HERMITIAN–YANG–MILLS METRICS ON HILBERT BUNDLES AND IN THE SPACE OF KÄHLER POTENTIALS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Kuang-Ru Wu

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2020

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Professor Steven Bell

Department of Mathematics

Professor László Lempert, Chair

Department of Mathematics

Professor Chi Li

Department of Mathematics

Professor Sai-Kee Yeung

Department of Mathematics

Approved by:

Professor Plamen Stefanov

Associate Head for Graduate Studies

To my parents.

ACKNOWLEDGMENTS

This thesis is a summary of my work for the past six years, and it could not have been done without the help and support of many people. My first and foremost thanks go to my family. My parents' love and wisdom shape who I am, and if I have any accomplishment, I accredit it to the education by them. My older brother, who I grew up with, has been playing an important part in my life, and I am grateful for having a brother like him. My family is the reason why I can do what I like the most, and I am truly grateful for having them.

There are many teachers who have influenced me during my education. I was first stunned by the rigor of mathematics in a calculus class taught by So-Chin Chen at National Tsing Hua University, and a large part of the reason I chose mathematics is because of him. My advisor, László Lempert, at Purdue is the person who makes me a mathematician. His ways of thinking and his style of mathematics are and will always be an inspiration to me. Tamás Darvas is a great source of ideas to me, and I appreciate the collaboration with him. Sai-Kee Yeung, Chi Li, and Victor Lie have taught me so much in various courses that genuinely broaden my view of mathematics. I would also like to thank Victor Lie for his warm encouragements over several occasions. I am grateful to Steven Bell, Sai-Kee Yeung, and Chi Li for being on the thesis committee. I am thankful to the instruction of my advisor Chiung-Ju Liu at National Taiwan University. Thanks also go to Jiyaun Han for spending time explaining many interesting results that I did not know.

Many friends back in Taiwan and here in the States have helped me during this journey. First I want to mention my former roommate "Lil Chen" whom I spent four years living with. It is unlikely he will read this work, but I appreciate the support from him over the four years. Carlos, Max, Vinh, and Hongshan are virtually my brothers at Purdue. I cannot imagine how life would have become without them. I would like to thank Kate for the help and joy she gave over the past few years. The friendship with Jun, Hengrong, Adam, and Warren was very important for me during my years of study, and it brought so much fun over the years. Special thanks are due to Turbo, a Taiwanese comrade, for all his advice and encouragements.

It is a corny saying in Taiwan that if one has too many people to thank, one may as well thank heaven. Well, I could not think of a better time than now to devote my thanks to heaven.

TABLE OF CONTENTS

ABSTRACT

Wu, Kuang-Ru Ph.D., Purdue University, August 2020. Hermitian–Yang–Mills Metrics on Hilbert Bundles and in the Space of Kähler Potentials. Major Professor: László Lempert.

The two main results in this thesis have a common point: Hermitian–Yang–Mills (HYM) metrics. In the first result, we address a Dirichlet problem for the HYM equations in bundles of infinite rank over Riemann surfaces. The solvability has been known since the work of Donaldson [Don92] and Coifman–Semmes [CS93], but only for bundles of finite rank. So the novelty of our first result is to show how to deal with infinite rank bundles. The key is an a priori estimate obtained from special feature of the HYM equation.

In the second result, we take on the topic of the so-called "geometric quantization." This is a vast subject. In one of its instances the aim is to approximate the space of Kähler potentials by a sequence of finite dimensional spaces. The approximation of a point or a geodesic in the space of Kähler potentials is well-known, and it has many applications in Kähler geometry. Our second result concerns the approximation of a Wess–Zumino–Witten type equation in the space of Kähler potentials via HYM equations, and it is an extension of the point/geodesic approximation.

1. INTRODUCTION

We will give a summary of results in the following two sections. The detailed accounts will be provided in later chapters.

1.1 A Dirichlet problem in noncommutative potential theory

A Hermitian metric on a line bundle can be locally represented by a scalar-valued function. For higher rank bundles, Hermitian metrics can be represented by matrix-valued functions, which is the origin of noncommutativity. One can push even further by considering bundles of infinite rank. In this part of the thesis, we will focus on bundles whose fibers are Hilbert spaces, and in particular trivial Hilbert bundles where the situation becomes clearest.

Let $(V, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Let $\mathrm{End}V$ be the set of bounded linear operators on V and End^+V be the set of positive invertible elements in $\mathrm{End}V$. Let $\overline{M}$ be a compact Riemann surface with boundary. A Hermitian metric h on the bundle $\overline{M} \times V \to \overline{M}$ can be represented by $h_z(v,w) = \langle P(z)v, w\rangle$ where $v, w \in V$ and $P : \overline{M} \to \mathrm{End}^+V$. If P is C^2, then the curvature R^P is $\bar{\partial}(P^{-1}\partial P)$. The Dirichlet problem we aim to solve is to extend a given metric on $\partial M \times V$ to a metric on $\overline{M} \times V$ with curvature zero. The main result is the following.

Theorem 1.1.1 *Let $\overline{M}$ be a compact Riemann surface with boundary and $F \in C^m(\partial M, End^+V)$, where $m = 0, \infty,$ or ω. There exists a unique $P \in C^m(\overline{M}, End^+V) \cap C^\omega(M, End^+V)$ such that $R^P = 0$ on M, and $P|_{\partial M} = F$. The same is true if we replace C^m by $C^{k,\alpha}$ for k a nonnegative integer and $0 < \alpha < 1$.*

We mention briefly previous work when $\dim V < \infty$. Masani and Wiener solve the Dirichlet problem over the unit disc in [WM57], with regularity weaker than continuous. In [Lem81], Lempert solves it in Hölder classes. More generally in [Don92], Donaldson solves a Dirichlet problem for the Hermitian–Yang–Mills equations over Kähler manifolds with boundary, and in [CS93] Coifman and Semmes solve it over domains in $\mathbb{C}^n$ which are regular for the Laplacian. When the base is one dimensional, Donaldson's and Coifman–Semmes' results reduce to the existence of flat Hermitian metrics. (Coifman and Semmes also solve a Dirichlet problem for norms more general than those coming from Hermitian metrics. See also a more recent related paper [BCEKR20].)

Devinatz [Dev61] and Douglas [Dou66] generalize Wiener amd Masani's result to infinite dimensional separable V, with the base still the unit disc (see also [Hel64, Lecture XI]). For a general V and various regularity classes, the Dirichlet problem over the unit disc is solved by Lempert in [Lem17]. Lempert's proof is by the continuity method and proceeds by a global factorization of flat metrics. However, such a factorization is not available when the base is multiply connected.

Our proof is also by the continuity method. Closedness is proved by a maximum principle and a local holomorphic factorization of flat metrics. Openness turns out to be harder than usual, because to deal with the linear partial differential equation originating from the implicit function theorem, Fredholm theory is not available. Nevertheless, the linear equation has various symmetries that we can exploit to obtain the requisite a priori estimates. This is the main novelty in our result.

1.2 A Wess–Zumino–Witten type equation in the space of Kähler potentials

Let X be a compact complex manifold of dimension n with a Kähler form ω. The space of Kähler potentials is

$$\mathcal{H}_\omega = \{\phi \in C^\infty(X, \mathbb{R}) : \omega + i\partial\bar{\partial}\phi > 0\},$$

and we will denote $\omega + i\partial\bar{\partial}\phi$ by ω_ϕ. We assume ω is the curvature of some Hermitian line bundle L. For a positive integer k, we denote by $\mathcal{H}_k$ the space of inner products on $H^0(X, L^k)$. Starting from a question asked by Yau [Yau87] and the work of Tian

[Tia90], Zelditch [Zel98], Catlin [Cat99], and many others, it is well-known that a given Kähler potential $\phi \in \mathcal{H}_\omega$ can be approximated by $\phi_k \in \mathcal{H}_\omega$ associated with $\mathcal{H}_k$ as $k \to \infty$. Furthermore, Mabuchi [Mab87], Semmes [Sem92], and Donaldson [Don99] discovered that $\mathcal{H}_\omega$ carries a Riemannian metric which allows one to talk about geometry, especially geodesics, of $\mathcal{H}_\omega$. Thanks to Phong–Sturm [PS06], Berndtsson [Ber13], and Darvas–Lu–Rubinstein [DLR18], geodesics in $\mathcal{H}_\omega$ can be approximated by geodesics in $\mathcal{H}_k$ as $k \to \infty$. More generally, one may wonder if harmonic maps into $\mathcal{H}_\omega$ can also be approximated by harmonic maps associated with $\mathcal{H}_k$. A version of this was confirmed by Rubinstein–Zelditch [RZ10] when X is toric, and the maps take values in toric Kähler metrics.

In this second part of the thesis, we focus on a Wess–Zumino–Witten (WZW) type equation for a map from $D \subset \mathbb{C}^m$ to $\mathcal{H}_\omega$, and we show that the solution to such an equation can be approximated by Hermitian–Yang–Mills metrics on certain direct image bundles. We will also see how this result recovers some of those mentioned in the previous paragraph.

We first explain how to derive this WZW equation. The space $\mathcal{H}_\omega$ is an open subset in the Fréchet space $C^\infty(X, R)$, and therefore it is a Fréchet manifold. The tangent space $T_\phi\mathcal{H}_\omega$ at $\phi \in \mathcal{H}_\omega$ is canonically isomorphic to $C^\infty(X, \mathbb{R})$, and tangent bundle $T\mathcal{H}_\omega$ is canonically isomorphic to $\mathcal{H}_\omega \times C^\infty(X, \mathbb{R})$ (These matters will be reviewed more rigorously in Chapter 2). Following Mabuchi [Mab87], Semmes [Sem92], and Donaldson [Don99], the Mabuchi metric g_M on $\mathcal{H}_\omega$ is the following. For a point $\phi \in \mathcal{H}_\omega$ and two vectors $\xi, \eta \in T_\phi\mathcal{H}_\omega \approx C^\infty(X, \mathbb{R})$, the Mabuchi metric is

$$g_M(\xi, \eta) = \int_X \xi\eta\, \omega_\phi^n.$$

Let D be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^m$. A map $\widehat{\Phi} : D \to \mathcal{H}_\omega$ will induce a map $\Phi : D \times X \to \mathbb{R}$ with $\Phi(z, \cdot) \in \mathcal{H}_\omega$ for $z \in D$, and vice versa. A map $\widehat{\Phi} : D \to \mathcal{H}_\omega$ is said to be harmonic if it is a critical point of the functional $E(\widehat{\Phi}) = \int_D |\widehat{\Phi}'|^2 dV$ where dV is the Euclidean volume form on D, $\widehat{\Phi}'$ is the tangent map of $\widehat{\Phi}$, and $|\widehat{\Phi}'|$ is the Hilbert–Schmidt norm of $\widehat{\Phi}'$, measured by Mabuchi metric

g_M and the Euclidean metric of D. A straightforward computation gives the harmonic map equation:

$$\sum_{j=1}^{m} \Phi_{z_j\bar{z}_j} - \frac{1}{2}|\nabla\Phi_{z_j}|^2 = 0, \tag{1.2.1}$$

where $\{z_j\}$ are coordinates on D and $\nabla\Phi_{z_j}(z)$ is the gradient of the function $\Phi_{z_j}(z)$ on X with respect to the metric $\omega_{\widehat{\Phi}(z)}$, and $|\nabla\Phi_{z_j}(z)|$ is its length computed using the metric $\omega_{\widehat{\Phi}(z)}$. The functional that we are looking for, which we denote by $\mathcal{E}$, is a perturbation of the harmonic functional E above. The construction of this perturbed functional $\mathcal{E}$ is similar to that of [Don99, Section 5] (see also [Wit83]), who dealt with one dimensional D. We will construct $\mathcal{E}$ in Chapter 4 and show in Lemma 4.3.4 that the Euler–Lagrange equation of $\mathcal{E}$ is

$$\sum_{j=1}^{m} \Phi_{z_j\bar{z}_j} - \frac{1}{2}|\nabla\Phi_{z_j}|^2 - \frac{i}{2}\{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_{\widehat{\Phi}}} = 0, \tag{1.2.2}$$

where $\{\cdot,\cdot\}_{\omega_{\widehat{\Phi}}}$ is the Poisson bracket on $C^\infty(X,\mathbb{R})$ determined by the symplectic form $\omega_{\widehat{\Phi}}$. In view of its connection with [Wit83] and following [Don99], we call the equation (1.2.2) the WZW equation for a map $\widehat{\Phi} : D \to \mathcal{H}_\omega$.

Donaldson showed in [Don99], when $m = 1$, the WZW equation is equivalent to a homogeneous complex Monge–Ampère equation. We have the following extended equivalence for $m \geq 1$ by a similar computation. Let $\pi : D\times X \to X$ be the projection onto X. Then the extended equivalence is

$$\Phi \text{ solves (1.2.2) if and only if } (i\partial\bar{\partial}\Phi + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} = 0. \tag{1.2.3}$$

This suggests that the proper generality of the WZW equation is for maps from a Kähler manifold D to $\mathcal{H}_\omega$. Nevertheless, in this thesis we restrict to $D \subset \mathbb{C}^m$.

The next step is to construct a solution of the WZW equation, and then we will show it can be approximated by the solutions of Hermitian–Yang–Mills equations. We first introduce the following definition

Definition 1.2.1 *We will say that a function $u : D\times X \to [-\infty,\infty)$ is ω-subharmonic on graphs if for any holomorphic map f from an open subset of D to X, $\psi(f(z)) + u(z, f(z))$ is subharmonic, where ψ is a local potential of ω.*

This definition does not depend on the choice of ψ since any two local potentials differ by a pluriharmonic function. (This definition has its origin in the works of Slodkowski [Slo88],[Slo90b],[Slo90a], and Coifman and Semmes [CS93]; however, they focus on functions u defined on $D \times V$ with a vector space V and $u(z, \cdot)$ are norms or quasi-norms, whereas we consider simply functions on $D \times X$. There is also a notion of k-subharmonicity, see [Bło05], but it is not equivalent to subharmonicity on graphs.)

Let v be a real-valued smooth function on $\partial D \times X$ and $\partial D \ni z \mapsto v(z, \cdot) = v_z \in \mathcal{H}_\omega$. Consider the Perron family

$$\begin{aligned} G_v := \{u \in \operatorname{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs}, \\ \text{and } \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \le v(\zeta, x)\}. \end{aligned}$$

As we will later see, the upper envelope $V = \sup\{u : u \in G_v\}$ is a weak solution of the WZW equation from D to $\mathcal{H}_\omega$.

There are two maps that connect $\mathcal{H}_\omega$ and $\mathcal{H}_k$, the Hilbert map $H_k : \mathcal{H}_\omega \to \mathcal{H}_k$ and the Fubini–Study map $FS_k : \mathcal{H}_k \to \mathcal{H}_\omega$. Their definitions will be given in Chapter 4.

The approximants are going to be the solutions of the Hermitian–Yang–Mills equation on the bundle $\overline{D} \times H^0(X, L^k)^* \to \overline{D}$. For $z \in \partial D$, the inner product $H_k(v_z)$ is defined on $H^0(X, L^k)$, and its dual inner product $H_k^*(v_z)$ on $H^0(X, L^k)^*$. Suppose V^k is a Hermitian metric on the bundle $\overline{D} \times H^0(X, L^k)^* \to \overline{D}$, and $\Theta(V^k)$ is its curvature, a $(1,1)$-form on D with values in endomorphisms of $H^0(X, L^k)^*$. Let Λ be the trace with respect to the Euclidean metric of D, so in general $\Lambda\Theta(V^k)$ takes values in endomorphisms of $H^0(X, L^k)^*$. The HYM equation is

$$\begin{cases} \Lambda\Theta(V^k) = 0 \\ V^k|_{\partial D} = H_k^*(v). \end{cases}$$

It has a unique solution by [Don92] and [CS93].

Denoting the dual metric by $(V^k)^*$, our main result is that the upper envelope V of G_v is the limit of Hermitian–Yang–Mills metrics:

Theorem 1.2.1 *$FS_k((V^k)^*)$ converges to V uniformly on $D \times X$, as $k \to \infty$.*

Now we turn to the interpretation of the upper envelope V and its relation with the WZW equation. The next theorem shows that V solves the WZW equation under a regularity assumption.

Theorem 1.2.2 *If the upper envelope V of G_v is in $C^2(D \times X)$, then*

$$(i\partial\bar{\partial}V + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} = 0.$$

As a result, Theorems 1.2.1 and 1.2.2 together show that the solution of the WZW equation can be approximated by the Hermitian–Yang–Mills metrics.

We mention briefly works related to our result. If $m = 1$, $D \subset \mathbb{C}$ is an annulus, and v is invariant under rotation of the annulus, then Theorems 1.2.1 and 1.2.2 recover the geodesic approximation result of Phong–Sturm [PS06] and Berndtsson [Ber13]. When X is toric, these theorems are reduced to the harmonic approximation of Rubinstein–Zelditch [RZ10], except that C^2 convergence is proved in their paper.

The proof of Theorem 1.2.1 hinges on a theorem regarding the positivity of direct image bundles. Consider a Hermitian holomorphic line bundle $(E, g) \to X^n$ over a compact complex manifold and assume the curvature η of the metric g is positive. We define a variant of the Hilbert map: given $s \in H^0(X, E \otimes K_X)$, then $g(s, s)$ is a real-valued (n, n)-form on X, and $\mathrm{Hilb}_{E\otimes K_X}(u)$, for a function $u : D \times X \to \mathbb{R}$, is defined by

$$\mathrm{Hilb}_{E\otimes K_X}(u)(s, s) = \int_X g(s, s)e^{-u(z,\cdot)}.$$

Therefore the map $z \mapsto \mathrm{Hilb}_{E\otimes K_X}(u)$ is a Hermitian metric on the bundle $D \times H^0(X, E \otimes K_X) \to D$.

Theorem 1.2.3 *If u is bounded and upper semicontinuous (usc) on $D \times X$, and η-subharmonic on graphs, then the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is a subharmonic norm function.*

A norm function being subharmonic means the logarithm of the length of any holomophic section is subharmonic.

Although Berndtsson's theorem [Ber09] has played a crucial role in approximation theorems similar to Theorem 1.2.1 (for example [Ber13], [BK12], [DLR18], and [DW19]), when it comes to approximating by Hermitian–Yang–Mills metrics, a subharmonic analogue of Berndtsson's theorem is desired. It is Theorem 1.2.3, where we prove a version of positivity of direct image bundles for weights that are subharmonic on graphs. This is perhaps the crux in this second part of the thesis. A corresponding result on Stein manifolds can be proved easily following the proof of Theorem 1.2.3.

2. DIFFERENTIABILITY OF VECTOR-VALUED FUNCTIONS

In this chapter we will review some facts about differentiability of functions with values in Banach spaces or Fréchet spaces. Section 2.1 serves background for Chapter 3, and Section 2.2 for Chapter 4.

2.1 Hölder classes of Banach-valued functions

Let Y be a Banach space with a norm $\|\cdot\|$. Let U be an open set in $\mathbb{R}^n$. For $k = 0, 1, 2, ...$ and $0 < \alpha < 1$, a function f is said to be in $C^{k,\alpha}(U, Y)$ if partial derivatives of f up to order k exist and are continuous in U, and all the k-th partial derivatives of f satisfy the α-Hölder condition

$$\sup_{x \neq y \in U} \frac{\|D^I f(x) - D^I f(y)\|}{|x-y|^\alpha} < \infty, \text{ for all } |I| = k,$$

where the $D^I f$ is the standard multi-index notation. Moreover, if T is a subset of ∂U, a function f is said to be in $C^{k,\alpha}(U \cup T, Y)$ if partial derivatives of f up to order k exist and are continuous up to T, and

$$\sup_{x \neq y \in U \cup T} \frac{\|D^I f(x) - D^I f(y)\|}{|x-y|^\alpha} < \infty, \text{ for all } |I| = k.$$

Similarly, we define $C^k(U, Y)$ and $C^k(U \cup T, Y)$ by removing the α-Hölder condition from above.

Let $\overline{N}$ be a compact smooth manifold, possibly with boundary. A function f is said to be in $C^{k,\alpha}(\overline{N}, Y)$ if f is $C^{k,\alpha}$ in charts. Fix a finite open cover $\{U_i\}$ of $\overline{N}$ such that each $\overline{U_i}$ is contained in a chart, we are going to equip $C^{k,\alpha}(\overline{N}, Y)$ with a

Banach algebra structure. For f in $C^{k,\alpha}(\overline{N}, Y)$, one first computes its Hölder norm in U_i using local coordinates, namely, if we ignore the coordinate map

$$\sum_{|I|=0}^{k} \sup_{U_i} \|D^I f\| + \sum_{|I|=k} \sup_{x \neq y \in U_i} \frac{\|D^I f(x) - D^I f(y)\|}{|x-y|^\alpha}.$$

Then $\|f\|_{k,\alpha}$ is defined to be the sum of these local Hölder norms. With a suitable scaling it can be arranged that $\|\cdot\|_{k,\alpha}$ is sub-multiplicative. It is straightforward to verify that the resulting space is indeed a Banach algebra, and we skip the verification. Likewise, $C^k(\overline{N}, Y)$ also carries a Banach algebra structure. We set $C^\infty = \bigcap_k C^k$, and also write C for C^0.

2.2 Fréchet spaces, Fréchet manifolds, and $C^\infty(X, \mathbb{R})$

In this subsection, we give a review on Fréchet spaces and Fréchet manifolds based on [Con90],[Mil84], and [Ham82]. We begin with topological vector spaces. A topological vector space is a vector space with a topology such that addition and scalar multiplication are continuous. Given a vector space V and a family $\mathcal{P}$ of seminorms, we can equip V with a topology by stipulating a set $U \subset V$ to be open if for any point $x_0 \in U$ there exist seminomrs $p_1, ..., p_n$ and positive numbers $\varepsilon_1, ..., \varepsilon_n$ such that

$$\bigcap_{i=1}^{n} \{x \in V : p_i(x - x_0) < \varepsilon_i\}$$

is in U. It is not hard to check $(V, \mathcal{P})$ is a topological vector space.

Definition 2.2.1 *A locally convex space (LCS) is a topological vector space whose topology is determined by a family $\mathcal{P}$ of seminorms and $\cap_{p\in\mathcal{P}}\{x : p(x) = 0\} = 0$. A Fréchet space is a locally convex space whose topology can be induced by a translation-invariant complete metric.*

Let V_1 and V_2 be two LCS, and let U be an open subset of V_1. For a map $f : U \to V_2$, we define the directional derivative of f at $x \in U$ in the direction $w \in V_1$ to be

$$f'(x, w) = f'_x(w) = \lim_{t\to 0} \frac{f(x + tw) - f(x)}{t}.$$

Ignoring the existence of the limit for a moment, we form the second directional derivative

$$f''(x, w_1, w_2) = \lim_{t \to 0} \frac{f'(x + tw_2, w_1) - f'(x, w_1)}{t},$$

with w_1 and w_2 in V_1. Similarly, the r-th directional derivative is

$$f^{(r)}(x, w_1, ..., w_r) = \lim_{t \to 0} \frac{f^{(r-1)}(x + tw_r, w_1, ..., w_{r-1}) - f^{(r-1)}(x, w_1, ..., w_{r-1})}{t},$$

with $w_1, ..., w_r \in V_1$.

The map f is said to be C^1 if f is continuous, and $f'(x, w)$ exists and is continuous on $U \times V_1$. The map f is said to be C^2 if f is C^1 and $f''(x, w_1, w_2)$ exists and is continuous on $U \times V_1 \times V_1$. Similarly, the map f is said to be C^r for a positive integer r if f is C^{r-1} and $f^{(r)}(x, w_1, ...w_r)$ exists and is continuous on $U \times V_1 \times ... \times V_1$. A map is C^∞ if it is C^r for every r.

Now we are able to define manifolds. A smooth manifold modeled on a LCS V is a Hausdorff and regular topological space M together with a collection of homeomorphisms $f_\alpha : V_\alpha \to M_\alpha$ (local coordinate systems), where V_α is open in V and M_α is open in M. Moreover $\bigcup_\alpha M_\alpha = M$ and the transition $f_\beta^{-1} \circ f_\alpha$ is required to be C^∞. (If the model space V is Fréchet, then M is called a Fréchet manifold).

If we are given two such smooth manifolds M_1 and M_2, modeled on LCS V_1 and V_2 respectively, then $f : M_1 \to M_2$ is called smooth if f is smooth after composing with local coordinate systems.

A tangent vector at $x_0 \in M$ can be defined as an equivalence class of paths through x_0 as follows. Let P_1 and P_2 be smooth maps from an open interval I to M with $P_i(0) = x_0$. Let $f_\alpha : V_\alpha \to M_\alpha$ be a local coordinate system with $x_0 \in M_\alpha$. We say that P_1 and P_2 are equivalent at $t = 0$ if $f_\alpha^{-1}(P_1(t))$ and $f_\alpha^{-1}(P_2(t))$ have the same first derivative at $t = 0$. (It is not hard to see that if P_1 and P_2 are equivalent for one local coordinate system then they are equivalent for all local coordinate systems). We denote the equivalence class of a path P by $[P]$. The set of all such equivalence classes of paths through x_0 is called the tangent space $T_{x_0}M$. Note that f_α induces a one-to-one correspondence between V and $T_{x_0}M$. In fact, if $f_\alpha(v_0) = x_0$ then every

$v \in V$ corresponds to the equivalence class of the path $t \mapsto f_\alpha(v_0 + tv)$. So we can equip $T_{x_0}M$ with a structure of LCS isomorphic to V.

The set $TM = \coprod_{x \in M} T_x M$ can be made into a smooth manifold, the tangent bundle of M. It is modeled on $V \times V$, and local coordinate systems are

$$\begin{aligned} V_\alpha \times V &\to TM_\alpha \subset TM \\ (u, v) &\mapsto [t \mapsto f_\alpha(u + tv)]. \end{aligned}$$

A smooth vector field v on M is simply a smooth map $v : M \to TM$ with $v(x) \in T_x M$.

Any smooth map $f : M_1 \to M_2$ induces a map $f'_x : T_x M_1 \to T_{f(x)} M_2$ by sending $[P]$ to $[f \circ P]$, and one can easily check the independence of the representative path P. Putting together f'_x over $x \in M_1$, we obtain a *tangent map* $f' : TM_1 \to TM_2$. Using local coordinate systems f_α on M_1 and g_β on M_2, we can see the map f'_x is the directional derivative of $g_\beta^{-1} \circ f \circ f_\alpha$. Because the map f is assumed smooth, the tangent map f' is smooth.

The simplest example of a smooth manifold modeled on a LCS V is perhaps an open subset U of V. There is a canonical local coordinate system, namely the identity map $\mathrm{Id} : U \to U \subset V$, and so the tangent space $T_x U$ at any point $x \in U$ is canonically isomorphic to V by

$$\begin{aligned} V &\to T_x U \\ v &\mapsto [t \mapsto x + tv]. \end{aligned}$$

The tangent bundle TU also has a canonical local coordinate system

$$\begin{aligned} U \times V &\to TU \\ (x, v) &\mapsto [t \mapsto x + tv], \end{aligned}$$

so TU is canonically isomorphic to $U \times V$.

The example we care the most about is the space of smooth functions from a compact smooth manifold X to $\mathbb{R}$, which we denote by $C^\infty(X, \mathbb{R})$. We first show that

$C^\infty(X,\mathbb{R})$ is a Fréchet space. Fix a finite open cover $\{U_i\}_{1\le i\le m}$ of X so that each $\overline{U_i}$ is in a chart ψ_i. Define seminorm p_l for positive integer l on $C^\infty(X,\mathbb{R})$ by setting

$$p_l(f) = \max_{1\le i\le m, |I|\le l} \sup_{\psi_i^{-1}(U_i)} |D^I(f\circ\psi_i)|.$$

These seminorms make $C^\infty(X,\mathbb{R})$ a locally convex space, and it is metrizable with the translation-invariant metric

$$d(f,g) = \sum_{1\le l} \frac{1}{2^l}\frac{p_l(f-g)}{1+p_l(f-g)}.$$

The last thing to show is completeness. Suppose $\{f_\mu\}$ is a Cauchy sequence in $C^\infty(X,\mathbb{R})$, then $p_l(f_\mu - f_\nu) \to 0$ as $\mu,\nu\to\infty$, for every seminorm p_l, and therefore, the sequence $\{D^I(f_\mu\circ\psi_i)\}_\mu$ converges uniformly on $\psi_i^{-1}(U_i)$ for every $1\le i\le m$ and $|I|\le l$. So there is a global function $f\in C^\infty(X,\mathbb{R})$, and it is straightforward to see f_μ converges to f. Hence $C^\infty(X,\mathbb{R})$ is a Fréchet space.

Assume X is additionally a complex manifold with a Kähler form ω. The space of Kähler potentials is $\mathcal{H}_\omega = \{\phi\in C^\infty(X,\mathbb{R}) : \omega + i\partial\bar\partial\phi > 0\}$. It is an open subset in $C^\infty(X,\mathbb{R})$. Indeed, for a fixed point $\phi_0\in\mathcal{H}_\omega$, since $\omega + i\partial\bar\partial\phi_0 > 0$ and X is compact, we can find $\varepsilon > 0$ such that if a function $\phi\in C^\infty(X,\mathbb{R})$ satisfies $p_2(\phi-\phi_0) < \varepsilon$, then $\omega + i\partial\bar\partial\phi = \omega + i\partial\bar\partial\phi_0 + i\partial\bar\partial(\phi-\phi_0)$ is still positive. Hence the point ϕ_0 has a neighborhood $\{\phi : p_2(\phi-\phi_0) < \varepsilon\}$ in $\mathcal{H}_\omega$.

The space $\mathcal{H}_\omega$ as an open subset of the Fréchet space $C^\infty(X,\mathbb{R})$ is a Fréchet manifold. As we have discussed, the tangent space $T_\phi\mathcal{H}_\omega$ at any point ϕ is canonically isomorphic to $C^\infty(X,\mathbb{R})$, and the tangent bundle $T\mathcal{H}_\omega$ is canonically isomorphic to $\mathcal{H}_\omega\times C^\infty(X,\mathbb{R})$.

Let Ω be a domain in $\mathbb{R}^m$. A map $\Phi:\Omega\times X\to\mathbb{R}$, smooth in the X-variables, induces a map $\widehat{\Phi}:\Omega\to C^\infty(X,\mathbb{R})$ by $\widehat{\Phi}(y) = \Phi(y,\cdot)$. Conversely, a map $\widehat{\Phi}:\Omega\to C^\infty(X,\mathbb{R})$ induces a map $\Phi:\Omega\times X\to\mathbb{R}$ smooth in X by setting $\Phi(y,x) = \widehat{\Phi}(y)(x)$. Suppose now we have a map $\widehat{\Phi}:\Omega\to C^\infty(X,\mathbb{R})$. Let $\{e_i\}$ be the canonical basis in $\mathbb{R}^m$. The r-th directional derivative along $\{e_i\}$ is the same as the partial derivative

$$\widehat{\Phi}^{(r)}(y, e_{i_1}, ..., e_{i_r}) = \widehat{\Phi}_{y_{i_1}\cdots y_{i_r}}(y),$$

and it is a map from Ω to $C^\infty(X,\mathbb{R})$. It is not hard to check that a map $\widehat{\Phi}$ is C^r in our earlier definition if and only if its partial derivatives up to order r exist and are continuous on Ω.

Lemma 2.2.1 *A map $\widehat{\Phi}:\Omega\to C^\infty(X,\mathbb{R})$ is C^∞ if and only if the corresponding Φ is smooth on $\Omega\times X$ jointly.*

Proof We start with Φ smooth on $\Omega\times X$. By smoothness of Φ, $\widehat{\Phi}$ is continuous on Ω. The function Φ_{y_i} induces $\widehat{(\Phi_{y_i})}:\Omega\to C^\infty(X,\mathbb{R})$, which is continuous by the smoothness of Φ on $\Omega\times X$. It is easy to see

$$\lim_{h\to 0}\frac{\widehat{\Phi}(y+he_i)-\widehat{\Phi}(y)}{h}$$

converges to $\widehat{(\Phi_{y_i})}(y)$ in $C^\infty(X,\mathbb{R})$ by the definition of the seminorms p_l, the mean value theorem, and the smoothness of Φ. Therefore, $\widehat{\Phi}_{y_i}=\widehat{(\Phi_{y_i})}$ and $\widehat{\Phi}$ is C^1. Similarly, since Φ_{y_i} is smooth on $\Omega\times X$, the map $\widehat{(\Phi_{y_i})}$, hence $\widehat{\Phi}_{y_i}$ is C^1, and so $\widehat{\Phi}$ is C^2 and eventually C^∞.

For the converse direction, We first observe the following lemma

Lemma 2.2.2 *If $\widehat{F}:\Omega\to C^\infty(X,\mathbb{R})$ is continuous, then all partial derivatives in the X-variables D^I_xF are continuous on $\Omega\times X$.*

Proof For $(y_\mu,x_\mu)\to(y,x)$ in $\Omega\times X$, we need to show $D^I_xF(y_\mu,x_\mu)\to D^I_xF(y,x)$ as $\mu\to\infty$. The continuity of $\widehat{F}$ says $\widehat{F}(y_\mu)\to\widehat{F}(y)$ in $C^\infty(X,\mathbb{R})$, which implies $D^I_xF(y_\mu,x)\to D^I_xF(y,x)$ uniformly in $x\in X$. By $|D^I_xF(y_\mu,x_\mu)-D^I_xF(y,x)|\le |D^I_xF(y_\mu,x_\mu)-D^I_xF(y,x_\mu)|+|D^I_xF(y,x_\mu)-D^I_xF(y,x)|$, the lemma follows ■

Now suppose $\widehat{\Phi}:\Omega\to C^\infty(X,\mathbb{R})$ is C^∞. The map $\widehat{\Phi}_{y_i}:\Omega\to C^\infty(X,\mathbb{R})$ induces a map on $\Omega\times X$ by $\widehat{\Phi}_{y_i}(y)(x)$. We know

$$\lim_{h\to 0}\frac{\widehat{\Phi}(y+he_i)-\widehat{\Phi}(y)}{h}=\widehat{\Phi}_{y_i}(y) \tag{2.2.1}$$

in $C^\infty(X,\mathbb{R})$, so by the definition of the seminorms p_l we have

$$\lim_{h\to 0}\frac{\Phi(y+he_i,x)-\Phi(y,x)}{h}=\widehat{\Phi}_{y_i}(y)(x).$$

Therefore, $\Phi_{y_i}(y,x) = \widehat{\Phi}_{y_i}(y)(x)$. So far, we have shown Φ_{y_i} exists and is smooth in X; moreover, $\widehat{(\Phi_{y_i})} = \widehat{\Phi}_{y_i}$, and $\widehat{(\Phi_{y_i x_j})} : \Omega \to C^\infty(X,\mathbb{R})$ is continuous. Meanwhile, the map $\widehat{(\Phi_{x_j})} : \Omega \to C^\infty(X,\mathbb{R})$ is continuous by Lemma 2.2.2 and $\widehat{\Phi} \in C^\infty$. $\widehat{(\Phi_{x_j})}$ is C^1 because

$$\lim_{h\to 0} \frac{\widehat{(\Phi_{x_j})}(y+he_i) - \widehat{(\Phi_{x_j})}(y)}{h}$$

converges to $\widehat{(\Phi_{y_i x_j})}(y)$ in $C^\infty(X,\mathbb{R})$ by (2.2.1). So we see if $\widehat{\Phi}$ is C^∞ then $\widehat{(\Phi_{x_j})}$ is C^1 and $\widehat{(\Phi_{x_j})}_{y_i} = \widehat{(\Phi_{y_i x_j})}$. Apply the same idea to $\widehat{\Phi}_{y_i}$, we have $\widehat{(\Phi_{y_i x_j})}$ is C^1, which implies $\widehat{(\Phi_{x_j})}$ is C^2 and by induction C^∞. All in all, $\widehat{(\Phi_{x_j})}$ and $\widehat{(\Phi_{y_i})} = \widehat{\Phi}_{y_i}$ are C^∞ if $\widehat{\Phi}$ is C^∞. By Lemma 2.2.2 Φ is $C^\infty(\Omega \times X)$. ■

One final remark, from the above proof, we see that $\widehat{(\Phi_{y_i})} = \widehat{\Phi}_{y_i}$ if Φ is smooth.

3. NONCOMMUTATIVE POTENTIAL THEORY

Let $(V, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space, $\operatorname{End} V$ the set of bounded linear operators on V, and $\operatorname{End}^+ V$ the set of all positive invertible elements of $\operatorname{End} V$. Let $\overline{M}$ be a compact Riemann surface with boundary. On the bundle $\overline{M} \times V \to \overline{M}$, a Hermitian metric h is a collection of Hermitian inner products h_z on V for $z \in \overline{M}$ that can be written as $h_z(v, w) = \langle P(z)v, w \rangle$ with $P : \overline{M} \to \operatorname{End}^+ V$, v and $w \in V$. Assuming P is C^2, the Chern connection of the metric is $P^{-1}\partial P$, and its curvature $R^P = \bar{\partial}(P^{-1}\partial P) = P^{-1}(P_{z\bar{z}} - P_{\bar{z}}P^{-1}P_z)d\bar{z} \wedge dz$ in a chart.

We digress here to argue that a Riemann surface with boundary $\overline{M}$ automatically gives a real analytic structure to its boundary ∂M (compactness is not needed here). By definition, the transition function between boundary charts of $\overline{M}$ is holomorphic in interior and continuous up to boundary. According to the reflection principle, such a function is holomorphic across the boundary, and therefore real analytic up to the boundary. In particular, the restriction of these transition functions to the boundary endows ∂M with a real analytic structure. It is this analytic structure that defines the smoothness classes in the theorem below.

Theorem 3.0.1 *Let $\overline{M}$ be a compact Riemann surface with boundary and $F \in C^m(\partial M, End^+ V)$, where $m = 0, \infty,$ or ω. There exists a unique $P \in C^m(\overline{M}, End^+ V) \cap C^\omega(M, End^+ V)$ such that $R^P = 0$ on M, and $P|_{\partial M} = F$. The same is true if we replace C^m by a Hölder class $C^{k,\alpha}$ for k a nonnegative integer and $0 < \alpha < 1$.*

The space $\operatorname{End} V$ with the operator norm $\| \cdot \|_{\text{op}}$ is a Banach space. So the spaces $C^{k,\alpha}(\overline{M}, \operatorname{End} V)$ and $C^{k,\alpha}(\partial M, \operatorname{End} V)$ are Banach algebras as defined in Chapter 2, and $C^{k,\alpha}(\overline{M}, \operatorname{End}^+ V)$ and $C^{k,\alpha}(\partial M, \operatorname{End}^+ V)$ are subspaces with values in $\operatorname{End}^+ V$. We denote by C^ω the space of real analytic maps, those that can be expanded at each point of its domain in a power series in a chart. Later on, we will use $\mathcal{O}(M, \operatorname{End} V)$

to denote the space of holomorphic maps, those that are complex differentiable in charts.

3.1 Preliminary lemmas

We denote the space of invertible elements in $\mathrm{End}\, V$ by $\mathrm{End}^{\times} V$. The first lemma is a standard fact in finite rank bundles, and the case of infinite rank bundles is proved similarly.

Lemma 3.1.1 *If M is a simply connected Riemann surface and $P \in C^2(M, End^{+}V)$ is flat, namely $R^P = 0$, then $P = H^*H$ where $H \in \mathcal{O}(M, End^{\times}V)$. If $P = K^*K$ is also such a factorization, then $H = UK$, where $U \in End V$ is unitary.*

Proof This lemma is actually true for M a simply connected complex manifold, and we will prove this general case. Fix a point $a \in M$, we define a map $\Psi : M \times V \to M \times V$ by setting $\Psi(z, v)$ equal to the parallel translation of v along a curve γ from a to z. Since M is simply-connected and $R^P = 0$, the map Ψ is independent of the choice of γ. For each $z \in M$, we denote by Ψ_z the map $v \mapsto \Psi(z, v)$, so Ψ_z is in $\mathrm{End}^{\times} V$. We claim that $z \mapsto \Psi_z$ is holomorphic. It suffices to show that for a fixed $v \in V$ the map $z \mapsto \Psi_z(v)$ is holomorphic. Indeed, the covariant derivative $D\Psi_z(v)$ is zero due to parallel translation, so $\bar{\partial}\Psi_z(v) = 0$ and $z \mapsto \Psi_z(v)$ is holomorphic.

Since parallel translation is an isometry, $\langle P(a)v, w\rangle = \langle P(z)\Psi_z(v), \Psi_z(w)\rangle$ for any $v, w \in V$, and hence $P(z) = (\Psi_z^{-1})^* P(a) \Psi_z^{-1}$. Because $P(a) \in \mathrm{End}^{+} V$, we see

$$P(z) = \left(P(a)^{1/2}\Psi_z^{-1}\right)^* P(a)^{1/2}\Psi_z^{-1},$$

and the lemma follows by setting $H(z) = P(a)^{1/2}\Psi_z^{-1}$.

If P has two factorizations $P = H^*H = K^*K$, then $HK^{-1} = H^{*-1}K^*$, which is holomorphic on one side while antiholomorphic on the other. Hence HK^{-1} must be a constant operator, say U. Then $U = U^{*-1}$ and $H = UK$. ∎

Lemma 3.1.2 *Let $\overline{M}$ be a compact Riemann surface with boundary. Let P, Q be in $C(\overline{M}, End^+V) \cap C^2(M, End^+V)$, and $R^P = R^Q = 0$. If $P \geq Q$ on ∂M, then $P \geq Q$ on M.*

Proof This is a special case of the maximum principle proved in [Lem15]. See also Lemma 3.2 in [Lem17]. ■

Lemma 3.1.3 *Let $\overline{M}$ be a compact Riemann surface with boundary. Let P_j be in $C(\overline{M}, End^+V) \cap C^2(M, End^+V)$, and $R^{P_j} = 0$, $j \in \mathbb{N}$. If $P_j|_{\partial M}$ converges in $C(\partial M, End^+V)$, then P_j converges in $C(\overline{M}, End^+V)$.*

Proof Suppose the limit of $P_j|_{\partial M}$ is P. Then there exists $\delta > 0$ such that $P(z) \geq \delta$ for $z \in \partial M$. We can find j_0 such that $\|P_j - P\|_{C^0(\partial M)} < \delta/2$ for $j \geq j_0$, which implies $-\delta/2 < P_j - P < \delta/2$ for $j \geq j_0$ and $z \in \partial M$. So $P_j > P - \delta/2 \geq \delta/2$ for $j \geq j_0$ and $z \in \partial M$, and by Lemma 3.1.2 this is true on M. Given $\varepsilon > 0$, we can find i_0 such that $\|P_i - P_j\|_{C^0(\partial M)} < \varepsilon\delta/2$ for $i, j \geq i_0$, and hence $-\varepsilon\delta/2 < P_i - P_j < \varepsilon\delta/2$ for $i, j \geq i_0$ and $z \in \partial M$. So for $z \in \partial M$ and $i, j \geq \max\{i_0, j_0\} := k_0$, we have $(1+\varepsilon)P_j > P_i > (1-\varepsilon)P_j$, and by Lemma 3.1.2 again, this is true on M. Hence $\|P_i - P_j\|_{C^0(M)} \leq \varepsilon \|P_j\|_{C^0(M)}$ for $i, j \geq k_0$. P_j is a bounded sequence on ∂M, and by Lemma 3.1.2 P_j is a bounded sequence on M. Therefore, P_j is a Cauchy sequence in $C(\overline{M}, \mathrm{End}^+V)$, and there exists $f \in C(\overline{M}, \mathrm{End}\, V)$ such that $P_j \to f$ in sup-norm. Since $P_j \geq \delta/2$ on M, the same holds for f, and hence $f \in C(\overline{M}, \mathrm{End}^+V)$.

■

Lemma 3.1.4 *Let $D \subset \mathbb{C}$ be the unit disc, $H_j \in \mathcal{O}(D, End^\times V)$, and $H_j(0) \in End^+V$. If $H_j^* H_j$ converges locally uniformly to some $P \in C(D, End^+V)$, then there exists $H \in \mathcal{O}(D, End^\times V)$ such that H_j converges, locally uniformly, to H on D.*

Proof Because $\|H_j\|^2 = \|H_j^* H_j\| \leq \|H_j^* H_j - P\| + \|P\|$, we see H_j is uniformly bounded on any compact set. Let C_r be the circle of radius $r < 1$ centered at the origin, so we have $\sup_{C_r} \|H_j\| \leq M_r$ for some positive M_r. The Cauchy estimate gives

$$\|\frac{\partial^k H_j(z)}{\partial z^k}\| \leq \frac{k!}{2\pi} M_r \frac{1}{(r - |z|)^{k+1}}.$$

It is straightforward to check that the partial derivatives of $P_j := H_j^* H_j$ are locally uniformly equicontinuous, and by [Lem17, Proposition 2.5] these partial derivatives converge locally uniformly. Meanwhile

$$P_j(0)^{-1/2}\frac{\partial^k P_j}{\partial z^k}(0) = \big(H_j^*(0)H_j(0)\big)^{-1/2} H_j^*(0)\frac{\partial^k H_j}{\partial z^k}(0) = \frac{\partial^k H_j}{\partial z^k}(0),$$

and as $j \to \infty$, the first term converges, say to A_k, so $\lim_{j\to\infty} \frac{\partial^k H_j}{\partial z^k}(0) = A_k$. Hence $\|A_k\| \leq k! M_r/(2\pi r^{k+1})$, and the power series $\sum_{k=0}^{\infty} A_k z^k/k! := H(z)$ has radius of convergence ≥ 1, so $H \in \mathcal{O}(D, \operatorname{End} V)$. Choose $R < r$, for $|z| \leq R$,

$$\|\frac{\partial^k H_j}{\partial z^k}(0)\frac{z^k}{k!}\| \leq \frac{M_r R^k}{2\pi r^{k+1}},$$

so $H_j(z) = \sum \frac{\partial^k H_j}{\partial z^k}(0)\frac{z^k}{k!}$ converges uniformly to H on $|z| \leq R$. Finally, since $H_j(z) \in \operatorname{End}^{\times} V$ and

$$\|H_j(z)^{-1}\|^2 = \|H_j(z)^{-1}H_j(z)^{-1*}\| = \|(H_j^* H_j)^{-1}(z)\| \to \|P^{-1}(z)\|,$$

$H(z)$ is actually in $\operatorname{End}^{\times} V$.

■

3.2 A priori estimates

Fix a smooth positive $(1,1)$-form ω on $\overline{M}$ and define a map Λ sending $(1,1)$-forms to functions: $\Lambda(\phi) = -\phi/\omega$, for a $(1,1)$-form ϕ. Locally, $\omega = \sqrt{-1} g dz \wedge d\bar{z}$, where g is a positive smooth function, so if $\phi = v dz \wedge d\bar{z}$ locally, then $\Lambda(\phi) = \sqrt{-1} v/g$.

Fix $0 < \alpha < 1$, assume $P \in C^{2,\alpha}(\overline{M}, \operatorname{End}^{+} V)$ has zero curvature, i.e. flat, and $A = P^{-1}\partial P$. We associate the following differential operator with P:

$$\begin{aligned} L :& C^{2,\alpha}(\overline{M}, \operatorname{End}^{\mathrm{self}} V) \longrightarrow C^{\alpha}(\overline{M}, \operatorname{End}^{\mathrm{self}} V) \\ h \longmapsto& \sqrt{-1}\Lambda(\bar{\partial}\partial h - A^* \wedge \partial h - \bar{\partial} h \wedge A + A^* \wedge h \wedge A). \end{aligned}$$

On a chart, $Lh = (1/g)\mathcal{L}h$, where

$$\mathcal{L}h = h_{z\bar{z}} - P_{\bar{z}}P^{-1}h_z - h_{\bar{z}}P^{-1}P_z + P_{\bar{z}}P^{-1}hP^{-1}P_z.$$

The reason for studying L is that it is the linearization of curvature, as we shall see in section 3.3. The main result in this section is

Theorem 3.2.1 *If* $h \in C^{2,\alpha}(\overline{M}, End^{\text{self}} V)$ *and* $h|_{\partial M} = 0$*, then*

$$\|h\|_{2,\alpha,M} \leq C\|Lh\|_{0,\alpha,M}$$

where $C = C(\|P\|_{2,\alpha}, \|P^{-1}\|_{0,\alpha})$.

We begin with a somewhat standard estimate.

Lemma 3.2.2 *If* $h \in C^{2,\alpha}(\overline{M}, End^{\text{self}} V)$ *and* $h|_{\partial M} = 0$*, then*

$$\|h\|_{2,\alpha,M} \leq C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M})$$

where $C = C(\|P\|_{2,\alpha}, \|P^{-1}\|_{0,\alpha})$.

The prominent feature of L is the following. On a simply connected open set, we have $H^*PH = 1$ with holomorphic H by Lemma 3.1.1, and it turns out that

$$\frac{1}{2}\Delta(H^*hH) = -H^*(Lh)H.$$

Here Δ is the Laplace operator with respect to ω, and we use the fact that Δ when acting on functions is the same as $2\sqrt{-1}\Lambda\bar{\partial}\partial$. Therefore, modulo a gauge transformation H, L is the Laplace operator, locally. In a chart, the above equality becomes $(H^*hH)_{z\bar{z}} = H^*(\mathcal{L}h)H$. We will exploit this to reduce Lemma 3.2.2 to the corresponding estimates for scalar-valued elliptic partial differential equations.

If L had nonpositive zero order term, general theory would imply $\|h\|_{0,M} \leq C\|Lh\|_{0,M}$, which together with Lemma 3.2.2 would give Theorem 3.2.1. However, the zero order term of L is nonnegative. To get around this problem we first prove a maximum principle, Lemma 3.2.3, and observe that for $u \in C^2(\overline{M}, \mathbb{C})$,

$$L(u \cdot P) = \sqrt{-1}\Lambda(-\bar{\partial}\partial u \cdot P + uP \cdot R^P) = (-\frac{1}{2}\Delta u)P,$$

as $R^P = 0$. A suitable choice of u will put us in the position of using Lemma 3.2.3, and Theorem 3.2.1 will follow quickly.

Proof [Proof of Lemma 3.2.2] Consider two finite open covers $\{U_i\}, \{V_i\}$ of $\overline{M}$, such that $\overline{U_i}, \overline{V_i}$ are in a chart ϕ_i for each i, and

$$\text{for interior chart, } \begin{cases} \phi_i(U_i) = B(0,1) \\ \phi_i(V_i) = B(0,2). \end{cases}$$

$$\text{for boundary chart, } \begin{cases} \phi_i(U_i) = B(0,1) \cap \overline{H} \\ \phi_i(V_i) = B(0,2) \cap \overline{H} \end{cases} \quad \text{where } H \subset \mathbb{C} \text{ is the upper-half plane.}$$

We use $\{U_i\}$ to define the norm on $C^{2,\alpha}(\overline{M}, \operatorname{End}^{\text{self}} V)$ and $\{V_i\}$ on $C^{\alpha}(\overline{M}, \operatorname{End}^{\text{self}} V)$.

Since our arguments will be local, we can assume U_i, V_i are already in $\mathbb{C}$ and ϕ_i is the identity. We first consider a boundary chart ϕ_i. As mentioned above, $(H^*hH)_{z\bar{z}} = H^*(\mathcal{L}h)H$, where H is a holomorphic function in the interior of this chart with $H^*PH = 1$. As P is $C^{2,\alpha}$ up to the boundary of $\overline{M}$, so is H, according to [Lem17, Theorem 3.7]. Consider a bounded linear functional $l \in (\operatorname{End} V)^*$ of norm one, and apply l to the equation obtaining $[l(H^*hH)]_{z\bar{z}} = l(H^*(\mathcal{L}h)H)$, a scalar-valued equation. Denote $\phi_i(U_i) = B'$ and $\phi_i(V_i) = B''$. By [GT01, Lemma 6.5 or Corollary 6.7]

$$\|l(H^*hH)\|_{2,\alpha,B'} \leq C(\|l(H^*hH)\|_{0,B''} + \|l(H^*(\mathcal{L}h)H)\|_{0,\alpha,B''}) \tag{3.2.1}$$

where C is a uniform constant.We can get rid of l and H to have

$$\|h\|_{2,\alpha,B'} \leq C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}).$$

Indeed, at each point in B'',

$$|l(H^*hH)| \leq \|H^*hH\|_{\text{op}} \leq \|H\|_{\text{op}}^2 \|h\|_{\text{op}} \leq C\|h\|_{\text{op}}.$$

The last inequality follows from $P^{-1} = HH^*$. Similarly,

$$\begin{aligned}
\|l(H^*(\mathcal{L}h)H)\|_{0,\alpha,B''} &\le \|H^*(\mathcal{L}h)H\|_{0,\alpha,B''} \\
&\le \|H\|^2_{0,\alpha,B''} \cdot \|\mathcal{L}h\|_{0,\alpha,B''} \\
&\le C\|H\|^2_{0,\alpha,B''} \cdot \|Lh\|_{0,\alpha,M} \\
&\le C(\|H\|_0^2 + \|H_z\|_0^2) \cdot \|Lh\|_{0,\alpha,M} \\
&\le C\|Lh\|_{0,\alpha,M}
\end{aligned}$$

The third inequality is by the definition of the C^α norm on M. The last inequality follows from $H_z = -P^{-1}P_zH$. Therefore, the right hand side of (3.2.1) is dominated by $C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M})$, which gives $C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}) \ge \|l(H^*hH)\|_{2,\alpha,B'}$. Let D stand for $(\partial_z, \partial_{\bar{z}})$, and D^2 for $(\partial_z^2,\ \partial_z\partial_{\bar{z}},\ \partial_{\bar{z}}^2)$. Notice that $\|l(H^*hH)\|_{2,\alpha,B'}$ is comparable with

$$\|l(H^*hH)\|_{0,B'} + \|Dl(H^*hH)\|_{0,B'} + \|D^2l(H^*hH)\|_{0,\alpha,B'}.$$

We obtain, for $x \in B'$,

$$|Dl(H^*hH)(x)| \le C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M})$$

and then take supremum over l of norm one to get

$$\|D(H^*hH)(x)\|_{\mathrm{op}} \le C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}).$$

As a consequence,

$$\begin{aligned}
C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}) &\ge \|DH^*hH + H^*DhH + H^*hDH\|_{0,B'} \\
&\ge \|H^*DhH\|_{0,B'} - \|H^*hDH\|_{0,B'} - \|DH^*hH\|_{0,B'} \\
&\ge \|H^*DhH\|_{0,B'} - C\|h\|_{0,B'}.
\end{aligned}$$

So

$$C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}) \ge \|H^*DhH\|_{0,B'}.$$

Since

$$\|Dh\|_{0,B'} \le \|H^*DhH\|_{0,B'}\|H^{-1}\|^2_{0,B'} = \|H^*DhH\|_{0,B'}\|P\|_{0,B'} \le C\|H^*DhH\|_{0,B'},$$

we have

$$\|Dh\|_{0,B'} \leq C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}).$$

We can estimate the second derivatives and their Hölder norms similarly, and obtain

$$\|h\|_{2,\alpha,B'} \leq C(\|h\|_{0,M} + \|Lh\|_{0,\alpha,M}). \tag{3.2.2}$$

We next consider an interior chart ϕ_i. As before $[l(H^*hH)]_{z\bar{z}} = l(H^*(\mathcal{L}h)H)$. We let $\phi_i(U_i) = B'$ and $\phi_i(V_i) = B''$. By [GT01, Corollary 6.3],

$$\begin{aligned} &\|Dl(H^*hH)\|_{0,B'} + \|D^2l(H^*hH)\|_{0,B'} + [D^2l(H^*hH)]_{\alpha,B'} \\ &\qquad \leq C\big[\|l(H^*hH)\|_{0,B''} + \|l(H^*(\mathcal{L}h)H)\|_{0,\alpha,B''}\big]. \end{aligned}$$

Using the same method as in boundary charts, we can get rid of l and H to obtain the same estimate (3.2.2). Hence the lemma follows. ■

We next prove a maximum principle, which in turn gives rise to C^0 estimates. Recall that $\langle\cdot,\cdot\rangle$ is the inner product of V, and denote $\|v\|^2_{P(z)} = \langle P(z)v, v\rangle$.

Lemma 3.2.3 *Suppose $h \in C^2(M, End^{\mathrm{self}}V)$. Define*

$$S_{P,h}(z) = \sup_{\|v\|_{P(z)}=1} \langle h(z)v, v\rangle.$$

If $Lh \geq 0$, then $S_{P,h}(z)$ is subharmonic. As a result, if additionally h is continuous on $\overline{M}$, then

$$\sup_{\overline{M}} S_{P,h} = \sup_{\partial M} S_{P,h}.$$

Proof First,

$$\begin{aligned} S_{P,h}(z) &= \sup_{\langle P(z)v,v\rangle=1} \langle h(z)v, v\rangle = \sup_{\|P(z)^{1/2}v\|=1} \langle P(z)^{-1/2}h(z)P(z)^{-1/2}P(z)^{1/2}v, P(z)^{1/2}v\rangle \\ &= \sup_{\|u\|=1} \langle P(z)^{-1/2}h(z)P(z)^{-1/2}u, u\rangle \end{aligned}$$

is continuous, as the sup of a family of equicontinuous functions. Locally, we have $H^*PH = 1$ and $(H^*hH)_{z\bar{z}} = H^*(\mathcal{L}h)H$; furthermore, $0 \leq Lh = (1/g)\cdot\mathcal{L}h$ means $\mathcal{L}h \geq 0$. Since

$$0 \leq \langle(\mathcal{L}h)Hv, Hv\rangle = \langle(H^*hH)_{z\bar{z}}v, v\rangle,$$

$\langle (H^*hH)v, v\rangle$ is subharmonic for any $v \in V$. Thus,

$$S_{P,h}(z) = \sup_{\langle P(z)v,v\rangle=1} \langle h(z)v, v\rangle = \sup_{\langle H^{-1}v,H^{-1}v\rangle=1} \langle h(z)v, v\rangle = \sup_{\langle u,u\rangle=1} \langle H^*hH(z)u, u\rangle$$

is the sup of a family of subharmonic functions. As we already know $S_{P,h}(z)$ is continuous, it is subharmonic. ■

Theorem 3.2.4 *If $h \in C^{2,\alpha}(\overline{M}, End^{\text{self}}V)$ and $h|_{\partial M} = 0$, then*

$$\|h\|_{0,M} \le C\|Lh\|_{0,M}$$

where $C = C(\|P\|_0, \|P^{-1}\|_0)$.

Proof Recall if $u \in C^2(\overline{M}, \mathbb{C})$, then

$$L(u \cdot P) = (-\frac{1}{2}\Delta u)P.$$

Let Φ be the function vanishing on ∂M such that $\Delta\Phi = 2$, and let $G = (\Phi - \inf \Phi)\|P^{-1}\|_0 P$. Then $G \ge 0$ with $L(G) = -\|P^{-1}\|_0 P \le -1$. Besides, $G \le C$, where C depends on $\|P\|_0$ and $\|P^{-1}\|_0$. With $F = G \cdot \|Lh\|_0$, we have $h \le F$ on ∂M. Moreover,

$$L(h - F) = Lh - \|Lh\|_0 \cdot LG \ge Lh + \|Lh\|_0 \ge 0.$$

By Lemma 3.2.3, $h - F \le 0$ on M. Therefore,

$$h \le G \cdot \|Lh\|_0 \le C\|Lh\|_0.$$

Replacing h by $-h$, the theorem follows. ■

Theorem 3.2.1 is a consequence of Lemma 3.2.2 and Theorem 3.2.4.

3.3 Proof of the main theorem

We start with a regularity result.

Lemma 3.3.1 *Let $P \in C(\overline{M}, End^{+}V) \cap C^2(M, End^{+}V)$ be flat. If $P|_{\partial M}$ is $C^{k,\alpha}$, C^∞, or C^ω, then P has the corresponding regularity on $\overline{M}$.*

Proof By Lemma 3.1.1, $P = H^*H$ with a holomorphic map H locally, so P is always C^ω in M regardless of its boundary values. Denote $P|_{\partial M}$ by F. If $F \in C^{k,\alpha}$, by [Lem17, Theorem 3.7] on a boundary chart $P = H^*H$ with H of class $C^{k,\alpha}$ up to ∂M; therefore, P is $C^{k,\alpha}$ up to ∂M. Next suppose F is C^∞, then by the $C^{k,\alpha}$ result, P is C^k up to ∂M for any positive integer k, hence C^∞.

Finally, suppose $F \in C^\omega$. On a boundary chart, that we identify with the upper-half disc in $\mathbb{C}$, $P = H^*H$ with H continuous up to the real axis by [Lem17, Theorem 3.7]. Since $F \in C^\omega$, it has a holomorphic extension in a neighborhood of the real axis in the disc, so the map $H^{*-1}(\bar{z}) \cdot F(z)$ provides H a holomorphic extension across the real axis, and it follows that P is real analytic across the real axis. ■

Proof [Proof of Theorem 3.0.1] The uniqueness follows from the maximum principle (see [Lem17, Lemma 3.2] or [Lem15]). We consider first the case $F \in C^\omega$ and prove the existence by the continuity method. Fix $0 < \alpha < 1$, let $\phi_t = tF + (1-t)\mathrm{Id}$, and

$$T = \left\{ t \in [0,1] \;\middle|\; \begin{array}{c} \text{If } 0 \le s \le t, \text{ then } \phi_s = P_s|_{\partial M}, \\ \text{for some } P_s \in C^{2,\alpha}(\overline{M}, \mathrm{End}^{+}V), \text{ and } R^{P_s} = 0 \end{array} \right\}.$$

We will say those ϕ_s "have an extension." The goal is to show $T = [0,1]$. If so, $\phi_1 = F$ has a $C^{2,\alpha}$ extension, and we can improve the regularity from $C^{2,\alpha}$ to C^ω by Lemma 3.3.1. Because 0 is in T, T is nonempty. First we prove T is closed.

Suppose $T \ni t_j \to t_0$. For $s < t_0$, we can find $t_j > s$, therefore ϕ_s has an extension. We have to show ϕ_{t_0} extends. For brevity, we write P_j instead of P_{t_j}. Since $P_j|_{\partial M} = \phi_{t_j} \to \phi_{t_0}$, P_j converges by Lemma 3.1.3, say to $P_\infty \in C(\overline{M}, \mathrm{End}^{+}V)$, and $P_\infty|_{\partial M} = \phi_{t_0}$. For any interior point of $\overline{M}$, choose a chart with image the unit disc D in $\mathbb{C}$. Thus, $P_j = H_j^* H_j$, where $H_j \in \mathcal{O}(D, \mathrm{End}^{\times}V)$ by Lemma 3.1.1, and after multiplying with the unitary operator $(H_j^*(0)H_j(0))^{1/2}H_j^{-1}(0)$ we can assume $H_j(0) \in \mathrm{End}^{+}V$. By Lemma 3.1.4, there exists H holomorphic on D such that $H_j \to H$ locally uniformly. Hence, $P_\infty = \lim H_j^* H_j = H^*H$ on D which implies

$P_\infty \in C^\infty(M, \mathrm{End}^+ V)$ and $R^{P_\infty} = 0$. By Lemma 3.3.1, P_∞ is C^ω, especially $C^{2,\alpha}$ on $\overline{M}$. Hence, t_0 is in T and T is closed.

Now we prove that T is open. If $t_0 \in T$ then ϕ_t has an extension P_t, for $0 \leq t \leq t_0$. Consider the smooth map

$$\begin{aligned}\Psi : C^{2,\alpha}(\overline{M}, \mathrm{End}^+ V) &\to C^\alpha(\overline{M}, \mathrm{End}^{\mathrm{self}} V) \times C^{2,\alpha}(\partial M, \mathrm{End}^+ V)\\ h &\mapsto (\sqrt{-1}\Lambda(h\bar{\partial}(h^{-1}\partial h)), h|_{\partial M}).\end{aligned}$$

Then $\Psi(P_{t_0}) = (0, \phi_{t_0})$. We denote $P_t^{-1}\partial P_t = A_t$, so the linearization of Ψ at P_{t_0} is

$$\begin{aligned}C^{2,\alpha}(\overline{M}, \mathrm{End}^{\mathrm{self}} V) &\to C^\alpha(\overline{M}, \mathrm{End}^{\mathrm{self}} V) \times C^{2,\alpha}(\partial M, \mathrm{End}^{\mathrm{self}} V)\\ h &\mapsto (\sqrt{-1}\Lambda(\bar{\partial}\partial h - A_{t_0}^* \wedge \partial h - \bar{\partial} h \wedge A_{t_0} + A_{t_0}^* \wedge h \wedge A_{t_0}), h|_{\partial M}).\end{aligned}$$

It is here the operator in section 3.2 turns up. We will show that the linearization is an isomorphism. Then Ψ is a diffeomorphism in a neighborhood of P_{t_0} by the implicit function theorem, and that implies T is open.

To show that the linearization is an isomorphism, it suffices to prove it is bijective because of the Open Mapping Theorem. That is, given

$$(f_1, f_2) \in C^\alpha(\overline{M}, \mathrm{End}^{\mathrm{self}} V) \times C^{2,\alpha}(\partial M, \mathrm{End}^{\mathrm{self}} V),$$

the equation

$$\begin{cases}\sqrt{-1}\Lambda(\bar{\partial}\partial h - A_{t_0}^* \partial h - \bar{\partial} h A_{t_0} + A_{t_0}^* h A_{t_0}) = f_1\\ h|_{\partial M} = f_2\end{cases} \tag{3.3.1}$$

has a unique solution. That there is at most one solution easily follows from the maximum principle, Lemma 3.2.3 or Theorem 3.2.1. If $\dim V < \infty$, existence follows from uniqueness by Fredholm alternative. However, if $\dim V = \infty$, Fredholm alternative is not available, because the embedding $C^{2,\alpha}(\overline{M}, \mathrm{End}\, V) \to C^\alpha(\overline{M}, \mathrm{End}\, V)$ is no longer compact. The way we solve (3.3.1) is again the continuity method, based on the next lemma:

Lemma 3.3.2 *Let B, V be two Banach spaces, and $\{L_t\}_{0\leq t\leq 1}$ a family of bounded linear operators from B to V. Suppose $t \mapsto L_t$ is continuous in operator norm; moreover, there exists a constant C such that*

$$\|x\| \leq C\|L_t x\| \tag{3.3.2}$$

for any $x \in B$ and any t. Then L_1 is onto if and only if L_0 is onto.

Proof This is a variant of [GT01, Theorem 5.2]. Suppose L_s is onto for some $s \in [0,1]$. By (3.3.2), L_s is one-to-one, and hence the inverse $L_s^{-1} : V \to B$ exists; moreover, for $y \in V$,

$$\|L_s^{-1} y\| \leq C\|L_s L_s^{-1} y\| = C\|y\|.$$

For $t \in [0,1]$ and $y \in V$, we are looking for $x \in B$ such that $L_t x = y$. The equation $L_t x = y$ is equivalent to $x = L_s^{-1} y + L_s^{-1}(L_s - L_t)x$. Define a map $T : B \to B$ by $Tx = L_s^{-1} y + L_s^{-1}(L_s - L_t)x$. For $x_1, x_2 \in B$,

$$\|Tx_1 - Tx_2\| = \|L_s^{-1}(L_s - L_t)(x_1 - x_2)\| \leq C\|L_s - L_t\|\|x_1 - x_2\|.$$

Because of the continuity of $t \mapsto L_t$, there exists $\delta > 0$ indepent of s, such that if $|s - t| < \delta$ then $\|L_s - L_t\| < 1/2C$. Therefore, if $|s - t| < \delta$, then $\|Tx_1 - Tx_2\| < \|x_1 - x_2\|/2$, a contraction on B, and hence T has a unique fixed point. So if $|s - t| < \delta$, L_t is onto. By dividing $[0,1]$ into subintervals of length less than δ, the lemma follows.

■

Unsurprisingly, we are going to deform our equation to the Laplace equation. The naive way of deforming is by convex combination, but this breaks the symmetry of our equation (after all we want to use the a priori estimates from Theorem 3.2.1). It is here the solution set T plays its role; it tells us how to deform.

First in equation (3.3.1), f_2 can be extended to $C^{2,\alpha}(\overline{M}, \mathrm{End}^{\,\mathrm{self}} V)$. If we subtract f_2 from h, we only need to consider the case of zero boundary value. In other words, we have to show that

$$L_t : \{h \in C^{2,\alpha}(\overline{M}, \mathrm{End}^{\,\mathrm{self}} V) : h|_{\partial M} = 0\} \to C^{\alpha}(\overline{M}, \mathrm{End}^{\,\mathrm{self}} V) \tag{3.3.3}$$

$$h \mapsto \sqrt{-1}\Lambda(\bar{\partial}\partial h - A_t^* \wedge \partial h - \bar{\partial} h \wedge A_t + A_t^* \wedge h \wedge A_t) \tag{3.3.4}$$

is surjective when $t = t_0$. Note that L_0 is the Laplace operator, for $P_0 = 1$. We start with the following lemma, which is stronger than what we need.

Lemma 3.3.3 *Let k be a nonnegative integer. If $t, s \in [0, t_0]$ and $t \to s$, then $\|P_t - P_s\|_{C^k} \to 0$, and $\|P_t^{-1} - P_s^{-1}\|_{C^k} \to 0$.*

Proof By Lemma 3.3.1, $P_t \in C^k(\overline{M}, \operatorname{End}^+ V)$. Since $P_t|_{\partial M} = \phi_t \to \phi_s$, P_t converges to P_s in $C(\overline{M}, \operatorname{End}^+ V)$ by Lemma 3.1.3. For the derivatives, we do estimates on charts and consider ∂_z only, as $\partial_{\bar{z}}$ can be done in the same way. On an interior chart, $P_t = H_t^* H_t, P_s = H^* H$ where H_t, H are holomorphic. As in the proof of closedness, $H_t \to H$ locally uniformly, and so do all their derivatives. Therefore,

$$(P_t)_z = H_t^*(H_t)_z \to H^* H_z = (P_s)_z$$

locally uniformly. On a boundary chart, that again we identify with the upper-half disc in $\mathbb{C}$, we similarly have $(P_t)_z \to (P_s)_z$ locally uniformly but only away from the boundary. The convergence near the boundary can be resolved as follows. $P_t = H_t^* H_t$ with H_t continuous up to boundary of $\overline{M}$ (in the current situation, this means the real axis of the unit disc) by [Lem17, Theorem 3.7]. Similarly, $P_s = H^* H$ with H continuous up to boundary of $\overline{M}$. As in the proof of Lemma 3.3.1, since ϕ_t is C^ω, it has a holomorphic extension in a neighborhood of the real axis of the unit disc, the map

$$H_t^{*-1}(\bar{z}) \cdot \phi_t(z)$$

provides an analytic continuation of H_t across the real axis that we continue denoting H_t. For a compact set in the unit disc, consider a contour around it. By Cauchy's Integral Formula and the fact $\|H_t\|$ has a uniform upper bound, the Bounded Convergence Theorem implies that H_t converges to H uniformly on this compact set, and the same holds for derivatives of all orders. Hence, $P_t \to P_s$ in C^k for any nonnegative integer k, locally uniformly in this boundary chart. Therefore, we conclude the C^k convergence on $\overline{M}$. Since $C^k(\overline{M}, \operatorname{End} V)$ is a Banach algebra, $P_t^{-1} \to P_s^{-1}$ in C^k. ■

This lemma implies that $\|L_t - L_s\| \to 0$ as $t \to s$, where the norm on L_t is the operator norm from (4.2). From Theorem 3.2.1 and the continuity Lemma 3.3.3, we get the desired estimates: if $h \in C^{2,\alpha}(\overline{M}, \mathrm{End}^{\mathrm{self}} V)$ and $h|_{\partial M} = 0$, then

$$\|h\|_{2,\alpha,M} \leq C\|L_t h\|_{0,\alpha,M}$$

where C is independent of t. Therefore, by Lemma 3.3.2 and the fact $L_0 = \Delta/2$ is onto, L_{t_0} is also onto, which implies the equation (3.3.1) is uniquely solvable, so T is open and therefore $T = [0, 1]$. This completes the proof of Theorem 3.0.1 for C^ω case.

If the boundary data F is only C^0, it can be approximated by a sequence $F_j \in C^\omega(\partial M, \mathrm{End}^+ V)$ in sup norm, for the following reason: ∂M as a real analytic manifold can be real analytically embedded in some $\mathbb{R}^N$ by an embedding theorem of Grauert and Morrey [Gra58] [Mor58]; F has a continuous extension to $\mathbb{R}^N$, which can be approximated by polynomials P_j; after composing P_j with the embedding, we have the desired F_j. Each F_j has a real analytic flat extension P_j according to the C^ω case. By Lemma 3.1.3, P_j converges in $C(\overline{M}, \mathrm{End}^+ V)$, say to P. As in the proof of closedness, P is C^2 in the interior and has curvature 0.

If F is $C^{k,\alpha}$ or C^∞, the P constructed in the previous paragraph is $C^{k,\alpha}$, respectively C^∞ on $\overline{M}$, by Lemma 3.3.1. ■

4. THE SPACE OF KÄHLER POTENTIALS

Let us recall the setup from Chapter 1 and give a more detailed account. Let X be a compact complex manifold of dimension n with a Kähler form ω. The space of Kähler potentials is

$$\mathcal{H}_\omega = \{\phi \in C^\infty(X, \mathbb{R}) : \omega + i\partial\bar{\partial}\phi > 0\},$$

and we will denote $\omega + i\partial\bar{\partial}\phi$ by ω_ϕ.

By the discussion in Chapter 2, the space $\mathcal{H}_\omega$ is an open subset in the Fréchet space $C^\infty(X, R)$, and therefore it is a Fréchet manifold. The tangent space $T_\phi\mathcal{H}_\omega$ at $\phi \in \mathcal{H}_\omega$ is canonically isomorphic to $C^\infty(X, \mathbb{R})$, and tangent bundle $T\mathcal{H}_\omega$ is canonically isomorphic to $\mathcal{H}_\omega \times C^\infty(X, \mathbb{R})$. Following Mabuchi [Mab87], Semmes [Sem92], and Donaldson [Don99], the Mabuchi metric g_M on $\mathcal{H}_\omega$ is the following. For a point $\phi \in \mathcal{H}_\omega$ and two vectors $\xi, \eta \in T_\phi\mathcal{H}_\omega \approx C^\infty(X, \mathbb{R})$, the Mabuchi metric is

$$g_M(\xi, \eta) = \int_X \xi\eta\, \omega_\phi^n.$$

On the right hand side ξ and η are viewed as smooth functions on X, and we are integrating a product of two functions.

Let D be a bounded smooth strongly pseudoconvex domain in $\mathbb{C}^m$. A map $\widehat{\Phi} : D \to \mathcal{H}_\omega$ will induce a map $\Phi : D \times X \to \mathbb{R}$ with $\Phi(z, \cdot) \in \mathcal{H}_\omega$ for $z \in D$, and vice versa. Equipping D with the Euclidean metric and $\mathcal{H}_\omega$ with Mabuchi metric, we can compute the Hilbert–Schmidt norm of the tangent map $\widehat{\Phi}'_z : T_zD \to T_{\widehat{\Phi}(z)}\mathcal{H}_\omega$ that we denote by $|\widehat{\Phi}'_z|$ for $z \in D$. If $\{v_1, ..., v_{2m}\}$ is an orthonormal basis for $T_zD \approx \mathbb{R}^{2m}$, then $|\widehat{\Phi}'_z|^2 = \sum_{j=1}^{2m} g_M(\widehat{\Phi}'_z v_j, \widehat{\Phi}'_z v_j)$. The harmonic energy functional is defined to be $E(\widehat{\Phi}) = \int_D |\widehat{\Phi}'_z|^2 dV$ where dV is the Euclidean volume form on D. A map $\widehat{\Phi} : D \to \mathcal{H}_\omega$ is said to be harmonic if it is a critical point of the harmonic energy functional. Let $\{z_j\}$ be coordinates on D. For $z \in D$, $\Phi_{z_j}(z, \cdot)$ is a function on X.

We compute the gradient of $\Phi_{z_j}(z,\cdot)$ with respect to the Kähler metric $\omega_{\widehat{\Phi}(z)}$ and denote the complexified vector field on X by $\nabla\Phi_{z_j}(z)$. The length of this vector field with respect to the metric $\omega_{\widehat{\Phi}(z)}$ is denoted by $|\nabla\Phi_{z_j}(z)|$. A computation, as in finite dimensions, gives that a map $\widehat{\Phi}: D \to \mathcal{H}_\omega$ is a critical point of E if the corresponding Φ satisfies the harmonic map equation:

$$\sum_{j=1}^{m} \Phi_{z_j\bar{z}_j} - \frac{1}{2}|\nabla\Phi_{z_j}|^2 = 0. \tag{4.0.1}$$

The functional that we are looking for, which we denote by $\mathcal{E}$, is a perturbation of the harmonic functional E above. In order to define $\mathcal{E}$, we recall first the Poisson bracket $\{\cdot,\cdot\}_{\omega_\phi}$ on $C^\infty(X,\mathbb{R})$ determined by the symplectic form ω_ϕ with $\phi \in \mathcal{H}_\omega$. For $\xi,\eta \in T_\phi\mathcal{H}_\omega \approx C^\infty(X,\mathbb{R})$, the Poisson bracket $\{\xi,\eta\}_{\omega_\phi}$ is a smooth function on X characterized by $\{\xi,\eta\}_{\omega_\phi}\omega_\phi^n = nd\xi\wedge d\eta\wedge\omega_\phi^{n-1}$. Next we define a three-form θ on $\mathcal{H}_\omega$: for $\phi\in\mathcal{H}_\omega$ and $\xi_1,\xi_2,\xi_3 \in T_\phi\mathcal{H}_\omega$,

$$\theta(\xi_1,\xi_2,\xi_3) := g_M(\{\xi_1,\xi_2\}_{\omega_\phi},\xi_3) = \int_X \{\xi_1,\xi_2\}_{\omega_\phi}\xi_3\omega_\phi^n. \tag{4.0.2}$$

This three-form θ is d-closed (see Lemma 4.3.3 below), and therefore there is a two-form α on $\mathcal{H}_\omega$ such that $d\alpha = \theta$. (This is because $\mathcal{H}_\omega \subset C^\infty(X,\mathbb{R})$ is convex, and Poincare's exactness lemma holds in Fréchet manifolds too, by the same proof as in finite dimensions.) For a map $\widehat{\Phi}: D\to\mathcal{H}_\omega$, we define

$$\mathcal{E}(\widehat{\Phi}) := E(\widehat{\Phi}) + 4i\sum_j \int_D \alpha(\Phi_{\bar{z}_j},\Phi_{z_j})dV. \tag{4.0.3}$$

We will show in Lemma 4.3.4 that the Euler–Lagrange equation of $\mathcal{E}$ is

$$\sum_{j=1}^{m} \Phi_{z_j\bar{z}_j} - \frac{1}{2}|\nabla\Phi_{z_j}|^2 - \frac{i}{2}\{\Phi_{\bar{z}_j},\Phi_{z_j}\}_{\omega_{\widehat{\Phi}}} = 0, \tag{4.0.4}$$

In view of its connection with [Wit83] and following [Don99], we call the equation (4.0.4) the WZW equation for a map $\widehat{\Phi}: D\to\mathcal{H}_\omega$.

Donaldson showed in [Don99], when $m=1$, the WZW equation is equivalent to a homogeneous complex Monge–Ampère equation. We have the following extended equivalence for $m \geq 1$ by a similar computation. Let $\pi : D \times X \to X$ be the projection onto X. Then the extended equivalence is

$$\Phi \text{ solves (4.0.4) if and only if } (i\partial\bar{\partial}\Phi + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} = 0. \quad (4.0.5)$$

After seeing the WZW equation, we are now trying to solve it and study properties of the solution. Recall the definition of ω-subharmonicity on graphs.

Definition 4.0.1 *A function $u : D \times X \to [-\infty, \infty)$ is called ω-subharmonic on graphs if for any holomorphic map f from an open subset of D to X, $\psi(f(z)) + u(z, f(z))$ is subharmonic, where ψ is a local potential of ω.*

Let v be a real-valued smooth function on $\partial D \times X$ and $\partial D \ni z \mapsto v(z, \cdot) = v_z \in \mathcal{H}_\omega$. Consider the Perron family

$$G_v := \{u \in \text{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs,}$$
$$\text{and } \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \leq v(\zeta, x)\}.$$

As we will later see, the upper envelope $V = \sup\{u : u \in G_v\}$ is a weak solution of the WZW equation from D to $\mathcal{H}_\omega$. Recall the philosophy that when ω is the curvature of some Hermitian line bundle $(L, h) \to X$, then the infinite dimensional space $\mathcal{H}_\omega$ can be approximated by the spaces $\mathcal{H}_k$ of inner products on $H^0(X, L^k)$. There are two maps going between $\mathcal{H}_\omega$ and $\mathcal{H}_k$. The Hilbert map $H_k : \mathcal{H}_\omega \to \mathcal{H}_k$ is

$$H_k(\phi)(s, s) = \int_X h^k(s, s) e^{-k\phi} \omega^n, \text{ for } \phi \in \mathcal{H}_\omega \text{ and } s \in H^0(X, L^k).$$

In the other direction, the Fubini–Study map $FS_k : \mathcal{H}_k \to \mathcal{H}_\omega$ is

$$FS_k(G)(x) = \frac{1}{k} \log \sup_{s \in H^0(X, L^k), G(s,s) \leq 1} h^k(s, s)(x), \text{ for } G \in \mathcal{H}_k \text{ and } x \in X.$$

An equally good name would be Bergman map, for the sup above defines the Bergman kernel of $H^0(X, L^k)$ with an inner product G.

Following the definitions from [CS93], let $\mathcal{N}_k^*$ be the set of norms on $H^0(X, L^k)^*$.

Definition 4.0.2 *A norm function $D \ni z \mapsto U_z \in \mathcal{N}_k^*$ is said to be subharmonic if $\log U_z(f(z))$ is subharmonic for any holomorphic function $f : W \subset D \to H^0(X, L^k)^*$.*

The *quantum* Perron family is

$$G_v^k := \{D \ni z \to U_z \in \mathcal{N}_k^* \text{ is subharmonic and}$$
$$\limsup_{D \ni z \to \zeta \in \partial D} U_z^2(s) \le H_k^*(v_\zeta)(s, s) \text{ for any } s \in H^0(X, L^k)^* \ \},$$

where $H_k^*(v)$ is the inner product dual to $H_k(v)$. A remarkable fact about the upper envelope $V^k = \sup\{U : U \in G_v^k\}$ is a theorem of Coifman and Semmes [CS93], which shows that V^k is not only a norm but an inner product (see [Slo90b, Corollary 2.7] for a different proof.); moreover it solves the Hermitian–Yang–Mills equation:

$$\begin{cases} \Lambda\Theta(V^k) = 0 \\ V^k|_{\partial D} = H_k^*(v). \end{cases} \tag{4.0.6}$$

Denoting the dual metric by $(V^k)^*$, our main result is that the upper envelope V of G_v is the limit of Hermitian–Yang–Mills metrics:

Theorem 4.0.1 *Let $v \in C^\infty(\partial D \times X, \mathbb{R})$ such that $v(z, \cdot) \in \mathcal{H}_\omega$ for $z \in \partial D$. If V^k is the solution of (4.0.6), then $FS_k((V^k)^*)$ converges to V uniformly on $D \times X$, as $k \to \infty$.*

Now we turn to the interpretation of the upper envelope V and its relation with the WZW equation. The next theorem shows that V solves the WZW equation under a regularity assumption.

Theorem 4.0.2 *If the upper envelope V of G_v is in $C^2(D \times X)$, then*

$$(i\partial\bar{\partial}V + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} = 0.$$

We expect that the converse of Theorem 4.0.2 also holds, then together with Theorem 4.0.1 this would show that the solution of the WZW equation can be approximated by the Hermitian–Yang–Mills metrics.

4.1 Positivity of direct image bundles

Consider a Hermitian holomorphic line bundle $(E, g) \to X^n$ over a compact complex manifold and assume the curvature η of the metric g is positive. For two sections $s, t \in H^0(X, E \otimes K_X)$, we write locally

$$s = \sigma \otimes s', t = \tau \otimes t'$$

where $\sigma, \tau \in E$ and $s', t' \in K_X$. We extend the metric g to acting on sections of $E \otimes K_X$ by setting $g(s, t) = g(\sigma, \tau) s' \wedge \bar{t}'$, which is an (n, n)-form. It is not hard to see this (n, n)-form is globally defined on X.

We define a variant of the Hilbert map: $\mathrm{Hilb}_{E \otimes K_X}(u)$, for a function $u : D \times X \to \mathbb{R}$, is given by

$$\mathrm{Hilb}_{E \otimes K_X}(u)(s, s) = \int_X g(s, s) e^{-u(z, \cdot)}$$

with $s \in H^0(X, E \otimes K_X)$. In the following, suitable assumptions will be made on u to make sure the integral converges. Then the map $D \ni z \mapsto \mathrm{Hilb}_{E \otimes K_X}(u(z, \cdot))$ defines a Hermitian metric on the bundle $D \times H^0(X, E \otimes K_X) \to D$. We will call this metric simply $\mathrm{Hilb}_{E \otimes K_X}(u)$. The main result of this section is the following positivity theorem.

Theorem 4.1.1 *If u is bounded and upper semicontinuous (usc) on $D \times X$, and η-subharmonic on graphs, then the dual metric $Hilb^*_{E \otimes K_X}(u)$ is a subharmonic norm function.*

The following approximation lemma is somewhat technical and we postpone its proof to section 4.4.

Lemma 4.1.1 *Let u be a bounded usc function on $D \times X$, η-subharmonic on graphs. Then for D' relatively compact open in D, there exist positive $\varepsilon_j \searrow 0$ and $u_j \in C^\infty(D' \times X)$ decreasing to u such that for any holomorphic map f from an open subset of D' to X, $\Delta(\psi(f(z)) + u_j(z, f(z))) \geq \varepsilon_j \Delta(\psi(f(z))$, where $\eta = i\partial\bar{\partial}\psi$ locally.*

Proof [Proof of Theorem 4.1.1] Since being a subharmonic norm function is a local property, we focus on D', a relatively compact open set in D. Take ε_j and u_j as in Lemma 4.1.1. Assuming the theorem holds for such a u_j, namely, the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u_j)$ is a subharmonic norm function, it follows that $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is also a subharmonic norm function because $\mathrm{Hilb}^*_{E\otimes K_X}(u_j)$ decreases to $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ as $j \to \infty$.

As a result, we only need to prove the theorem for $u \in C^\infty(D' \times X)$ with the property that there exists $\varepsilon > 0$ such that for any holomorphic function f from an open subset of D' to X,

$$\Delta(\psi(f(z)) + u(z, f(z))) \geq \varepsilon\Delta(\psi(f(z)), \text{ where } \eta = i\partial\bar{\partial}\psi \text{ locally.} \tag{4.1.1}$$

In a coordinate system $\Omega \subset \mathbb{C}^n$ on X, we will use Greek letters μ, λ for indices of coordinates on X, and Roman letters i, j for indices of coordinates on D; moreover, f^μ means the μ-th component of f, whereas $\psi_{\mu\bar{\lambda}}$, $u_{i\bar{i}}$, and $u_{i\bar{\lambda}}$ mean partial derivatives $\partial^2\psi/\partial x_\mu \partial \bar{x}_\lambda$, $\partial^2 u/\partial z_i \partial \bar{z}_i$, and $\partial^2 u/\partial z_i \partial \bar{x}_\lambda$ respectively. In this coordinate system $\Omega \subset \mathbb{C}^n$ on X, the inequality (4.1.1) becomes

$$\begin{aligned}\varepsilon \sum_{i,\lambda,\mu} \psi_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} \leq & \sum_{i,\lambda,\mu} \psi_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_i u_{i\bar{i}} \\ & + \sum_{i,\lambda} u_{i\bar{\lambda}} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_{i,\mu} u_{\bar{i}\mu} \frac{\partial f^\mu}{\partial z_i} + \sum_{i,\lambda,\mu} u_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i}.\end{aligned} \tag{4.1.2}$$

In (4.1.2), choose $f(z) = N(\xi_1, \xi_2, ..., \xi_n)z_1$ where N is a positive number and $(\xi_1, \xi_2, ..., \xi_n) \in \mathbb{C}^n$, divide the resulting (4.1.2) by N^2 and send N to infinity, to obtain $(\psi_{\mu\bar{\lambda}} + u_{\mu\bar{\lambda}}) \geq \varepsilon(\psi_{\mu\bar{\lambda}})$ as matrices, and hence $(\psi_{\mu\bar{\lambda}} + u_{\mu\bar{\lambda}})$ is positive definite.

Let $L^2(X, E\otimes K_X)$ be the space of measurable sections s whose L^2 norm $\int_X g(s,s)e^{-u(z,\cdot)}$ is finite. Since different z will give rise to comparable L^2 norms, the space $L^2(X, E \otimes K_X)$ does not change with z, and so we have a Hermitian Hilbert bundle $D' \times L^2(X, E\otimes K_X) \to D'$, with the metric $\mathrm{Hilb}_{E\otimes K_X}(u)$, which has $D' \times H^0(X, E\otimes K_X) \to$

D' as a subbundle. Denote the curvature of the subbundle by $\Theta = \sum \Theta_{j\bar{k}} dz_j \wedge d\bar{z_k}$. By the computations in [Ber09, P. 540] we deduce

$$\sum_j (\Theta_{j\bar{j}} s, s) \geq \int_X K(z,\cdot) g(s,s) e^{-u(z,\cdot)} \tag{4.1.3}$$

where $s \in H^0(X, E \otimes K_X)$, and $K : D' \times X \to \mathbb{R}$ is a smooth function, given in local coordinates on X by

$$K = \sum_j (u_{j\bar{j}} - \sum_{\lambda,\mu} (\psi + u)^{\bar{\lambda}\mu} u_{j\bar{\lambda}} u_{\bar{j}\mu});$$

here $(\psi + u)^{\bar{\lambda}\mu}$ stands for the inverse matrix of $(\psi + u)_{\bar{\lambda}\mu}$.

We claim that $K \geq 0$. First notice that ψ is independent of z, so if we denote $\psi(x) + u(z,x)$ by $\phi(z,x)$, then $K = \sum_j (\phi_{j\bar{j}} - \sum_{\lambda,\mu} \phi_{j\bar{\lambda}} \phi^{\bar{\lambda}\mu} \phi_{\bar{j}\mu})$. Fix $(z_0, x_0) \in D' \times X$, since the matrix $(\phi_{\mu\bar{\lambda}})$ is positive definite, we can choose local coordinates on X around x_0 such that $(\phi_{\mu\bar{\lambda}})$ is the identity matrix at (z_0, x_0), and therefore $K(z_0, x_0) = \sum_j (\phi_{j\bar{j}} - \sum_\lambda |\phi_{j\bar{\lambda}}|^2)(z_0, x_0)$. For a holomorphic function f from an open subset of D' to X, the subharmonicity of $\phi(z, f(z))$ reads

$$\sum_i \phi_{i\bar{i}} + \sum_{i,\lambda} \phi_{i\bar{\lambda}} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_{i,\mu} \phi_{\bar{i}\mu} \frac{\partial f^\mu}{\partial z_i} + \sum_{i,\lambda,\mu} \phi_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i} \frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} \geq 0. \tag{4.1.4}$$

Without loss of generality, we assume $(z_0, x_0) = (0,0)$ and choose $f^\lambda = -\sum_i \phi_{i\bar{\lambda}}(0,0) z_i$ in (4.1.4), and it becomes $\sum_j (\phi_{j\bar{j}} - \sum_\lambda |\phi_{j\bar{\lambda}}|^2)(0,0) \geq 0$. Therefore, $K \geq 0$. (See also the remark after Lemma 4.3.1 for a slightly different proof of this claim, and an invariant meaning of K).

As a result, (4.1.3) implies $\sum_j (\Theta_{j\bar{j}} s, s) \geq 0$, and hence the curvature of the dual metric $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ satisfies the opposite inequality; according to [CS93, Theorem 4.1], this implies $\mathrm{Hilb}^*_{E\otimes K_X}(u)$ is a subharmonic norm function. ■

Now let $(E, g) = (L^k \otimes K_X^*, h^k \otimes \omega^n)$, which is positively curved for large k since $\Theta(h^k \otimes \omega^n) = k\omega + \mathrm{Ric}\,\omega$. Note that $E \otimes K_X \approx L^k$. We have the following proposition regarding the metric $H_k(u)$ on the bundle $D \times H^0(X, L^k)$. Recall that

$$H_k(u(z,\cdot))(s,s) = \int_X h^k(s,s) e^{-ku(z,\cdot)} \omega^n, \ \ s \in H^0(X, L^k).$$

Proposition 4.1.2 *Suppose u is a bounded usc function on $D \times X$ and with some $\varepsilon \in (0,1)$ u is $(1-\varepsilon)\omega$-subharmonic on graphs. Then there exists $k_0 = k_0(\varepsilon, \omega)$, independent of u, such that, for $k \geq k_0$, the dual metric $H_k^*(u)$ is a subharmonic norm function.*

Proof In order to use Theorem 4.1.1, we check if ku is $(k\omega + \mathrm{Ric}\,\omega)$-subharmonic on graphs. Suppose $\omega = i\partial\bar{\partial}\psi$ and $\mathrm{Ric}\,\omega = i\partial\bar{\partial}\phi$ locally, then we want to see if $k\psi(f(z)) + \phi(f(z)) + ku(z, f(z))$ is subharmonic for any holomorphic map f. Note that $k\psi + \phi + ku = k(1-\varepsilon)\psi + ku + \varepsilon k\psi + \phi$, and $k(1-\varepsilon)\psi(f(z)) + ku(z, f(z))$ is subharmonic by the assumption. On the other hand, there exists k_0 depending on ε, ω such that $\varepsilon k\psi + \phi$ is plurisubharmonic (psh) for $k \geq k_0$. Therefore, ku is $(k\omega + \mathrm{Ric}\,\omega)$-subharmonic on graphs for $k \geq k_0$. By Theorem 4.1.1, the metric $\mathrm{Hilb}_{L^k}^*(ku)$ is a subharmonic norm function for $k \geq k_0$. The proposition follows since $\mathrm{Hilb}_{L^k}(ku) = H_k(u)$.

■

4.2 Approximation by Hermitian–Yang–Mills metrics

Recall that D is in $\mathbb{C}^m$, and $(L, h) \to X^n$ is a positive line bundle with curvature ω. A function $f : X \to [-\infty, \infty)$ is called ω-psh if f is usc on X, and for any coordinate system where $\omega = i\partial\bar{\partial}\psi$, the function $f + \psi$ is psh in the local coordinates. We denote the set of all ω-psh functions by $\mathrm{PSH}(X, \omega)$.

Lemma 4.2.1 *Let u be an usc function on $D \times X$, ω-subharmonic on graphs. Then for any fixed $z \in D$, $u(z, x)$ is ω-psh on X, and for any fixed $x \in X$, $u(z, x)$ is subharmonic on D.*

This can be seen as a special case of an abstract theorem in [Slo90b, Section 1], whose proof we translate to our setting.

Proof By choosing the holomorphic map f constant in the definition of ω-subharmonic on graphs, it follows immediately that $u(z, x)$ is subharmonic in z.

For a fixed $z_0 \in D$, we want to show $x \mapsto \psi(x) + u(z_0, x)$ is psh in a coordinate system on X, where ψ is a local potential of ω. Without loss of generality, it suffices to prove that $\mathbb{C} \ni \lambda \mapsto \psi(\lambda e_1) + u(0, \lambda e_1)$ is subharmonic, where $e_1 = (1, 0, ..., 0) \in \mathbb{C}^n$. Let $U = \{\lambda \in \mathbb{C} : |\lambda - a| < R\}$ and $h(\lambda)$ harmonic on U and continuous up to boundary. We will be done if

$$\psi(ae_1) + u(0, ae_1) + h(a) \leq \max_{\lambda \in \partial U} \psi(\lambda e_1) + u(0, \lambda e_1) + h(\lambda).$$

Suppose the inequality is not true. By [Slo86, Lemma 4.5], there is an $\mathbb{R}$-linear function $l : \mathbb{C} \to \mathbb{R}$ and $b \in U$ such that, if we denote

$$v(z, \lambda) = \psi(\lambda e_1) + u(z, \lambda e_1) + h(\lambda) + l(\lambda) \tag{4.2.1}$$

then

$$v(0, b) > v(0, \lambda), \text{ for } \lambda \in U - \{b\}.$$

Now define $W(z, \lambda_1, ..., \lambda_m) := v(z, \lambda_1) + ... + v(z, \lambda_m)$ in a neighborhood of $(0, b^*) := (0, b, ..., b)$ in $\mathbb{C}^m \times \mathbb{C}^m$. As $W(0, b^*) > W(0, \lambda_1, ..., \lambda_m)$ for $(\lambda_1, ..., \lambda_m) \neq b^*$, there exists a ball $B \subset \mathbb{C}^m$ of radius r centered at b^* such that

$$W(0, b^*) > \max_{\{0\} \times \partial B} W.$$

Since W is usc, there exists $\varepsilon > 0$ such that $W(z, \lambda_1, ..., \lambda_m) < W(0, b^*)$, for $|z| \leq \varepsilon$ and $(\lambda_1, ..., \lambda_m) \in \partial B$. Let $S = r/\varepsilon \mathrm{Id}_{\mathbb{C}^m}$. We have $W(z, b^* + S(z)) < W(0, b^*)$ for $|z| = \varepsilon$, which contradicts the maximum principle because $W(z, b^* + S(z)) = \sum_{i=1}^m v(z, b + r/\varepsilon z_i)$ is subharmonic by (4.2.1).

∎

Although in the introduction the boundary data v is in $C^\infty(\partial D, \mathcal{H}_\omega)$, we will prove a lemma for a broader class of boundary data ν. Let ν be a continuous map $\partial D \times X \to \mathbb{R}$ such that $\nu_z(\cdot) := \nu(z, \cdot) \in \mathrm{PSH}(X, \omega)$ for $z \in \partial D$. Let

$$G_\nu = \{u \in \mathrm{usc}(D \times X) : u \text{ is } \omega\text{-subharmonic on graphs, and } \limsup_{D \ni z \to \zeta \in \partial D} u(z, x) \leq \nu(\zeta, x)\}.$$

In order to study the properties of the upper envelope $\mathcal{V}$ of G_ν, we introduce a closely related family. With $\pi : D \times X \to X$ the projection, let

$$F_\nu := \{u : u \in \mathrm{PSH}(D \times X, \pi^*\omega) \text{ and } \limsup_{D \ni z \to \zeta \in \partial D} u(z,x) \leq \nu(\zeta, x)\}.$$

The upper envelope of F_ν extends to a solution $\mathcal{U} \in C(\overline{D} \times X)$ of

$$\begin{cases} (\pi^*\omega + i\partial\bar{\partial}\mathcal{U})^{n+m} = 0 \text{ on } D \times X \\ \pi^*\omega + i\partial\bar{\partial}\mathcal{U} \geq 0 \text{ on } D \times X \\ \mathcal{U}|_{\partial D \times X} = \nu, \end{cases}$$

see for example [Bou12, DW19]. Even though $\mathcal{U}$ is only continuous, by the work of Bedford and Taylor [BT76, BT82] the left hand side of the first equation above can be given sense as a Borel measure on $D \times X$. In addition, we also need the solution $h \in C(\overline{D} \times X) \cap C^2(D \times X)$ to the Dirichlet problem (see [Aub98, P. 112, Theorem 4.17])

$$\begin{cases} \sum_j h_{j\bar{j}} + \Delta_\omega h + 2n = 0 \text{ on } D \times X \\ h|_{\partial D \times X} = \nu. \end{cases}$$

Lemma 4.2.2 *If we denote the upper envelopes of G_ν and F_ν by $\mathcal{V}$ and $\mathcal{U}$ respectively, then $\mathcal{U} \leq \mathcal{V} \leq h$ and $\lim_{(z,x)\to(z_0,x_0)\in\partial D\times X} \mathcal{V}(z,x) = \nu(z_0, x_0)$. Moreover, if ν is negative, then so is $\mathcal{V}$.*

Proof Unraveling the definitions of F_ν and G_ν, we see $F_\nu \subset G_\nu$, so $\mathcal{U} \leq \mathcal{V}$. For any $u \in G_\nu$, by Lemma 4.2.1, $u(z, \cdot)$ is ω-psh for fixed z, hence $\Delta_\omega u + 2n \geq 0$; in addition, $u(\cdot, x)$ is subharmonic for fixed x. By the maximum principle, $u \leq h$ and hence $\mathcal{V} \leq h$. Since $\mathcal{U}$ and h are both equal to ν on $\partial D \times X$, the limit of $\mathcal{V}$ on $\partial D \times X$ must also be ν.

For a fixed $x_0 \in X$, let $H_0(z)$ be the harmonic function on D with boundary values $\nu(z, x_0)$. For $u \in G_\nu$, we have $u(z, x_0) \leq H_0(z)$, and therefore $\mathcal{V}(z, x_0) \leq H_0(z)$. The second statement follows at once. ■

With Proposition 4.1.2 at hand, we can start to prove Theorem 4.0.1. The following envelope will be used in the proof: for an usc function F on X, we introduce $P(F) := \sup\{h \in \mathrm{PSH}(X,\omega) \text{ such that } h \le F\} \in \mathrm{PSH}(X,\omega)$ (see [Ber19]).

Proof [Proof of Theorem 4.0.1] Without loss of generality, we assume $v \le 0$. Fix $\delta > 1$, and for $z \in \partial D$, define $v_z^\delta = P(\delta v_z)$. By [DW19, Lemma 4.9], $\partial D \times X \ni (z,x) \mapsto v_z^\delta(x)$ is continuous. Let V^δ be the upper envelope of G_{v^δ}. By Lemma 4.2.2, $V^\delta \le 0$, and so $u \le 0$ for $u \in G_{v^\delta}$. The next step is to have a better upper bound for $u \in G_{v^\delta}$. To that end, we can look instead at $\max\{u, c\}$, which is still in G_{v^δ} as long as the constant $c \le \min v^\delta$. Since $\max\{u, c\}$ is bounded, we will assume u is bounded. Moreover, u/δ is ω/δ-subharmonic on graphs. According to Proposition 4.1.2, there exists $k_0 = k_0(\delta)$ such that for $k \ge k_0$, $H_k^*(u/\delta)$ is a subharmonic norm function. Because $\limsup_{\partial D} H_k^*(u/\delta) \le H_k^*(v)$, it follows that $H_k^*(u/\delta) \in G_v^k$ and therefore $H_k^*(u/\delta) \le V^k$ on D and $FS_k(H_k(u/\delta)) \le FS_k((V^k)^*)$. By Lemma 4.2.1, we have $\omega + i\partial\bar\partial u(z,\cdot)/\delta \ge (1 - 1/\delta)\omega$ for all $z \in D$, ($\partial\bar\partial$ on X). By [DW19, Lemma 4.10] (a consequence of the Ohsawa–Takegoshi extension theorem), there exist $C > 0$ and $k_0(\delta)$ such that, for $k \ge k_0$,

$$\frac{1}{\delta}u - \frac{C}{k} \le FS_k \circ H_k\Big(\frac{1}{\delta}u\Big) \le FS_k((V^k)^*).$$

As we saw $u \le 0$, this implies $u - C/k \le FS_k((V^k)^*)$; this is true for any $u \in G_{v^\delta}$, so we actually have $V^\delta - C/k \le FS_k((V^k)^*)$. In addition, since $v_z + (\delta - 1)\inf_{\partial D \times X}(v_z)$ is a competitor in $P(\delta v_z)$,

$$V + (\delta - 1)\inf_{\partial D \times X}(v) \le V^\delta.$$

Putting things together, we conclude

$$V + (\delta - 1)\inf_{\partial D \times X}(v) - \frac{C}{k} \le FS_k((V^k)^*), \text{ for } k \ge k_0(\delta). \tag{4.2.2}$$

Next we claim that $FS_k((V_z^k)^*)(x)$ is ω-subharmonic on graphs. Some preparation is needed. Let s be a non-vanishing holomorphic section of L^k over an open set $Y \subset X$. Let $e^{-k\phi} := h^k(s,s)$ and $s_k^* : Y \to (L^k)^*$ be defined by $s_k^*(x)(\cdot) = h^k(\cdot, e^{k\phi(x)/2}s(x))$ for

$x \in Y$. Suppose $\hat{s}_k^* : Y \to H^0(X, L^k)^*$ is the pointwise evaluation map of s_k^*, namely $\hat{s}_k^*(x)(\sigma) := s_k^*(x)(\sigma(x))$ for $\sigma \in H^0(X, L^k)$. Then we have the following formula, which is taken from [DW19, Lemma 4.1],

$$FS_k((V_z^k)^*)(x) = \frac{2}{k} \log \left[V_z^k(\hat{s}_k^*(x))\right], \ x \in Y. \tag{4.2.3}$$

Meanwhile, for $\sigma \in H^0(X, L^k)$, $e^{k\phi(x)/2}\hat{s}_k^*(x)(\sigma) = \sigma(x)/s(x)$ is holomorphic, so $e^{k\phi/2}\hat{s}_k^*$ is holomorphic. Hence for any holomorphic map g from an open subset of D to X

$$\Delta\big(\phi(g(z)) + FS_k((V_z^k)^*)(g(z))\big) = \Delta(\frac{1}{k} \log \left[V_z^k((e^{k\phi/2}\hat{s}_k^*) \circ g(z))\right]^2). \tag{4.2.4}$$

By [CS93, Theorem 4.1] the Hermitian–Yang–Mills metric V_z^k is a subharmonic norm function, so the last term of (4.2.4) is nonnegative, which means $FS_k((V_z^k)^*)(x)$ is ω-subharmonic on graphs as we claimed. Further, according to the Tian–Catlin–Zelditch asymptotic theorem or by [DW19, Lemma 4.10] an easier but cruder estimate, $FS_k((V^k)^*|_{\partial D}) = FS_k(H_k(v)) \le v + O(\log k/k)$, so $FS_k((V^k)^*) \in G_{v+O(\log k/k)}$ and $FS_k((V^k)^*) \le V + O(\log k/k)$. This last inequality together with (4.2.2) concludes the proof. ■

It is natural to ask if V belongs to G_v. A standard approach to show the envelope belongs to a family is to take upper regularization, and the case at hand is very similar to Coifman and Semmes' [CS93, Lemma 11.11], where upper regularization is taken in the z-variables. The reason it works in their lemma is because their function in the x-variables is a norm, but ours is not and regularization does not seem to work. Nevertheless, with Theorem 4.0.1 one can easily show $V \in G_v$. It would be interesting to prove $V \in G_v$ directly without using Theorem 4.0.1, after all G_v and V can be defined on any Kähler manifold (X, ω) without reference to a line bundle.

Corollary 4.2.3 *The upper envelope V is continuous and $V \in G_v$.*

Proof The first statment is a direct consequence of Theorem 4.0.1. As to the second statement, let ψ be a local potential of ω and f a holomorphic map from an open subset of D to X. For any $u \in G_v$, $\psi(f(z)) + u(z, f(z))$ is subharmonic; hence $\psi(f(z)) + V(z, f(z))$, the supremum over $u \in G_v$, is also subharmonic since V is continuous. By Lemma 4.2.2, it follows $V \in G_v$. ■

4.3 The WZW equation

We will prove Theorem 4.0.2 and compute the Euler–Lagrange equation of $\mathcal{E}$ (4.0.3) in this section. We begin with an observation. Suppose u is a C^2 function on $D \times X$ and ψ is a local potential of ω. Consider the complex Hessian of $u+\psi$ with respect to a fixed coordinate z_j in D and local coordinates x in X where ψ is defined

$$\begin{pmatrix} (u+\psi)_{z_j\bar{z}_j} & (u+\psi)_{z_j\bar{x}_1} & \cdots & (u+\psi)_{z_j\bar{x}_n} \\ (u+\psi)_{x_1\bar{z}_j} & (u+\psi)_{x_1\bar{x}_1} & \cdots & (u+\psi)_{x_1\bar{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ (u+\psi)_{x_n\bar{z}_j} & (u+\psi)_{x_n\bar{x}_1} & \cdots & (u+\psi)_{x_n\bar{x}_n} \end{pmatrix}, \tag{4.3.1}$$

which we will denote by $(u+\psi)_j$. Then

$$\begin{aligned} &(i\partial\bar{\partial}u + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} \\ =&(n+1)!(m-1)!\sum_{j=1}^{m} \det(u+\psi)_j\big(\bigwedge_{k=1}^{m} idz_k \wedge d\bar{z}_k \wedge \bigwedge_{k=1}^{n} idx_k \wedge d\bar{x}_k\big). \end{aligned} \tag{4.3.2}$$

Lemma 4.3.1 *Suppose u is a C^2 function on $D \times X$ and $\omega + i\partial\bar{\partial}u(z,\cdot) > 0$ on X for all $z \in D$. Then u is ω-subharmonic on graphs if and only if*

$$(i\partial\bar{\partial}u + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} \geq 0.$$

Proof Let ψ be a local potential of ω and denote the complex Hessian of $u+\psi$ with respect to z_j and x by $(u+\psi)_j$, as in the matrix (4.3.1). Due to (4.3.2), we will focus on $\sum_{j=1}^{m} \det(u+\psi)_j$.

Let f be a holomorphic function from an open subset of D to X, then in a coordinate system on X

$$\begin{aligned} &\Delta(\psi(f(z)) + u(z, f(z))) = \\ &\sum_{i,\lambda,\mu} \psi_{\mu\bar{\lambda}} \frac{\partial f^\mu}{\partial z_i}\frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_i u_{i\bar{i}} + \sum_{i,\lambda} u_{i\bar{\lambda}}\frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i} + \sum_{i,\mu} u_{\bar{i}\mu}\frac{\partial f^\mu}{\partial z_i} + \sum_{i,\lambda,\mu} u_{\mu\bar{\lambda}}\frac{\partial f^\mu}{\partial z_i}\frac{\partial \bar{f}^\lambda}{\partial \bar{z}_i}. \end{aligned}$$

If we denote the matrix $(\psi_{\mu\bar{\lambda}} + u_{\mu\bar{\lambda}})$ by A and the column vector $(u_{i\bar{\lambda}})$ by B_i, then the above is the same as

$$\sum_i \left(\langle A \frac{\partial f}{\partial z_i}, \frac{\partial f}{\partial z_i} \rangle + \langle B_i, \frac{\partial f}{\partial z_i} \rangle + \overline{\langle B_i, \frac{\partial f}{\partial z_i} \rangle} + u_{i\bar{i}} \right), \tag{4.3.3}$$

where the angled inner product is the usual Euclidean inner product and $\partial f / \partial z_i$ is the column vector $(\partial f^\mu / \partial z_i)$. The matrix form can be further written as

$$\sum_i \left(\| \sqrt{A} \frac{\partial f}{\partial z_i} + \sqrt{A}^{-1} B_i \|^2 - \| \sqrt{A}^{-1} B_i \|^2 + u_{i\bar{i}} \right). \tag{4.3.4}$$

Notice that

$$\begin{aligned} \sum_i (-\| \sqrt{A}^{-1} B_i \|^2 + u_{i\bar{i}}) = \sum_i (u_{i\bar{i}} - \langle A^{-1} B_i, B_i \rangle) = \sum_i (u_{i\bar{i}} - \sum_{\lambda,\mu} u_{i\bar{\lambda}} (\psi + u)^{\bar{\lambda}\mu} u_{\bar{i}\mu}) \\ = \sum_i \frac{\det(u + \psi)_i}{\det(\psi_{\mu\bar{\lambda}} + u_{\mu\bar{\lambda}})}, \end{aligned} \tag{4.3.5}$$

where the last equality can be deduced from Schur's formula for determinants of block matrices as follows (see also [Sem92] and [Ber09] for a different computation). We examine the complex Hessian of $u + \psi$

$$(u+\psi)_j = \begin{pmatrix} (u+\psi)_{z_j \bar{z}_j} & (u+\psi)_{z_j \bar{x}_1} & \cdots & (u+\psi)_{z_j \bar{x}_n} \\ (u+\psi)_{x_1 \bar{z}_j} & (u+\psi)_{x_1 \bar{x}_1} & \cdots & (u+\psi)_{x_1 \bar{x}_n} \\ \vdots & \vdots & \ddots & \vdots \\ (u+\psi)_{x_n \bar{z}_j} & (u+\psi)_{x_n \bar{x}_1} & \cdots & (u+\psi)_{x_n \bar{x}_n} \end{pmatrix},$$

and the Schur complement of the trailing $n \times n$ minor $((u+\psi)_{\mu\bar{\lambda}})$ is precisely $u_{j\bar{j}} - \sum_{\lambda,\mu} u_{j\bar{\lambda}} (u+\psi)^{\bar{\lambda}\mu} u_{\bar{j}\mu}$, which is also equal to $\det(u+\psi)_j / \det((u+\psi)_{\mu\bar{\lambda}})$ by Schur's formula, for example see [HZ05].

u is ω-subharmonic on graphs if and only if (4.3.4) is nonnegative for any holomorphic maps f, and it is equivalent to the last term in (4.3.5) being nonnegative. The lemma follows by (4.3.2) since the matrix $(\psi_{\mu\bar{\lambda}} + u_{\mu\bar{\lambda}})$ is positive. ■

From (4.3.2) and (4.3.5), the function K in the proof of Theorem 4.1.1 has the following invariant expression

$$K = \frac{m!n!}{(m-1)!(n+1)!} \frac{(\pi^*\omega + i\partial\bar{\partial}u)^{n+1} \wedge (i \sum_{j=1}^m dz_j \wedge d\bar{z}_j)^{m-1}}{(\pi^*\omega + i\partial\bar{\partial}u)^n \wedge (i \sum_{j=1}^m dz_j \wedge d\bar{z}_j)^m},$$

and one can see $K \geq 0$ if u is ω-subharmonic on graphs.

Proof [Proof of Theorem 4.0.2] By the equality (4.3.2), the equation

$$(i\partial\bar{\partial}V + \pi^*\omega)^{n+1} \wedge (i\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j)^{m-1} = 0$$

is equivalent to $\sum_j \det(\psi + V)_j = 0$, so we will prove that the upper envelope V of G_v satisfies the latter equation.

By Corollary 4.2.3, V is ω-subharmonic on graphs, and hence $V(z, x)$ is ω-psh on X by Lemma 4.2.1. Take a coordinate chart Ω of X, then for $\varepsilon > 0$ and $x \in \Omega$, the function $V(z, x) + \varepsilon|x|^2$ satisfies the assumption of Lemma 4.3.1, so $\sum_i \det(\psi + V + \varepsilon|x|^2)_i \geq 0$ and $\sum_i \det(\psi + V)_i \geq 0$.

Suppose $\sum_i \det(\psi + V)_i$ is positive at a point p in $D \times X$. We may assume $\det(\psi + V)_1$ is positive at p, so it is positive in a neighborhood B of p in $D \times X$. For small $\varepsilon > 0$, $\det(\psi + V + \varepsilon|x|^2)_1 > 0$ on B, then by Sylvester's criterion for positive matrices or a property of Schur complement for positive matrices (see [HZ05, Theorem 1.12]) we deduce that the matrix $(\psi + V + \varepsilon|x|^2)_1$ is positive on B, so the matrix $(\psi+V)_1$ is semi-positive on B, but since $\det(\psi+V)_1$ is positive on B, $(\psi+V)_1$ is actually positive on B; in particular, the $n\times n$ trailing minor $(\psi_{\lambda\bar{\mu}} + V_{\lambda\bar{\mu}})$ is positive on B.

So if we pick a suitably small smooth cutoff function ρ supported in B, then the function $V + \rho$ satisfies the assumption of Lemma 4.3.1 on B, and hence $V + \rho$ is ω-subharmonic on graphs and is in G_v, which contradicts $V = \sup G_v$. Therefore, $\sum_j \det(\psi + V)_j = 0$. ∎

As in (4.0.2), for $\phi \in \mathcal{H}_\omega$ and $\xi_1, \xi_2, \xi_3 \in T_\phi\mathcal{H}_\omega$, θ on $\mathcal{H}_\omega$ is

$$\theta(\xi_1, \xi_2, \xi_3) := g_M(\{\xi_1, \xi_2\}_{\omega_\phi}, \xi_3) = \int_X \{\xi_1, \xi_2\}_{\omega_\phi}\xi_3\omega_\phi^n. \tag{4.3.6}$$

In light of $\{\xi_1, \xi_2\}_{\omega_\phi}\omega_\phi^n = n d\xi_1\wedge d\xi_2\wedge\omega_\phi^{n-1}$, an integration by parts shows $\int_X \{\xi_1, \xi_2\}_{\omega_\phi}\xi_3\omega_\phi^n = \int_X \xi_1\{\xi_2, \xi_3\}_{\omega_\phi}\omega_\phi^n$, and therefore θ is indeed skew-symmetric and a three form. The rest of this section is devoted to showing that the three form θ is d-closed, and to the derivation of Euler–Lagrange equation of $\mathcal{E}$.

Lemma 4.3.2 *Let β be a k-form on $\mathcal{H}_\omega$, and let $\xi_0, ...\xi_k$ be vector fields on $\mathcal{H}_\omega$, which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$. Then*

$$d\beta(\xi_0, ...\xi_k) = \sum_{j=0}^{k} (-1)^j \xi_j \beta(\xi_0, ..., \hat{\xi_j}, ..., \xi_k), \tag{4.3.7}$$

where $\hat{\xi_j}$ means ξ_j is to be omitted. (This formula is true if $\mathcal{H}_\omega \subset C^\infty(X)$ is replaced by an open subset of a Fréchet space.)

Proof This is a well-known formula except there should be terms involving Lie brackets on the right hand side, but since ξ_j are constant vector fields, their Lie brackets are zero. ■

Lemma 4.3.3 *The three-form θ is d-closed.*

Proof This is similar to the derivation of the Aubin–Yau functional and the Mabuchi energy (see e.g. [Bło13, Sectioin 4]). Consider four vector fields $\xi_1, \xi_2, \xi_3, \xi_4$ on $\mathcal{H}_\omega$, which are constant in the canonical trivialization $T\mathcal{H}_\omega \approx \mathcal{H}_\omega \times C^\infty(X)$. By Lemma 4.3.2

$$d\theta(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1\theta(\xi_2, \xi_3, \xi_4) - \xi_2\theta(\xi_1, \xi_3, \xi_4) + \xi_3\theta(\xi_1, \xi_2, \xi_4) - \xi_4\theta(\xi_1, \xi_2, \xi_3). \tag{4.3.8}$$

Using $\{\xi_3, \xi_4\}_{\omega_\phi}\omega_\phi^n = nd\xi_3 \wedge d\xi_4 \wedge \omega_\phi^{n-1}$ and $\frac{d}{dt}\Big|_{t=0} \omega_{\phi+t\xi_1}^{n-1} = (n-1)i\partial\bar{\partial}\xi_1 \wedge \omega_\phi^{n-2}$,

$$\begin{aligned}
\xi_1\theta(\xi_2, \xi_3, \xi_4) = \xi_1\theta(\xi_3, \xi_4, \xi_2) = d(\theta(\xi_3, \xi_4, \xi_2))(\xi_1) &= \frac{d}{dt}\Big|_{t=0} \theta(\xi_3, \xi_4, \xi_2)(\phi + t\xi_1) \\
&= \frac{d}{dt}\Big|_{t=0} \int_X \{\xi_3, \xi_4\}_{\omega_{\phi+t\xi_1}} \xi_2 \omega_{\phi+t\xi_1}^n \\
&= \frac{d}{dt}\Big|_{t=0} \int_X \xi_2 n d\xi_3 \wedge d\xi_4 \wedge \omega_{\phi+t\xi_1}^{n-1} \\
&= \int_X \xi_2 n d\xi_3 \wedge d\xi_4 \wedge (n-1)i\partial\bar{\partial}\xi_1 \wedge \omega_\phi^{n-2} \\
&= \int_X \xi_1 n d\xi_3 \wedge d\xi_4 \wedge (n-1)i\partial\bar{\partial}\xi_2 \wedge \omega_\phi^{n-2} = \xi_2\theta(\xi_1, \xi_3, \xi_4),
\end{aligned}$$

where the second to last equality is due to integration by parts. Because of this symmetry in index, (4.3.8) is 0 and therefore $d\theta = 0$. ■

Since θ is d-closed, there exists a two-form α on $\mathcal{H}_\omega$ such that $d\alpha = \theta$. For a map $\widehat{\Phi} : D \to \mathcal{H}_\omega$, the derivative $\Phi_{z_j} = 1/2(\Phi_{\mathrm{Re}\, z_j} - i\Phi_{\mathrm{Im}\, z_j})$ is a section of $\mathbb{C} \otimes T\mathcal{H}_\omega$ along $\widehat{\Phi}$, and $\alpha(\Phi_{\bar{z}_j}, \Phi_{z_j})$ is a function on D. We define $\mathcal{E}$ by

$$\mathcal{E}(\widehat{\Phi}) := E(\widehat{\Phi}) + 4i \sum_j \int_D \alpha(\Phi_{\bar{z}_j}, \Phi_{z_j}) dV = \int_D |\widehat{\Phi}'|^2 dV + 4i \sum_j \int_D \alpha(\Phi_{\bar{z}_j}, \Phi_{z_j}) dV,$$

with dV the Euclidean volume form on D. Recall that $|\widehat{\Phi}'|$ is the Hilbert–Schmidt norm, see page 29.

Lemma 4.3.4 *The Euler–Lagrange equation for the critical points of $\mathcal{E}$ is*

$$\sum_{j=1}^{m} \Phi_{z_j \bar{z}_j} - \frac{1}{2} |\nabla \Phi_{z_j}|^2 - \frac{i}{2} \{\Phi_{\bar{z}_j}, \Phi_{z_j}\}_{\omega_{\widehat{\Phi}}} = 0, \tag{4.3.9}$$

where $\nabla \Phi_{z_j}$ is the gradient of Φ_{z_j} with respect to the metric $\omega_{\widehat{\Phi}}$.

Proof Suppose $\widehat{\Phi} : D \to \mathcal{H}_\omega$ is a critical point of $\mathcal{E}$. Let $\widehat{\Psi}$ be a smooth map from D to $C^\infty(X)$ with compact support. The variational equation is

$$0 = \frac{d}{dt}\Big|_{t=0} \Big(\int_D |(\widehat{\Phi} + t\widehat{\Psi})'|^2 dV + 4i \sum_j \int_D \alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) dV \Big). \tag{4.3.10}$$

An extension of the computation in [Don99, Section 2] shows that the first term in (4.3.10) is equal to

$$\frac{d}{dt}\Big|_{t=0} \int_D |(\widehat{\Phi} + t\widehat{\Psi})'|^2 dV = \int_D \int_X 4(\sum_j |\nabla \Phi_{z_j}|^2 - 2 \sum_j \Phi_{z_j \bar{z}_j}) \Psi \omega_{\widehat{\Phi}}^n dV. \tag{4.3.11}$$

So the remaining task is to compute the second term in (4.3.10).

We denote $C^\infty(X, \mathbb{C})$ by $C^\infty_{\mathbb{C}}(X)$. Introduce $A : \mathcal{H}_\omega \times C^\infty_{\mathbb{C}}(X) \times C^\infty_{\mathbb{C}}(X) \to \mathbb{C}$ as follows. If $(u, \xi), (u, \eta) \in \mathcal{H}_\omega \times C^\infty_{\mathbb{C}}(X) \approx \mathbb{C} \otimes T\mathcal{H}_\omega$, then $A(u, \xi, \eta) := \alpha((u, \xi), (u, \eta))$. Therefore, for fixed small $t \in \mathbb{R}$, $\alpha((\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) = A(\Phi + t\Psi, (\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) : D \to \mathbb{C}$. By chain rule,

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} A(\Phi + t\Psi, (\Phi + t\Psi)_{\bar{z}_j}, (\Phi + t\Psi)_{z_j}) \\ =& d_1 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi) + d_2 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{\bar{z}_j}) + d_3 A(\Phi, \Phi_{\bar{z}_j}, \Phi_{z_j})(\Psi_{z_j}), \end{aligned} \tag{4.3.12}$$

where d_1A, d_2A, and d_3A are partial differentials of A. Since A is linear in the second and the third variables, $d_2A(\Phi,\Phi_{\bar z_j},\Phi_{z_j})(\Psi_{\bar z_j}) = A(\Phi,\Psi_{\bar z_j},\Phi_{z_j})$ and $d_3A(\Phi,\Phi_{\bar z_j},\Phi_{z_j})(\Psi_{z_j}) = A(\Phi,\Phi_{\bar z_j},\Psi_{z_j})$. Hence (4.3.12) becomes

$$d_1A(\Phi,\Phi_{\bar z_j},\Phi_{z_j})(\Psi) + A(\Phi,\Psi_{\bar z_j},\Phi_{z_j}) + A(\Phi,\Phi_{\bar z_j},\Psi_{z_j}). \tag{4.3.13}$$

By similar computations,

$$\begin{aligned}\frac{\partial}{\partial \bar z_j}A(\Phi,\Psi,\Phi_{z_j}) =& d_1A(\Phi,\Psi,\Phi_{z_j})(\Phi_{\bar z_j}) + A(\Phi,\Psi_{\bar z_j},\Phi_{z_j}) + A(\Phi,\Psi,\Phi_{z_j\bar z_j}), \text{ and also}\\ \frac{\partial}{\partial z_j}A(\Phi,\Phi_{\bar z_j},\Psi) =& d_1A(\Phi,\Phi_{\bar z_j},\Psi)(\Phi_{z_j}) + A(\Phi,\Phi_{\bar z_j z_j},\Psi) + A(\Phi,\Phi_{\bar z_j},\Psi_{z_j}).\end{aligned} \tag{4.3.14}$$

So integration by parts gives

$$\begin{aligned}\int_D A(\Phi,\Psi_{\bar z_j},\Phi_{z_j})dV =& -\int_D \Big(d_1A(\Phi,\Psi,\Phi_{z_j})(\Phi_{\bar z_j}) + A(\Phi,\Psi,\Phi_{z_j\bar z_j})\Big)dV,\\ \int_D A(\Phi,\Phi_{\bar z_j},\Psi_{z_j})dV =& -\int_D \Big(d_1A(\Phi,\Phi_{\bar z_j},\Psi)(\Phi_{z_j}) + A(\Phi,\Phi_{\bar z_j z_j},\Psi)\Big)dV.\end{aligned} \tag{4.3.15}$$

Combining (4.3.13) and (4.3.15)

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0}\int_D \alpha((\Phi+t\Psi)_{\bar z_j},(\Phi+t\Psi)_{z_j})dV\\ =&\int_D d_1A(\Phi,\Phi_{\bar z_j},\Phi_{z_j})(\Psi) - d_1A(\Phi,\Psi,\Phi_{z_j})(\Phi_{\bar z_j}) - d_1A(\Phi,\Phi_{\bar z_j},\Psi)(\Phi_{z_j})dV.\end{aligned} \tag{4.3.16}$$

For a fixed point $z_0 \in D$, $\Psi(z_0), \Phi_{\bar z_j}(z_0)$, and $\Phi_{z_j}(z_0)$ define three constant vector fields on $\mathcal{H}_\omega$, and we denote them by ξ_1,ξ_2, and ξ_3 respectively. By Lemma 4.3.2, $d\alpha(\xi_1,\xi_2,\xi_3) = \xi_1\alpha(\xi_2,\xi_3) - \xi_2\alpha(\xi_1,\xi_3) + \xi_3\alpha(\xi_1,\xi_2)$. Meanwhile, for constant vector fields ξ_1,ξ_2,ξ_3, the function $\xi_1\alpha(\xi_2,\xi_3)$ evaluated at $u \in \mathcal{H}_\omega$ is $d_1A(u,\xi_2,\xi_3)(\xi_1)$. So at $\Phi(z_0) \in \mathcal{H}_\omega$,

$$\begin{aligned}d\alpha(\xi_1,\xi_2,\xi_3) =& d_1A(\Phi(z_0),\xi_2,\xi_3)(\xi_1) - d_1A(\Phi(z_0),\xi_1,\xi_3)(\xi_2) + d_1A(\Phi(z_0),\xi_1,\xi_2)(\xi_3)\\ =& d_1A(\Phi(z_0),\xi_2,\xi_3)(\xi_1) - d_1A(\Phi(z_0),\xi_1,\xi_3)(\xi_2) - d_1A(\Phi(z_0),\xi_2,\xi_1)(\xi_3).\end{aligned} \tag{4.3.17}$$

Hence (4.3.16) becomes

$$\int_D d\alpha(\Psi,\Phi_{\bar z_j},\Phi_{z_j})dV = \int_D \theta(\Psi,\Phi_{\bar z_j},\Phi_{z_j})dV = \int_D\int_X \{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_{\hat\Phi}}\Psi\omega^n_{\hat\Phi}dV. \tag{4.3.18}$$

Finally, with (4.3.11) and (4.3.18), the variational equation (4.3.10) becomes

$$0=\int_D\int_X\Big(4(\sum_j|\nabla\Phi_{z_j}|^2-2\sum_j\Phi_{z_j\bar z_j})+4i\sum_j\{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_{\widehat\Phi}}\Big)\Psi\omega_{\widehat\Phi}^n dV, \tag{4.3.19}$$

and we obtain the Euler–Lagrange equation

$$\sum_j|\nabla\Phi_{z_j}|^2-2\sum_j\Phi_{z_j\bar z_j}+i\sum_j\{\Phi_{\bar z_j},\Phi_{z_j}\}_{\omega_{\widehat\Phi}}=0.$$

■

4.4 Lemma 4.1.1

This section is mainly devoted to the proof of Lemma 4.1.1, and we will follow closely the ideas in [BK07]. The first two lemmas, concerning smooth approximation of continuous η-subharmonic functions, are based on Demailly's exposition [Dem12, Chapter I, Section 5E] of Richberg's paper [Ric68].

Let $\theta\in C^\infty(\mathbb{R},\mathbb{R})$ be a nonnegative function having support in $[-1,1]$ with $\int_{\mathbb{R}}\theta(h)dh=1$ and $\int_{\mathbb{R}}h\theta(h)dh=0$. For arbitrary $\xi=(\xi_1,...,\xi_p)\in(0,\infty)^p$, the regularized maximal function is

$$M_\xi(t_1,...,t_p):=\int_{\mathbb{R}^p}\max\{t_1+h_1,...,t_p+h_p\}\prod_{j=1}^p\theta(\frac{h_j}{\xi_j})\frac{dh_1}{\xi_1}...\frac{dh_p}{\xi_p}.$$

Lemma 4.4.1 *Fix a closed smooth positive* $(1,1)$*-form* η *on* X*. Let* $\Omega_\alpha\subset\subset D\times X$ *be a locally finite open cover of* $D\times X$*,* c *be a real number, and* $u_\alpha\in C^\infty(\overline{\Omega}_\alpha)$ *such that* $u_\alpha(z,x)+c|z|^2$ *is* η*-subharmonic on graphs. Assume that there exists a family* $\{\xi_\alpha\}$ *of positive numbers such that, for all* β *and* $(z,x)\in\partial\Omega_\beta$*,*

$$u_\beta(z,x)+\xi_\beta\le\max\{u_\alpha(z,x)-\xi_\alpha:\alpha\text{ such that }(z,x)\in\Omega_\alpha\}.$$

Define a function $\tilde u$ *on* $D\times X$ *as follows. Given* $(z,x)\in D\times X$*, let* $A=\{\alpha:(z,x)\in\Omega_\alpha\}$*,* $\xi_A=(\xi_\alpha)_{\alpha\in A}$*,* $u_A(z,x)=\{u_\alpha(z,x):\alpha\in A\}$*, and*

$$\tilde u(z,x):=M_{\xi_A}(u_A(z,x)).$$

Then $\tilde u$ *is in* $C^\infty(D\times X)$ *and* $\tilde u(z,x)+c|z|^2$ *is* η*-subharmonic on graphs.*

Proof As in the proof of [Dem12, Chapter I, Lemma 5.17 and Corollary 5.19], one can deduce that for a fixed point in $D\times X$, there exist a neighborhood V and a finite set I of indices α such that $V \subset \bigcap_{\alpha\in I}\Omega_\alpha$ on which $\tilde{u} = M_{\xi_I}(u_I)$. As a result, by [Dem12, Lemma 5.18 (a)], $\tilde{u}$ is smooth on $D \times X$. Now for a holomorphic map f from an open subset of D to X,

$$\begin{aligned}\tilde{u}(z, f(z)) + c|z|^2 + \psi(f(z)) &= c|z|^2 + \psi(f(z)) + M_{\xi_I}(u_I(z, f(z)))\\ &= M_{\xi_I}\big(c|z|^2 + \psi(f(z)) + u_I(z, f(z))\big),\end{aligned}$$

where $\eta = i\partial\bar{\partial}\psi$, and we use [Dem12, Lemma 5.18 (d)] in the last equality; furthermore, since $c|z|^2 + \psi(f(z)) + u_\alpha(z, f(z))$ is subharmonic by assumption, so is $M_{\xi_I}(c|z|^2+\psi(f(z))+u_I(z, f(z)))$ by [Dem12, Lemma 5.18 (a)], and therefore $\tilde{u}+c|z|^2$ is η-subharmonic on graphs. ∎

We introduce here notation that will be used later. Let ρ_1, ρ_2 be nonnegative radial smooth functions with support in the unit ball that have integral one in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. For $\varepsilon > 0$, $\rho_{1,\varepsilon}(\cdot) := \varepsilon^{-2m}\rho_1(\cdot/\varepsilon)$, and $\rho_{2,\varepsilon}$ is similarly defined.

The proof of the following lemma is very similar to that of [Dem12, Chapter 1, Theorem 5.21].

Lemma 4.4.2 *Let $u \in C(D \times X)$ be η-subharmonic on graphs. For any number $\lambda > 0$, there exists $\tilde{u} \in C^\infty(D \times X)$ such that $u \le \tilde{u} \le u + M\lambda$, where M depends only on the diameter of D, and $\tilde{u}$ is $(1 + \lambda)\eta$-subharmonic on graphs.*

Proof Let $\{\Omega_\alpha\}$ be a locally finite open cover of $D\times X$ by relatively compact open balls, with $\overline{\Omega}_\alpha$ contained in coordinate patches of $D \times X$. Choose concentric balls $\Omega''_\alpha \subset \Omega'_\alpha \subset \Omega_\alpha$ of radii $r''_\alpha < r'_\alpha < r_\alpha$ and center $(c_\alpha, 0)$ in the given coordinates (z, x) near $\overline{\Omega}_\alpha$, such that Ω''_α still cover $D \times X$, and η has a local potential ψ_α in a neighborhood of $\overline{\Omega}_\alpha$. For small $\varepsilon_\alpha > 0$ and $\delta_\alpha > 0$, we set

$$u_\alpha(z, x) =\big((u + \psi_\alpha) * \rho_{\varepsilon_\alpha}\big)(z, x) - \psi_\alpha(x) + \delta_\alpha(r'^2_\alpha - |z - c_\alpha|^2 - |x|^2) \text{ on } \overline{\Omega}_\alpha,$$

where $*\rho_{\varepsilon_\alpha}$ is the convolution with $\rho_{\varepsilon_\alpha} := \rho_{1,\varepsilon_\alpha}\rho_{2,\varepsilon_\alpha}$. Since $\psi_\alpha(x) + u(z,x)$ is subharmonic in z and psh in x by Lemma 4.2.1, the functions $(\psi_\alpha + u) * \rho_{\varepsilon_\alpha}$ decrease to $\psi_\alpha + u$ as ε_α goes to 0, locally uniformly because u is continuous. For ε_α and δ_α small enough, we have $u_\alpha \le u + \lambda/2$ on $\overline{\Omega}_\alpha$; moreover, for any holomorphic map f from an open subset of D to X,

$$\begin{aligned}\Delta\big(u_\alpha(z,f(z)) + \psi_\alpha(f(z))\big) &= \Delta\big((u+\psi_\alpha) * \rho_{\varepsilon_\alpha}\big)(z,f(z)) - \delta_\alpha\Delta(|z-c_\alpha|^2 + |f(z)|^2)\\ &\ge -\delta_\alpha\Delta(|z-c_\alpha|^2 + |f(z)|^2)\\ &\ge -\lambda\Delta|z|^2 - \lambda\Delta\psi_\alpha(f(z)),\end{aligned}$$

where the first inequality is due to the fact $(u+\psi_\alpha)*\rho_{\varepsilon_\alpha}$ is subharmonic on holomorphic graphs, which can be verified easily using that $u+\psi_\alpha$ is subharmonic on holomorphic graphs. So $u_\alpha(z,x) + \lambda|z|^2$ is $(1+\lambda)\eta$-subharmonic on graphs. Set

$$\xi_\alpha = \delta_\alpha \min\{r_\alpha'^2 - r_\alpha''^2, (r_\alpha^2 - r_\alpha'^2)/2\}.$$

Choose first δ_α such that $\xi_\alpha < \lambda/2$, and then ε_α so small that $u \le (u+\psi_\alpha)*\rho_{\varepsilon_\alpha}(z,x) - \psi_\alpha(x) < u + \xi_\alpha$ on $\overline{\Omega}_\alpha$. As $\delta_\alpha(r_\alpha'^2 - |z-c_\alpha|^2 - |x|^2)$ is $\le -2\xi_\alpha$ on $\partial\Omega_\alpha$ and $> \xi_\alpha$ on Ω''_α, we have $u_\alpha < u - \xi_\alpha$ on $\partial\Omega_\alpha$ and $u_\alpha > u + \xi_\alpha$ on Ω''_α, so that the assumption in Lemma 4.4.1 is satisfied, and the function

$$U(z,x) := M_{\xi_A}(u_A(z,x)), \text{ for } A = \{\alpha\colon\, \Omega_\alpha \ni (z,x)\}\ ,$$

is in $C^\infty(D\times X)$ and $U(z,x) + \lambda|z|^2$ is $(1+\lambda)\eta$-subharmonic on graphs. By [Dem12, Lemma 5.18 (b)], $u \le U \le u + \lambda$. Then the function defined by $\tilde{u} := U + \lambda|z|^2$ is what we need. ■

The following lemma is proved in the same way as Lemmas 4 and 5 in [BK07]. The only issue is keeping track of uniformity.

Lemma 4.4.3 *Let U, V be two open sets in $\mathbb{C}^n$ and F a biholomorphic map from U to V. Let u be usc, bounded, and subharmonic on holomorphic graphs in $D\times U$. Let*

$\rho_1, \rho_2, \rho_{1,\delta_1}, \rho_{2,\delta_2}$ be defined as in the remark before Lemma 4.4.2. Define u_{δ_1,δ_2} to be the convolution

$$u_{\delta_1,\delta_2}(z,x) = \int\int u(z-a, x-b)\rho_{1,\delta_1}(a)\rho_{2,\delta_2}(b)dadb.$$

On the other hand, define

$$u^F_{\delta_1,\delta_2} = (u \circ (Id \times F^{-1}))_{\delta_1,\delta_2} \circ (Id \times F). \tag{4.4.1}$$

Then given a compact set $K \subset D \times U$, there exists $\delta(K) > 0$ such that as $\delta_2 \to 0$, $(u^F_{\delta_1,\delta_2} - u_{\delta_1,\delta_2})(z,x)$ goes to 0 uniformly for $(z,x) \in K$, and $\delta_1 < \delta(K)$.

Proof Define

$$\begin{aligned}
\hat{u}_{\delta_2}(z,x) &= \max_{\{z\}\times\overline{B(x,\delta_2)}} u, \\
\tilde{u}_{\delta_2}(z,x) &= \frac{1}{|\partial B(x,\delta_2)|}\int_{\partial B(x,\delta_2)} u(z,b)db, \\
u_{\delta_2}(z,x) &= \int u(z, x-b)\rho_{2,\delta_2}(b)db,
\end{aligned}$$

where $|\partial B(x,\delta_2)|$ is the Lebesgue measure of the sphere $\partial B(x,\delta_2)$. Their counterparts under $\mathrm{Id} \times F^{-1}$ and $\mathrm{Id} \times F$ as in (4.4.1) are denoted by $\hat{u}^F_{\delta_2}(z,x)$, $\tilde{u}^F_{\delta_2}(z,x)$, and $u^F_{\delta_2}(z,x)$ respectively.

By Lemma 4.2.1, $u(z,\cdot)$ is psh in U, so $\hat{u}_{\delta_2}(z,x)$ is a convex function of $\log\delta_2$. Fixing $a \geq 1$ and $r > 0$, choose δ_2 so small that $0 \leq \frac{\log a}{\log\frac{r}{\delta_2}} \leq 1$, then by convexity

$$0 \leq \hat{u}_{a\delta_2}(z,x) - \hat{u}_{\delta_2}(z,x) \leq \frac{\log a}{\log\frac{r}{\delta_2}}(\hat{u}_r(z,x) - \hat{u}_{\delta_2}(z,x)).$$

Since u is assumed to be bounded, it follows that for any $a > 0$ (for the case $1 > a > 0$, use $1/a$ instead), $\hat{u}_{a\delta_2}(z,x) - \hat{u}_{\delta_2}(z,x)$ goes to 0 as $\delta_2 \to 0$, locally uniformly in z and x. Then following the same argument as in [BK07, Lemma 4], we see $\hat{u}^F_{\delta_2} - \hat{u}_{\delta_2}$ goes to 0 locally uniformly in z and x, as $\delta_2 \to 0$.

Since $u(z,\cdot)$ is psh in U, $\tilde{u}_{\delta_2}(z,x)$ is convex in $\log\delta_2$. By the argument [BK07, Lemma 5] and the fact that u is bounded, we see both $\hat{u}_{\delta_2} - \tilde{u}_{\delta_2}$ and $\tilde{u}_{\delta_2} - u_{\delta_2}$ go to 0 locally uniformly in z, x, as $\delta_2 \to 0$, and as a result, so does $u^F_{\delta_2} - u_{\delta_2}$. Given a compact set $K \subset D \times U$, there exists $\delta(K) > 0$ such that if $(z, x) \in K$ and $(a, b) \in \mathbb{C}^m \times \mathbb{C}^n$ wiht $|(a, b)| < \delta(K)$, then $(z + a, x + b)$ is still in $D \times U$. Since $(u^F_{\delta_1,\delta_2} - u_{\delta_1,\delta_2})$ is the convolution of $(u^F_{\delta_2} - u_{\delta_2})$ in z, we see at once the conclusion of the lemma. ■

Proof [Proof of Lemma 4.1.1] Fix a finite number of charts $U_\alpha \supset\supset V_\alpha$ such that V_α covers X, and η has a local potential ψ_α in a neighborhood of $\overline{U_\alpha}$. For each α, let $f_\alpha : U_\alpha \to \mathbb{C}^n$ be the coordinate map, we consider the convolution $((\psi_\alpha + u) \circ f_\alpha^{-1})_{\delta_1,\delta_2} \circ f_\alpha$, which we simply denote by $(\psi_\alpha + u)_{\delta_1,\delta_2}$ on $D \times U_\alpha$. Because u added by a constant still satisfies the same assumption in Lemma 4.1.1, we will assume u is so negative that $(\psi_\alpha + u)_{\delta_1,\delta_2} - \psi_\alpha < -a$ for some $a > 0$ and all α. At the same time, we consider the convolution of $(\psi_\alpha + u)$ under f_β, namely $((\psi_\alpha + u) \circ f_\beta^{-1})_{\delta_1,\delta_2} \circ f_\beta$, which can be written as

$$((\psi_\alpha + u) \circ f_\alpha^{-1} \circ F^{-1})_{\delta_1,\delta_2} \circ F \circ f_\alpha, \tag{4.4.2}$$

if $F^{-1} = f_\alpha \circ f_\beta^{-1}$. We denote (4.4.2) by $(\psi_\alpha + u)^F_{\delta_1,\delta_2}$ (the notation is consistent with Lemma 4.4.3 except we do not write out the identity map of D here). By Lemma 4.4.3 on $D \times (U_\alpha \cap U_\beta)$

$$\begin{aligned}(\psi_\alpha + u)_{\delta_1,\delta_2} - (\psi_\beta + u)_{\delta_1,\delta_2} &= (\psi_\alpha + u)_{\delta_1,\delta_2} - (\psi_\alpha + u)^F_{\delta_1,\delta_2} + (\psi_\alpha + u - (\psi_\beta + u))^F_{\delta_1,\delta_2} \\ &\to \psi_\alpha - \psi_\beta\end{aligned} \tag{4.4.3}$$

locally uniformly in z and x, as $\delta_2, \delta_1 \to 0$.

Let χ_α be a smooth function in U_α that is 0 in V_α and -1 near ∂U_α. We have $i\partial\bar{\partial}\chi_\alpha \geq -C\eta$ for some constant C. For $0 < \varepsilon < 1$, according to (4.4.3) we can find δ_1, δ_2 so small that for any β and for any $(z, x) \in \overline{D'} \times \partial U_\beta$,

$$((\psi_\beta + u)_{\delta_1,\delta_2} - \psi_\beta + \frac{\varepsilon}{C}\chi_\beta)(z, x) < \max_{(z,x)\in\overline{D'}\times U_\alpha} ((\psi_\alpha + u)_{\delta_1,\delta_2} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha)(z, x),$$

where the maximum is taken over all $\overline{D'} \times U_\alpha$ that contain (z, x). Let $\delta = \min\{\delta_1, \delta_2\}$. By [Dem12, Chapter I, Lemma 5.17], the function

$$u^\varepsilon_\delta(z, x) := \max_{(z,x)\in\overline{D'}\times U_\alpha} ((\psi_\alpha + u)_{\delta,\delta} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha)(z, x), \tag{4.4.4}$$

is continuous on $\overline{D'} \times X$. Notice that $u^\varepsilon_\delta(z,x) < -a$ for any $0 < \varepsilon < 1$. Since $\psi_\alpha(x) + u(z,x)$ is subharmonic in z and psh in x by Lemma 4.2.1, the function $(\psi_\alpha + u)_{\delta,\delta}$ is decreasing to $\psi_\alpha + u$ as $\delta \to 0$, and hence u^ε_δ is decreasing to

$$\max_{(z,x)\in\overline{D'}\times U_\alpha}(u + \frac{\varepsilon}{C}\chi_\alpha)$$

as $\delta \to 0$. Because the maximum of χ_α is zero, u^ε_δ is decreasing to u as $\delta \to 0$.

We already know that $\psi_\alpha + u$ is subharmonic on graphs, and a straightforward verification shows so is $(\psi_\alpha + u)_{\delta,\delta}$. This fact together with $i\partial\bar{\partial}\chi_\alpha \geq -C\eta$ shows, for any holomorphic function f from an open subset of D' to X,

$$\Delta((\psi_\alpha + u)_{\delta,\delta} - \psi_\alpha + \frac{\varepsilon}{C}\chi_\alpha)(z, f(z)) \geq (-1-\varepsilon)\Delta\psi_\alpha(f(z)),$$

so u^ε_δ is $(1+\varepsilon)\eta$-subharmonic on graphs.

So far we have shown that given $1 < p \in \mathbb{N}$, there exist $q_0 \in \mathbb{N}$ such that, for $q > q_0$, the functions u^ε_δ with $(\varepsilon,\delta) = (1/p, 1/q)$ are in $C(\overline{D'} \times X)$, $(1+1/p)\eta$-subharmonic on graphs, and decrease to u as $q \to \infty$. For simplicity, we will denote u^ε_δ with $(\varepsilon,\delta) = (1/p,1/q)$ by $u_{(p,q)}$. Let M be the constant in Lemma 4.4.2. We will construct inductively $u_{(k,j_k)}$ with $j_k > k^2$ and $\tilde{u}_k \in C^\infty(D' \times X)$ such that

$$u_{(k,j_k)} + 1/j_k \leq \tilde{u}_k \leq u_{(k,j_k)} + 1/j_k + M/j_k. \tag{4.4.5}$$

Moreover $\tilde{u}_k$ is $(1+1/k)(1+1/j_k)\eta$-subharmonic on graphs, and $u_{(k,j_k)} + 1/j_k + M/j_k$ is less than both $u_{(k-1,j_{k-1})} + 1/j_{k-1}$ and $u_{(2,j_{k-1})} + 1/j_{k-1}$.

Suppose that this is true at $(k-1)$-th step. As $u_{(k-1,j_{k-1})} + 1/j_{k-1}$ and $u_{(2,j_{k-1})} + 1/j_{k-1}$ are both greater than u, we can find $j_k > \max\{j_{k-1}, k^2\}$ such that $u_{(k,j_k)} + 1/j_k + M/j_k$ is less than both $u_{(k-1,j_{k-1})} + 1/j_{k-1}$ and $u_{(2,j_{k-1})} + 1/j_{k-1}$ by continuity on the compact set $\overline{D'} \times X$. Applying Lemma 4.4.2 with $\lambda = 1/j_k$, we find a function $\tilde{u}_k \in C^\infty(D' \times X)$ with

$$u_{(k,j_k)} + 1/j_k \leq \tilde{u}_k \leq u_{(k,j_k)} + 1/j_k + M/j_k$$

and $\tilde{u}_k$ is $(1+1/k)(1+1/j_k)\eta$-subharmonic on graphs. So the induction process is true at k-th step. (One can begin the induction process with $u_{(2,j_2)} + 1/j_2$ with j_2 large enough that $u_{(2,j_2)} + 1/j_2 < 0$).

One can see that $\tilde{u}_k$ is decreasing to u. Since $\tilde{u}_k < 0$, $(1-1/k)\tilde{u}_k$ is still decreasing to u. The function $(1-1/k)\tilde{u}_k$ is $(1-1/k^2)(1+1/j_k)\eta$-subharmonic on graphs, which is also $(1-1/k^2 j_k)\eta$-subharmonic on graphs because $j_k > k^2$. So $(1-1/k)\tilde{u}_k$ are the desired approximants. ■

REFERENCES

REFERENCES

[Aub98] Thierry Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR 1636569

[BCEKR20] Bo Berndtsson, Dario Cordero-Erausquin, Bo'az Klartag, and Yanir A. Rubinstein, *Complex interpolation of $\mathbb{R}$-norms, duality and foliations*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 2, 477–505. MR 4049223

[Ber09] Bo Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2) **169** (2009), no. 2, 531–560. MR 2480611

[Ber13] ______, *Probability measures associated to geodesics in the space of Kähler metrics*, Algebraic and Analytic Microlocal Analysis, Springer, 2013, pp. 395–419.

[Ber19] Robert J. Berman, *From Monge-Ampère equations to envelopes and geodesic rays in the zero temperature limit*, Math. Z. **291** (2019), no. 1-2, 365–394. MR 3936074

[BK07] Zbigniew Błocki and Sławomir Kołodziej, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2089–2093. MR 2299485

[BK12] Robert Berman and Julien Keller, *Bergman geodesics*, Complex Monge-Ampère equations and geodesics in the space of Kähler metrics, Lecture Notes in Math., vol. 2038, Springer, Heidelberg, 2012, pp. 283–302. MR 2932447

[Bło05] Zbigniew Błocki, *Weak solutions to the complex Hessian equation*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 5, 1735–1756. MR 2172278

[Bło13] ______, *The complex Monge-Ampère equation in Kähler geometry*, Pluripotential theory, Lecture Notes in Math., vol. 2075, Springer, Heidelberg, 2013, pp. 95–141. MR 3089069

[Bou12] Sébastien Boucksom, *Monge-Ampère equations on complex manifolds with boundary*, Complex Monge-Ampère equations and geodesics in the space of Kähler metrics, Lecture Notes in Math., vol. 2038, Springer, Heidelberg, 2012, pp. 257–282. MR 2932446

[BT76] Eric Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), no. 1, 1–44. MR 445006

[BT82] ______, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), no. 1-2, 1–40. MR 674165

[Cat99] David Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1–23. MR 1699887

[Con90] John B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR 1070713

[CS93] Ronald Coifman and Stephen Semmes, *Interpolation of Banach spaces, Perron processes, and Yang-Mills*, Amer. J. Math. **115** (1993), no. 2, 243–278. MR 1216432

[Dem12] Jean-Pierre Demailly, *Complex analytic and differential geometry*, Université de Grenoble I Grenoble, 2012.

[Dev61] Allen Devinatz, *The factorization of operator valued functions*, Annals of Mathematics (1961), 458–495.

[DLR18] Tamás Darvas, Chinh H Lu, and Yanir A Rubinstein, *Quantization in geometric pluripotential theory*, arXiv preprint arXiv:1806.03800 (2018).

[Don92] Simon Donaldson, *Boundary value problems for Yang-Mills fields*, J. Geom. Phys. **8** (1992), no. 1-4, 89–122. MR 1165874

[Don99] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, vol. 196, Amer. Math. Soc., Providence, RI, 1999, pp. 13–33. MR 1736211

[Dou66] Ronald Douglas, *On factoring positive operator functions*, Journal of Mathematics and Mechanics **16** (1966), no. 2, 119–126.

[DW19] Tamás Darvas and Kuang-Ru Wu, *Griffiths extremality, interpolation of norms, and Kähler quantization*, arXiv preprint arXiv:1910.01782 (2019).

[Gra58] Hans Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472. MR 0098847

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364

[Ham82] Richard S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65–222. MR 656198

[Hel64] Henry Helson, *Lectures on invariant subspaces*, Academic Press, New York-London, 1964. MR 0171178

[HZ05] Roger A. Horn and Fuzhen Zhang, *Basic properties of the Schur complement*, pp. 17–46, Springer US, Boston, MA, 2005.

[Lem81] László Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), no. 4, 427–474. MR 660145

[Lem15] ______, *A maximum principle for Hermitian (and other) metrics*, Proc. Amer. Math. Soc. **143** (2015), no. 5, 2193–2200. MR 3314125

[Lem17] ______, *Noncommutative potential theory*, Anal. Math. **43** (2017), no. 4, 603–627. MR 3738363

[Mab87] Toshiki Mabuchi, *Some symplectic geometry on compact Kähler manifolds. I*, Osaka J. Math. **24** (1987), no. 2, 227–252. MR 909015

[Mil84] J. Milnor, *Remarks on infinite-dimensional Lie groups*, Relativity, groups and topology, II (Les Houches, 1983), North-Holland, Amsterdam, 1984, pp. 1007–1057. MR 830252

[Mor58] Charles B. Morrey, *The analytic embedding of abstract real-analytic manifolds*, Annals of Mathematics **68** (1958), no. 1, 159–201.

[PS06] D. H. Phong and Jacob Sturm, *The Monge-Ampère operator and geodesics in the space of Kähler potentials*, Invent. Math. **166** (2006), no. 1, 125–149. MR 2242635

[Ric68] Rolf Richberg, *Stetige streng pseudokonvexe Funktionen*, Math. Ann. **175** (1968), 257–286. MR 222334

[RZ10] Yanir A. Rubinstein and Steve Zelditch, *Bergman approximations of harmonic maps into the space of Kähler metrics on toric varieties*, J. Symplectic Geom. **8** (2010), no. 3, 239–265. MR 2684507

[Sem92] Stephen Semmes, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. **114** (1992), no. 3, 495–550. MR 1165352

[Slo86] Zbigniew Slodkowski, *Local maximum property and q-plurisubharmonic functions in uniform algebras*, J. Math. Anal. Appl. **115** (1986), no. 1, 105–130. MR 835588

[Slo88] ______, *Complex interpolation of normed and quasinormed spaces in several dimensions. I*, Trans. Amer. Math. Soc. **308** (1988), no. 2, 685–711. MR 951623

[Slo90a] ______, *Complex interpolation for normed and quasi-normed spaces in several dimensions. III. Regularity results for harmonic interpolation*, Trans. Amer. Math. Soc. **321** (1990), no. 1, 305–332. MR 991968

[Slo90b] ______, *Complex interpolation of normed and quasinormed spaces in several dimensions. II. Properties of harmonic interpolation*, Trans. Amer. Math. Soc. **317** (1990), no. 1, 255–285. MR 949900

[Tia90] Gang Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130. MR 1064867

[Wit83] Edward Witten, *Global aspects of current algebra*, Nuclear Phys. B **223** (1983), no. 2, 422–432. MR 717915

[WM57] Norbert Wiener and Pesi Rustom Masani, *The prediction theory of multivariate stochastic processes*, Acta Mathematica **98** (1957), no. 1-4, 111–150.

[Yau87] Shing-Tung Yau, *Nonlinear analysis in geometry*, Enseign. Math. (2) **33** (1987), no. 1-2, 109–158. MR 896385

[Zel98] Steve Zelditch, *Szegö kernels and a theorem of Tian*, Internat. Math. Res. Notices (1998), no. 6, 317–331. MR 1616718