ which have power series $f(z) = \sum_{i,j=0} a_{ij} z^i z^j$ such that

$$||f||^2_{H^2_{2,\beta}} = ||f||^2 := \sum_{i,j=0} |a_{ij}|^2 \frac{(\beta-1)!i!j!}{(\beta+i+j-1)!} < \infty.$$

The Drury-Arveson space is obtained by setting $\beta = 1$ and the Hardy space on the ball is obtained by setting $\beta = m + 1$. The spectral radii of linear fractional maps has been determined for these spaces [20]. Preliminary results suggesting the following is true for this more generalized class of Hilbert function spaces.

Conjecture 33 *Assuming some natural hypotheses, similar to those given in Theorem 28, given an analytic map ϕ, from the ball into the ball, with no interior fixed point and boundary fixed point ζ (without multiplicity), the spectrum of C_ϕ acting on $H^2_{2,\beta}(\mathbb{B}_2)$ is given by*

$$\sigma(C_\phi) = \{\lambda \mid |\lambda| < \alpha^{-\frac{\beta}{2}}\}$$

where α is the radial limit of the complex directional derivative $D_\zeta \phi$, i.e. the dilation coefficient.

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