GAUDIN MODELS ASSOCIATED TO CLASSICAL LIE ALGEBRAS

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Dedicated to my parents and my wife Ziting Tang.

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PREFACE

This thesis is mainly based on three publications.

Chapters 2 and 3 are based on the publications jointly with Evgeny Mukhin and Alexander Varchenko

[LMV16] Kang Lu, E. Mukhin, A. Varchenko, *On the Gaudin model associated to Lie algebras of classical types*, J. Math. Phys. **57** (2016), no. 101703, 1-22.

[LMV17] Kang Lu, E. Mukhin, A. Varchenko, *Self-dual Grassmannian, Wronski map, and representations of* $\mathfrak{gl}_N$, $\mathfrak{sp}_{2r}$, $\mathfrak{so}_{2r+1}$, Pure Appl. Math. Q., **13** (2017), no. 2, 291–335.

Chapter 4 contains the sole work of the author

[Lu18] Kang Lu, *Lower bounds for numbers of real self-dual spaces in problems of Schubert calculus*, SIGMA **14** (2018), 046, 15 pages.

I am the main author of these articles, and was responsible for the majority of the manuscript composition and proof writing.

TABLE OF CONTENTS

Page

Page

Page

ABSTRACT

Kang Lu Ph.D., Purdue University, August 2020. Gaudin models associated to classical Lie algebras. Major Professor: Evgeny Mukhin.

We study the Gaudin model associated to Lie algebras of classical types.

First, we derive explicit formulas for solutions of the Bethe ansatz equations of the Gaudin model associated to the tensor product of one arbitrary finite-dimensional irreducible module and one vector representation for all simple Lie algebras of classical type. We use this result to show that the Bethe Ansatz is complete in any tensor product where all but one factor are vector representations and the evaluation parameters are generic. We also show that except for the type D, the joint spectrum of Gaudin Hamiltonians in such tensor products is simple.

Second, using the result from [MTV09b], we define a new stratification of the Grassmannian of N planes $\mathrm{Gr}(N, d)$. Following [MV04], we introduce a new subvariety of Grassmannian, called self-dual Grassmannian, using the connections between self-dual spaces and Gaudin model associated to Lie algebras of types B and C. Then we use the result from [Ryb18] to obtain a stratification of self-dual Grassmannian.

1. INTRODUCTION

1.1 Motivation

Quantum spin chains are one of the most important models in integrable system. They have connections with mathematics in many different aspects. To name a few, for example

(1) **Quantum groups**, see [Dri85,CP94]: finite-dimensional irreducible representations of quantum affine algebras were classified in [CP91,CP95]. The character theory of quantum group, was introduced for Yangians in [Kni95] and for quantum affine algebras in [FR99]. It turns out to be one of the most important tool for studying the representation theory of quantum groups. As described in [FR99], the q-character of quantum affine algebras is essentially the Harish-Chandra image of transfer matrices which are generating series of Hamiltonians of quantum spin chain. Conversely, the q-character itself also carries information about the spectrum of transfer matrices when acting on finite-dimensional irreducible representations, see [FH15, Theorem 5.11] and [FJMM17, Theorem 7.5].

(2) **Algebraic geometry**: in the work [MV04], it is shown that the Bethe ansatz for Gaudin model of type A is related to the Schubert calculus in Grassmannian. This connection was further established in the work [MTV09b], where the algebra of Hamiltonians (Bethe subalgebra) acting on finite-dimensional irreducible representation of the current algebra is identified with the scheme-theoretic intersection of suitable Schubert varieties. This result gives the proof of the strong Shapiro-Shapiro conjecture and transversality conjecture of intersection of Schubert varieties. Moreover, a lower bound for the numbers of real solutions in problems appearing in Schubert calculus for Grassmannian is given in [MT16].

(3) **Center of vertex algebra and $\mathcal{W}$-algebras**: the algebra of Hamiltonians (Bethe algebra) for Gaudin model was described by Feigin-Frenkel center, see [FFR94, Fre07, Mol18], which is the center of vacuum model over $\hat{\mathfrak{g}}$ at critical level. This commutative algebra is also isomorphic to the classical $\mathcal{W}$-algebra associated to ${}^L\mathfrak{g}$, the Langlands dual of $\mathfrak{g}$, via affine Harish-Chandra isomorphism. There are also quantum analogues of relations between quantum integrable systems and centers of quantum vertex algebra over quantum affine algebras and Yangians, see [FJMR16, JKMY18] and references therein. Remarkably, shifted Yangians are also deeply related to finite $\mathcal{W}$-algebras, see e.g. [BK06].

(4) **Quantum cohomology and quantum K-theory**: it is shown in [GRTV12] that the quantum cohomology algebra of the cotangent bundle of a partial flag variety can be identified as the Bethe subalgebra of Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. Moreover, the idempotents of the quantum cohomology algebra can be determined by the XXX Bethe ansatz method. There are also parallel results for equivariant cohomology and quantum K-theory corresponding to Gaudin model and XXZ spin chains, respectively, see [RSTV11, RTV15]. The literature on the connections between quantum integrable system and quantum cohomology becomes immense and keeps growing.

(5) **Combinatorics**: the alternating sign matrix conjecture is proved by studying six-vertex model using the Izergin-Korepin determinant for a partition function for square ice with domain wall boundary, see [Kup96]. The number of alternating sign matrices can also be described as the largest coefficient of the normalized ground state eigenvector of the XXZ spin chain of size $2n+1$, see [RS01, RSZJ07]. Quantum spin chains are also related to standard Young tableaux. A bijective correspondence between the set of standard Young tableaux (bitableaux) and rigged configurations was constructed in [KR86] , where rigged configurations "parameterize" the solutions of Bethe ansatz equations.

(6) There are also other directions, for example orthogonal polynomial [MV07], hypergeometric functions, qKZ equations, Selberg type Integrals, arrangement of hyperplanes [SV91], etc, see [Var03] for a review.

All these connections and applications show that quantum spin chains play a central role in mathematics. It is important to study quantum spin chains in a mathematical and rigorous way. A modern approach to describe quantum integrable systems is using the representation theory of various quantum algebras [FRT88]. For example, enveloping algebras of current algebras, Yangians, quantum affine algebras, and elliptic quantum groups correspond to Gaudin model, XXX, XXZ, and XYZ spin chains, respectively. We discuss the formulation of the problem below in more detail.

1.2 Main Problems

In general, Gaudin model and XXX spin chains can be described as follows. The XXZ and XYZ spin chains can be described similarly with certain modifications.

Let $\mathfrak{g}$ be a simple (or reductive) Lie algebra (or superalgebra). Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathcal{A}(\mathfrak{g})$ be an algebra associated to $\mathfrak{g}$ such that $\mathrm{U}(\mathfrak{g})$ can be identified as a Hopf subalgebra of $\mathcal{A}(\mathfrak{g})$. For example, $\mathcal{A}(\mathfrak{g})$ is the universal enveloping algebra of the current algebra $\mathrm{U}(\mathfrak{g}[t])$ for Gaudin model and Yangian $\mathrm{Y}(\mathfrak{g})$ associated to $\mathfrak{g}$ for XXX spin chains. The Bethe algebra $\mathcal{B}(\mathfrak{g})$ is a certain unitial commutative subalgebra of $\mathcal{A}(\mathfrak{g})$. The Bethe subalgebra depends on an element $\mu \in \mathfrak{g}^*$. Here we only concentrate on the periodic case for simplicity of exposition, namely $\mu = 0$. In general, similar results are also expected when μ regular semi-simple or even simply regular. The Bethe algebra commutes with the algebra $\mathrm{U}(\mathfrak{g})$. Take any finite-dimensional irreducible representation V of $\mathcal{A}(\mathfrak{g})$. Since $\mathcal{B}(\mathfrak{g})$ commutes with $\mathrm{U}(\mathfrak{g})$, the Bethe subalgebra $\mathcal{B}(\mathfrak{g})$ acts naturally on V^{sing}, the singular subspace with respect to the $\mathfrak{g}$-action. One would like to study the spectrum of $\mathcal{B}(\mathfrak{g})$ acting on V^{sing}.

Let $\mathcal{E} : \mathcal{B}(\mathfrak{g}) \to \mathbb{C}$ be a character, then the $\mathcal{B}(\mathfrak{g})$-*eigenspace* and *generalized* $\mathcal{B}(\mathfrak{g})$-*eigenspace associated to* $\mathcal{E}$ in V^{sing} are defined by $\bigcap_{a\in\mathcal{B}(\mathfrak{g})} \ker(a|_{V^{\mathrm{sing}}} - \mathcal{E}(a))$ and $\bigcap_{a\in\mathcal{B}(\mathfrak{g})} \left(\bigcup_{m=1}^{\infty} \ker(a|_{V^{\mathrm{sing}}} - \mathcal{E}(a))^m\right)$, respectively. We call $\mathcal{E}$ an *eigenvalue* of

$\mathcal{B}(\mathfrak{g})$ acting on V^{sing} if the $\mathcal{B}(\mathfrak{g})$-eigenspace associated to $\mathcal{E}$ is non-trivial. We call a non-zero vector in a $\mathcal{B}(\mathfrak{g})$-eigenspace an *eigenvector* of $\mathcal{B}(\mathfrak{g})$.

Question 1.2.1. *Find eigenvalues and eigenvectors of* $\mathcal{B}(\mathfrak{g})$ *acting on* V^{sing}*.*

The main approach to address Question 1.2.1 is the *Bethe ansatz* method, which was introduced by H. Bethe back in 1931 [Bet31]. The Bethe ansatz usually works well for the generic situation. For the degenerate situation, the problem is more subtle.

Let $\mathcal{B}_V(\mathfrak{g})$ be the image of $\mathcal{B}(\mathfrak{g})$ in $\mathrm{End}(V^{\mathrm{sing}})$. A *Frobenius algebra* is a finite-dimensional unital commutative algebra whose regular and coregular representations are isomorphic. Based on the extensive study of quantum integrable systems, the following conjecture is expected.

Conjecture 1.2.2 ([Lu20])**.** *The* $\mathcal{B}_V(\mathfrak{g})$*-module* V^{sing} *is isomorphic to a regular representation of a Frobenius algebra.*

When Conjecture 1.2.2 holds, we call the corresponding integrable system *perfect integrable.* This conjecture has been proved for the following cases, (1) Gaudin model of type A in [MTV08b, MTV09b]; (2) Gaudin model of all types in [Lu20] with the help of [FF92,FFR10,Ryb18]; (3) XXX spin chains of type A associated to irreducible tensor products of evaluation vector representations in [MTV14]; (4) XXX spin chains of Lie superalgebra $\mathfrak{gl}_{1|1}$ associated to cyclic tensor products of evaluation polynomial modules in [LM19b].

The notion of perfect integrability (or Conjecture 1.2.2) is motivated by the following corollary about general facts of regular and coregular representations, geometric Langlands correspondence, and Bethe ansatz conjecture.

Corollary 1.2.3. *For each eigenvalue* $\mathcal{E}$*, the corresponding* $\mathcal{B}(\mathfrak{g})$*-eigenspace has dimension one. There exists a bijection between* $\mathcal{B}(\mathfrak{g})$*-eigenspaces and closed points in* $\mathrm{spec}(\mathcal{B}_V(\mathfrak{g}))$*. Moreover, each generalized* $\mathcal{B}(\mathfrak{g})$*-eigenspace is a cyclic* $\mathcal{B}(\mathfrak{g})$*-module. The image of Bethe algebra in* $\mathrm{End}(V^{\mathrm{sing}})$ *is a maximal commutative subalgebra of dimension equal to* $\dim V^{\mathrm{sing}}$*.*

By the philosophy of geometric Langlands correspondence, one would like to understand the following question.

Question 1.2.4. *Describe the finite-dimensional algebra* $\mathcal{B}_V(\mathfrak{g})$ *and the corresponding scheme* $\operatorname{spec}(\mathcal{B}_V(\mathfrak{g}))$. *Find the geometric object parameterizing the eigenspace of* $\mathcal{B}(\mathfrak{g})$ *when* V *runs over all finite-dimensional irreducible representations.*

It is well-known that if $\operatorname{spec}(\mathcal{B}_V(\mathfrak{g}))$ is a complete intersection, then $\mathcal{B}_V(\mathfrak{g})$ is a Frobenius algebra. Conversely if $\mathcal{B}_V(\mathfrak{g})$ is Frobenius, it would be interesting to check if $\operatorname{spec}(\mathcal{B}_V(\mathfrak{g}))$ is a complete intersection, see [MTV09b].

1.3 Gaudin Model

The Gaudin model was introduced by M. Gaudin in [Gau76] for the simple Lie algebra $\mathfrak{sl}_2$ and later generalized to arbitrary semi-simple Lie algebras in [Gau83, Section 13.2.2].

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^n$ be a sequence of dominant integral weight. Let $\boldsymbol{z} = (z_i)_{i=1}^n$ be a sequence of pair-wise distinct complex numbers. Let $V_{\boldsymbol{\lambda}}$ be the tensor product of finite-dimensional irreducible representations of highest weights λ_s, $s = 1, \dots, n$. Let $\{X_i\}_{i=1}^{\dim \mathfrak{g}}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. For $X \in \mathfrak{g}$, denote by $X^{(a)}$ the operator $1^{\otimes(a-1)} \otimes X \otimes 1^{\otimes(n-a)} \in \mathrm{U}(\mathfrak{g})^{\otimes n}$.

The *Gaudin Hamiltonians* are given by

$$\mathcal{H}_i = \sum_{j, j \neq i} \frac{\sum_{k=1}^{\dim \mathfrak{g}} X_k^{(i)} \otimes X_k^{(j)}}{z_i - z_j}, \qquad i = 1, \dots, n. \tag{1.3.1}$$

The Gaudin Hamiltonians commute, $[\mathcal{H}_i, \mathcal{H}_j] = 0$. In Gaudin model, we study the spectrum of Gaudin Hamiltonians acting on $V_{\boldsymbol{\lambda}}$. The Gaudin Hamiltonians also commute with the diagonal action $\mathfrak{g}$.

1.3.1 Feigin-Frenkel Center and Bethe Subalgebra

In the seminal work [FFR94], Feigin, Frenkel, and Reshetikhin established a connection between the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of affine vertex algebra at the critical level and higher Gaudin Hamiltonians in the Gaudin model. Let us discuss $\mathfrak{z}(\widehat{\mathfrak{g}})$ in more detail.

Let $\mathfrak{g}$ be a simple Lie algebra. Consider the affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$, $\mathfrak{g}[t,t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$. We simply write $X[s]$ for $X \otimes t^s$ for $X \in \mathfrak{g}$ and $s \in \mathbb{Z}$. Let $\mathfrak{g}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ and $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. Let $h^\vee$ be the *dual Coxeter number* of $\mathfrak{g}$. Define the module $V_{-h^\vee}(\mathfrak{g})$ as the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $\mathfrak{g}[t]$ and $K + h^\vee$. We call the module $V_{-h^\vee}(\mathfrak{g})$ the *Vaccum module at the critical level over* $\widehat{\mathfrak{g}}$. The vacuum module $V_{-h^\vee}(\mathfrak{g})$ has a vertex algebra structure.

Define the *center* $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $V_{-h^\vee}(\mathfrak{g})$ by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V_{-h^\vee}(\mathfrak{g}) \mid \mathfrak{g}[t]v = 0\}.$$

Using the PBW theorem, it is clear that $V_{-h^\vee}(\mathfrak{g})$ is isomorphic to $\mathrm{U}(\mathfrak{g}_-)$ as vector spaces. There is an injective homomorphism from $\mathfrak{z}(\widehat{\mathfrak{g}})$ to $\mathrm{U}(\mathfrak{g}_-)$. Hence $\mathfrak{z}(\widehat{\mathfrak{g}})$ is identified as a commutative subalgebra of $\mathrm{U}(\mathfrak{g}_-)$. The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called the *Feigin-Frenkel center*, see [FF92]. An element in $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a *Segal-Sugawara vector*. There is a distinguished element $S_1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$ given by

$$S_1 = \sum_{a=1}^{\dim \mathfrak{g}} X_a[-1]^2.$$

To obtain the Bethe subalgebra of $\mathfrak{g}[t]$, one applies an anti-homomorphism to $\mathfrak{z}(\widehat{\mathfrak{g}})$ which sends $X[-s-1]$ to $\partial_u^s X(u)/s!$, where $X(u) = X[0]u^{-1} + X[1]u^{-2} + \cdots \in \mathrm{U}(\mathfrak{g}[t])[[u^{-1}]]$. One obtains generating series in u^{-1}. Then the *Bethe subalgebra* $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g}[t]$ is the unital subalgebra of $\mathrm{U}(\mathfrak{g}[t])$ generated by all coefficients of generating series corresponding to elements in $\mathfrak{z}(\widehat{\mathfrak{g}})$. The Bethe algebra is considered as the algebra of Hamiltonians. For instance, the Gaudin Hamiltonians $\mathcal{H}_i$ (1.3.1) can be obtained by taking the residues of the generating series corresponding to S_1 at z_i acting on $V_{\lambda_1}(z_1) \otimes \cdots \otimes V_{\lambda_n}(z_n)$, where $V_{\lambda_i}(z_i)$ is the evaluation module of $\mathfrak{g}[t]$ with evaluation parameter z_i. This procedure can be found for e.g. in [Mol13, MR14].

Let V be a finite-dimensional irreducible representation of $\mathfrak{g}[t]$, namely a tensor product of evaluation modules $V_{\lambda_1}(z_1) \otimes \cdots \otimes V_{\lambda_n}(z_n)$, where $\boldsymbol{\lambda} = (\lambda^{(i)})_{i=1}^n$ and $\boldsymbol{z} = (z_i)_{i=1}^n$ as before. We are interested in the spectrum of $\mathcal{B}(\mathfrak{g})$ acting on V^{sing}.

There is also a generalization of Gaudin model, which is called Gaudin model with irregular singularities, see [Ryb06, FFTL10]. In this case, the Bethe algebra also depends on an element $\mu \in \mathfrak{g}^*$.

1.3.2 Opers and Perfect Integrability

In this section, we discuss the known results posed in the introduction.

It was shown in [Fre05, Theorem 2.7] that $\mathcal{B}_V(\mathfrak{g})$ is isomorphic to the algebra of functions on the space of monodromy-free ${}^L\mathfrak{g}$-opers on $\mathbb{P}^1$ which has regular singularities at the point z_i of residues described by λ_i and also at infinity. Moreover, the joint eigenvalues of the Bethe algebra acting on V^{sing} are encoded by these ${}^L\mathfrak{g}$-opers. It was also conjectured there that there exists a bijection between joint eigenvalues of Bethe algebra acting on V^{sing} and monodromy-free ${}^L\mathfrak{g}$-opers on $\mathbb{P}^1$ stated above.

Similar statements are also obtained for Gaudin model with irregular singularities in [FFTL10]. In this case, the difference is that the corresponding ${}^L\mathfrak{g}$-opers now have irregular singularities at infinity. It is then shown in [FFR10, Corollary 5] for Gaudin model with irregular singularities associated to regular $\mu \in \mathfrak{g}^*$ that the Bethe algebra acts on V cyclically and there exists a bijection between joint eigenvalues of Bethe algebra acting on V with monodromy-free ${}^L\mathfrak{g}$-opers on $\mathbb{P}^1$ which has regular singularities at the point z_i of residues described by λ_i and a irregular singularity at infinity.

Using the results of [FFTL10] and taking μ to be the principal nilpotent element, Rybnikov managed to prove the conjecture in [Fre05] for Gaudin model, see [Ryb18]. Namely, the Bethe algebra $\mathcal{B}_V(\mathfrak{g})$ acts on V^{sing} cyclically and there exists a bijection between joint eigenvalues of Bethe algebra acting on V^{sing} with monodromy-free ${}^L\mathfrak{g}$-

opers on $\mathbb{P}^1$ which has regular singularities at the point z_i of residues described by λ_i and also at infinity.

These results give answers for Questions 1.2.1, 1.2.4 and the essential parts of Conjecture 1.2.2 for Gaudin model, that is the $\mathcal{B}_V(\mathfrak{g})$-module V^{sing} is isomorphic to the regular representation of $\mathcal{B}_V(\mathfrak{g})$.

To show Conjecture 1.2.2, it remains to show that $\mathcal{B}_V(\mathfrak{g})$ is a Frobenius algebra. Combining the results [FF92,FFR10,Ryb18] and using the Shapovalov form on V, we are able to construct an invariant nondegenerate symmetric bilinear form on $\mathcal{B}_V(\mathfrak{g})$, which in turn shows that $\mathcal{B}_V(\mathfrak{g})$ is Frobenius. Hence we obtain

Theorem 1.3.1 ([Lu20]). *Gaudin model for $\mu = 0$ and regular $\mu \in \mathfrak{h}^*$ is perfectly integrable.*

In other words, we obtain the perfect integrability for Gaudin model with periodic and regular quasi periodic boundaries. As a corollary, we also obtain that there exists a bijection between common eigenvectors of Bethe algebra acting on V^{sing} with aforementioned ${}^L\mathfrak{g}$-opers. This can be thought as the proof of Bethe ansatz conjecture of eigenvector form.

1.3.3 Grassmannian and Gaudin Model

A remarkable observation is the connections between Gaudin model of type A and Grassmannian. This was first observed in [MV04] by studying the reproduction procedure of solutions of Bethe ansatz equation. An invariant object for reproduction procedure is a differential operator whose kernel is a space of polynomials with prescribed exponents at z_i described by the corresponding partitions λ_i (dominant weights). This differential operator can be explicitly written in terms of the corresponding solution of Bethe ansatz equation. It is essentially the same as the $\mathfrak{sl}_N$-opers, namely it describes the joint eigenvalues of the Bethe algebra acting on the corresponding Bethe vector constructed from the solution of Bethe ansatz equation, see [FFR94, MTV06].

This connection leads to a proof of Shapiro-Shapiro conjecture in real algebraic geometry, see [MTV09c]. This connection was made precise in [MTV09b] by interpreting the Bethe algebra $\mathcal{B}_V(\mathfrak{g})$ as the space of functions on the intersection of suitable Schubert cycles in a Grassmannian variety. This interpretation gives a relation between representation theory of $\mathfrak{gl}_N$ and Schubert calculus useful in both directions. In particular, the proofs of a strong form of Shapiro-Shapiro conjecture and the transversality conjecture of intersection of Schubert varieties are deduced from that, see [MTV09b].

We further study this connection in [LMV17]. To state our result, we make the statement in [MTV09b] more precise. Let $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ be the intersection of Schubert cells Ω_{λ_i,z_i} with respect to the osculating flag at z_i and the partition λ_i, see [LMV17, Section 3.1] for more detail.

Theorem 1.3.2 ([MTV09b])**.** *There exists a bijection between eigenvectors of the Bethe algebra $\mathcal{B}_V(\mathfrak{gl}_N)$ in V^{sing} and $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$.*

Note that, for generic $\boldsymbol{z}$, $\mathcal{B}_V(\mathfrak{gl}_N)$ is diagonalizable and has simple spectrum on V^{sing}. Let $\Omega_{\boldsymbol{\lambda}}$ be the disjoint union of all $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ with $\boldsymbol{z}$ running over all tuples of distinct coordinates. These $\Omega_{\boldsymbol{\lambda}}$ are constructible subsets in the Grassmannian. We show that these $\Omega_{\boldsymbol{\lambda}}$ form a stratification of Grassmannian, see [LMV17, Section 3.3], similar to the well-known stratification consisting of Schubert cells. By taking closure of $\Omega_{\boldsymbol{\lambda}}$, it means we allow distinct z_i and z_j coinciding. Note that

$$V_\mu(z) \otimes V_\nu(z) = \bigoplus_{\lambda} C^{\lambda}_{\mu,\nu} V_\lambda(z),$$

where $C^{\lambda}_{\mu,\nu}$ are the Littlewood-Richardson coefficients, therefore we know how V decomposes if several z_i coincide. Using Theorem 1.3.2, it tells us that the closure of $\Omega_{\boldsymbol{\lambda}}$ is exact a disjoint union of $\Omega_{\boldsymbol{\mu}}$ and those $\boldsymbol{\mu}$ are determined by the representation theory of $\mathfrak{gl}_N$ and $\boldsymbol{\lambda}$. Therefore this shows these $\Omega_{\boldsymbol{\lambda}}$ form a new stratification of the Grassmannian. This generalizes the standard stratification of the swallowtail, see for example [AGZV85, Section 2.5 of Part 1].

Since this connection is so important, it would be interesting to explore similar connections by studying Gaudin model of other types. We are able to deal with types B, C, G_2 with the following reasons. Since the Bethe algebra can be obtained from Feigin-Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$, we need a complete set of explicitly generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$. These generators are obtained for types A [CT06], BCD [Mol13], and G_2 [MRR16]. This method for type D is not applicable as the Dynkin diagram has branch. As a result, after using the Miura transformation to the ${}^L\mathfrak{g}$-opers, one obtains pseudo-differential operators.

Let $\mathfrak{g}$ be a simple Lie algebra of types B and C. Identifying a ${}^L\mathfrak{g}$-oper as a $\mathfrak{sl}_N$-oper of special form and using Miura transformation, one obtains a differential operator in a symmetric form as follows depending type B or C,

$$(\partial_x - f_1(x))\cdots(\partial_x - f_n(x))(\partial_x + f_n(x))\cdots(\partial_x + f_1(x)),$$

$$(\partial_x - f_1(x))\cdots(\partial_x - f_n(x))\partial_x(\partial_x + f_n(x))\cdots(\partial_x + f_1(x)).$$

Therefore, the kernels of these differential operators have certain symmetry, which are the same as the ones introduced in [MV04, Section 6]. Such spaces are coming from the reproduction procedure for types BC and called *self-dual spaces*. The subset of all self-dual spaces in the Grassmannian is called *self-dual Grassmannian*. The self-dual Grassmannian is a new geometric object which is an algebraic subset in Grassmannian and different from the orthogonal Grassmannian.

Using the main result of [Ryb18], we managed to obtain a stratification for self-dual Grassmannian described by the representation theory of Lie algebras of types B and C similar to the one of type A for Grassmannian, see [LMV17, Section 4.4]. Combining [LMV17, Theorem 4.5] and the perfect integrability of Gaudin model, we have

Theorem 1.3.3 ([Lu20]). *There exists a bijection between eigenvectors of $\mathcal{B}_V(\mathfrak{g})$ in V^{sing} and the subset of all self-dual spaces in $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$.*

To be more precise, $\boldsymbol{\lambda}$ has to be changed to the corresponding partitions, see [LMV17, Section 4] for the definition. There is a similar study for type G_2 in this direction. The corresponding geometric object is called *self-self-dual Grassmannian* due to a further symmetry, see [LM19a].

Following the idea of [MT16], we obtain a lower bound for number of real self-dual spaces in $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ by analizing a modified Shapovalov form on $V_{\boldsymbol{\lambda}}$, see [Lu18].

1.3.4 Bethe Ansatz

The Bethe ansatz is the main method to find eigenvectors of quantum integrable systems, see [Bet31]. The Bethe ansatz construction provides an eigenvector of the Gaudin model from a solution to the Bethe ansatz equation. The Bethe ansatz equation is a system of algebraic equations which is in general very difficult to solve. We solve certain Bethe ansatz equation associated to tensor products of defining representations for Gaudin model of types BCD [LMV16] and G_2 [LM19a]. To solve the Bethe ansatz equation, we use the reproduction procedure which allows us to solve Bethe ansatz equation recursively. In particular, we obtain that for generic evaluation parameters, the Bethe ansatz construction provides a basis of $V^{\rm sing}$ when $V^{\rm sing}$ is a tensor product of defining representations. Moreover, when $\mathfrak{g}$ is Lie algebras of types BC and G_2, the spectrum of Gaudin Hamiltonians (1.3.1) on $V^{\rm sing}$ is simple. We remark that for type D, the spectrum may not be simple as the Dynkin diagram admits symmetry.

2. ON THE GAUDIN MODEL ASSOCIATED TO LIE ALGEBRAS OF CLASSICAL TYPES

2.1 Introduction

The Gaudin Hamiltonians are an important example of a family of commuting operators. We study the case when the Gaudin Hamiltonians possess a symmetry given by the diagonal action of $\mathfrak{g}$. In this case the Gaudin Hamiltonians depend on a choice of a simple Lie algebra $\mathfrak{g}$, $\mathfrak{g}$-modules $V_1, \dots, V_n$ and distinct complex numbers $z_1, \dots, z_n$, see (2.2.1).

The problem of studying the spectrum of the Gaudin Hamiltonians has received a lot of attention. However, the majority of the work has been done in type A. In this paper we study the cases of types B, C and D.

The main approach is the Bethe ansatz method. Our goal is to establish the method when all but one modules V_i are isomorphic to the first fundamental representation V_{ω_1}. Namely, we show that the Bethe ansatz equations have sufficiently many solutions and that the Bethe vectors constructed from those solutions form a basis in the space of singular vectors of $V_1 \otimes \cdots \otimes V_n$.

The solution of a similar problem in type A in [MV05b] led to several important results, such as a proof of the strong form of the Shapiro-Shapiro conjecture for Grassmanians, simplicity of the spectrum of higher Gauding Hamiltonians, the bijection betweem Fuchsian differential operators without monodromy with the Bethe vectors, etc, see [MTV09b] and references therein. We hope that this paper will give a start to similar studies in type B. In addition, the explicit formulas for simplest examples outside type A are important as experimental data for testing various conjectures.

By the standard methods, the problem is reduced to the case of $n = 2$, with V_1 being an arbitrary finite-dimensional module, $V_2 = V_{\omega_1}$ and $z_1 = 0$, $z_2 = 1$. The reduction involves taking appropriate limits, when all points z_i go to the same number with different rates. Then the $n = 2$ problems are observed in the leading order and the generic situation is recovered from the limiting case by the usual argument of deformations of isolated solutions of algebraic systems, see [MV05b] and Section 2.4 for details.

For the 2-point case when one of the modules is the defining representation V_{ω_1}, the spaces of singular vectors of a given weight are either trivial or one-dimensional. Then, according to the general philosophy, see [MV00], one would expect to solve the Bethe ansatz equations explicitly. In type A it was done in [MV00]. In the supersymmetric case of $\mathfrak{gl}(p|q)$ the corresponding Bethe ansatz equations are solved in [MVY15]. The other known cases with one dimensional spaces include tensor products of two arbitrary irreducible $\mathfrak{sl}_2$ modules, see [Var95] and tensor products of an arbitrary module with a symmetric power $V_{k\omega_1}$ of the vector representation in the case of $\mathfrak{sl}_{r+1}$, see [MV07]. Interestingly, in the latter case the solutions of the Bethe ansatz equations are related to zeros of Jacobi-Pineiro polynomials which are multiple orthogonal polynomials.

In all previously known cases when the dimension of the space of singular vectors of a given weight is one, the elementary symmetric functions of solutions of Bethe ansatz equations completely factorize into products of linear functions of the parameters. This was one of the main reasons the formulas were found essentially by brute force. However, unexpectedly, the computer experiments showed that in types B, C, D, the formulas do not factorize, see also Theorem 5.5 in [MV04], and therefore, the problem remained unsolved. In this paper we present a method to compute the answer systematically.

Our idea comes from the reproduction procedure studied in [MV08]. Let $V_1 = V_\lambda$ be the irreducible module of highest weight λ, let $V_2, \dots, V_n$ be finite-dimensional irreducible modules, and let $l_1, \dots, l_r$ be nonnegative integers, where r is the rank of $\mathfrak{g}$. Fix distinct complex numbers $z_1 = 0, z_1, \dots, z_n$. Consider the Bethe ansatz equation, see (2.2.2), associated to these data. Set $V = V_2 \otimes \cdots \otimes V_n$, denote the highest weight vector of V by v^+, the weight of v^+ by μ^+, and set $\mu = \mu^+ - \sum_{i=1}^r l_i\alpha_i$. Here α_i are simple roots of $\mathfrak{g}$.

Given an isolated solution of the Bethe ansatz equations we can produce two Bethe vectors: one in the space of singular vectors in $V_\lambda \otimes V$ of weight $\mu + \lambda$ and another one in the space of vectors in V of weight μ. The first Bethe vector, see (2.2.3), is an eigenvector of the standard Gaudin Hamiltonians, see (2.2.1), acting in $V_\lambda \otimes V$ and the second Bethe vector is an eigenvector of trigonometric Gaudin Hamiltonians, see [MV07]. The second vector is a projection of the first vector to the space $v^+ \otimes V \simeq V$.

Then the reproduction procedure of [MV07] in the j-th direction allows us to construct a new solution of the Bethe ansatz equation associated to new data: representations $V_1 = V_{s_j\cdot\lambda}$, $V_2, \dots, V_n$ and integers $l_1, \dots, \tilde{l}_j, \dots, l_r$ so that the new weight $\tilde{\mu} = \mu^+ - \sum_{i\neq j} l_i\alpha_i - \tilde{l}_j\alpha_j$ is given by $\tilde{\mu} = s_j\mu$. This construction is quite general, it works for all symmetrizable Kac-Moody algebras provided that the weight λ is generic, see Theorem 2.2.6 below. It gives a bijection between solutions corresponding to weights μ of V in the same Weyl orbit.

Note that in the case $\mu = \mu^+$, the Bethe ansatz equations are trivial. Therefore, using the trivial solution and the reproduction procedure, we, in principal, can obtain solutions for all weights of the form: $\mu = w\mu^+$. Note also that in the case of the vector representation, $V = V_{\omega_1}$, all weights in V are in the Weyl orbit of $\mu^+ = \omega_1$ (with the exception of weight $\mu = 0$ in type B). Therefore, we get all the solutions we need that way (the exceptional weight is easy to treat separately).

In contrast to [MV07], we do not have the luxury of generic weight λ, and we have to check some technical conditions on each reproduction step. It turns out, such checks are easy when going to the trivial solution, but not the other way, see Section 2.3.3. We manage to solve the recursion and obtain explicit formulas, see Corollary 2.3.10 for type B, Theorem 2.5.1 for type C and Theorem 2.5.4 for type D. We complete the check using these formulas, see Section 2.3.5.

To each solution of Bethe ansatz, one can associate an oper. For types A, B, C the oper becomes a scalar differential operator with rational coefficients, see [MV04], and Sections 2.3.6, 2.5.1. In fact, the coefficients of this operator are eigenvalues of higher Gaudin Hamiltonians, see [MTV06] for type A and [MM17] for types B, C. The differential operators for the solutions obtained via the reproduction procedure are closely related. It allows us to give simple formulas for the differential operators related to our solutions, see Propositions 2.3.11 and 2.5.3. According to [MV04], the kernel of the differential operator is a space of polynomials with a symmetry, called a self-dual space. We intend to discuss the self-dual spaces related to our situation in detail elsewhere.

The paper is constructed as follows. In Section 2.2 we describe the problem and set our notation. We study in detail the case of type B in Sections 2.3 and 2.4. In Section 2.3 we solve the Bethe ansatz equation for $n = 2$ when one of the modules is V_{ω_1}. In Section 2.4, we use the results of Section 2.3 to show the completeness and simplicity of the spectrum of Gaudin Hamiltonians acting in tensor products where all but one factors are V_{ω_1}, for generic values of z_i. In Section 2.5 we give the corresponding formulas and statements in types C and D.

2.2 The Gaudin Model and Bethe Ansatz

2.2.1 Simple Lie Algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{i,j})_{i,j=1}^{r}$. Denote the universal enveloping algebra of $\mathfrak{g}$ by $\mathcal{U}(\mathfrak{g})$. Let $D = \mathrm{diag}\{d_1, \dots, d_r\}$ be the diagonal matrix with positive relatively prime integers d_i such that $B = DA$ is symmetric.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra. Fix simple roots $\alpha_1, \dots, \alpha_r$ in $\mathfrak{h}^*$. Let $\alpha_1^\vee, \dots, \alpha_r^\vee \in \mathfrak{h}$ be the corresponding coroots. Fix a nondegenerate invariant bilinear form $(,)$ in $\mathfrak{g}$ such that $(\alpha_i^\vee, \alpha_j^\vee) = a_{i,j}/d_j$. Define the corresponding invariant bilinear forms in $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = d_i a_{i,j}$. We have $\langle \lambda, \alpha_i^\vee \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ for $\lambda \in$

$\mathfrak{h}^*$. In particular, $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. Let $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ be the fundamental weights, $\langle \omega_j, \alpha_i^\vee \rangle = \delta_{i,j}$.

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$ and $\mathcal{P}^+ = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}\}$ be the weight lattice and the set of dominant integral weights. The dominance order $>$ on $\mathfrak{h}^*$ is defined by: $\mu > \nu$ if and only if $\mu - \nu = \sum_{i=1}^r a_i \alpha_i$, $a_i \in \mathbb{Z}_{\geqslant 0}$ for $i = 1, \dots, r$.

Let $\rho \in \mathfrak{h}^*$ be such that $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 1, \dots, r$. We have $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$.

For $\lambda \in \mathfrak{h}^*$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ. We denote $\langle \lambda, \alpha_i^\vee \rangle$ by λ_i and sometimes write $V_{(\lambda_1, \lambda_2, \dots, \lambda_r)}$ for V_λ.

The Weyl group $\mathcal{W} \subset \mathrm{Aut}(\mathfrak{h}^*)$ is generated by reflections s_i, $i = 1, \dots, r$,

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

We use the notation

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \in \mathcal{W}, \lambda \in \mathfrak{h}^*,$$

for the shifted action of the Weyl group.

Let $E_1, \dots, E_r \in \mathfrak{n}_+$, $H_1, \dots, H_r \in \mathfrak{h}$, $F_1, \dots, F_r \in \mathfrak{n}_-$ be the Chevalley generators of $\mathfrak{g}$.

The coproduct $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is defined to be the homomorphism of algebras such that $\Delta x = 1 \otimes x + x \otimes 1$, for all $x \in \mathfrak{g}$.

Let $(x_i)_{i \in O}$ be an orthonormal basis with respect to the bilinear form $(,)$ in $\mathfrak{g}$.

Let $\Omega_0 = \sum_{i \in O} x_i^2 \in \mathcal{U}(\mathfrak{g})$ be the Casimir element. For any $u \in \mathcal{U}(\mathfrak{g})$, we have $u\Omega_0 = \Omega_0 u$. Let $\Omega = \sum_{i \in O} x_i \otimes x_i \in \mathfrak{g} \otimes \mathfrak{g} \subset \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. For any $u \in \mathcal{U}(\mathfrak{g})$, we have $\Delta(u)\Omega = \Omega\Delta(u)$.

The following lemma is well-known, see for example [Hum78], Ex. 23.4.

Lemma 2.2.1. *Let V_λ be an irreducible module of highest weight λ. Then Ω_0 acts on V_λ by the constant $(\lambda + \rho, \lambda + \rho) - (\rho, \rho)$.* □

Let V be a $\mathfrak{g}$-module. Let $\mathrm{Sing}\, V = \{v \in V \mid \mathfrak{n}_+ v = 0\}$ be the subspace of singular vectors in V. For $\mu \in \mathfrak{h}^*$ let $V[\mu] = \{v \in V \mid hv = \langle \mu, h \rangle v\}$ be the subspace of V of vectors of weight μ. Let $\mathrm{Sing}\, V[\mu] = (\mathrm{Sing}\, V) \cap (V[\mu])$ be the subspace of singular vectors in V of weight μ.

2.2.2 Gaudin Model

Let n be a positive integer and $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_i \in \mathfrak{h}^*$, a sequence of weights. Denote by $V_{\boldsymbol{\Lambda}}$ the $\mathfrak{g}$-module $V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$.

If $X \in \mathrm{End}(V_{\Lambda_i})$, then we denote by $X^{(i)} \in \mathrm{End}(V_{\boldsymbol{\Lambda}})$ the operator $\mathrm{id}^{\otimes i-1} \otimes X \otimes \mathrm{id}^{\otimes n-i}$ acting non-trivially on the i-th factor of the tensor product. If $X = \sum_k X_k \otimes Y_k \in \mathrm{End}(V_{\Lambda_i} \otimes V_{\Lambda_j})$, then we set $X^{(i,j)} = \sum_k X_k^{(i)} \otimes Y_k^{(j)} \in \mathrm{End}(V_{\boldsymbol{\Lambda}})$.

Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be a point in $\mathbb{C}^n$ with distinct coordinates. Introduce linear operators $\mathcal{H}_1(\boldsymbol{z}), \dots, \mathcal{H}_n(\boldsymbol{z})$ on $V_{\boldsymbol{\Lambda}}$ by the formula

$$\mathcal{H}_i(\boldsymbol{z}) = \sum_{j,\ j \neq i} \frac{\Omega^{(i,j)}}{z_i - z_j}, \quad i = 1, \dots, n. \tag{2.2.1}$$

The operators $\mathcal{H}_1(\boldsymbol{z}), \dots, \mathcal{H}_n(\boldsymbol{z})$ are called the *Gaudin Hamiltonians* of the Gaudin model associated with $V_{\boldsymbol{\Lambda}}$. One can check that the Hamiltonians commute,

$$[\mathcal{H}_i(\boldsymbol{z}), \mathcal{H}_j(\boldsymbol{z})] = 0$$

for all i, j. Moreover, the Gaudin Hamiltonians commute with the action of $\mathfrak{g}$, $[\mathcal{H}_i(\boldsymbol{z}), x] = 0$ for all i and $x \in \mathfrak{g}$. Hence for any $\mu \in \mathfrak{h}^*$, the Gaudin Hamiltonians preserve the subspace $\mathrm{Sing}\, V_{\boldsymbol{\Lambda}}[\mu] \subset V_{\boldsymbol{\Lambda}}$.

2.2.3 Bethe Ansatz

Fix a sequence of weights $\boldsymbol{\Lambda} = (\Lambda_i)_{i=1}^n$, $\Lambda_i \in \mathfrak{h}^*$, and a sequence of non-negative integers $\boldsymbol{l} = (l_1, \dots, l_r)$. Denote $l = l_1 + \cdots + l_r$, $\Lambda = \Lambda_1 + \cdots + \Lambda_n$, and $\alpha(\boldsymbol{l}) = l_1\alpha_1 + \cdots + l_r\alpha_r$.

Let c be the unique non-decreasing function from $\{1, \dots, l\}$ to $\{1, \dots, r\}$, such that $\# c^{-1}(i) = l_i$ for $i = 1, \dots, r$. The *master function* $\Phi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{l})$ is defined by

$$\Phi(\boldsymbol{t}, \boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{l}) = \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{(\Lambda_i, \Lambda_j)} \prod_{i=1}^{l} \prod_{s=1}^{n} (t_i - z_s)^{-(\alpha_{c(i)}, \Lambda_s)} \prod_{1 \leqslant i < j \leqslant l} (t_i - t_j)^{(\alpha_{c(i)}, \alpha_{c(j)})}.$$

The function Φ is a function of complex variables $\boldsymbol{t} = (t_1, \dots, t_l)$, $\boldsymbol{z} = (z_1, \dots, z_n)$, weights $\boldsymbol{\Lambda}$, and discrete parameters $\boldsymbol{l}$. The main variables are $\boldsymbol{t}$, the other variables will be considered as parameters.

We call Λ_i *the weight at a point* z_i, and we also call $c(i)$ *the color of variable* t_i.

A point $\boldsymbol{t} \in \mathbb{C}^l$ is called a *critical point associated to* $\boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{l}$, if the following system of algebraic equations is satisfied,

$$-\sum_{s=1}^{n} \frac{(\alpha_{c(i)}, \Lambda_s)}{t_i - z_s} + \sum_{j, j \neq i} \frac{(\alpha_{c(i)}, \alpha_{c(j)})}{t_i - t_j} = 0, \quad i = 1, \ldots, l. \tag{2.2.2}$$

In other words, a point $\boldsymbol{t}$ is a critical point if

$$\left(\Phi^{-1} \frac{\partial \Phi}{\partial t_i}\right)(\boldsymbol{t}) = 0, \quad \text{for } i = 1, \ldots, l.$$

Equation (2.2.2) is called the *Bethe ansatz equation associated to* $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$.

By definition, if $\boldsymbol{t} = (t_1, \ldots, t_l)$ is a critical point and $(\alpha_{c(i)}, \alpha_{c(j)}) \neq 0$ for some i, j, then $t_i \neq t_j$. Also if $(\alpha_{c(i)}, \Lambda_s) \neq 0$ for some i, s, then $t_i \neq z_s$.

Let Σ_l be the permutation group of the set $\{1, \ldots, l\}$. Denote by $\boldsymbol{\Sigma_l} \subset \Sigma_l$ the subgroup of all permutations preserving the level sets of the function c. The subgroup $\boldsymbol{\Sigma_l}$ is isomorphic to $\Sigma_{l_1} \times \cdots \times \Sigma_{l_r}$. The action of the subgroup $\boldsymbol{\Sigma_l}$ preserves the set of critical points of the master function. All orbits of $\boldsymbol{\Sigma_l}$ on the critical set have the same cardinality $l_1! \ldots l_r!$. In what follows we do not distinguish between critical points in the same $\boldsymbol{\Sigma_l}$-orbit.

The following lemma is known.

Lemma 2.2.2 ([MV04]). *If weight $\Lambda - \alpha(\boldsymbol{l})$ is dominant integral, then the set of critical points is finite.* $\square$

2.2.4 Weight Function

Consider highest weight irreducible $\mathfrak{g}$-modules $V_{\Lambda_1}, \ldots, V_{\Lambda_n}$, the tensor product $V_{\boldsymbol{\Lambda}}$, and its weight subspace $V_{\boldsymbol{\Lambda}}[\Lambda - \alpha(\boldsymbol{l})]$. Fix a highest weight vector v_{Λ_i} in V_{Λ_i} for $i = 1, \ldots, n$.

Following [SV91], we consider a rational map

$$\omega : \mathbb{C}^n \times \mathbb{C}^l \to V_{\boldsymbol{\Lambda}}[\Lambda - \alpha(\boldsymbol{l})]$$

called *the canonical weight function.*

Let $P(\boldsymbol{l}, n)$ be the set of sequences $I = (i_1^1, \dots, i_{j_1}^1; \dots; i_1^n, \dots, i_{j_n}^n)$ of integers in $\{1, \dots, r\}$ such that for all $i = 1, \dots, r$, the integer i appears in I precisely l_i times. For $I \in P(\boldsymbol{l}, n)$, and a permutation $\sigma \in \Sigma_l$, set $\sigma_1(i) = \sigma(i)$ for $i = 1, \dots, j_1$ and $\sigma_s(i) = \sigma(j_1 + \dots + j_{s-1} + i)$ for $s = 2, \dots, n$ and $i = 1, \dots, j_s$. Define

$$\Sigma(I) = \{\sigma \in \Sigma_l \mid c(\sigma_s(j)) = i_s^j \text{ for } s = 1, \dots, n \text{ and } j = 1, \dots, j_s\}.$$

To every $I \in P(\boldsymbol{l}, n)$ we associate a vector

$$F_I v = F_{i_1^1} \dots F_{i_{j_1}^1} v_{\Lambda_1} \otimes \dots \otimes F_{i_1^n} \dots F_{i_{j_n}^n} v_{\Lambda_n}$$

in $V_{\boldsymbol{\Lambda}}[\Lambda - \alpha(\boldsymbol{l})]$, and rational functions

$$\omega_{I,\sigma} = \omega_{\sigma_1(1), \dots, \sigma_1(j_1)}(z_1) \dots \omega_{\sigma_n(1), \dots, \sigma_n(j_n)}(z_n),$$

labeled by $\sigma \in \Sigma(I)$, where

$$\omega_{i_1, \dots, i_j}(z) = \frac{1}{(t_{i_1} - t_{i_2}) \dots (t_{i_{j-1}} - t_{i_j})(t_{i_j} - z)}.$$

We set

$$\omega(\boldsymbol{z}, \boldsymbol{t}) = \sum_{I \in P(\boldsymbol{l}, n)} \sum_{\sigma \in \Sigma(I)} \omega_{I,\sigma} F_I v. \tag{2.2.3}$$

Let $\boldsymbol{t} \in \mathbb{C}^l$ be a critical point of the master function $\Phi(\cdot, \boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{l})$. Then the value of the weight function $\omega(\boldsymbol{z}, \boldsymbol{t}) \in V_{\boldsymbol{\Lambda}}[\Lambda - \alpha(\boldsymbol{l})]$ is called *the Bethe vector.* Note that the Bethe vector does not depend on a choice of the representative in the $\boldsymbol{\Sigma_l}$-orbit of critical points.

The following facts about Bethe vectors are known. Assume that $\boldsymbol{z} \in \mathbb{C}^n$ has distinct coordinates. Assume that $\boldsymbol{t} \in \mathbb{C}^l$ is an isolated critical point of the master function $\Phi(\cdot, \boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{l})$.

Lemma 2.2.3 ([MV05b]). *The Bethe vector $\omega(\boldsymbol{z}, \boldsymbol{t})$ is well defined.* □

Theorem 2.2.4 ([Var11]). *The Bethe vector $\omega(\boldsymbol{z}, \boldsymbol{t})$ is non-zero.* □

Theorem 2.2.5 ([RV95])**.** *The Bethe vector $\omega(\boldsymbol{z},\boldsymbol{t})$ is singular, $\omega(\boldsymbol{z},\boldsymbol{t}) \in \operatorname{Sing} V_{\boldsymbol{\Lambda}}[\Lambda-\alpha(\boldsymbol{l})]$. Moreover, $\omega(\boldsymbol{z},\boldsymbol{t})$ is a common eigenvector of the Gaudin Hamiltonians,*

$$\mathcal{H}_i(\boldsymbol{z})\omega(\boldsymbol{z},\boldsymbol{t}) = \left(\Phi^{-1}\frac{\partial\Phi}{\partial z_i}\right)(\boldsymbol{t},\boldsymbol{z})\omega(\boldsymbol{z},\boldsymbol{t}), \qquad i = 1,\ldots,n.$$

□

2.2.5 Polynomials Representing Critical Points

Let $\boldsymbol{t} = (t_1,\ldots,t_l)$ be a critical point of a master function $\Phi(\boldsymbol{t},\boldsymbol{z},\boldsymbol{\Lambda},\boldsymbol{l})$. Introduce a sequence of polynomials $\boldsymbol{y} = (y_1(x),\ldots,y_r(x))$ in a variable x by the formula

$$y_i(x) = \prod_{j,c(j)=i}(x-t_j).$$

We say that the r-tuple of polynomials $\boldsymbol{y}$ *represents a critical point* $\boldsymbol{t}$ of the master function $\Phi(\boldsymbol{t},\boldsymbol{z},\boldsymbol{\Lambda},\boldsymbol{l})$. Note that the r-tuple $\boldsymbol{y}$ does not depend on a choice of the representative in the $\boldsymbol{\Sigma_l}$-orbit of the critical point $\boldsymbol{t}$.

We have $l = \sum_{i=1}^r \deg\, y_i = \sum_{i=1}^r l_i$. We call l *the length of* $\boldsymbol{y}$. We use notation $\boldsymbol{y}^{(l)}$ to indicate the length of $\boldsymbol{y}$.

Introduce functions

$$T_i(x) = \prod_{s=1}^n (x-z_s)^{\langle\Lambda_s,\alpha_i^\vee\rangle}, \quad i = 1,\ldots,r. \tag{2.2.4}$$

We say that a given r-tuple of polynomials $\boldsymbol{y} \in \boldsymbol{P}(\mathbb{C}[x])^r$ is *generic with respect to* $\boldsymbol{\Lambda},\boldsymbol{z}$ if

G1 polynomials $y_i(x)$ have no multiple roots;

G2 roots of $y_i(x)$ are different from roots and singularities of the function T_i;

G3 if $a_{ij} < 0$ then polynomials $y_i(x)$, $y_j(x)$ have no common roots.

If $\boldsymbol{y}$ represents a critical point of Φ, then $\boldsymbol{y}$ is generic.

Following [MV07], we reformulate the property of $\boldsymbol{y}$ to represent a critical point for the case when all but one weights are dominant integral.

We denote by $W(f,g)$ the Wronskian of functions f and g, $W(f,g) = f'g - fg'$.

Theorem 2.2.6 ([MV07]). *Assume that $\boldsymbol{z} \in \mathbb{C}^n$ has distinct coordinates and $z_1 = 0$. Assume that $\Lambda_i \in \mathcal{P}^+$, $i = 2, \ldots, n$. A generic r-tuple $\boldsymbol{y}$ represents a critical point associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$ if and only if for every $i = 1, \ldots, r$ there exists a polynomial $\tilde{y}_i$ satisfying*

$$W(y_i, x^{\langle \Lambda_1 + \rho, \alpha_i^\vee \rangle} \tilde{y}_i) = T_i \prod_{j \neq i} y_j^{-\langle \alpha_j, \alpha_i^\vee \rangle}. \tag{2.2.5}$$

Moreover, if the r-tuple $\tilde{\boldsymbol{y}}_{\boldsymbol{i}} = (y_1, \ldots, \tilde{y}_i, \ldots, y_r)$ is generic, then it represents a critical point associated to data $(s_i \cdot \Lambda_1, \Lambda_2, \ldots, \Lambda_n), \boldsymbol{z}, \boldsymbol{l}_{\boldsymbol{i}}$, where $\boldsymbol{l}_{\boldsymbol{i}}$ is determined by equation

$$\Lambda - \Lambda_1 - \alpha(\boldsymbol{l}_{\boldsymbol{i}}) = s_i(\Lambda - \Lambda_1 - \alpha(\boldsymbol{l})).$$

□

We say that the r-tuple $\tilde{\boldsymbol{y}}_{\boldsymbol{i}}$ (and the critical point it represents) is obtained from the r-tuple $\boldsymbol{y}$ (and the critical point it represents) by *the reproduction procedure in the i-th direction*.

Note that reproduction procedure can be iterated. The reproduction procedure in the i-th direction applied to r-tuple $\tilde{\boldsymbol{y}}_i$ returns back the r-tuple $\boldsymbol{y}$. More generally, it is shown in [MV07], that the r-tuples obtained from $\boldsymbol{y}$ by iterating a reproduction procedure are in a bijective correspondence with the elements of the Weyl group.

We call a function $f(x)$ a *quasi-polynomial* if it has the form $x^a p(x)$, where $a \in \mathbb{C}$ and $p(x) \in \mathbb{C}[x]$. Under the assumptions of Theorem 2.2.6, all T_i are quasi-polynomials.

2.3 Solutions of Bethe Ansatz Equation in the Case of $V_\lambda \otimes V_{\omega_1}$ for Type B_r

In Sections 2.3, 2.4 we work with Lie algebra of type B_r.

Let $\mathfrak{g} = \mathfrak{so}(2r+1)$. We have $(\alpha_i, \alpha_i) = 4$, $i = 1, \ldots, r-1$, and $(\alpha_r, \alpha_r) = 2$.

In this section we work with data $\boldsymbol{\Lambda} = (\lambda, \omega_1)$, $\boldsymbol{z} = (0, 1)$. The main result of the section is the explicit formulas for the solutions of the Bethe ansatz equations, see Corollary 2.3.10.

2.3.1 Parameterization of Solutions

One of our goals is to diagonalize the Gaudin Hamiltonians associated to $\mathbf{\Lambda} = (\lambda, \omega_1)$, $\boldsymbol{z} = (0, 1)$. It is sufficient to do that in the spaces of singular vectors of a given weight.

Let $\lambda \in \mathcal{P}^+$. We write the decomposition of finite-dimensional $\mathfrak{g}$-module $V_\lambda \otimes V_{\omega_1}$. We have

$$\begin{aligned}
V_\lambda \otimes V_{\omega_1} =& V_{\lambda+\omega_1} \oplus V_{\lambda+\omega_1-\alpha_1} \oplus \cdots \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_r} \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_{r-1}-2\alpha_r} \\
& \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_{r-2}-2\alpha_{r-1}-2\alpha_r} \oplus \cdots \oplus V_{\lambda+\omega_1-2\alpha_1-\cdots-2\alpha_{r-1}-2\alpha_r} \\
=& V_{(\lambda_1+1,\lambda_2,\dots,\lambda_r)} \oplus V_{(\lambda_1-1,\lambda_2+1,\lambda_3,\dots,\lambda_r)} \oplus V_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k-1,\lambda_{k+1}+1,\dots,\lambda_r)} \\
& \oplus \cdots \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2},\lambda_{r-1}-1,\lambda_r+2)} \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-1},\lambda_r)} \\
& \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2},\lambda_{r-1}+1,\lambda_r-2)} \\
& \oplus V_{(\lambda_1,\dots,\lambda_{r-2}+1,\lambda_{r-1}-1,\lambda_r)} \oplus \cdots \oplus V_{(\lambda_1-1,\lambda_2,\dots,\lambda_r)},
\end{aligned} \tag{2.3.1}$$

with the convention that the summands with non-dominant highest weights are omitted. In addition, if $\lambda_r = 0$, then the summand $V_{\lambda-\alpha_1-\cdots-\alpha_r} = V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-1},\lambda_r)}$ is absent.

Note, in particular, that all multiplicities are 1.

By Theorem 2.2.5, to diagonalize the Gaudin Hamiltonians, it is sufficient to find a solution of the Bethe ansatz equation (2.2.2) associated to $\mathbf{\Lambda}$, $\boldsymbol{z}$ and $\boldsymbol{l}$ corresponding to the summands in the decomposition (2.3.1). We call an r-tuple of integers $\boldsymbol{l}$ *admissible* if $V_{\lambda+\omega_1-\alpha(\boldsymbol{l})} \subset V_\lambda \otimes V_{\omega_1}$.

The admissible r-tuples $\boldsymbol{l}$ have the form

$$\boldsymbol{l} = (\underbrace{1,\dots,1}_{k \text{ ones}}, 0, \dots, 0) \qquad \text{or} \qquad \boldsymbol{l} = (\underbrace{1,\dots,1}_{k \text{ ones}}, 2, \dots, 2), \tag{2.3.2}$$

where $k = 0, \dots, r$. In the first case the length $l = l_1 + \cdots + l_r$ is k and in the second case $2r - k$. It follows that different admissible r-tuples have different length and, therefore, admissible tuples $\boldsymbol{l}$ are parameterized by length $l \in \{0, 1, \dots, 2r\}$. We call a nonnegative integer l *admissible* if it is the length of an admissible r-tuple $\boldsymbol{l}$. More precisely, a nonnegative integer l is admissible if $l = 0$ or if $l \leqslant r$, $\lambda_l > 0$ or if $l = r+1$, $\lambda_r > 1$ or if $r + 1 < l \leqslant 2r$, $\lambda_{2r-l+1} > 0$.

In terms of $\boldsymbol{y} = (y_1, \dots, y_r)$, we have the following cases, corresponding to (2.3.2).

For $l \leqslant r$, the polynomials $y_1, \dots, y_l$ are linear and $y_{l+1}, \dots, y_r$ are all equal to one.

For $l > r$, the polynomials $y_1, \dots, y_{2r-l}$ are linear and $y_{2r-l+1}, \dots, y_r$ are quadratic.

Remark 2.3.1. For $l \leqslant r$ the Bethe ansatz equations for type B_r coincide with the Bethe ansatz equations for type A_r which were solved directly in [MV00]. In what follows, we recover the result for $l < r$, and we refer to [MV00] for the case of $l = r$.

2.3.2 Example of B_2

We illustrate our approach in the case of B_2, $l = 4$. We have $n = 2$, $\Lambda_1 = \lambda \in \mathcal{P}^+$, $\Lambda_2 = \omega_1$, $z_1 = 0, z_2 = 1$. We write $\lambda = (\lambda_1, \lambda_2)$, where $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}$.

Suppose the Bethe ansatz equation has a solution with $l = 4$. Then it is represented by quadratic polynomials $y_1^{(4)}$ and $y_2^{(4)}$. By Theorem 2.2.6, it means that there exist polynomials $\tilde{y}_1, \tilde{y}_2$ such that

$$W(y_1^{(4)}, \tilde{y}_1) = x^{\lambda_1}(x-1)y_2^{(4)}, \qquad W(y_2^{(4)}, \tilde{y}_2) = x^{\lambda_2}\big(y_1^{(4)}\big)^2.$$

Note we have $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geqslant 0}$, but for $\lambda_1 = 0$ the first equation is impossible for degree reasons. Therefore, there are no solutions with $l = 4$ for $\lambda_1 = 0$ which is exactly when the corresponding summand is absent in (2.3.1) and when $l = 4$ is not admissible.

Step 1: There exists a unique monic linear polynomial u_1 such that $-\lambda_1 \tilde{y}_1 = x^{\lambda_1+1}u_1$. Clearly, the only root of u_1 cannot coincide with the roots of $x^{\lambda_1}(x-1)y_2^{(4)}$, therefore the pair $(u_1, y_2^{(4)})$ is generic. It follows from Theorem 2.2.6, that the pair $(u_1, y_2^{(4)})$ solves Bethe ansatz equation with $l = 3$ and λ replaced by $s_1 \cdot \lambda = (-\lambda_1 - 2, 2\lambda_1 + \lambda_2 + 2)$.

In terms of Wronskians, it means that there exist quasi-polynomials $\hat{y}_1$ and $\hat{y}_2$ such that

$$W(u_1, \hat{y}_1) = x^{-\lambda_1-2}(x-1)y_2^{(4)}, \qquad W(y_2^{(4)}, \hat{y}_2) = x^{2\lambda_1+\lambda_2+2}u_1^2.$$

The procedure we just described corresponds to the reproduction in the first direction, we have $s_1(\omega_1 - 2\alpha_1 - 2\alpha_2) = \omega_1 - \alpha_1 - 2\alpha_2$.

Note that $s_2(\omega_1 - 2\alpha_1 - 2\alpha_2) = \omega_1 - 2\alpha_1 - 2\alpha_2$ and the reproduction in the second direction applied to $(y_1^{(4)}, y_2^{(4)})$ does not change $l = 4$. We do not use it.

Step 2: We apply the reproduction in the second direction to the $l = 3$ solution $(u_1, y_2^{(4)})$.

By degree reasons, we have $-(\lambda_2 + 2\lambda_1 + 1)\hat{y}_2 = x^{\lambda_2+2\lambda_1+3} \cdot 1$. Set $u_2 = 1$. Clearly, the pair (u_1, u_2) is generic. By Theorem 2.2.6, the pair (u_1, u_2) solves Bethe ansatz equation with $l = 1$ and $\Lambda_1 = (s_2 s_1) \cdot \lambda = (\lambda_1 + \lambda_2 + 1, -2\lambda_1 - \lambda_2 - 4)$.

It means, we have $s_2(\omega_1 - \alpha_1 - 2\alpha_2) = \omega_1 - \alpha_1$ and there exist quasi-polynomials $\bar{y}_1, \bar{y}_2$ such that

$$W(u_1, \bar{y}_1) = x^{\lambda_1+\lambda_2+1}(x-1)u_2 = x^{\lambda_1+\lambda_2+1}(x-1), \qquad W(u_2, \bar{y}_2) = x^{-2\lambda_1-\lambda_2-4}u_1^2.$$

Note that we also have $\lambda_1 \hat{y}_1 = x^{-\lambda_1-1}y_1^{(4)}$. Therefore, we can recover the initial solution $(y_1^{(4)}, y_2^{(4)})$ from $(u_1, y_2^{(4)})$. In general, if we start with an arbitrary $l = 3$ solution and use the reproduction in the first direction, we obtain a pair of quadratic polynomials. If this pair is generic, then it represents an $l = 4$ solution associated to the data $\boldsymbol{\Lambda} = (\lambda, \omega_1), \boldsymbol{z}, \boldsymbol{l} = (2, 2)$. However, we have no easy argument to show that it is generic. Thus, our procedure gives an inclusion of all $l = 4$ solutions to the $l = 3$ solutions and we need an extra argument to show this inclusion is a bijection.

Step 3: Finally, we apply the reproduction in the first direction to the $l = 1$ solution (u_1, u_2).

We have $-(\lambda_1 + \lambda_2 + 1)\bar{y}_1 = x^{\lambda_1+\lambda_2+2} \cdot 1$. Set $v_1 = 1$. Clearly, the pair $(v_1, u_2) = (1, 1)$ is generic and represents the only solution of the Bethe ansatz equation with $l = 0$ and $\Lambda_1 = (s_1 s_2 s_1) \cdot \lambda = (-\lambda_1 - \lambda_2 - 3, \lambda_2)$. We denote the final weight $(s_1 s_2 s_1) \cdot \lambda$ by $\theta = (\theta_1, \theta_2)$.

It means, we have $s_1(\omega_1 - \alpha_1) = \omega_1$, and there exist quasi-polynomials $\mathring{y}_1, \mathring{y}_2$ such that

$$W(v_1, \mathring{y}_1) = x^{-\lambda_1-\lambda_2-3}(x-1)u_2^2, \qquad W(u_2, \mathring{y}_2) = x^{\lambda_2}v_1 = x^{\lambda_2}.$$

As before, we have $(\lambda_1+\lambda_2+1)\overset{\circ}{y}_1 = x^{-\lambda_1-\lambda_2-2}u_1$, and therefore using reproduction in the first direction to pair (v_1, u_2) we recover the pair (u_1, u_2).

To sum up, we have the inclusions of solutions for $l = 4$ to $l = 3$ to $l = 1$ to $l = 0$ with the Λ_1 varying by the shifted action of the Weyl group. Since for $l = 0$ the solution is unique, it follows that for $l = 1, 3, 4$ the solutions are at most unique. Moreover, if it exists, it can be computed recursively.

We proceed with the direct computation of $y_1^{(4)}, y_2^{(4)}$. From step 3, we have $v_1 = u_2 = 1$. Then we compute

$$u_1 = x - \frac{\lambda_1+\lambda_2+1}{\lambda_1+\lambda_2+2}.$$

From step 2, we get

$$y_2^{(4)} = x^2 - \frac{2(2\lambda_1+\lambda_2+1)(\lambda_1+\lambda_2+1)}{(2\lambda_1+\lambda_2+2)(\lambda_1+\lambda_2+2)}x + \frac{(2\lambda_1+\lambda_2+1)(\lambda_1+\lambda_2+1)^2}{(2\lambda_1+\lambda_2+3)(\lambda_1+\lambda_2+2)^2}.$$

Finally, from Step 1,

$$\begin{aligned} y_1^{(4)} =& x^2 - \frac{(2\lambda_1+\lambda_2+1)(2\lambda_1^2+2\lambda_1\lambda_2+4\lambda_1+\lambda_2+2)}{(\lambda_1+1)(\lambda_1+\lambda_2+2)(2\lambda_1+\lambda_2+2)}x \\ &+ \frac{\lambda_1(\lambda_1+\lambda_2+1)(2\lambda_1+\lambda_2+1)}{(\lambda_1+1)(\lambda_1+\lambda_2+2)(2\lambda_1+\lambda_2+3)}. \end{aligned}$$

From the formula it is easy to check that the pair $(y_1^{(4)}, y_2^{(4)})$ is generic if $\lambda_1 > 0$ and therefore represents a solution of the Bethe ansatz equation associated to $\boldsymbol{\Lambda}, \boldsymbol{z}$ and $l = 4$.

Thus the Bethe ansatz equation associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l} = (2, 2)$ has a unique solution given by the formulas above.

2.3.3 The Recursion Lemmas

Let $l \in \{0, \ldots, r-1, r+2, \ldots, 2r\}$, we establish a reproduction procedure which produces solutions of length $l-1$ from the ones of length l. For $l = r+1$, the reproduction procedure goes from $l = r+1$ to $l = r-1$. We recover the special case $l = r$ directly from [MV00], see Remark 2.3.1. By Theorem 2.2.6 it is sufficient to check that the new r-tuple of polynomial is generic with respect to new data. It is done with the help of following series of lemmas.

For brevity, we denote $x-1$ by y_0 for this section.

The first lemma describes the reproduction in the k-th direction from $l=2r-k+1$ to $l=2r-k$, where $k=1,\dots,r-1$.

Lemma 2.3.2. *Let $k \in \{1,\dots,r-1\}$. Let $\nu=(\nu_1,\dots,\nu_r)$ be an integral weight such that $\nu_k \geqslant 0$. Let $y_1,\dots,y_{k-1}$ be linear polynomials and $y_k,\dots,y_r$ be quadratic polynomials. Suppose the r-tuple of polynomials $\boldsymbol{y}^{(2r-k+1)}=(y_1,\dots,y_r)$ represents a critical point associated to $(\nu,\omega_1),\boldsymbol{z}$ and $l=2r-k+1$. Then there exists a unique monic linear polynomial u_k such that $W(y_k,x^{\nu_k+1}u_k)=-\nu_k x^{\nu_k}y_{k-1}y_{k+1}$. Moreover, $\nu_k>0$ and the r-tuple of polynomials $\boldsymbol{y}^{(2r-k)}=(y_1,\dots,y_{k-1},u_k,y_{k+1},\dots,y_r)$ represents a critical point associated to $(s_k\cdot\nu,\omega_1),\boldsymbol{z}$ and $l=2r-k$.*

Proof. The existence of polynomial $\tilde{y}_k$ such that $W(y_k,\tilde{y}_k)=x^{\nu_k}y_{k-1}y_{k+1}$ implies $\nu_k>0$. Indeed, if $\deg\tilde{y}_k\geqslant 3$, then $\deg W(y_k,\tilde{y}_k)\geqslant 4$; if $\deg\tilde{y}_k\leqslant 2$, then $\deg W(y_k,\tilde{y}_k)\leqslant 2$. Hence $\deg x^{\nu_k}y_{k-1}y_{k+1}\neq 3$, it follows that $\nu_k\neq 0$.

By Theorem 2.2.6, it is enough to show $\boldsymbol{y}^{(2r-k)}$ is generic. If $y_{k-1}y_{k+1}$ is divisible by u_k, then y_k has common root with $y_{k-1}y_{k+1}$ which is impossible since $(y_1,\dots,y_r)$ is generic. Since u_k is linear, it cannot have a multiple root. $\square$

Note that we do not have such a lemma for the reproduction in the k-th direction which goes from $l-1$ to l since unlike u_k the new polynomial is quadratic and we cannot immediately conclude that it has distinct roots. We overcome this problem using the explicit formulas in Section 2.3.5.

The next lemma describes the reproduction in the r-th direction from $l=r+1$ to $l=r-1$.

Lemma 2.3.3. *Let $\nu=(\nu_1,\dots,\nu_r)$ be an integral weight such that $\nu_r\geqslant 0$. Let $y_1,\dots,y_{r-1}$ be linear polynomials and y_r be a quadratic polynomial. Suppose the r-tuple of polynomials $\boldsymbol{y}^{(r+1)}=(y_1,\dots,y_r)$ represents a critical point associated to $(\nu,\omega_1),\boldsymbol{z}$ and $l=r+1$. Then $W(y_r,x^{\nu_r+1})=-(\nu_r-1)x^{\nu_r}y_{r-1}^2$. Moreover, $\nu_r\geqslant 2$ and* *the r-tuple of polynomials $\boldsymbol{y}^{(r-1)} = (y_1, \dots, y_{r-2}, y_{r-1}, 1)$ represents a critical point associated to $(s_r \cdot \nu, \omega_1), \boldsymbol{z}$ and $l = r-1$.* $\square$

Finally, we disuss the reproduction in the k-th direction from $l = k$ to $l = k-1$, where $k = 1, \dots, r-1$.

Lemma 2.3.4. *Let $k \in \{1, \dots, r-1\}$. Let $\nu = (\nu_1, \dots, \nu_r)$ be an integral weight such that $\nu_k \geqslant 0$. Let $y_1, \dots, y_k$ be linear polynomials and $y_{k+1} = \cdots = y_r = 1$. Suppose the r-tuple of polynomials $\boldsymbol{y}^{(k)} = (y_1, \dots, y_r)$ represents a critical point associated to $(\nu, \omega_1), \boldsymbol{z}$ and $l = k$. Then $W(y_k, x^{\nu_k+1}) = -\nu_k x^{\nu_k} y_{k-1} y_{k+1}$. Moreover, $\nu_k > 0$ and the r-tuple of polynomials $\boldsymbol{y}^{(k-1)} = (y_1, \dots, y_{k-1}, 1, 1, \dots, 1)$ represents a critical point associated to $(s_k \cdot \nu, \omega_1), \boldsymbol{z}$ and $l = k-1$.* $\square$

2.3.4 At Most One Solution

In this section, we show that there exists at most one solution of the Bethe ansatz equation (2.2.2).

We start with the explicit formulas for the shifted action of the Weyl group.

Lemma 2.3.5. *Let $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathfrak{h}^*$.*

We have

$$(s_1 \dots s_k) \cdot \lambda = (-\lambda_1 - \cdots - \lambda_k - k - 1, \lambda_1, \dots, \lambda_{k-1}, \lambda_k + \lambda_{k+1} + 1, \lambda_{k+2}, \dots, \lambda_r),$$

where $k = 1, \dots, r-2$,

$$(s_1 \dots s_{r-1}) \cdot \lambda = (-\lambda_1 - \cdots - \lambda_{r-1} - r, \lambda_1, \dots, \lambda_{r-2}, 2\lambda_{r-1} + \lambda_r + 2),$$

$$(s_1 \dots s_r) \cdot \lambda = (-\lambda_1 - \cdots - \lambda_r - r - 1, \lambda_1, \dots, \lambda_{r-2}, 2\lambda_{r-1} + \lambda_r + 2),$$

and

$$\begin{aligned}(s_1 \dots s_r s_{r-1} \dots s_{2r-k}) \cdot \lambda \\ = (- \lambda_1 - \cdots - \lambda_{2r-k-1} - 2\lambda_{2r-k} - \cdots - 2\lambda_{r-1} - \lambda_r - k - 1, \\ \lambda_1, \dots, \lambda_{2r-k-2}, \lambda_{2r-k-1} + \lambda_{2r-k} + 1, \lambda_{2r-k+1}, \dots, \lambda_r),\end{aligned}$$

where $k = r+1, \dots, 2r-1$.

Proof. If $k = 1, \dots, r-2, r$, the action of a simple reflection is given by

$$s_k \cdot \lambda = (\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1} + \lambda_k + 1, -\lambda_k - 2, \lambda_k + \lambda_{k+1} + 1, \lambda_{k+2}, \dots, \lambda_r).$$

In addition,

$$s_{r-1} \cdot \lambda = (\lambda_1, \dots, \lambda_{r-3}, \lambda_{r-2} + \lambda_{r-1} + 1, -\lambda_{r-1} - 2, 2\lambda_{r-1} + \lambda_r + 2).$$

The lemma follows. □

We also prepare the inverse formulas.

Lemma 2.3.6. *Let $\theta = (\theta_1, \dots, \theta_r) \in \mathfrak{h}^*$. We have*

$$(s_k \dots s_1) \cdot \theta = (\theta_2, \dots, \theta_k, -\theta_1 - \dots - \theta_k - k - 1, \theta_1 + \dots + \theta_{k+1} + k, \theta_{k+2}, \dots, \theta_r),$$

where $k = 1, \dots, r-2$,

$$(s_{r-1} \dots s_1) \cdot \theta = (\theta_2, \dots, \theta_{r-1}, -\theta_1 - \dots - \theta_{r-1} - r, 2\theta_1 + \dots + 2\theta_{r-1} + \theta_r + 2r - 2).$$

$$(s_r \dots s_1) \cdot \theta = (\theta_2, \dots, \theta_{r-1}, \theta_1 + \dots + \theta_r + r - 1, -2\theta_1 - \dots - 2\theta_{r-1} - \theta_r - 2r),$$

and

$$\begin{aligned}(s_{2r-k} s_{2r-k+1} \dots s_r s_{r-1} \dots s_1) \cdot \theta \\ = (\theta_2, \dots, \theta_{2r-k-1}, \theta_1 + \dots + \theta_{2r-k-1} + 2\theta_{2r-k} + \dots + 2\theta_{r-1} + \theta_r + k - 1, \\ -\theta_1 - \dots - \theta_{2r-k} - 2\theta_{2r-k+1} - \dots - 2\theta_{r-1} - \theta_r - k, \theta_{2r-k+1}, \dots, \theta_r),\end{aligned}$$

where $k = r+1, \dots, 2r-1$. In particular,

$$(s_1 s_2 \dots s_r s_{r-1} \dots s_1) \cdot \theta = (-\theta_1 - 2\theta_2 - \dots - 2\theta_{r-1} - \theta_r - 2r + 1, \theta_2, \dots, \theta_r).$$

□

Lemma 2.3.7. *Let $\lambda \in \mathcal{P}^+$ and let $\boldsymbol{l}$ be as in (2.3.2). Suppose the Bethe ansatz equation associated to $\boldsymbol{\Lambda} = (\lambda, \omega_1), \boldsymbol{z} = (0, 1), \boldsymbol{l}$, where $\lambda \in \mathcal{P}^+$, has solutions. Then $\boldsymbol{l}$ is admissible. Moreover, if $l \geqslant r+1$, then we can perform the reproduction procedure in the $(2r-l+1)$-th, $(2r-l+2)$-th, ..., $(r-1)$-th, r-th, $(r-1)$-th, ..., 1-st directions successively. If $l < r$, we can perform the reproduction procedure in the l-th, $(l-1)$-th, ..., 1-st directions successively.*

Proof. We use Lemmas 2.3.2-2.3.4. The condition of the lemmas of the form $\nu_k \geqslant 0$ follows from Lemmas 2.3.5 and 2.3.6. $\square$

Corollary 2.3.8. *Let $\lambda \in \mathcal{P}^+$ and $\boldsymbol{l}$ as in* (2.3.2)*. The Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$, has at most one solution. If l is not admissible it has no solutions.*

Proof. If $l \neq r$, then by Lemma 2.3.7, every solution of the Bethe ansatz equations by a series of reproduction procedures produces a solution for $l = 0$. These reproduction procedures are invertible, and for $l = 0$ we clearly have only one solution $(1, \ldots, 1)$. Therefore the conclusion.

For $l = r$ the corollary follows from Theorem 2 in [MV00], see also Remark 2.3.1. $\square$

2.3.5 Explicit Solutions

In this section, we give explicit formulas for the solution of the Bethe ansatz equation corresponding to data $\boldsymbol{\Lambda} = (\lambda, \omega_1)$, $\boldsymbol{z} = (0, 1)$ and l, $\lambda \in \mathcal{P}^+$, $l \in \{0, \ldots, 2r\}$.

We denote by θ the weight obtained from λ after the successive reproduction procedures as in Lemma 2.3.7. Explicitly, if $l \leqslant r - 1$, then $\theta = (s_1 \ldots s_{l-1} s_l) \cdot \lambda$; if $l \geqslant r + 1$, then $\theta = (s_1 \ldots s_{r-1} s_r s_{r-1} \ldots s_{2r-l+1}) \cdot \lambda$. We recover the solution starting from data (θ, ω_1), $z = (0, 1)$ and $l = 0$, where the solution is $(1, \ldots, 1)$ by applying the reproduction procedures in the opposite direction explicitly. In the process we obtain monic polynomials $(y_1^{(l)}, \ldots, y_r^{(l)})$ representing a critical point.

Recall that for $l \leqslant r$, $y_1, \ldots, y_l$ are linear polynomials and $y_{l+1}, \ldots, y_r$ are all equal to one. We use the notation: $y_i^{(l)} = x - c_i^{(l)}$, $i = 1, \ldots, l$.

Recall further that for $l > r$, the polynomials $y_1, \ldots, y_{2r-l}$ are linear and $y_{2r-l+1}, \ldots, y_r$ are quadratic. We use the notation: $y_i^{(l)} = x - c_i^{(l)}$, $i = 1, \ldots, 2r - l$ and $y_i^{(l)} = (x - a_i^{(l)})(x - b_i^{(l)})$, $i = 2r - l + 1, \ldots, r$.

Formulas for $c_i^{(l)}$, $a_i^{(l)}$ and $b_i^{(l)}$ in terms of θ_i, clearly, do not depend on l, in such cases we simply write c_i, a_i and b_i.

Denote $y_0^{(k)} = x - 1$, $c_0 = 1$ and $T_1(x) = x^{\lambda_1}$. Also let

$$A^{(k)}(\theta) = \begin{cases} (s_k \dots s_1) \cdot \theta & \text{if } k \leqslant r, \\ (s_{2r-k} \dots s_{r-1} s_r s_{r-1} \dots s_1) \cdot \theta & \text{if } k \geqslant r+1. \end{cases}$$

Explicitly, $A^{(k)}(\theta)$ are given in Lemma 2.3.6.

Constant Term of y_i in Terms of θ

For brevity, we write simply $A^{(k)}$ for $A^{(k)}(\theta)$. We also use $A_i^{(k)}$ for components of the weight $A^{(k)}$: $A^{(k)} = (A_1^{(k)}, \dots, A_r^{(k)})$.

For $l \leqslant r-1$, we have $\boldsymbol{y}^{(l-1)} = (x - c_1, \dots, x - c_{l-1}, 1, \dots, 1)$. It is easy to check that if l is admissible and λ is dominant then $A_l^{(l-1)} = \theta_1 + \dots + \theta_l + (l-1)$ is a negative integer.

We solve for $\tilde{y}_l^{(l-1)}$,

$$W(y_l^{(l-1)}, \tilde{y}_l^{(l-1)}) = T_l^{(l-1)} y_{l-1}^{(l-1)} y_{l+1}^{(l-1)} = x^{A_l^{(l-1)}}(x - c_{l-1}).$$

In other words

$$-(\tilde{y}_l^{(l-1)})' = x^{A_l^{(l-1)}+1} - c_{l-1} x^{A_l^{(l-1)}}.$$

Choosing the solution which is a quasi-polynomial, we obtain

$$\tilde{y}_l^{(l-1)} = \frac{-x^{A_l^{(l-1)}+1}}{A_l^{(l-1)}+2} \left(x - \frac{A_l^{(l-1)}+2}{A_l^{(l-1)}+1} c_{l-1} \right).$$

Therefore, the reproduction procedure in the l-th direction gives $\boldsymbol{y}^{(l)} = (x-c_1, \dots, x-c_l, 1, \dots, 1)$, where $c_l = \dfrac{A_l^{(l-1)}+2}{A_l^{(l-1)}+1} c_{l-1}$. Substituting the value for $A_l^{(l-1)}$ and using induction, we have

$$c_k = \prod_{j=1}^{k} \frac{\theta_1 + \dots + \theta_j + j + 1}{\theta_1 + \dots + \theta_j + j},$$

for $k = 1, \dots, r-1$.

For $l = r+1$ we have $\boldsymbol{y}^{(r-1)} = (x - c_1, \dots, x - c_{r-1}, 1)$ and $A_r^{(r-1)} = 2\theta_1 + \dots + 2\theta_{r-1} + \theta_r + 2r - 2 \in \mathbb{Z}_{<0}$. We solve for $\tilde{y}_r^{(r-1)}$,

$$W(y_r^{(r-1)}, \tilde{y}_r^{(r-1)}) = T_r^{(r-1)} (y_{r-1}^{(r-1)})^2 = x^{A_r^{(r-1)}}(x - c_{r-1})^2.$$

This implies

$$\tilde{y}_r^{(r-1)} = \frac{-x^{A_r^{(r-1)}+1}}{A_r^{(r-1)}+3}\left(x^2 - \frac{2(A_r^{(r-1)}+3)}{A_r^{(r-1)}+2}c_{r-1}x + \frac{A_r^{(r-1)}+3}{A_r^{(r-1)}+1}c_{r-1}^2\right).$$

Therefore, after performing the reproduction procedure in r-th direction to $\boldsymbol{y}^{(r-1)}$, we obtain the r-tuple $\boldsymbol{y}^{(r+1)} = (x-c_1,\ldots,x-c_{r-1},(x-a_r)(x-b_r))$, where

$$a_rb_r = \frac{A_r^{(r-1)}+3}{A_r^{(r-1)}+1}c_{r-1}^2 = \left(\prod_{j=1}^{r-1}\frac{\theta_1+\cdots+\theta_j+j+1}{\theta_1+\cdots+\theta_j+j}\right)^2\frac{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r+1}{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r-1}.$$

For l such that $r+2 \leqslant l \leqslant 2r$, let $k = 2r-l$, then $\boldsymbol{y}^{(2r-k)} = (x-c_1,\ldots,x-c_k,(x-a_{k+1})(x-b_{k+1}),\ldots,(x-a_r)(x-b_r))$ and $A_k^{(2r-k-1)} = \theta_1+\cdots+\theta_k+2\theta_{k+1}+\cdots+2\theta_{r-1}+\theta_r+2r-k-2 \in \mathbb{Z}_{<0}$.

We have

$$W(y_k^{(2r-k)},\tilde{y}_k^{(2r-k)}) = x^{A_k^{(2r-k-1)}}y_{k-1}^{(2r-k)}y_{k+1}^{(2r-k)},$$

substituting $-(A_k^{(2r-k-1)}+2)\tilde{y}_k^{(2r-k)} = x^{A_k^{(2r-k-1)}+1}(x-a_k)(x-b_k)$, we get

$$\begin{aligned}&(A_k^{(2r-k-1)}+1)(x-c_k)(x-a_k)(x-b_k)+x(x-c_k)(x-a_k)\\&+x(x-c_k)(x-b_k)-x(x-a_k)(x-b_k)\\=&(A_k^{(2r-k-1)}+2)(x-a_{k+1})(x-b_{k+1})(x-c_{k-1}).\end{aligned}\tag{2.3.3}$$

Substituting $x=0$ into (2.3.3), we obtain

$$(A_k^{(2r-k-1)}+1)c_ka_kb_k = (A_k^{(2r-k-1)}+2)c_{k-1}a_{k+1}b_{k+1}.\tag{2.3.4}$$

It results in

$$\begin{aligned}a_kb_k =&c_{r-1}c_{k-1}\frac{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r+1}{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r-1}\\&\times\prod_{i=1}^{r-k}\frac{(\theta_1+\cdots+\theta_{r-1})+(\theta_r+\theta_{r-1}+\cdots+\theta_{r+1-i})+r+i}{(\theta_1+\cdots+\theta_{r-1})+(\theta_r+\theta_{r-1}+\cdots+\theta_{r+1-i})+r+i-1}\\=&\frac{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r+1}{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r-1}\prod_{j=1}^{r-1}\frac{\theta_1+\cdots+\theta_j+j+1}{\theta_1+\cdots+\theta_j+j}\prod_{j=1}^{k-1}\frac{\theta_1+\cdots+\theta_j+j+1}{\theta_1+\cdots+\theta_j+j}\\&\times\prod_{i=1}^{r-k}\frac{(\theta_1+\cdots+\theta_{r-1})+(\theta_r+\theta_{r-1}+\cdots+\theta_{r+1-i})+r+i}{(\theta_1+\cdots+\theta_{r-1})+(\theta_r+\theta_{r-1}+\cdots+\theta_{r+1-i})+r+i-1}.\end{aligned}\tag{2.3.5}$$

The Formula for $a_k + b_k$ in Terms of θ

Comparing the coefficient of x^2 in (2.3.3), we obtain

$$(A_k^{(2r-k-1)} + 1)(a_k + b_k + c_k) + 2c_k = (A_k^{(2r-k-1)} + 2)(a_{k+1} + b_{k+1} + c_{k-1}). \quad (2.3.6)$$

Comparing the coefficient of x in (2.3.3), we obtain

$$\begin{aligned} &(A_k^{(2r-k-1)} + 1)(c_k(a_k + b_k) + a_k b_k) + c_k(a_k + b_k) - a_k b_k \\ =&(A_k^{(2r-k-1)} + 2)(c_{k-1}(a_{k+1} + b_{k+1}) + a_{k+1}b_{k+1}). \end{aligned} \quad (2.3.7)$$

Solving (2.3.6) and (2.3.7) for $a_k + b_k$, one has

$$a_k + b_k = \frac{(A_k^{(2r-k-1)} + 2)(a_{k+1}b_{k+1} - c_{k-1}^2) + (A_k^{(2r-k-1)} + 3)c_{k-1}c_k - A_k^{(2r-k-1)} a_k b_k}{(A_k^{(2r-k-1)} + 2)c_k - (A_k^{(2r-k-1)} + 1)c_{k-1}}.$$

This gives the explicit formulas,

$$\begin{aligned} a_k + b_k =& \frac{2\theta_1 + \cdots + 2\theta_{r-1} + 2r + 1}{2\theta_1 + \cdots + 2\theta_{r-1} + 2r} \prod_{j=1}^{k-1} \frac{\theta_1 + \cdots + \theta_j + j + 1}{\theta_1 + \cdots + \theta_j + j} \\ &\times \left(1 + \prod_{j=k}^{r-1} \frac{\theta_1 + \cdots + \theta_j + j + 1}{\theta_1 + \cdots + \theta_j + j} \times \right. \\ &\left. \prod_{j=k}^{r-1} \frac{\theta_1 + \cdots + \theta_j + 2\theta_{j+1} + \cdots + 2\theta_{r-1} + \theta_r + 2r - k}{\theta_1 + \cdots + \theta_j + 2\theta_{j+1} + \cdots + 2\theta_{r-1} + \theta_r + 2r - k - 1} \right). \end{aligned}$$

These solutions indeed satisfy (2.3.3) for each k. This can be checked by a direct computation.

Final Formulas

We use Lemma 2.3.5 to express θ_i by λ_j. Here are the final formulas.

If $l < r$, then

$$c_j^{(l)} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_l + l - i}{\lambda_i + \cdots + \lambda_l + l - i + 1}, \quad j = 1, \ldots, l. \quad (2.3.8)$$

We also borrow from [MV00] the $l = r$ result.

$$c_j^{(r)} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_{r-1} + \lambda_r/2 + r - i}{\lambda_i + \cdots + \lambda_{r-1} + \lambda_r/2 + r - i + 1}, \quad j = 1, \ldots, r. \tag{2.3.9}$$

If $l \geqslant r+1$, then

$$c_k^{(l)} = \prod_{j=1}^{k} \frac{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j}, \tag{2.3.10}$$

for $k = 1, \ldots, 2r-l$. Finally, for $2r-l+1 \leqslant k \leqslant r$, we have

$$\begin{aligned} a_k^{(l)} b_k^{(l)} = &\left(\prod_{j=1}^{2r-l} \frac{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j} \right)^2 \\ &\times \prod_{j=2r-l+1}^{r-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1} \\ &\times \prod_{j=2r-l+1}^{k-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1} \\ &\times \prod_{i=1}^{r-k} \frac{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i - 1}{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i} \\ &\times \frac{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 3}{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 1}, \end{aligned} \tag{2.3.11}$$

and

$$\begin{aligned} a_k^{(l)} + b_k^{(l)} = &\frac{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 3}{2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2l - 2r - 2} \\ &\times \prod_{j=1}^{2r-l} \frac{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1}{\lambda_j + \cdots + \lambda_{2r-l} + 2\lambda_{2r-l+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j} \\ &\times \left(\prod_{j=2r-l+1}^{k-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1} \right. \\ &+ \prod_{j=2r-l+1}^{r-1} \frac{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 2}{\lambda_{2r-l+1} + \cdots + \lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{r-1} + \lambda_r + l - j - 1} \\ &\left. \times \prod_{i=1}^{r-k} \frac{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i - 1}{\lambda_{2r-l+1} + \cdots + \lambda_{r-i} + l - r - i} \right). \end{aligned} \tag{2.3.12}$$

The Solutions Are Generic

In this section we show the solutions are generic.

Theorem 2.3.9. *Suppose $\lambda \in \mathcal{P}^+$ and $\boldsymbol{l}$ is admissible, then $\boldsymbol{y}^{(l)}$ in Section 2.3.5 represents a critical point associated to $\boldsymbol{\Lambda} = (\lambda, \omega_1)$, $\boldsymbol{z} = (0,1)$, and $\boldsymbol{l}$.*

Proof. It is sufficient to show that $\boldsymbol{y}$ is generic with respect to $\boldsymbol{\Lambda}, \boldsymbol{z}$.

Let us first consider **G2**. For $\lambda \in \mathcal{P}^+$, **G2** is equivalent to $y_1^{(l)}(1) \neq 0$ and $y_i^{(l)}(0) \neq 0$ if $\lambda_i \neq 0$.

If $l \leqslant r-1$, then the admissibility of $\boldsymbol{l}$ implies $\lambda_l > 0$. To prove **G2**, it suffices to show $c_j^{(l)} \neq 0$ if $\lambda_l \neq 0$ and $c_1^{(l)} \neq 1$, see (2.3.8). Note that if $\lambda_l > 0$, then

$$0 < \frac{\lambda_i + \cdots + \lambda_l + l - i}{\lambda_i + \cdots + \lambda_l + l - i + 1} < 1$$

for all $i \in \{1, \ldots, l\}$, therefore all $c_j^{(l)} \in (0,1)$.

If $l = r$, this is similar to the previous situation.

If $l = r+1$, the admissibility of $\boldsymbol{l}$ implies $\lambda_r \geqslant 2$. **G2** is obviously true.

If $l \geqslant r+2$, the admissibility of $\boldsymbol{l}$ implies $\lambda_{2r-l+1} > 0$. One has $y_k^{(l)}(0) \neq 0$ since we have $a_k^{(l)} b_k^{(l)} \neq 0$. As for $y_1^{(l)}(1) \neq 0$ in the case $l = 2r$, we delay the proof until after the case **G1**.

Now, we consider **G1**. Suppose $a_k^{(l)} = b_k^{(l)}$ for some $2r - l + 1 \leqslant k \leqslant r$. Observe that

$$W(y_k^{(l)}, \tilde{y}_k^{(l)}) = T_k^{(l)} y_{k-1}^{(l)} y_{k+1}^{(l)}.$$

By **G2**, $y_k^{(l)}$ and $T_k^{(l)}$ have no common roots. In addition if $l = 2r$ and $k = 1$, we have $y_1^{(l)}(1) = 0$, then $a_1^{(l)} b_1^{(l)} = 1$, while as above we have $a_1^{(l)} b_1^{(l)} \in (0,1)$. It follows that we must have $a_k^{(l)} = a_{k+1}^{(l)}$ or $a_k^{(l)} = a_{k-1}^{(l)}$ ($a_k^{(l)} = c_{k-1}^{(l)}$, if $k = 2r - l + 1$).

We work in terms of θ. We have $a_k = b_k = a_{k+1}$ or $a_k = b_k = a_{k-1}$ or $a_{2r-l+1} = c_{2r-l}$. If $a_k = b_k = a_{k+1}$, then substituting $x = c_k$ into (2.3.3), we get

$$-c_k(c_k - a_{k+1}) = (c_k - b_{k+1})(c_k - c_{k-1})(A_k^{(2r-k-1)} + 2). \tag{2.3.13}$$

Solving (2.3.4) and (2.3.13) for $a_{k+1} = a_k = b_k$ and b_{k+1} in terms of c_k, c_{k-1} and $A_k^{(2r-k-1)}$, we obtain

$$a_{k+1}b_{k+1} = (A_k^{(2r-k-1)}+2)(A_k^{(2r-k-1)}+1)c_{k-1}c_k\left(\frac{(A_k^{(2r-k-1)}+3)c_k - (A_k^{(2r-k-1)}+2)c_{k-1}}{(A_k^{(2r-k-1)}+1)c_k - A_k^{(2r-k-1)}c_{k-1}}\right)^2.$$

Comparing it with (2.3.5) and canceling common factors, we obtain

$$\begin{aligned}&(A_k^{(2r-k-1)}+2)(A_k^{(2r-k-1)}+1)\frac{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r+1}{2\theta_1+\cdots+2\theta_{r-1}+\theta_r+2r-1}\\ =&\prod_{i=k}^{r}\frac{\theta_1+\cdots+\theta_i+i+1}{\theta_1+\cdots+\theta_i+i}\prod_{i=k+2}^{r-1}\frac{\theta_1+\cdots+\theta_{i-1}+2\theta_i+\cdots+2\theta_{r-1}+\theta_r+2r-i+1}{\theta_1+\cdots+\theta_{i-1}+2\theta_i+\cdots+2\theta_{r-1}+\theta_r+2r-i}.\end{aligned}$$

Substituting θ_i in terms of λ_j, we have

$$\begin{aligned}&(\lambda_{2r-l+1}+\cdots+\lambda_k+k+l-2r-1)(\lambda_{2r-l+1}+\cdots+\lambda_k+k+l-2r)\\ &\times\frac{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-3}{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-1}\\ =&\prod_{j=k}^{r-1}\frac{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-2}{\lambda_{2r-l+1}+\cdots+\lambda_j+2\lambda_{j+1}+\cdots+2\lambda_{r-1}+\lambda_r+l-j-1}\\ &\times\prod_{i=1}^{r-k-1}\frac{\lambda_{2r-l+1}+\cdots+\lambda_{r-i}+l-r-i-1}{\lambda_{2r-l+1}+\cdots+\lambda_{r-i}+l-r-i}.\end{aligned}\tag{2.3.14}$$

By our assumption, we have $\lambda_{2r-l+1} \geqslant 1$, $k \geqslant 2r-l+1$ and $l \geqslant r+2$. It is easily seen that

$$\begin{aligned}&(\lambda_{2r-l+1}+\cdots+\lambda_k+k+l-2r-1)(\lambda_{2r-l+1}+\cdots+\lambda_k+k+l-2r)\\ &\times\frac{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-3}{2\lambda_{2r-l+1}+\cdots+2\lambda_{r-1}+\lambda_r+2l-2r-1} \geqslant 1\times 2\times\frac{3}{5} > 1.\end{aligned}$$

Therefore (2.3.14) is impossible. Similarly, we can exclude $a_k^{(l)} = a_{k-1}^{(l)}$. As for $a_{2r-l+1}^{(l)} = c_{2r-l}^{(l)}$, by (2.3.4), it is impossible since each fractional factor is strictly less than 1.

Finally, we prove **G3**. The nontrivial cases are $a_k^{(l)} = a_{k+1}^{(l)}$ and $a_{2r-l+1}^{(l)} = c_{2r-l}^{(l)}$ for $l \geqslant r+1$, where $k \geqslant 2r-l+1$.

If $a_k = a_{k+1}$, then by (2.3.3) we have that $x - a_k$ divides $x(x-c_k)(x-b_k)$. As we already proved $a_k \neq b_k$ and $a_k \neq 0$, it follows that $a_k = c_k$. This again implies that

$(x-a_k)^2$ divides $(x-a_{k+1})(x-b_{k+1})(x-c_{k-1})$ as $A_k^{(2r-k+1)}+2 \neq 0$. If $b_{k+1}=a_k=a_{k+1}$, then we are done. If $a_k=c_{k-1}$, then $c_{k-1}=c_k$. It is impossible by the argument used in **G2**.

If $a_k = c_{k-1}$, then by (2.3.3) one has $x-a_k$ divides $x(x-c_k)(x-b_k)$. Since l is admissible, $a_k^{(l)} \neq 0$. Then $a_k \neq b_k$ implies $c_{k-1}=c_k$. It is also a contradiction.

In particular, this shows that $y_1^{(l)}$ and $y_0^{(l)}$ have no common roots, i.e., $y_1^{(l)}(1) \neq 0$. □

Corollary 2.3.10. *Suppose $\lambda \in \mathcal{P}^+$. Then the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$, where $\boldsymbol{l}$ is admissible, has exactly one solution. Explicitly, for $l \leqslant r-1$, the corresponding r-tuple $\boldsymbol{y}^{(l)}$ which represents the solution is described by* (2.3.8)*, for $l=r$ by* (2.3.9)*, for $2r \geqslant l \geqslant r+1$ by* (2.3.10)*,* (2.3.11) *and* (2.3.12)*.* □

2.3.6 Associated Differential Operators for Type B

Let $\boldsymbol{y}$ be an r-tuple of quasi-polynomials. Following [MV04], we introduce a linear differential operator $D(\boldsymbol{y})$ of order $2r$ by the formula

$$\begin{aligned} D_\lambda(\boldsymbol{y}) = &\left(\partial - \ln'\left(\frac{T_1^2\dots T_{r-1}^2 T_r}{y_1}\right)\right)\left(\partial - \ln'\left(\frac{y_1T_1^2\dots T_{r-1}^2T_r}{y_2T_1}\right)\right)\\ &\times\left(\partial - \ln'\left(\frac{y_2T_1^2\dots T_{r-1}^2T_r}{y_3T_1T_2}\right)\right)\dots\left(\partial - \ln'\left(\frac{y_{r-1}T_1\dots T_{r-1}T_r}{y_r}\right)\right)\\ &\times\left(\partial - \ln'\left(\frac{y_rT_1\dots T_{r-1}}{y_{r-1}}\right)\right)\left(\partial - \ln'\left(\frac{y_{r-1}T_1\dots T_{r-2}}{y_{r-2}}\right)\right)\dots\\ &\times(\partial - \ln'(y_1)), \end{aligned}$$

where T_i, $i=1,\dots,r$, are given by (2.2.4).

If $\boldsymbol{y}$ is an r-tuple of polynomials representing a critical point associated to integral dominant weights $\Lambda_1,\dots,\Lambda_n$ and points $z_1,\dots,z_n$ of type B_r, then by [MV04], the kernel of $D_\lambda(\boldsymbol{y})$ is a self-dual space of polynomials. By [MM17] the coefficients of $D_\lambda(\boldsymbol{y})$ are eigenvalues of higher Gaudin Hamiltonians acting on the Bethe vector related to $\boldsymbol{y}$.

For admissible l and $\lambda \in \mathfrak{h}^*$, define $a_\lambda^l(1),\dots,a_\lambda^l(r)$ as the following.

For $l = 0, \dots, r-1$, $i = 1, \dots, l$, set $a^l_\lambda(i) = \lambda_i + \cdots + \lambda_l + l + 1 - i$. For $l = 0, \dots, r-1$, $i = l+1, \dots, r$, set $a^l_\lambda(i) = 0$.

For $l = r+1, \dots, 2r$, set $k = 2r - l$. Then for $i = 1, \dots, k$, set

$$a^l_\lambda(i) = \lambda_i + \cdots + \lambda_k + 2\lambda_{k+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2r - k - i$$

and for $i = k+1, \dots, r$, set $a^l_\lambda(i) = 2\lambda_{k+1} + \cdots + 2\lambda_{r-1} + \lambda_r + 2r - 2k - 1$.

Proposition 2.3.11. *Let the r-tuple $\boldsymbol{y}$ represent the solution of the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}$, $\boldsymbol{z}$ and admissible $\boldsymbol{l}$, where $\lambda \in \mathcal{P}^+$ and $l \neq r$. Then $D_\lambda(\boldsymbol{y}) = D_\lambda(x^{a^l_\lambda(1)}, \dots, x^{a^l_\lambda(r)})$.*

Proof. The $(2r-1)$-tuple $(y_1, \dots, y_{r-1}, y_r, y_{r-1}, \dots, y_1)$ represents a critical point of type A_{2r-1}. Then the reproduction procedure in direction i of type B_r corresponds to a composition of reproduction procedures of type A_{2r-1} in directions i and $2r - i$ for $i = 1, \dots, r-1$, and to reproduction procedure of type A_{2r-1} in direction r for $i = r$, see [MV04], [MV07]. Proposition follows from Lemma 4.2 in [MV07]. □

2.4 Completeness of Bethe Ansatz for Type B

In this section we continue to study the case of $\mathfrak{g} = \mathfrak{so}(2r+1)$. The main result of the section is Theorem 2.4.5.

2.4.1 Completeness of Bethe Ansatz for $V_\lambda \otimes V_{\omega_1}$

Let $\lambda \in \mathcal{P}^+$. Consider the tensor product of a finite-dimensional irreducible module with highest weight λ, V_λ, and the vector representation V_{ω_1}.

Recall that the value of the weight function $\omega(z_1, z_2, \boldsymbol{t})$ at a solution of the Bethe ansatz equations (2.2.2) is called the Bethe vector. We have the following result, which is usually referred to as completeness of Bethe ansatz.

Theorem 2.4.1. *The set of Bethe vectors $\omega(z_1, z_2, \boldsymbol{t})$, where $\boldsymbol{t}$ runs over the solutions to the Bethe ansatz equations* (2.2.2) *with admissible length l, forms a basis of* Sing $(V_\lambda \otimes V_{\omega_1})$.

Proof. All multiplicities in the decomposition of $V_\lambda \otimes V_{\omega_1}$ are 1. By Corollary 2.3.10 for each admissible length l we have a solution of the Bethe ansatz equation. The theorem follows from Theorems 2.2.4 and 2.2.5. $\square$

2.4.2 Simple Spectrum of Gaudin Hamiltonians for $V_\lambda \otimes V_{\omega_1}$

We have the following standard fact.

Lemma 2.4.2. *Let $\mu, \nu \in \mathcal{P}^+$. If $\mu > \nu$ then $(\mu+\rho, \mu+\rho) > (\nu+\rho, \nu+\rho)$.*

Proof. The lemma follows from the proof of Lemma 13.2B in [Hum78]. $\square$

Proposition 2.4.3. *Let $\omega, \omega' \in V_\lambda \otimes V_{\omega_1}$ be Bethe vectors corresponding to solutions to the Bethe ansatz equations of two different lengths. Then ω, ω' are eigenvectors of the Gaudin Hamiltonian $\mathcal{H} := \mathcal{H}_1 = -\mathcal{H}_2$ with distinct eigenvalues.*

Proof. Recall the relation

$$\Omega^{(1,2)} = \frac{1}{2}\left(\Delta\Omega_0 - 1 \otimes \Omega_0 - \Omega_0 \otimes 1\right).$$

Since Ω_0 acts as a constant in any irreducible module, $1 \otimes \Omega_0 + \Omega_0 \otimes 1$ acts as a constant on $V_\lambda \otimes V_{\omega_1}$. It remains to consider the spectrum of the diagonal action of $\Delta\Omega_0$. By Theorem 2.2.5, ω and ω' are highest weight vectors of two non-isomorphic irreducible submodules of $V_\lambda \otimes V_{\omega_1}$. By Lemmas 2.2.1 and 2.4.2 the values of $\Delta\Omega_0$ on ω and ω' are different. $\square$

2.4.3 The Generic Case

We use the following well-known lemma from algebraic geometry.

Lemma 2.4.4. *Let $n \in \mathbb{Z}_{\geqslant 1}$ and suppose $f_k^{(\epsilon)}(x_1, \ldots, x_l) = 0$, $k = 1, \ldots, n$, is a system of n algebraic equations for l complex variables $x_1, \ldots, x_l$, depending on a complex parameter ϵ algebraically. Let $(x_1^{(0)}, \ldots, x_l^{(0)})$ be an isolated solution with*

$\epsilon = 0$. *Then for sufficiently small* ϵ, *there exists an isolated solution* $(x_1^{(\epsilon)}, \ldots, x_l^{(\epsilon)})$, *depending algebraically on* ϵ, *such that*

$$x_k^{(\epsilon)} = x_k^{(0)} + o(1).$$

$\square$

Our main result is the following theorem.

Theorem 2.4.5. *Let* $\mathfrak{g} = \mathfrak{so}(2r+1)$, $\lambda \in \mathcal{P}^+$ *and* $N \in \mathbb{Z}_{\geqslant 0}$. *For a generic* $(N+1)$*-tuple of distinct complex numbers* $\boldsymbol{z} = (z_0, z_1, \ldots, z_N)$, *the Gaudin Hamiltonians* $(\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_N)$ *acting in* $\operatorname{Sing}\left(V_\lambda \otimes V_{\omega_1}^{\otimes N}\right)$ *are diagonalizable and have simple joint spectrum. Moreover, for generic* $\boldsymbol{z}$ *there exists a set of solutions* $\{\boldsymbol{t}_i,\ i \in I\}$ *of the Bethe ansatz equation* (2.2.2) *such that the corresponding Bethe vectors* $\{\omega(\boldsymbol{z}, \boldsymbol{t}_i),\ i \in I\}$ *form a basis of* $\operatorname{Sing}\left(V_\lambda \otimes V_{\omega_1}^{\otimes N}\right)$.

Proof. Our proof follows that of Theorem 5.2 of [MVY15], see also of Section 4 in [MV05b].

Pick distinct non-zero complex numbers $\tilde{z}_1, \ldots, \tilde{z}_N$. We use Theorem 2.4.1 to define a basis in the space of singular vectors $\operatorname{Sing}(V_\lambda \otimes V_{\omega_1}^{\otimes N})$ as follows.

We call a $(k+1)$-tuple of weights $\mu_0, \mu_1, \ldots, \mu_k \in \mathcal{P}^+$ *admissible* if $\mu_0 = \lambda$ and for $i = 1, \ldots, k$, we have a submodule $V_{\mu_i} \subset V_{\mu_{i-1}} \otimes V_{\omega_1}$, see (2.3.1).

For an admissible tuple of weights, we define a singular vector $v_{\mu_0,\ldots,\mu_k} \in V_\lambda \otimes V_{\omega_1}^{\otimes k}$ of weight μ_k using induction on k as follows. Let $v_{\mu_0} = v_\lambda$ be the highest weight vector for module V_λ. Let k be such that $1 \leqslant k \leqslant N$. Suppose we have the singular vector $v_{\mu_0,\ldots,\mu_{k-1}} \in V_\lambda \otimes V_{\omega_1}^{\otimes k-1}$. It generates a submodule $V_{\mu_0,\ldots,\mu_{k-1}} \subset V_\lambda \otimes V_{\omega_1}^{\otimes k-1}$ of highest weight μ_{k-1}.

Let $\bar{\boldsymbol{t}}_k = (\bar{t}_{k,j}^{(b)})$, where $b = 1, \ldots, r$ and $j = 1, \ldots, l_{k,b}$, be the solution of the Bethe ansatz equation associated to $V_{\mu_{k-1}} \otimes V_{\omega_1}$, $\boldsymbol{z} = (0, \tilde{z}_k)$ and $\boldsymbol{l}_k = (l_{k,1}, \ldots, l_{k,r})$ such that $\mu_{k-1} + \omega_1 - \alpha(\boldsymbol{l}_k) = \mu_k$. Note that $\bar{\boldsymbol{t}}_k$ depends on μ_{k-1} and μ_k, even though we do not indicate this dependence explicitly. Note also that in all cases $l_{k,b} \in \{0, 1, 2\}$.

Then, define $v_{\mu_0,\ldots,\mu_k}$ to be the Bethe vector

$$v_{\mu_0,\ldots,\mu_k} = \omega(0, \tilde{z}_k, \bar{\boldsymbol{t}}_k) \in V_{\mu_0,\ldots,\mu_{k-1}} \otimes V_{\omega_1} \subset V_\lambda \otimes V_{\omega_1}^{\otimes k}.$$

We denote by $V_{\mu_0,\dots,\mu_k}$ the submodule of $V_\lambda \otimes V_{\omega_1}^{\otimes k}$ generated by $v_{\mu_0,\dots,\mu_k}$.

The vectors $v_{\mu_0,\dots,\mu_N} \in V_\lambda \otimes V_{\omega_1}^{\otimes N}$ are called the *iterated singular vectors*. To each iterated singular vector $v_{\mu_0,\dots,\mu_N}$ we have an associated collection $\bar{\boldsymbol{t}} = (\bar{\boldsymbol{t}}_1, \dots, \bar{\boldsymbol{t}}_N)$ consisting of all the Bethe roots used in its construction.

Clearly, the iterated singular vectors corresponding to all admissible $(N+1)$-tuples of weights form a basis in $\operatorname{Sing}(V_\lambda \otimes V_{\omega_1}^{\otimes N})$, so we have

$$V_\lambda \otimes V_{\omega_1}^{\otimes N} = \bigoplus_{\mu_0,\dots,\mu_N} V_{\mu_0,\mu_1,\dots,\mu_N},$$

where the sum is over all admissible $(N+1)$-tuples of weights.

To prove the theorem, we show that in some region of parameters $\boldsymbol{z}$ for any admissible $(N+1)$-tuple of weights $\mu_0, \dots, \mu_N$, there exists a Bethe vector $\omega_{\mu_1,\dots,\mu_N}$ which tends to $v_{\mu_1,\dots,\mu_N}$ when approaching a certain point (independent on μ_i) on the boundary of the region.

To construct the Bethe vector $\omega_{\mu_1,\dots,\mu_N}$ associated to $v_{\mu_1,\dots,\mu_N}$, we need to find a solution to the Bethe equations associated to $V_\lambda \otimes V_{\omega_1}^{\otimes N}$ with Bethe roots, $\boldsymbol{t} = (t_j^{(b)})$, where $b = 1, \dots, r$ and $j = 1, \dots, \sum_{k=1}^N l_{k,b}$.

We do it for $\boldsymbol{z}$ of the form

$$z_0 = z, \quad \text{and} \quad z_k = z + \varepsilon^{N+1-k} \tilde{z}_k, \quad k = 1, \dots, N, \tag{2.4.1}$$

for sufficiently small $\varepsilon \in \mathbb{C}^\times$. Here $z \in \mathbb{C}$ is an arbitrary fixed number and $\tilde{z}_k$ are as above.

Then, similarly to $\bar{\boldsymbol{t}}$ we write $\boldsymbol{t} = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_N)$ where $\boldsymbol{t}_k = (t_{k,j}^{(b)})$, $b = 1, \dots, r$ and $j = 1, \dots, l_{k,b}$, is constructed in the form

$$t_{k,j}^{(b)} = z + \varepsilon^{N+1-k} \tilde{t}_{k,j}^{(b)}, \quad k = 1, \dots, N, \ j = 1, \dots, l_{k,b}, \ b = 1, \dots, r. \tag{2.4.2}$$

The variables $t_{k,j}^{(b)}$ satisfy the system of Bethe ansatz equations:

$$-\frac{(\lambda, \alpha_b)}{t_{k,j}^{(b)} - z_0} + \sum_{s=1}^N \left(\frac{-2\delta_{b,1}}{t_{k,j}^{(b)} - z_s} + \sum_{\substack{q=1 \\ (s,q)\neq(k,j)}}^{l_{s,b}} \frac{(\alpha_b, \alpha_b)}{t_{k,j}^{(b)} - t_{s,q}^{(b)}} \right.$$
$$\left. + \sum_{q=1}^{l_{s,b+1}} \frac{(\alpha_b, \alpha_{b+1})}{t_{k,j}^{(b)} - t_{s,q}^{(b+1)}} + \sum_{q=1}^{l_{s,b-1}} \frac{(\alpha_b, \alpha_{b-1})}{t_{k,j}^{(b)} - t_{s,q}^{(b-1)}} \right) = 0 \tag{2.4.3}$$

for $b = 1, \dots, r$, $k = 1, \dots, N$, $j = 1, \dots, l_{k,b}$. Here we agree that $l_{s,0} = l_{s,N+1} = 0$ for all s.

Consider the leading asymptotic behavior of the Bethe ansatz equations as $\varepsilon \to 0$. We claim that in the leading order, the Bethe ansatz equations for $\boldsymbol{t}$ reduce to the Bethe ansatz equations obeyed by the variables $\bar{\boldsymbol{t}}$.

Consider for example the leading order of the Bethe equation for $t^{(1)}_{k,j}$. Note that

$$\frac{(\lambda, \alpha_1)}{t^{(1)}_{k,j} - z_0} + \sum_{s=1}^{N} \frac{2}{t^{(1)}_{k,j} - z_s} = \left(\frac{(\lambda, \alpha_1)}{\tilde{t}^{(1)}_{k,j}} + \frac{2(k-1)}{\tilde{t}^{(1)}_{k,j}} + \frac{2}{\tilde{t}^{(1)}_{k,j} - \tilde{z}_k} + \mathcal{O}(\varepsilon) \right) \varepsilon^{-N-1+k},$$

$$\sum_{s=1}^{N} \sum_{\substack{q=1 \\ (s,q)\neq(k,j)}}^{l_{s,1}} \frac{(\alpha_1, \alpha_1)}{t^{(1)}_{k,j} - t^{(1)}_{s,q}} = \left(\sum_{\substack{q=1 \\ q\neq j}}^{l_{k,1}} \frac{(\alpha_1, \alpha_1)}{\tilde{t}^{(1)}_{k,j} - \tilde{t}^{(1)}_{k,q}} + \sum_{s=1}^{k-1} \sum_{q=1}^{l_{s,1}} \frac{(\alpha_1, \alpha_1)}{\tilde{t}^{(1)}_{k,j}} + \mathcal{O}(\varepsilon) \right) \varepsilon^{-N-1+k},$$

and similarly

$$\sum_{s=1}^{N} \sum_{q=1}^{l_{s,2}} \frac{(\alpha_1, \alpha_2)}{t^{(1)}_{k,j} - t^{(2)}_{s,q}} = \left(\sum_{q=1}^{l_{k,2}} \frac{(\alpha_1, \alpha_2)}{\tilde{t}^{(1)}_{k,j} - \tilde{t}^{(2)}_{k,q}} + \sum_{s=1}^{k-1} \sum_{q=1}^{l_{s,2}} \frac{(\alpha_1, \alpha_2)}{\tilde{t}^{(1)}_{k,j}} + \mathcal{O}(\varepsilon) \right) \varepsilon^{-N-1+k}.$$

Then by definition of the numbers $l_{s,b}$, we have

$$\mu_{k-1} = \lambda + (k-1)\omega_1 - \sum_{b=1}^{r} \sum_{s=1}^{k-1} \sum_{q=1}^{l_{s,b}} \alpha_b$$

and, in particular,

$$(\mu_{k-1}, \alpha_1) = (\lambda, \alpha_1) + 2(k-1) - \sum_{s=1}^{k-1} \left(\sum_{q=1}^{l_{s,1}} (\alpha_1, \alpha_1) - \sum_{q=1}^{l_{s,2}} (\alpha_1, \alpha_2) \right).$$

Therefore

$$-\frac{(\mu_{k-1}, \alpha_1)}{\tilde{t}^{(1)}_{k,j}} - \frac{2}{\tilde{t}^{(1)}_{k,j} - \tilde{z}_k} + \sum_{\substack{q=1 \\ q\neq j}}^{l_{k,1}} \frac{(\alpha_1, \alpha_1)}{\tilde{t}^{(1)}_{k,j} - \tilde{t}^{(1)}_{k,q}} + \sum_{q=1}^{l_{k,2}} \frac{(\alpha_1, \alpha_2)}{\tilde{t}^{(1)}_{k,j} - \tilde{t}^{(2)}_{k,q}} = \mathcal{O}(\varepsilon).$$

At leading order this is indeed the Bethe equation for $\tilde{t}^{(1)}_{k,j}$ from the set of Bethe equations for the tensor product $V_{\mu_{k-1}} \otimes V_{\omega_1}$, with the tensor factors assigned to the points 0 and $\tilde{z}_k$, respectively. The other equations work similarly.

By Lemma 2.4.4 it follows that for sufficiently small ε there exists a solution to the Bethe equations (2.4.3) of the form $\tilde{t}^{(a)}_{j,k} = \bar{t}^{(a)}_{j,k} + o(1)$.

Now we claim that the Bethe vector $\omega_{\mu_1,\dots,\mu_N} = \omega(\boldsymbol{z}, \boldsymbol{t})$ associated to $\boldsymbol{t}$ has leading asymptotic behavior

$$\omega_{\mu_1,\dots,\mu_N} = \varepsilon^K (v_{\mu_1,\dots,\mu_N} + o(1)), \tag{2.4.4}$$

as $\varepsilon \to 0$, for some K. Consider the definition (2.2.3) of $\omega(\boldsymbol{z}, \boldsymbol{t})$. We write $\omega_{\mu_1,\dots,\mu_N} = w_1 + w_2$ where w_1 contains only those summands in which every factor in the denominator is of the form

$$t^{(a)}_{k,j} - t^{(b)}_{k,q} \quad \text{or} \quad t^{(a)}_{k,j} - z_k.$$

The term w_2 contains terms where at least one factor is of the form $t^{(a)}_{k,j} - t^{(b)}_{s,q}$ or $t^{(a)}_{k,j} - z_s$, $s \neq k$. After substitution using (2.4.1) and (2.4.2), one finds that

$$w_1 = \left(\prod_{k=1}^{N} \prod_{j=1}^{r} \left(\varepsilon^{-N-1+k}\right)^{l_{k,j}} \right) v_{\mu_1,\dots,\mu_N},$$

and that w_2 is subleading to w_1, which establishes our claim.

Consider two distinct Bethe vectors $\omega_{\mu_1,\dots,\mu_N}$ and $\omega_{\mu'_1,\dots,\mu'_N}$ constructed as above. By Theorem 2.2.5 both are simultaneous eigenvectors of the quadratic Gaudin Hamiltonians $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_N$. Let k, be the largest possible number in $\{1, \dots, N\}$ such that $\mu_i = \mu'_i$ for all $i = 1, \dots, k-1$. Consider the Hamiltonian $\mathcal{H}_k$. When the z_i are chosen as in (2.4.1) then one finds

$$\mathcal{H}_k = \varepsilon^{-N-1+k} \left(\sum_{j=0}^{k-1} \frac{\Omega^{(k,j)}}{\tilde{z}_k} + o(1) \right). \tag{2.4.5}$$

The sum $\sum_{j=0}^{k-1} \frac{\Omega^{(k,j)}}{\tilde{z}_k}$ coincides with the action of the quadratic Gaudin Hamiltonian $\mathcal{H}$ of the spin chain $V_{\mu_{k-1}} \otimes V_{\omega_1}$ with sites at 0 and $\tilde{z}_k$, embedded in $V_\lambda \otimes (V_{\omega_1})^{\otimes k}$ via

$$V_{\mu_{k-1}} \otimes V_{\omega_1} \simeq V_{\mu_1,\dots,\mu_{k-1}} \otimes V_{\omega_1} \subset V_\lambda \otimes (V_{\omega_1})^{\otimes k}.$$

Since $\mu_k \neq \mu'_k$, $v_{\mu_1,\dots,\mu_k}$ and $v_{\mu'_1,\dots,\mu'_k}$ are eigenvectors of $\sum_{j=0}^{k-1} \frac{\Omega^{(k,j)}}{\tilde{z}_k}$ with distinct eigenvalues by Proposition 2.4.3. By (2.4.4) and (2.4.5), we have that the eigenvalues of $\mathcal{H}_k$ on $\omega_{\mu_1,\dots,\mu_N}$ and $\omega_{\mu'_1,\dots,\mu'_N}$ are distinct.

The argument above establishes that the set of points $\boldsymbol{z} = (z_0, z_1, \dots, z_N)$ for which the Gaudin Hamiltonians are diagonalizable with joint simple spectrum is non-empty. It is a Zariski-open set, therefore the theorem follows. $\square$

2.5 The Cases of C_r and D_r

2.5.1 The Case of C_r

Let $\mathfrak{g} = \mathfrak{sp}(2r)$, be the simple Lie algebra of type C_r, $r \geqslant 3$. We have $(\alpha_i, \alpha_i) = 2$, $i = 1, \dots, r-1$, and $(\alpha_r, \alpha_r) = 4$. We work with data $\boldsymbol{\Lambda} = (\lambda, \omega_1)$, $\boldsymbol{z} = (0, 1)$, where $\lambda \in \mathcal{P}^+$.

We have

$$\begin{aligned} V_\lambda \otimes V_{\omega_1} =& V_{\lambda+\omega_1} \oplus V_{\lambda+\omega_1-\alpha_1} \oplus \cdots \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_r} \\ & \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_{r-2}-2\alpha_{r-1}-\alpha_r} \oplus \cdots \oplus V_{\lambda+\omega_1-2\alpha_1-\cdots-2\alpha_{r-1}-\alpha_r} \\ =& V_{(\lambda_1+1,\lambda_2,\dots,\lambda_r)} \oplus V_{(\lambda_1-1,\lambda_2+1,\lambda_3,\dots,\lambda_r)} \oplus V_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k-1,\lambda_{k+1}+1,\dots,\lambda_r)} \\ & \oplus \cdots \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-1}-1,\lambda_r+1)} \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2},\lambda_{r-1}+1,\lambda_r-1)} \\ & \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-3},\lambda_{r-2}+1,\lambda_{r-1}-1,\lambda_r)} \oplus \cdots \oplus V_{(\lambda_1+1,\lambda_2-1,\lambda_3,\dots,\lambda_r)} \oplus V_{(\lambda_1-1,\lambda_2,\dots,\lambda_r)}, \end{aligned} \tag{2.5.1}$$

with the convention that the summands with non-dominant highest weights are omitted. Note, in particular, all multiplicities are 1.

We call an r-tuple of integers $\boldsymbol{l} = (l_1, \dots, l_r)$ *admissible* if the $V_{\lambda+\omega_1-\alpha(\boldsymbol{l})}$ appears in (2.5.1).

The admissible r-tuples $\boldsymbol{l}$ have the form

$$(\underbrace{1, \dots, 1}_{k_1 \text{ ones}}, 0, \dots, 0) \text{ or } (\underbrace{1, \dots, 1}_{k_2 \text{ ones}}, 2, \dots, 2, 1), \tag{2.5.2}$$

where $k_1 = 0, 1, \dots, r$ and $k_2 = 0, 1, \dots, r-2$. In the first case the length $l = l_1 + \cdots + l_r$ is k_1 and in the second case $2r - k_2 - 1$. It follows that different admissible r-tuples have different length and, therefore, admissible tuples $\boldsymbol{l}$ are parametrized by length $l \in \{0, 1, \dots, 2r-1\}$. We call a nonnegative integer l *admissible* if it is the length of an admissible r-tuple $\boldsymbol{l}$. More precisely, a nonnegative integer l is admissible if $l=0$ or if $l \leqslant r$, $\lambda_l > 0$ or if $r < l \leqslant 2r-1$, $\lambda_{2r-l} > 0$.

Similarly to the case of type B_r, see Theorem 2.3.9 and Corollary 2.3.10, we obtain the solutions to the Bethe ansatz equations for $V_\lambda \otimes V_{\omega_1}$.

Theorem 2.5.1. *Let $\mathfrak{g} = \mathfrak{sp}(2r)$. Let $\boldsymbol{l}$ be as in* (2.5.2)*. If $\boldsymbol{l}$ is not admissible then the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$ has no solutions. If $\boldsymbol{l}$ is admissible then the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$ has exactly one solution represented by the following r-tuple of polynomials $\boldsymbol{y}^{(l)}$.*

For $l = 0, 1, \dots, r-1$, we have $\boldsymbol{y}^{(l)} = (x - c_1^{(l)}, \dots, x - c_l^{(l)}, 1, \dots, 1)$, where $c_j^{(l)}$ are given by (2.3.8)*.*

For $l = r$, we have $\boldsymbol{y}^{(l)} = (x - c_1^{(r)}, \dots, x - c_r^{(r)})$, where

$$c_j^{(r)} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_{r-1} + 2\lambda_r + r + 1 - i}{\lambda_i + \cdots + \lambda_{r-1} + 2\lambda_r + r + 2 - i}, \qquad j = 1, \dots, r-1,$$

$$c_r^{(r)} = \frac{\lambda_r}{\lambda_r + 1} \prod_{i=1}^{r-1} \frac{\lambda_i + \cdots + \lambda_{r-1} + 2\lambda_r + r + 1 - i}{\lambda_i + \cdots + \lambda_{r-1} + 2\lambda_r + r + 2 - i}.$$

For $l = r+1, \dots, 2r-1$, we have $\boldsymbol{y}^{(l)} = (x - c_1^{(l)}, \dots, x - c_{2r-l-1}^{(l)}, (x - a_{2r-l}^{(l)})(x - b_{2r-l}^{(l)}), \dots, (x - a_{r-1}^{(l)})(x - b_{r-1}^{(l)}), x - c_r^{(l)})$, where

$$c_j^{(l)} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 1 - i}{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 2 - i}, \qquad j = 1, \dots, 2r-l-1,$$

$$\begin{aligned} c_r^{(l)} = &\prod_{i=1}^{2r-l-1} \frac{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 1 - i}{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 2 - i} \\ &\times \prod_{i=2r-l}^{r} \frac{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i}{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i + 1}, \end{aligned}$$

$$a_k^{(l)}b_k^{(l)} = \left(\prod_{i=1}^{2r-l-1} \frac{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 1 - i}{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 2 - i}\right)^2$$
$$\times \prod_{i=2r-l}^{k-1} \frac{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i}{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i + 1}$$
$$\times \prod_{i=2r-l}^{r} \frac{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i}{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i + 1}$$
$$\times \prod_{i=1}^{r+1-k} \frac{\lambda_{2r-l} + \cdots + \lambda_{r+1-i} + l + 1 - i - r}{\lambda_{2r-l} + \cdots + \lambda_{r+1-i} + l + 2 - i - r}$$

and

$$a_k^{(l)} + b_k^{(l)} = \prod_{i=1}^{2r-l-1} \frac{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 1 - i}{\lambda_i + \cdots + \lambda_{2r-1-l} + 2\lambda_{2r-l} + \cdots + 2\lambda_r + l + 2 - i}$$
$$\times \prod_{i=2r-l}^{k-1} \frac{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i}{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i + 1}$$
$$\times \left(\frac{2\lambda_{2r-l} + \cdots + 2\lambda_r + 2l - 2r}{2\lambda_{2r-l} + \cdots + 2\lambda_r + 2l + 1 - 2r} + \frac{2\lambda_{2r-l} + \cdots + 2\lambda_r + 2l + 2 - 2r}{2\lambda_{2r-l} + \cdots + 2\lambda_r + 2l + 1 - 2r}\right.$$
$$\times \prod_{i=k}^{r} \frac{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i}{\lambda_{2r-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_r + l - i + 1}$$
$$\left.\times \prod_{i=k}^{r} \frac{\lambda_{2r-l} + \cdots + \lambda_i + l + i - 2r}{\lambda_{2r-l} + \cdots + \lambda_i + l + i + 1 - 2r}\right),$$

for $k = 2r - l, \ldots, r - 1$. □

Therefore, in parallel to Theorem 2.4.5, we have the completeness of Bethe ansatz.

Theorem 2.5.2. *Let* $\mathfrak{g} = \mathfrak{sp}(2r)$ *and* $\lambda \in \mathcal{P}^+$. *For a generic* $(N+1)$*-tuple of distinct complex numbers* $\boldsymbol{z} = (z_0, z_1, \ldots, z_N)$, *the Gaudin Hamiltonians* $(\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_N)$ *acting in* $\mathrm{Sing}\left(V_\lambda \otimes V_{\omega_1}^{\otimes N}\right)$ *are diagonalizable and have simple joint spectrum. Moreover, for generic* $\boldsymbol{z}$ *there exists a set of solutions* $\{\boldsymbol{t}_i,\ i \in I\}$ *of the Bethe ansatz equation* (2.2.2) *such that the corresponding Bethe vectors* $\{\omega(\boldsymbol{z}, \boldsymbol{t}_i),\ i \in I\}$ *form a basis of* $\mathrm{Sing}\left(V_\lambda \otimes V_{\omega_1}^{\otimes N}\right)$. □

Similarly to Section 2.3.6, following [MV04], we introduce a linear differential operator $D(\boldsymbol{y})$ of order $2r+1$ by the formula

$$\begin{aligned} D_\lambda(\boldsymbol{y}) = &\left(\partial - \ln'\left(\frac{T_1^2 \dots T_{r-1}^2 T_r^2}{y_1}\right)\right)\left(\partial - \ln'\left(\frac{y_1 T_1^2 \dots T_{r-1}^2 T_r^2}{y_2 T_1}\right)\right) \dots \\ &\times \left(\partial - \ln'\left(\frac{y_{r-2} T_1^2 \dots T_{r-1}^2 T_r^2}{y_{r-1} T_1 \dots T_{r-2}}\right)\right) \dots \left(\partial - \ln'\left(\frac{y_{r-1} T_1^2 \dots T_{r-1}^2 T_r^2}{y_r^2 T_1 \dots T_{r-1}}\right)\right) \\ &\times \left(\partial - \ln'\left(T_1 \dots T_r\right)\right)\left(\partial - \ln'\left(\frac{y_r^2 T_1 \dots T_{r-1}}{y_{r-1}}\right)\right) \\ &\times \left(\partial - \ln'\left(\frac{y_{r-1} T_1 \dots T_{r-2}}{y_{r-2}}\right)\right) \dots \left(\partial - \ln'\left(\frac{y_2 T_1}{y_1}\right)\right)(\partial - \ln'(y_1)), \end{aligned}$$

where T_i, $i = 1, \dots, r$, are given by (2.2.4).

If $\boldsymbol{y}$ is an r-tuple of polynomials representing a critical point associated with integral dominant weights $\Lambda_1, \dots, \Lambda_n$ and points $z_1, \dots, z_n$ of type C_r, then by [MV04], the kernel of $D_\lambda(\boldsymbol{y})$ is a self-dual space of polynomials. By [MM17] the coefficients of $D_\lambda(\boldsymbol{y})$ are eigenvalues of higher Gaudin Hamiltonians acting on the Bethe vector related to $\boldsymbol{y}$.

For admissible l and $\lambda \in \mathfrak{h}^*$, define $a_\lambda^l(1), \dots, a_\lambda^l(r)$ as follows.

For $l = 0, \dots, r-1$, $i = 1, \dots, l$, set $a_\lambda^l(i) = \lambda_i + \dots + \lambda_l + l + 1 - i$. For $l = 0, \dots, r$, $i = l+1, \dots, r$, set $a_\lambda^l(i) = 0$.

For $l = r$, $i = 1, \dots, r-1$, set $a_\lambda^l(i) = \lambda_i + \dots + \lambda_{r-1} + 2\lambda_r + r + 2 - i$ and $a_\lambda^r(r) = \lambda_r + 1$.

For $l = r+1, \dots, 2r-1$, set $k = 2r - l - 1$. Then for $i = 1, \dots, k$, set

$$a_\lambda^l(i) = \lambda_i + \dots + \lambda_k + 2\lambda_{k+1} + \dots + 2\lambda_{r-1} + 2\lambda_r + 2r + 1 - k - i$$

and for $i = k+1, \dots, r-1$, set $a_\lambda^l(i) = 2\lambda_{k+1} + \dots + 2\lambda_{r-1} + 2\lambda_r + 2r - 2k$ and $a_\lambda^l(r) = \lambda_{k+1} + \dots + \lambda_{r-1} + \lambda_r + r - k$.

Proposition 2.5.3. *Let the r-tuple $\boldsymbol{y}$ represent the solution of the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}$, $\boldsymbol{z}$ and admissible $\boldsymbol{l}$, where $\lambda \in \mathcal{P}^+$. Then $D_\lambda(\boldsymbol{y}) = D_\lambda(x^{a_\lambda^l(1)}, \dots, x^{a_\lambda^l(r)})$.* □

2.5.2 The Case of D_r

Let $\mathfrak{g} = \mathfrak{so}(2r)$ be the simple Lie algebra of type D_r, where $r \geqslant 4$. We have $(\alpha_i, \alpha_i) = 2$, $i = 1, \dots, r$, $(\alpha_i, \alpha_{i-1}) = 1$, $i = 1, \dots, r-1$, and $(\alpha_r, \alpha_{r-2}) = 1$, $(\alpha_r, \alpha_{r-1}) = 0$. We work with data $\boldsymbol{\Lambda} = (\lambda, \omega_1)$, $\boldsymbol{z} = (0, 1)$, where $\lambda \in \mathcal{P}^+$.

We have

$$
\begin{aligned}
V_\lambda \otimes V_{\omega_1} =& V_{\lambda+\omega_1} \oplus V_{\lambda+\omega_1-\alpha_1} \oplus \cdots \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_r} \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_{r-2}-\alpha_r} \\
& \oplus V_{\lambda+\omega_1-\alpha_1-\cdots-\alpha_{r-3}-2\alpha_{r-2}-\alpha_{r-1}-\alpha_r} \oplus \cdots \oplus V_{\lambda+\omega_1-2\alpha_1-\cdots-2\alpha_{r-2}-\alpha_{r-1}-\alpha_r} \\
=& V_{(\lambda_1+1,\lambda_2,\dots,\lambda_r)} \oplus V_{(\lambda_1-1,\lambda_2+1,\lambda_3,\dots,\lambda_r)} \oplus \cdots \oplus V_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k-1,\lambda_{k+1}+1,\lambda_{k+2},\dots,\lambda_r)} \\
& \oplus \cdots \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2}-1,\lambda_{r-1}+1,\lambda_r+1)} \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2},\lambda_{r-1}-1,\lambda_r+1)} \\
& \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2}+1,\lambda_{r-1}-1,\lambda_r-1)} \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-2},\lambda_{r-1}+1,\lambda_r-1)} \\
& \oplus V_{(\lambda_1,\lambda_2,\dots,\lambda_{r-4},\lambda_{r-3}+1,\lambda_{r-2}-1,\lambda_{r-1},\lambda_r)} \oplus \cdots \oplus V_{(\lambda_1,\dots,\lambda_{k-2},\lambda_{k-1}+1,\lambda_k-1,\lambda_{k+1},\dots,\lambda_r)} \\
& \oplus \cdots \oplus V_{(\lambda_1+1,\lambda_2-1,\lambda_3,\dots,\lambda_r)} \oplus V_{(\lambda_1-1,\lambda_2,\dots,\lambda_r)},
\end{aligned} \tag{2.5.3}
$$

with the convention that the summands with non-dominant highest weights are omitted. Note, in particular, all multiplicities are 1.

We call an r-tuple of integers $\boldsymbol{l} = (l_1, \dots, l_r)$ *admissible* if the $V_{\lambda+\omega_1-\alpha(\boldsymbol{l})}$ appears in (2.5.3).

The admissible r-tuple $\boldsymbol{l}$ have the form

$$
(\underbrace{1,\dots,1}_{k_1 \text{ ones}}, 0, \dots, 0) \text{ or } (\underbrace{1,\dots,1}_{r-2 \text{ ones}}, 1, 0) \text{ or } (\underbrace{1,\dots,1}_{r-2 \text{ ones}}, 0, 1) \text{ or } (\underbrace{1,\dots,1}_{k_2 \text{ ones}}, 2, \dots, 2, 1, 1), \tag{2.5.4}
$$

where $k_1 = 0, \dots, r-2, r$ and $k_2 = 0, \dots, r-2$. In the first case the length $l = l_1 + \cdots + l_r$ is k_1, in the second and third cases $r-1$ and in the forth case $2r - k_2 - 2$. It follows that different admissible r-tuples in the first and forth cases have different length and, therefore, admissible tuples $\boldsymbol{l}$ of these types are parametrized by length $l \in \{0, 1, \dots, r-2, r, \dots, 2r-2\}$. We denote the lengths in the second and third cases by $r-1$ and $\overline{r-1}$, respectively. More precisely, for $l \in \{0, 1, \dots, r-1, \overline{r-1}, r, \dots, 2r-2\}$, l is a length of an admissible r-tuple $\boldsymbol{l}$ if $l = 0$ or $l \leqslant r-1$, $\lambda_l > 0$ or if $l = \overline{r-1}$,

$\lambda_r > 0$ or if $l = r$, $\lambda_{r-1} > 0$ and $\lambda_r > 0$ or if $l \geqslant r+1$, $\lambda_{2r-l-1} > 0$. We call such l *admissible.*

Similarly to the case of type B_r, see Theorem 2.3.9 and Corollary 2.3.10, we obtain the solutions to Bethe ansatz equations for $V_\lambda \otimes V_{\omega_1}$.

Theorem 2.5.4. *Let $\mathfrak{g} = \mathfrak{so}(2r)$. Let $\boldsymbol{l}$ be as in* (2.5.4)*. If $\boldsymbol{l}$ is not admissible then the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$ has no solutions. If $\boldsymbol{l}$ is admissible then the Bethe ansatz equation* (2.2.2) *associated to $\boldsymbol{\Lambda}, \boldsymbol{z}, \boldsymbol{l}$ has exactly one solution represented by the following r-tuple of polynomials $\boldsymbol{y}^{(l)}$.*

For $l = 0, 1, \dots, r-1$, we have $\boldsymbol{y}^{(l)} = (x - c_1^{(l)}, \dots, x - c_l^{(l)}, 1, \dots, 1)$, where $c_j^{(l)}$ are given by (2.3.8)*.*

For $l = \overline{r-1}$, we have $\boldsymbol{y}^{(\overline{r-1})} = (x - c_1^{(\overline{r-1})}, \dots, x - c_{r-2}^{(\overline{r-1})}, 1, x - c_r^{(\overline{r-1})})$, where

$$c_j^{(\overline{r-1})} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_{r-2} + \lambda_r + r - 1 - i}{\lambda_i + \cdots + \lambda_{r-2} + \lambda_r + r - i}, \quad j = 1, \dots, r-2,$$

and

$$c_r^{(\overline{r-1})} = \frac{\lambda_r}{\lambda_r + 1} \prod_{i=1}^{r-2} \frac{\lambda_i + \cdots + \lambda_{r-2} + \lambda_r + r - 1 - i}{\lambda_i + \cdots + \lambda_{r-2} + \lambda_r + r - i}.$$

For $l = r$, we have $\boldsymbol{y}^{(r)} = (x - c_1^{(r)}, \dots, x - c_r^{(r)})$, where

$$c_j^{(r)} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_r + r - i}{\lambda_i + \cdots + \lambda_r + r + 1 - i}, \quad j = 1, \dots, r-2,$$

$$c_{r-1}^{(r)} = \frac{\lambda_{r-1}}{\lambda_{r-1} + 1} \prod_{i=1}^{r-2} \frac{\lambda_i + \cdots + \lambda_r + r - i}{\lambda_i + \cdots + \lambda_r + r + 1 - i},$$

and

$$c_r^{(r)} = \frac{\lambda_r}{\lambda_r + 1} \prod_{i=1}^{r-2} \frac{\lambda_i + \cdots + \lambda_r + r - i}{\lambda_i + \cdots + \lambda_r + r + 1 - i}.$$

For $l = r+1, \dots, 2r-2$, we have

$$\boldsymbol{y}^{(l)} = (x - c_1^{(l)}, \dots, x - c_{2r-l-2}^{(l)}, (x - a_{2r-l-1}^{(l)})(x - b_{2r-l-1}^{(l)}), \dots,$$
$$(x - a_{r-2}^{(l)})(x - b_{r-2}^{(l)}), x - c_{r-1}^{(l)}, x - c_r^{(l)}),$$

where

$$c_j^{(l)} = \prod_{i=1}^{j} \frac{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i}{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l + 1 - i},$$

$j = 1, \ldots, 2r - l - 2,$

$$\begin{aligned} c_{r-1}^{(l)} = {} & \prod_{i=1}^{2r-2-l} \frac{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i}{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l + 1 - i} \\ & \times \prod_{i=2r-1-l}^{r-2} \frac{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i - 1}{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i} \\ & \times \frac{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_{r-1} + l - r}{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_{r-1} + l - r + 1}, \end{aligned}$$

$$\begin{aligned} c_{r}^{(l)} = {} & \prod_{i=1}^{2r-2-l} \frac{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i}{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l + 1 - i} \\ & \times \prod_{i=2r-1-l}^{r-2} \frac{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i - 1}{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i} \\ & \times \frac{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_r + l - r}{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_r + l - r + 1}, \end{aligned}$$

$$\begin{aligned} a_k^{(l)} b_k^{(l)} = {} & \left(\prod_{i=1}^{2r-l-2} \frac{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i}{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i + 1} \right)^2 \\ & \times \prod_{i=2r-1-l}^{r-2} \frac{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i - 1}{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i} \\ & \times \prod_{i=2r-1-l}^{k-1} \frac{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i - 1}{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i} \\ & \times \prod_{i=k}^{r-2} \frac{\lambda_{2r-l-1} + \cdots + \lambda_i + l + i + 1 - 2r}{\lambda_{2r-l-1} + \cdots + \lambda_i + l + i + 2 - 2r} \\ & \times \frac{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_{r-1} + l - r}{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_{r-1} + l - r + 1} \cdot \frac{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_r + l - r}{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_r + l - r + 1}, \end{aligned}$$

and

$$
\begin{aligned}
a_k^{(l)} + b_k^{(l)} = & \prod_{i=1}^{2r-l-2} \frac{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i}{\lambda_i + \cdots + \lambda_{2r-2-l} + 2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i + 1} \\
& \times \prod_{i=2r-1-l}^{k-1} \frac{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i - 1}{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i} \\
& \times \Bigg(\frac{2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + 2l - 2r}{2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + 2l - 2r + 1} \\
& + \frac{2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + 2l - 2r + 2}{2\lambda_{2r-l-1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + 2l - 2r + 1} \\
& \times \prod_{i=k}^{r-2} \frac{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i - 1}{\lambda_{2r-1-l} + \cdots + \lambda_i + 2\lambda_{i+1} + \cdots + 2\lambda_{r-2} + \lambda_{r-1} + \lambda_r + l - i} \\
& \times \frac{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_{r-1} + l - r}{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_{r-1} + l - r + 1} \cdot \frac{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_r + l - r}{\lambda_{2r-1-l} + \cdots + \lambda_{r-2} + \lambda_r + l - r + 1} \\
& \times \prod_{i=k}^{r-2} \frac{\lambda_{2r-l-1} + \cdots + \lambda_i + l + i + 1 - 2r}{\lambda_{2r-l-1} + \cdots + \lambda_i + l + i + 2 - 2r} \Bigg),
\end{aligned}
$$

$k = 2r - 1 - l, \ldots, r - 2.$ □

Note that the formulas above with $r = 3$ correspond to solutions of the Bethe ansatz equations of type A_3 and $\boldsymbol{\Lambda} = (\lambda, \omega_2)$. These formulas were given in Theorem 5.5, [MV05b].

Then we deduce the analog of Theorem 2.4.5.

Theorem 2.5.5. *Let $\mathfrak{g} = \mathfrak{so}(2r)$ and $\lambda \in \mathcal{P}^+$. For a generic $(N+1)$-tuple of distinct complex numbers $\boldsymbol{z} = (z_0, z_1, \ldots, z_N)$, the Gaudin Hamiltonians $(\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_N)$ acting in* $\mathrm{Sing}\left(V_\lambda \otimes V_{\omega_1}^{\otimes N}\right)$ *are diagonalizable. Moreover, for generic $\boldsymbol{z}$ there exists a set of solutions $\{\boldsymbol{t}_i,\ i \in I\}$ of the Bethe ansatz equation* (2.2.2) *such that the corresponding Bethe vectors $\{\omega(\boldsymbol{z}, \boldsymbol{t}_i),\ i \in I\}$ form a basis of* $\mathrm{Sing}\left(V_\lambda \otimes V_{\omega_1}^{\otimes N}\right)$. □

For type D, the algebra has a non-trivial diagram automorphism which leads to degeneracy of the spectrum. For example, if $\lambda_{r-1} = \lambda_r$, then the Bethe vectors corresponding to the critical points $\boldsymbol{y}^{(r-1)}$ and $\boldsymbol{y}^{(\overline{r-1})}$ are eigenvectors of the Gaudin Hamiltonian $\mathcal{H} := \mathcal{H}_1 = -\mathcal{H}_2$ with the same eigenvalue. In particular Proposition 2.4.3 is not applicable since the two corresponding summands in (2.5.3) have non-comparable highest weights.

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3. SELF-DUAL GRASSMANNIAN, WRONSKI MAP, AND REPRESENTATIONS OF $\mathfrak{gl}_N$, $\mathfrak{sp}_{2r}$, $\mathfrak{so}_{2r+1}$

3.1 Introduction

Grassmannian $\mathrm{Gr}(N,d)$ of N-dimensional subspaces of the complex d-dimensional vector space has the standard stratification by Schubert cells Ω_λ labeled by partitions $\lambda = (d-N \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_N \geqslant 0)$. A Schubert cycle is the closure of a cell Ω_λ. It is well known that the Schubert cycle $\overline{\Omega}_\lambda$ is the union of the cells Ω_ξ such that the Young diagram of λ is inscribed into the Young diagram of ξ. This stratification depends on a choice of a full flag in the d-dimensional space.

In this paper we introduce a new stratification of $\mathrm{Gr}(N,d)$ governed by representation theory of $\mathfrak{gl}_N$ and called the $\mathfrak{gl}_N$*-stratification*, see Theorem 3.3.5. The $\mathfrak{gl}_N$-strata $\Omega_{\boldsymbol{\Lambda}}$ are labeled by unordered sets $\boldsymbol{\Lambda} = (\lambda^{(1)},\ldots,\lambda^{(n)})$ of nonzero partitions $\lambda^{(i)} = (d-N \geqslant \lambda_1^{(i)} \geqslant \ldots \geqslant \lambda_N^{(i)} \geqslant 0)$ such that

$$(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N} \neq 0, \qquad \sum_{i=1}^n \sum_{j=1}^N \lambda_j^{(i)} = N(d-N), \tag{3.1.1}$$

where $V_{\lambda^{(i)}}$ is the irreducible $\mathfrak{gl}_N$-module with highest weight $\lambda^{(i)}$. We have $\dim \Omega_{\boldsymbol{\Lambda}} = n$. We call the closure of a stratum $\Omega_{\boldsymbol{\Lambda}}$ in $\mathrm{Gr}(N,d)$ a $\mathfrak{gl}_N$*-cycle*. The $\mathfrak{gl}_N$-cycle $\overline{\Omega}_{\boldsymbol{\Lambda}}$ is an algebraic set in $\mathrm{Gr}(N,d)$. We show that $\overline{\Omega}_{\boldsymbol{\Lambda}}$ is the union of the strata $\Omega_{\boldsymbol{\Xi}}$, $\boldsymbol{\Xi} = (\xi^{(1)},\ldots,\xi^{(m)})$, such that there is a partition $\{I_1,\ldots,I_m\}$ of $\{1,2,\ldots,n\}$ with $\mathrm{Hom}_{\mathfrak{gl}_N}\big(V_{\xi^{(i)}}, \otimes_{j\in I_i} V_{\lambda^{(j)}}\big) \neq 0$ for $i=1,\ldots,m$, see Theorem 3.3.8.

Thus we have a partial order on the set of sequences of partitions satisfying (3.1.1). Namely $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Xi}$ if there is a partition $\{I_1,\ldots,I_m\}$ of $\{1,2,\ldots,n\}$ with $\mathrm{Hom}_{\mathfrak{gl}_N}\big(V_{\xi^{(i)}}, \otimes_{j\in I_i} V_{\lambda^{(j)}}\big) \neq 0$ for $i=1,\ldots,m$. An example of the corresponding graph is given in Example 3.3.9. The $\mathfrak{gl}_N$-stratification can be viewed as the geometrization of this partial order.

Let us describe the construction of the strata in more detail. We identify the Grassmannian $\mathrm{Gr}(N,d)$ with the Grassmannian of N-dimensional subspaces of the d-dimensional space $\mathbb{C}_d[x]$ of polynomials in x of degree less than d. In other words, we always assume that for $X \in \mathrm{Gr}(N,d)$, we have $X \subset \mathbb{C}_d[x]$. Set $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then, for any $z \in \mathbb{P}^1$, we have the osculating flag $\mathcal{F}(z)$, see (3.3.3), (3.3.4). Denote the Schubert cells corresponding to $\mathcal{F}(z)$ by $\Omega_\lambda(\mathcal{F}(z))$. Then the stratum $\Omega_{\mathbf{\Lambda}}$ consists of spaces $X \in \mathrm{Gr}(N,d)$ such that X belongs to the intersection of Schubert cells $\Omega_{\lambda^{(i)}}(\mathcal{F}(z_i))$ for some choice of distinct $z_i \in \mathbb{P}^1$:

$$\Omega_{\mathbf{\Lambda}} = \bigcup_{\substack{z_1,\dots,z_n \\ z_i \neq z_j}} \Big(\bigcap_{i=1}^{n} \Omega_{\lambda^{(i)}}(\mathcal{F}(z_i)) \Big) \subset \mathrm{Gr}(N,d).$$

A stratum $\Omega_{\mathbf{\Lambda}}$ is a ramified covering over $(\mathbb{P}^1)^n$ without diagonals quotient by the free action of an appropriate symmetric group, see Proposition 3.3.4. The degree of the covering is $\dim(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N}$.

For example, if $N = 1$, then $\mathrm{Gr}(1,d)$ is the $(d-1)$-dimensional projective space of the vector space $\mathbb{C}_d[x]$. The strata $\Omega_{\boldsymbol{m}}$ are labeled by unordered sets $\boldsymbol{m} = (m_1,\dots,m_n)$ of positive integers such that $m_1+\dots+m_n = d-1$. A stratum $\Omega_{\boldsymbol{m}}$ consists of all polynomials $f(x)$ which have n distinct zeros of multiplicities $m_1,\dots,m_n$. In this stratum we also include the polynomials of degree $d-1-m_i$ with $n-1$ distinct roots of multiplicities $m_1,\dots,m_{i-1},m_{i+1},\dots,m_n$. We interpret these polynomials as having a zero of multiplicity m_i at infinity. The stratum $\Omega_{(1,\dots,1)}$ is open in $\mathrm{Gr}(1,d)$. The union of other strata is classically called the *swallowtail* and the $\mathfrak{gl}_1$-stratification is the standard stratification of the swallowtail, see for example Section 2.5 of Part 1 of [AGZV85].

The $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$ agrees with the Wronski map

$$\mathrm{Wr} : \mathrm{Gr}(N,d) \to \mathrm{Gr}(1, N(d-N)+1)$$

which sends an N-dimensional subspace of polynomials to its Wronskian

$$\det(d^{i-1}f_j/dx^{i-1})_{i,j=1}^N,$$

where $f_1(x),\dots,f_N(x)$ is a basis of the subspace. For any $\mathfrak{gl}_1$-stratum $\Omega_{\boldsymbol{m}}$ of Grassmannian $\mathrm{Gr}(1,N(d-N)+1)$, the preimage of $\Omega_{\boldsymbol{m}}$ under the Wronski map is the union of $\mathfrak{gl}_N$-strata of $\mathrm{Gr}(N,d)$ and the restriction of the Wronski map to each of those strata $\Omega_{\boldsymbol{\Lambda}}$ is a ramified covering over $\Omega_{\boldsymbol{m}}$ of degree $b(\boldsymbol{\Lambda})\dim(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{sl}_N}$, where $b(\boldsymbol{\Lambda})$ is some combinatorial symmetry coefficient of $\boldsymbol{\Lambda}$, see (3.3.9).

The main goal of this paper is to develop a similar picture for the new object $\mathrm{sGr}(N,d)\subset\mathrm{Gr}(N,d)$, called *self-dual Grassmannian*. Let $X\in\mathrm{Gr}(N,d)$ be an N-dimensional subspace of polynomials in x. Let $X^\vee$ be the N-dimensional space of polynomials which are Wronski determinants of $N-1$ elements of X:

$$X^\vee=\{\det\left(d^{i-1}f_j/dx^{i-1}\right)_{i,j=1}^{N-1},\ f_j(x)\in X\}.$$

The space X is called *self-dual* if $X^\vee=g\cdot X$ for some polynomial $g(x)$, see [MV04]. We define $\mathrm{sGr}(N,d)$ as the subset of $\mathrm{Gr}(N,d)$ of all self-dual spaces. It is an algebraic set.

The main result of this paper is the stratification of $\mathrm{sGr}(N,d)$ governed by representation theory of the Lie algebras $\mathfrak{g}_{2r+1}:=\mathfrak{sp}_{2r}$ if $N=2r+1$ and $\mathfrak{g}_{2r}:=\mathfrak{so}_{2r+1}$ if $N=2r$. This stratification of $\mathrm{sGr}(N,d)$ is called the $\mathfrak{g}_N$*-stratification*, see Theorem 3.4.11.

The $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$ consists of $\mathfrak{g}_N$-strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ labeled by unordered sets of dominant integral $\mathfrak{g}_N$-weights $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})$, equipped with nonnegative integer labels $\boldsymbol{k}=(k_1,\dots,k_n)$, such that $(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{g}_N}\neq 0$ and satisfying a condition similar to the second equation in (3.1.1), see Section 3.4.3. Here $V_{\lambda^{(i)}}$ is the irreducible $\mathfrak{g}_N$-module with highest weight $\lambda^{(i)}$. Different liftings of an $\mathfrak{sl}_N$-weight to a $\mathfrak{gl}_N$-weight differ by a vector $(k,\dots,k)$ with integer k. Our label k_i is an analog of this parameter in the case of $\mathfrak{g}_N$.

A $\mathfrak{g}_N$-stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is a ramified covering over $(\mathbb{P}^1)^n$ without diagonals quotient by the free action of an appropriate symmetric group. The degree of the covering is $\dim(\otimes_{i=1}^n V_{\lambda^{(i)}})^{\mathfrak{g}_N}$ and, in particular, $\dim\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}=n$, see Proposition 3.4.9. We call the closure of a stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ in $\mathrm{sGr}(N,d)$ a $\mathfrak{g}_N$*-cycle*. The $\mathfrak{g}_N$-cycle $\overline{\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}}$ is an algebraic set. We show that $\overline{\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}}$ is the union of the strata $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(m)})$, such that there is a partition $\{I_1, \dots, I_m\}$ of $\{1, 2, \dots, n\}$ satisfying $\mathrm{Hom}_{\mathfrak{g}_N}\big(V_{\xi^{(i)}}, \otimes_{j \in I_i} V_{\lambda^{(j)}}\big) \neq 0$ for $i = 1, \dots, m$, and the appropriate matching of labels, see Theorem 3.4.13.

If $N = 2r$, there is exactly one stratum of top dimension $2(d-N) = \dim \mathrm{sGr}(N, d)$. For example, the $\mathfrak{so}_5$-stratification of $\mathrm{sGr}(4, 6)$ consists of 9 strata of dimensions 4, 3, 3, 3, 2, 2, 2, 2, 1, see the graph of adjacencies in Example 3.4.14. If $N = 2r+1$, there are many strata of top dimension $d - N$ (except in the trivial cases of $d = 2r + 1$ and $d = 2r + 2$). For example, the $\mathfrak{sp}_4$-stratification of $\mathrm{sGr}(5, 8)$ has four strata of dimension 3, see Section 3.4.7. In all cases we have exactly one one-dimensional stratum corresponding to $n = 1$, $\boldsymbol{\Lambda} = (0)$, and $\boldsymbol{k} = (d - N)$.

Essentially, we obtain the $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N, d)$ by restricting the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N, d)$ to $\mathrm{sGr}(N, d)$.

For $X \in \mathrm{sGr}(N, d)$, the multiplicity of every zero of the Wronskian of X is divisible by r if $N = 2r$ and by N if $N = 2r + 1$. We define the reduced Wronski map $\overline{\mathrm{Wr}} : \mathrm{sGr}(N, d) \to \mathrm{Gr}(1, 2(d-N)+1)$ if $N = 2r$ and $\overline{\mathrm{Wr}} : \mathrm{sGr}(N, d) \to \mathrm{Gr}(1, d-N+1)$ if $N = 2r+1$ by sending X to the r-th root of its Wronskian if $N = 2r$ and to the N-th root if $N = 2r+1$. The $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N, d)$ agrees with the reduced Wronski map and swallowtail $\mathfrak{gl}_1$-stratification of $\mathrm{Gr}(1, 2(d - N) + 1)$ or $\mathrm{Gr}(1, d - N + 1)$. For any $\mathfrak{gl}_1$-stratum $\Omega_{\boldsymbol{m}}$ the preimage of $\Omega_{\boldsymbol{m}}$ under $\overline{\mathrm{Wr}}$ is the union of $\mathfrak{g}_N$-strata, see Proposition 3.4.17, and the restriction of the reduced Wronski map to each of those strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is a ramified covering over $\Omega_{\boldsymbol{m}}$, see Proposition 3.4.18.

Our definition of the $\mathfrak{gl}_N$-stratification is motivated by the connection to the Gaudin model of type A, see Theorem 3.3.2. Similarly, our definition of the self-dual Grassmannian and of the $\mathfrak{g}_N$-stratification is motivated by the connection to the Gaudin models of types B and C, see Theorem 3.4.5.

It is interesting to study the geometry and topology of strata, cycles, and of self-dual Grassmannian, see Section 3.4.7.

The exposition of the material is as follows. In Section 3.2 we introduce the $\mathfrak{gl}_N$ Bethe algebra. In Section 3.3 we describe the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$. In Section 3.4 we define the $\mathfrak{g}_N$-stratification of the self-dual Grassmannian $\mathrm{sGr}(N,d)$. In Section 3.5 we recall the interrelations of the Lie algebras $\mathfrak{sl}_N$, $\mathfrak{so}_{2r+1}$, $\mathfrak{sp}_{2r}$. In Section 3.6 we discuss $\mathfrak{g}$-opers and their relations to self-dual spaces. Section 3.7 contains proofs of theorems formulated in Sections 3.3 and 3.4. In Appendix A we describe the bijection between the self-dual spaces and the set of $\mathfrak{gl}_N$ Bethe vectors fixed by the Dynkin diagram automorphism of $\mathfrak{gl}_N$.

3.2 Lie Algebras

3.2.1 Lie Algebra $\mathfrak{gl}_N$

Let e_{ij}, $i,j=1,\dots,N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$, satisfying the relations $[e_{ij},e_{sk}]=\delta_{js}e_{ik}-\delta_{ik}e_{sj}$. We identify the Lie algebra $\mathfrak{sl}_N$ with the subalgebra of $\mathfrak{gl}_N$ generated by the elements $e_{ii}-e_{jj}$ and e_{ij} for $i\neq j$, $i,j=1,\dots,N$.

Let M be a $\mathfrak{gl}_N$-module. A vector $v\in M$ has weight $\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb{C}^N$ if $e_{ii}v=\lambda_i v$ for $i=1,\dots,N$. A vector v is called *singular* if $e_{ij}v=0$ for $1\leqslant i<j\leqslant N$.

We denote by $(M)_\lambda$ the subspace of M of weight λ, by $(M)^{\mathrm{sing}}$ the subspace of M of all singular vectors and by $(M)^{\mathrm{sing}}_\lambda$ the subspace of M of all singular vectors of weight λ.

Denote by V_λ the irreducible $\mathfrak{gl}_N$-module with highest weight λ.

The $\mathfrak{gl}_N$-module $V_{(1,0,\dots,0)}$ is the standard N-dimensional vector representation of $\mathfrak{gl}_N$, which we denote by L.

A sequence of integers $\lambda=(\lambda_1,\dots,\lambda_N)$ such that $\lambda_1\geqslant\lambda_2\geqslant\dots\geqslant\lambda_N\geqslant 0$ is called *a partition with at most N parts*. Set $|\lambda|=\sum_{i=1}^N\lambda_i$. Then it is said that λ is a partition of $|\lambda|$. The $\mathfrak{gl}_N$-module $L^{\otimes n}$ contains the module V_λ if and only if λ is a partition of n with at most N parts.

Let λ,μ be partitions with at most N parts. We write $\lambda\subseteq\mu$ if and only if $\lambda_i\leqslant\mu_i$ for $i=1,\dots,N$.

3.2.2 Simple Lie Algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{i,j})_{i,j=1}^r$. Let $D = \mathrm{diag}\{d_1, \dots, d_r\}$ be the diagonal matrix with positive relatively prime integers d_i such that DA is symmetric.

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the Cartan decomposition. Fix simple roots $\alpha_1, \dots, \alpha_r$ in $\mathfrak{h}^*$. Let $\check{\alpha}_1, \dots, \check{\alpha}_r \in \mathfrak{h}$ be the corresponding coroots. Fix a nondegenerate invariant bilinear form $(,)$ in $\mathfrak{g}$ such that $(\check{\alpha}_i, \check{\alpha}_j) = a_{i,j}/d_j$. The corresponding invariant bilinear form in $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = d_i a_{i,j}$. We have $\langle \lambda, \check{\alpha}_i \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ for $\lambda \in \mathfrak{h}^*$. In particular, $\langle \alpha_j, \check{\alpha}_i \rangle = a_{i,j}$. Let $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ be the fundamental weights, $\langle \omega_j, \check{\alpha}_i \rangle = \delta_{i,j}$.

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha}_i \rangle \in \mathbb{Z},\ i = 1, \dots, r\}$ and $\mathcal{P}^+ = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha}_i \rangle \in \mathbb{Z}_{\geqslant 0},\ i = 1, \dots, r\}$ be the weight lattice and the cone of dominant integral weights.

For $\lambda \in \mathfrak{h}^*$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ. We denote $\langle \lambda, \check{\alpha}_i \rangle$ by λ_i and sometimes write $(\lambda_1, \lambda_2, \dots, \lambda_r)$ for λ.

Let M be a $\mathfrak{g}$-module. Let $(M)^{\mathrm{sing}} = \{v \in M \mid \mathfrak{n}_+ v = 0\}$ be the subspace of singular vectors in M. For $\mu \in \mathfrak{h}^*$ let $(M)_\mu = \{v \in M \mid hv = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}$ be the subspace of M of vectors of weight μ. Let $(M)_\mu^{\mathrm{sing}} = M^{\mathrm{sing}} \cap (M)_\mu$ be the subspace of singular vectors in M of weight μ.

Given a $\mathfrak{g}$-module M, denote by $(M)^{\mathfrak{g}}$ the subspace of $\mathfrak{g}$-invariants in M. The subspace $(M)^{\mathfrak{g}}$ is the multiplicity space of the trivial $\mathfrak{g}$-module in M. The following facts are well known. Let λ, μ be partitions with at most N parts, $\dim(V_\lambda \otimes V_\mu)^{\mathfrak{sl}_N} = 1$ if $\lambda_i = k - \mu_{N+1-i}$, $i = 1, \dots, N$, for some integer $k \geqslant \mu_1$ and 0 otherwise. Let λ, μ be $\mathfrak{g}$-weights, $\dim(V_\lambda \otimes V_\mu)^{\mathfrak{g}} = \delta_{\lambda,\mu}$ for $\mathfrak{g} = \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}$.

For any Lie algebra $\mathfrak{g}$, denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

3.2.3 Current Agebra $\mathfrak{g}[t]$

Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{g}$-valued polynomials with the pointwise commutator. We call it the *current algebra* of $\mathfrak{g}$. We identify the Lie algebra $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1$ of constant polynomials in $\mathfrak{g}[t]$. Hence, any $\mathfrak{g}[t]$-module has the canonical structure of a $\mathfrak{g}$-module.

The standard generators of $\mathfrak{gl}_N[t]$ are $e_{ij} \otimes t^p$, $i, j = 1, \dots, N$, $p \in \mathbb{Z}_{\geqslant 0}$. They satisfy the relations $[e_{ij} \otimes t^p, e_{sk} \otimes t^q] = \delta_{js} e_{ik} \otimes t^{p+q} - \delta_{ik} e_{sj} \otimes t^{p+q}$.

It is convenient to collect elements of $\mathfrak{g}[t]$ in generating series of a formal variable x. For $g \in \mathfrak{g}$, set

$$g(x) = \sum_{s=0}^{\infty} (g \otimes t^s) x^{-s-1}. \tag{3.2.1}$$

For $\mathfrak{gl}_N[t]$ we have $(x_2 - x_1)[e_{ij}(x_1), e_{sk}(x_2)] = \delta_{js}(e_{ik}(x_1) - e_{ik}(x_2)) - \delta_{ik}(e_{sj}(x_1) - e_{sj}(x_2))$.

For each $a \in \mathbb{C}$, there exists an automorphism τ_a of $\mathfrak{g}[t]$, $\tau_a : g(x) \to g(x-a)$. Given a $\mathfrak{g}[t]$-module M, we denote by $M(a)$ the pull-back of M through the automorphism τ_a. As $\mathfrak{g}$-modules, M and $M(a)$ are isomorphic by the identity map.

We have the evaluation homomorphism, $\mathrm{ev} : \mathfrak{g}[t] \to \mathfrak{g}$, $\mathrm{ev} : g(x) \to g x^{-1}$. Its restriction to the subalgebra $\mathfrak{g} \subset \mathfrak{g}[t]$ is the identity map. For any $\mathfrak{g}$-module M, we denote by the same letter the $\mathfrak{g}[t]$-module, obtained by pulling M back through the evaluation homomorphism. For each $a \in \mathbb{C}$, the $\mathfrak{g}[t]$-module $M(a)$ is called an *evaluation module*.

For $\mathfrak{g} = \mathfrak{sl}_N$, $\mathfrak{sp}_{2r}$, $\mathfrak{so}_{2r+1}$, it is well known that finite-dimensional irreducible $\mathfrak{g}[t]$-modules are tensor products of evaluation modules $V_{\lambda^{(1)}}(z_1) \otimes \dots \otimes V_{\lambda^{(n)}}(z_n)$ with dominant integral $\mathfrak{g}$-weights $\lambda^{(1)}, \dots, \lambda^{(n)}$ and distinct evaluation parameters $z_1, \dots, z_n$.

3.2.4 Bethe Algebra

Let S_l be the permutation group of the set $\{1, \dots, l\}$. Given an $N \times N$ matrix B with possibly noncommuting entries b_{ij}, we define its *row determinant* to be

$$\mathrm{rdet}\, B = \sum_{\sigma \in S_N} (-1)^{\sigma} b_{1\sigma(1)} b_{2\sigma(2)} \dots b_{N\sigma(N)}.$$

Define *the universal differential operator* $\mathcal{D}^{\mathcal{B}}$ by

$$\mathcal{D}^{\mathcal{B}} = \mathrm{rdet}(\delta_{ij}\partial_x - e_{ji}(x))_{i,j=1}^{N}. \tag{3.2.2}$$

It is a differential operator in variable x, whose coefficients are formal power series in x^{-1} with coefficients in $\mathcal{U}(\mathfrak{gl}_N[t])$,

$$\mathcal{D}^{\mathcal{B}} = \partial_x^N + \sum_{i=1}^{N} B_i(x)\partial_x^{N-i}, \tag{3.2.3}$$

where

$$B_i(x) = \sum_{j=i}^{\infty} B_{ij}x^{-j}$$

and $B_{ij} \in \mathcal{U}(\mathfrak{gl}_N[t])$, $i = 1, \dots, N$, $j \in \mathbb{Z}_{\geqslant i}$. We call the unital subalgebra of $\mathcal{U}(\mathfrak{gl}_N[t])$ generated by $B_{ij} \in \mathcal{U}(\mathfrak{gl}_N[t])$, $i = 1, \dots, N$, $j \in \mathbb{Z}_{\geqslant i}$, the *Bethe algebra* of $\mathfrak{gl}_N$ and denote it by $\mathcal{B}$.

The Bethe algebra $\mathcal{B}$ is commutative and commutes with the subalgebra $\mathcal{U}(\mathfrak{gl}_N) \subset \mathcal{U}(\mathfrak{gl}_N[t])$, see [Tal06]. As a subalgebra of $\mathcal{U}(\mathfrak{gl}_N[t])$, the algebra $\mathcal{B}$ acts on any $\mathfrak{gl}_N[t]$-module M. Since $\mathcal{B}$ commutes with $\mathcal{U}(\mathfrak{gl}_N)$, it preserves the subspace of singular vectors $(M)^{\mathrm{sing}}$ as well as weight subspaces of M. Therefore, the subspace $(M)^{\mathrm{sing}}_{\lambda}$ is $\mathcal{B}$-invariant for any weight λ.

We denote $M(\infty)$ the $\mathfrak{gl}_N$-module M with the trivial action of the Bethe algebra $\mathcal{B}$. More generally, for a $\mathfrak{gl}_N[t]$-module M', we denote by $M' \otimes M(\infty)$ the $\mathfrak{gl}_N$-module where we define the action of $\mathcal{B}$ so that it acts trivially on $M(\infty)$. Namely, the element $b \in \mathcal{B}$ acts on $M' \otimes M(\infty)$ by $b \otimes 1$.

Note that for $a \in \mathbb{C}$ and $\mathfrak{gl}_N$-module M, the action of $e_{ij}(x)$ on $M(a)$ is given by $e_{ij}/(x-a)$ on M. Therefore, the action of series $B_i(x)$ on the module $M' \otimes M(\infty)$ is the limit of the action of the series $B_i(x)$ on the module $M' \otimes M(z)$ as $z \to \infty$ in the sense of rational functions of x. However, such a limit of the action of coefficients B_{ij} on the module $M' \otimes M(z)$ as $z \to \infty$ does not exist.

Let $M = V_\lambda$ be an irreducible $\mathfrak{gl}_N$-module and let M' be an irreducible finite-dimensional $\mathfrak{gl}_N[t]$-module. Let c be the value of the $\sum_{i=1}^{N} e_{ii}$ action on M'.

Lemma 3.2.1. *We have an isomorphism of vector spaces:*

$$\pi: \ (M' \otimes V_\lambda)^{\mathfrak{sl}_N} \to (M')^{\mathrm{sing}}_{\bar{\lambda}}, \ \text{where } \bar{\lambda}_i = \frac{c + |\lambda|}{N} - \lambda_{N+1-i},$$

given by the projection to a lowest weight vector in V_λ. The map π is an isomorphism of $\mathcal{B}$-modules $(M' \otimes V_\lambda(\infty))^{\mathfrak{sl}_N} \to (M')^{\mathrm{sing}}_{\bar{\lambda}}$. □

Consider $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$. Set

$$\mathring{\mathbb{P}}_n := \{\boldsymbol{z} = (z_1, \dots, z_n) \in (\mathbb{P}^1)^n \mid z_i \neq z_j \ \text{ for } \ 1 \leqslant i < j \leqslant n\},$$

$$\mathbb{R}\mathring{\mathbb{P}}_n := \{\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n \mid z_i \in \mathbb{R} \text{ or } z_i = \infty, \ \text{ for } \ 1 \leqslant i \leqslant n\}.$$

We are interested in the action of the Bethe algebra $\mathcal{B}$ on the tensor product $\bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s)$, where $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$. By Lemma 3.2.1, it is sufficient to consider spaces of invariants $(\bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s))^{\mathfrak{sl}_N}$. For brevity, we write $V_{\boldsymbol{\Lambda},\boldsymbol{z}}$ for the $\mathcal{B}$-module $\bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s)$ and $V_{\boldsymbol{\Lambda}}$ for the $\mathfrak{gl}_N$-module $\bigotimes_{s=1}^n V_{\lambda^{(s)}}$.

Let $v \in V_{\boldsymbol{\Lambda},\boldsymbol{z}}$ be a common eigenvector of the Bethe algebra $\mathcal{B}$, $B_i(x)v = h_i(x)v$, $i = 1, \dots, N$. Then we call the scalar differential operator

$$\mathcal{D}_v = \partial_x^N + \sum_{i=1}^N h_i(x)\partial_x^{N-i}$$

the *differential operator associated with the eigenvector v*.

3.3 The $\mathfrak{gl}_N$-Stratification of Grassmannian

Let N, $d \in \mathbb{Z}_{>0}$ such that $N \leqslant d$.

3.3.1 Schubert Cells

Let $\mathbb{C}_d[x]$ be the space of polynomials in x with complex coefficients of degree less than d. We have $\dim \mathbb{C}_d[x] = d$. Let $\mathrm{Gr}(N, d)$ be the Grassmannian of all N-dimensional subspaces in $\mathbb{C}_d[x]$. The Grassmannian $\mathrm{Gr}(N, d)$ is a smooth projective complex variety of dimension $N(d - N)$.

Let $\mathbb{R}_d[x] \subset \mathbb{C}_d[x]$ be the space of polynomials in x with real coefficients of degree less than d. Let $\mathrm{Gr}^{\mathbb{R}}(N,d) \subset \mathrm{Gr}(N,d)$ be the set of subspaces which have a basis consisting of polynomials with real coefficients. For $X \in \mathrm{Gr}(N,d)$ we have $X \in \mathrm{Gr}^{\mathbb{R}}(N,d)$ if and only if $\dim_{\mathbb{R}}(X \cap \mathbb{R}_d[x]) = N$. We call such points X *real*.

For a full flag $\mathcal{F} = \{0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = \mathbb{C}_d[x]\}$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda_1 \leqslant d - N$, the Schubert cell $\Omega_\lambda(\mathcal{F}) \subset \mathrm{Gr}(N,d)$ is given by

$$\begin{aligned}\Omega_\lambda(\mathcal{F}) = \{X \in \mathrm{Gr}(N,d) \mid\ & \dim(X \cap F_{d-j-\lambda_{N-j}}) = N-j,\\ & \dim(X \cap F_{d-j-\lambda_{N-j}-1}) = N-j-1\}.\end{aligned}$$

We have $\operatorname{codim} \Omega_\lambda(\mathcal{F}) = |\lambda|$.

The Schubert cell decomposition associated to a full flag $\mathcal{F}$, see for example [GH94], is given by

$$\mathrm{Gr}(N,d) = \bigsqcup_{\lambda,\ \lambda_1 \leqslant d-N} \Omega_\lambda(\mathcal{F}). \tag{3.3.1}$$

The Schubert cycle $\overline{\Omega}_\lambda(\mathcal{F})$ is the closure of a Schubert cell $\Omega_\lambda(\mathcal{F})$ in the Grassmannian $\mathrm{Gr}(N,d)$. Schubert cycles are algebraic sets with very rich geometry and topology. It is well known that Schubert cycle $\overline{\Omega}_\lambda(\mathcal{F})$ is described by the formula

$$\overline{\Omega}_\lambda(\mathcal{F}) = \bigsqcup_{\substack{\lambda \subseteq \mu,\\ \mu_1 \leqslant d-N}} \Omega_\mu(\mathcal{F}). \tag{3.3.2}$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda_1 \leqslant d-N$, introduce a new partition

$$\bar{\lambda} = (d-N-\lambda_N, d-N-\lambda_{N-1}, \ldots, d-N-\lambda_1).$$

We have $|\lambda| + |\bar{\lambda}| = N(d-N)$.

Let $\mathcal{F}(\infty)$ be the full flag given by

$$\mathcal{F}(\infty) = \{0 \subset \mathbb{C}_1[x] \subset \mathbb{C}_2[x] \subset \cdots \subset \mathbb{C}_d[x]\}. \tag{3.3.3}$$

The subspace X is a point of $\Omega_\lambda(\mathcal{F}(\infty))$ if and only if for every $i = 1, \ldots, N$, it contains a polynomial of degree $\bar{\lambda}_i + N - i$.

For $z \in \mathbb{C}$, consider the full flag

$$\mathcal{F}(z) = \{0 \subset (x-z)^{d-1}\mathbb{C}_1[x] \subset (x-z)^{d-2}\mathbb{C}_2[x] \subset \cdots \subset \mathbb{C}_d[x]\}. \tag{3.3.4}$$

The subspace X is a point of $\Omega_\lambda(\mathcal{F}(z))$ if and only if for every $i = 1, \ldots, N$, it contains a polynomial with a root at z of order $\lambda_i + N - i$.

A point $z \in \mathbb{C}$ is called a *base point* for a subspace $X \subset \mathbb{C}_d[x]$ if $g(z) = 0$ for every $g \in X$.

3.3.2 Intersection of Schubert Cells

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ be a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathring{\mathbb{P}}_n$. Set $|\boldsymbol{\Lambda}| = \sum_{s=1}^{n} |\lambda^{(s)}|$.

The following lemma is elementary.

Lemma 3.3.1. *If* $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} > 0$, *then* $|\boldsymbol{\Lambda}|$ *is divisible by* N. *Suppose further* $|\boldsymbol{\Lambda}| = N(d-N)$, *then* $\lambda_1^{(s)} \leqslant d - N$ *for* $s = 1, \ldots, n$. □

Assuming $|\boldsymbol{\Lambda}| = N(d-N)$, denote by $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ the intersection of the Schubert cells:

$$\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}} = \bigcap_{s=1}^{n} \Omega_{\lambda^{(s)}}(\mathcal{F}(z_s)). \tag{3.3.5}$$

Note that due to our assumption, $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is a finite subset of $\mathrm{Gr}(N,d)$. Note also that $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is non-empty if and only if $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} > 0$.

Theorem 3.3.2. *Suppose* $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} > 0$. *Let* $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ *be an eigenvector of the Bethe algebra* $\mathcal{B}$. *Then* $\operatorname{Ker} \mathcal{D}_v \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$. *Moreover, the assignment* $\kappa : v \mapsto \operatorname{Ker} \mathcal{D}_v$ *is a bijective correspondence between the set of eigenvectors of the Bethe algebra in* $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ *(considered up to multiplication by nonzero scalars) and the set* $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$.

Proof. The first statement is Theorem 4.1 in [MTV09c] and the second statement is Theorem 6.1 in [MTV09b]. □

We also have the following lemma, see for example [MTV06].

Lemma 3.3.3. *Let $\boldsymbol{z}$ be a generic point in $\mathring{\mathbb{P}}_n$. Then the action of the Bethe algebra $\mathcal{B}$ on $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ is diagonalizable. In particular, this statement holds for any sequence $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$.* □

3.3.3 The $\mathfrak{gl}_N$-Stratification of $\mathrm{Gr}(N,d)$

The following definition plays an important role in what follows.

Define a partial order $\geqslant$ on the set of sequences of partitions with at most N parts as follows. Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(m)})$ be two sequences of partitions with at most N parts. We say that $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Xi}$ if and only if there exists a partition $\{I_1, \dots, I_m\}$ of the set $\{1, 2, \dots, n\}$ such that

$$\mathrm{Hom}_{\mathfrak{gl}_N}\big(V_{\xi^{(i)}}, \bigotimes_{j\in I_i} V_{\lambda^{(j)}}\big) \neq 0, \quad i = 1, \dots, m.$$

Note that $\boldsymbol{\Lambda}$ and $\boldsymbol{\Xi}$ are comparable only if $|\boldsymbol{\Lambda}| = |\boldsymbol{\Xi}|$.

We say that $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is *nontrivial* if and only if $(V_{\boldsymbol{\Lambda}})^{\mathfrak{sl}_N} \neq 0$ and $|\lambda^{(s)}| > 0$, $s = 1, \dots, n$. The sequence $\boldsymbol{\Lambda}$ will be called *d-nontrivial* if $\boldsymbol{\Lambda}$ is nontrivial and $|\boldsymbol{\Lambda}| = N(d-N)$.

Suppose $\boldsymbol{\Xi}$ is d-nontrivial. If $\boldsymbol{\Lambda} \geqslant \boldsymbol{\Xi}$ and $|\lambda^{(s)}| > 0$ for all $s = 1, \dots, n$, then $\boldsymbol{\Lambda}$ is also d-nontrivial.

Recall that $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is the intersection of Schubert cells for each given $\boldsymbol{z}$, see (3.3.5), define $\Omega_{\boldsymbol{\Lambda}}$ by the formula

$$\Omega_{\boldsymbol{\Lambda}} := \bigcup_{\boldsymbol{z}\in\mathring{\mathbb{P}}_n} \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}} \subset \mathrm{Gr}(N,d). \tag{3.3.6}$$

By definition, $\Omega_{\boldsymbol{\Lambda}}$ does not depend on the order of $\lambda^{(s)}$ in the sequence

$$\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}).$$

Note that $\Omega_{\boldsymbol{\Lambda}}$ is a constructible subset of the Grassmannian $\mathrm{Gr}(N,d)$ in Zariski topology. We call $\Omega_{\boldsymbol{\Lambda}}$ with a d-nontrivial $\boldsymbol{\Lambda}$ a $\mathfrak{gl}_N$*-stratum* of $\mathrm{Gr}(N,d)$.

Let $\mu^{(1)}, \dots, \mu^{(a)}$ be the list of all distinct partitions in $\boldsymbol{\Lambda}$. Let n_i be the number of occurrences of $\mu^{(i)}$ in $\boldsymbol{\Lambda}$, $i = 1, \dots, a$, then $\sum_{i=1}^a n_i = n$. Denote $\boldsymbol{n} = (n_1, \dots, n_a)$.

We shall write $\mathbf{\Lambda}$ in the following order: $\lambda^{(i)} = \mu^{(j)}$ for $\sum_{s=1}^{j-1} n_s + 1 \leqslant i \leqslant \sum_{s=1}^{j} n_s$, $j = 1, \dots, a$.

Let $S_{\boldsymbol{n};n_i}$ be the subgroup of the symmetric group S_n permuting $\{n_1+\dots+n_{i-1}+1, \dots, n_1+\dots+n_i\}$, $i = 1, \dots, a$. Then the group $S_{\boldsymbol{n}} = S_{\boldsymbol{n};n_1} \times S_{\boldsymbol{n};n_2} \times \dots \times S_{\boldsymbol{n};n_a}$ acts freely on $\mathring{\mathbb{P}}_n$ and on $\mathbb{R}\mathring{\mathbb{P}}_n$. Denote by $\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$ and $\mathbb{R}\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$ the sets of orbits.

Proposition 3.3.4. *Suppose $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is d-nontrivial. The stratum $\Omega_{\mathbf{\Lambda}}$ is a ramified covering of $\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$. Moreover, the degree of the covering is equal to $\dim(V_{\mathbf{\Lambda}})^{\mathfrak{sl}_N}$. In particular, $\dim \Omega_{\mathbf{\Lambda}} = n$. Over $\mathbb{R}\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$, this covering is unramified of the same degree, moreover all points in fibers are real.*

Proof. The statement follows from Theorem 3.3.2, Lemma 3.3.3, and Theorem 1.1 of [MTV09c]. □

Clearly, we have the following theorem.

Theorem 3.3.5. *We have*

$$\mathrm{Gr}(N, d) = \bigsqcup_{d\text{-nontrivial } \mathbf{\Lambda}} \Omega_{\mathbf{\Lambda}}. \tag{3.3.7}$$

□

Next, for a d-nontrivial $\mathbf{\Lambda}$, we call the closure of $\Omega_{\mathbf{\Lambda}}$ inside $\mathrm{Gr}(N, d)$, a $\mathfrak{gl}_N$*-cycle*. The $\mathfrak{gl}_N$-cycle $\overline{\Omega}_{\mathbf{\Lambda}}$ is an algebraic set. We describe the $\mathfrak{gl}_N$-cycles as unions of $\mathfrak{gl}_N$-strata.

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ and $\mathbf{\Xi} = (\xi^{(1)}, \dots, \xi^{(n-1)})$ be such that $\mathbf{\Xi} \leqslant \mathbf{\Lambda}$. We call $\Omega_{\mathbf{\Xi}}$ a *simple degeneration* of $\Omega_{\mathbf{\Lambda}}$ if and only if both $\mathbf{\Lambda}$ and $\mathbf{\Xi}$ are d-nontrivial. In view of Theorem 3.3.2, taking a simple degeneration is equivalent to making two coordinates of $\boldsymbol{z}$ collide.

Theorem 3.3.6. *If $\Omega_{\mathbf{\Xi}}$ is a simple degeneration of $\Omega_{\mathbf{\Lambda}}$, then $\Omega_{\mathbf{\Xi}}$ is contained in the $\mathfrak{gl}_N$-cycle $\overline{\Omega}_{\mathbf{\Lambda}}$.*

Theorem 3.3.6 is proved in Section 3.7.1.

Suppose $\mathbf{\Theta} = (\theta^{(1)}, \dots, \theta^{(l)})$ is d-nontrivial and $\mathbf{\Lambda} \geqslant \mathbf{\Theta}$. Then, it is clear that $\Omega_{\mathbf{\Theta}}$ is obtained from $\Omega_{\mathbf{\Lambda}}$ by a sequence of simple degenerations. We call $\Omega_{\mathbf{\Theta}}$ a *degeneration* of $\Omega_{\mathbf{\Lambda}}$.

Corollary 3.3.7. *If $\Omega_{\mathbf{\Theta}}$ is a degeneration of $\Omega_{\mathbf{\Lambda}}$, then $\Omega_{\mathbf{\Theta}}$ is contained in the $\mathfrak{gl}_N$-cycle $\overline{\Omega}_{\mathbf{\Lambda}}$.* □

Theorem 3.3.8. *For d-nontrivial $\mathbf{\Lambda}$, we have*

$$\overline{\Omega}_{\mathbf{\Lambda}} = \bigsqcup_{\substack{\mathbf{\Xi} \leqslant \mathbf{\Lambda}, \\ d\text{-nontrivial } \mathbf{\Xi}}} \Omega_{\mathbf{\Xi}}. \tag{3.3.8}$$

Theorem 3.3.8 is proved in Section 3.7.1.

Theorems 3.3.5 and 3.3.8 imply that the subsets $\Omega_{\mathbf{\Lambda}}$ with d-nontrivial $\mathbf{\Lambda}$ give a stratification of $\mathrm{Gr}(N,d)$. We call it the $\mathfrak{gl}_N$*-stratification of* $\mathrm{Gr}(N,d)$.

Example 3.3.9. We give an example of the $\mathfrak{gl}_2$-stratification for $\mathrm{Gr}(2,4)$ in the following picture. In the picture, we simply write $\mathbf{\Lambda}$ for $\Omega_{\mathbf{\Lambda}}$. We also write tuples of numbers with bold font for 4-nontrivial tuples of partitions, solid arrows for simple degenerations between 4-nontrivial tuples of partitions. The dashed arrows go between comparable sequences where the set $\Omega_{\mathbf{\Xi}}$ corresponding to the smaller sequence is empty.

$$\begin{tikzcd}
 & & \mathbf{((1,0),(1,0),(1,0),(1,0))} \arrow[dl] \arrow[dr] & & \\
 & \mathbf{((2,0),(1,0),(1,0))} \arrow[dl, dashed] \arrow[d] \arrow[dr, dashed] \arrow[drr] & & \mathbf{((1,1),(1,0),(1,0))} \arrow[dl, dashed] \arrow[d] \arrow[dr] & \\
((3,0),(1,0)) \arrow[dr, dashed] \arrow[drr, dashed] & \mathbf{((2,0),(2,0))} \arrow[d, dashed] \arrow[drr] & ((2,0),(1,1)) \arrow[d, dashed] & \mathbf{((2,1),(1,0))} \arrow[dl, dashed] \arrow[d] & \mathbf{((1,1),(1,1))} \arrow[dl] \\
 & ((4,0)) & ((3,1)) & \mathbf{((2,2))} &
\end{tikzcd}$$

In particular, $\Omega_{((1,0),(1,0),(1,0),(1,0))}$ is dense in $\mathrm{Gr}(2,4)$.

Remark 3.3.10. In general, for $\mathrm{Gr}(N,d)$, let $\epsilon_1 = (1,0,\dots,0)$ and let

$$\boldsymbol{\Lambda} = (\underbrace{\epsilon_1, \epsilon_1, \dots, \epsilon_1}_{N(d-N)}).$$

Then $\boldsymbol{\Lambda}$ is d-nontrivial, and $\Omega_{\boldsymbol{\Lambda}}$ is dense in $\mathrm{Gr}(N,d)$. Clearly, $\Omega_{\boldsymbol{\Lambda}}$ consists of spaces of polynomials whose Wronskian (see Section 3.3.4) has only simple roots.

Remark 3.3.11. The group of affine translations acts on $\mathbb{C}_d[x]$ by changes of variable. Namely, for $a \in \mathbb{C}^*, b \in \mathbb{C}$, we have a map sending $f(x) \mapsto f(ax+b)$ for all $f(x) \in \mathbb{C}_d[x]$. This group action preserves the Grassmannian $\mathrm{Gr}(N,d)$ and the strata $\Omega_{\boldsymbol{\Lambda}}$.

3.3.4 The Case of $N = 1$ and the Wronski Map

We show that the decomposition in Theorems 3.3.5 and 3.3.8 respects the Wronski map.

From now on, we use the convention that $x - z_s$ is considered as the constant function 1 if $z_s = \infty$.

Consider the Grassmannian of lines $\mathrm{Gr}(1,\tilde{d})$. By Theorem 3.3.5, the decomposition of $\mathrm{Gr}(1,\tilde{d})$ is parameterized by unordered sequences of positive integers $\boldsymbol{m} = (m_1,\dots,m_n)$ such that $|\boldsymbol{m}| = \tilde{d} - 1$.

Let $\boldsymbol{z} = (z_1,\dots,z_n) \in \mathring{\mathbb{P}}_n$. We have $\mathbb{C}f \in \Omega_{\boldsymbol{m},\boldsymbol{z}}$ if and only if

$$f(x) = a\prod_{s=1}^{n}(x - z_s)^{m_s}, \quad a \neq 0.$$

In other words, the stratum $\Omega_{\boldsymbol{m}}$ of the $\mathfrak{gl}_1$-stratification (3.3.7) of $\mathrm{Gr}(1,\tilde{d})$ consists of all points in $\mathrm{Gr}(1,\tilde{d})$ whose representative polynomials have n distinct roots (one of them can be ∞) of multiplicities $m_1,\dots,m_n$.

Therefore the $\mathfrak{gl}_1$-stratification is exactly the celebrated swallowtail stratification.

For $g_1(x),\dots,g_l(x) \in \mathbb{C}[x]$, denote by $\mathrm{Wr}(g_1(x),\dots,g_l(x))$ the *Wronskian*,

$$\mathrm{Wr}(g_1(x),\dots,g_l(x)) = \det(d^{i-1}g_j/dx^{i-1})_{i,j=1}^{l}.$$

Let $X \in \mathrm{Gr}(N,d)$. The Wronskians of two bases of X differ by a multiplication by a nonzero number. We call the monic polynomial representing the Wronskian the *Wronskian* of X and denote it by $\mathrm{Wr}(X)$. It is clear that $\deg_x \mathrm{Wr}(X) \leqslant N(d-N)$.

The *Wronski* map

$$\mathrm{Wr} : \mathrm{Gr}(N,d) \to \mathrm{Gr}(1, N(d-N)+1)$$

is sending $X \in \mathrm{Gr}(N,d)$ to $\mathbb{C}\mathrm{Wr}(X)$.

The Wronski map is a finite algebraic map, see for example Propositions 3.1 and 4.2 in [MTV09a], of degree $\dim(L^{\otimes N(d-N)})^{\mathfrak{sl}_n}$, which is explicitly given by

$$(N(d-N))!\ \frac{0!\ 1!\ 2! \dots (d-N-1)!}{N!\ (N+1)!\ (N+2)! \dots (d-1)!},$$

see [Sch86].

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be d-nontrivial and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$. If $X \in \Omega_{\mathbf{\Lambda},\boldsymbol{z}}$, then one has

$$\mathrm{Wr}(X) = \prod_{s=1}^{n} (x - z_s)^{|\lambda^{(s)}|}.$$

Set $\tilde{d} = N(d-N)+1$. Therefore, we have the following proposition.

Proposition 3.3.12. *The preimage of the stratum $\Omega_{\boldsymbol{m}}$ of $\mathrm{Gr}(1, N(d-N)+1)$ under the Wronski map is a union of all d-nontrivial strata $\Omega_{\mathbf{\Lambda}}$ of $\mathrm{Gr}(N,d)$ such that $|\lambda^{(s)}| = m_s$, $s = 1, \dots, n$.* □

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be an unordered sequence of partitions with at most N parts. Let a be the number of distinct partitions in $\mathbf{\Lambda}$. We can assume that $\lambda^{(1)}, \dots, \lambda^{(a)}$ are all distinct and let $n_1, \dots, n_a$ be their multiplicities in $\mathbf{\Lambda}$, $n_1 + \dots + n_a = n$. Define the *symmetry coefficient* of $\mathbf{\Lambda}$ as the product of multinomial coefficients:

$$b(\mathbf{\Lambda}) = \prod_i \frac{\left(\sum_{s=1,\dots,a,\ |\lambda^{(s)}|=i} n_s\right)!}{\prod_{s=1,\dots,a,\ |\lambda^{(s)}|=i} (n_s)!}. \tag{3.3.9}$$

Proposition 3.3.13. *Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be d-nontrivial. Then the Wronski map $\mathrm{Wr}|_{\Omega_{\mathbf{\Lambda}}} : \Omega_{\mathbf{\Lambda}} \to \Omega_{\boldsymbol{m}}$ is a ramified covering of degree $b(\mathbf{\Lambda}) \dim(V_{\mathbf{\Lambda}})^{\mathfrak{sl}_N}$.*

Proof. The statement follows from Theorem 3.3.2, Lemma 3.3.3, and Proposition 3.3.12. □

In other words, the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$ given by Theorems 3.3.5 and 3.3.8, is adjacent to the swallowtail $\mathfrak{gl}_1$-stratification of $\mathrm{Gr}(1, N(d-N)+1)$ and the Wronski map.

3.4 The $\mathfrak{g}_N$-Stratification of Self-Dual Grassmannian

It is convenient to use the notation: $\mathfrak{g}_{2r+1} = \mathfrak{sp}_{2r}$, and $\mathfrak{g}_{2r} = \mathfrak{so}_{2r+1}$, $r \geqslant 2$. We also set $\mathfrak{g}_3 = \mathfrak{sl}_2$. The case of $\mathfrak{g}_3 = \mathfrak{sl}_2$ is discussed in detail in Section 3.4.6.

3.4.1 Self-Dual Spaces

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a tuple of partitions with at most N parts such that $|\boldsymbol{\Lambda}| = N(d-N)$ and let $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$.

Define a tuple of polynomials $\boldsymbol{T} = (T_1, \dots, T_N)$ by

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{\lambda_i^{(s)} - \lambda_{i+1}^{(s)}}, \quad i = 1, \dots, N, \tag{3.4.1}$$

where $\lambda_{N+1}^{(s)} = 0$. We say that $\boldsymbol{T}$ is *associated with* $\boldsymbol{\Lambda}, \boldsymbol{z}$.

Let $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and $g_1, \dots, g_i \in X$. Define the *divided Wronskian* $\mathrm{Wr}^\dagger$ *with respect to* $\boldsymbol{\Lambda}, \boldsymbol{z}$ by

$$\mathrm{Wr}^\dagger(g_1, \dots, g_i) = \mathrm{Wr}(g_1, \dots, g_i) \prod_{j=1}^{i} T_{N+1-j}^{j-i-1}, \quad i = 1, \dots, N.$$

Note that $\mathrm{Wr}^\dagger(g_1, \dots, g_i)$ is a polynomial in x.

Given $X \in \mathrm{Gr}(N,d)$, define the *dual space* $X^\dagger$ of X by

$$X^\dagger = \{\mathrm{Wr}^\dagger(g_1, \dots, g_{N-1}) \mid g_i \in X,\ i = 1, \dots, N-1\}.$$

Lemma 3.4.1. *If* $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, *then* $X^\dagger \in \Omega_{\tilde{\boldsymbol{\Lambda}},\boldsymbol{z}} \subset \mathrm{Gr}(N, \tilde{d})$, *where*

$$\tilde{d} = \sum_{s=1}^{n} \lambda_1^{(s)} - d + 2N,$$

and $\tilde{\mathbf{\Lambda}} = (\tilde{\lambda}^{(1)}, \dots, \tilde{\lambda}^{(n)})$ *is a sequence of partitions with at most* N *parts such that*

$$\tilde{\lambda}_i^{(s)} = \lambda_1^{(s)} - \lambda_{N+1-i}^{(s)}, \quad i = 1, \dots, N, \quad s = 1, \dots, n.$$

□

Note that we always have $\tilde{\lambda}_N^{(s)} = 0$ for every $s = 1, \dots, n$, hence $X^\dagger$ has no base points.

Given a space of polynomials X and a rational function g in x, denote by $g \cdot X$ the space of rational functions of the form $g \cdot f$ with $f \in X$.

A self-dual space is called a *pure self-dual space* if $X = X^\dagger$. A space of polynomials X is called *self-dual* if $X = g \cdot X^\dagger$ for some polynomial $g \in \mathbb{C}[x]$. In particular, if $X \in \Omega_{\mathbf{\Lambda},\boldsymbol{z}}$ is self-dual, then $X = T_N \cdot X^\dagger$, where T_N is defined in (3.4.1). Note also, that if X is self-dual then $g \cdot X$ is also self-dual.

It is obvious that every point in $\mathrm{Gr}(2, d)$ is a self-dual space.

Let $\mathrm{sGr}(N, d)$ be the set of all self-dual spaces in $\mathrm{Gr}(N, d)$. We call $\mathrm{sGr}(N, d)$ the *self-dual Grassmannian*. The self-dual Grassmannian $\mathrm{sGr}(N, d)$ is an algebraic subset of $\mathrm{Gr}(N, d)$.

Let $\Omega_{\mathbf{\Lambda},\boldsymbol{z}}$ be the finite set defined in (3.3.5) and $\Omega_{\mathbf{\Lambda}}$ the set defined in (3.3.6). Denote by $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{z}}$ the set of all self-dual spaces in $\Omega_{\mathbf{\Lambda},\boldsymbol{z}}$ and by $\mathrm{s}\Omega_{\mathbf{\Lambda}}$ the set of all self-dual spaces in $\Omega_{\mathbf{\Lambda}}$:

$$\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{z}} = \Omega_{\mathbf{\Lambda},\boldsymbol{z}} \bigcap \mathrm{sGr}(N, d) \quad \text{and} \quad \mathrm{s}\Omega_{\mathbf{\Lambda}} = \Omega_{\mathbf{\Lambda}} \bigcap \mathrm{sGr}(N, d).$$

We call the sets $\mathrm{s}\Omega_{\mathbf{\Lambda}}$ $\mathfrak{g}_N$*-strata* of the self-dual Grassmannian. A stratum $\mathrm{s}\Omega_{\mathbf{\Lambda}}$ does not depend on the order of the set of partitions $\mathbf{\Lambda}$. Note that each $\mathrm{s}\Omega_{\mathbf{\Lambda}}$ is a constructible subset of the Grassmannian $\mathrm{Gr}(N, d)$ in Zariski topology.

A partition λ with at most N parts is called N*-symmetric* if $\lambda_i - \lambda_{i+1} = \lambda_{N-i} - \lambda_{N-i+1}$, $i = 1, \dots, N-1$. If the stratum $\mathrm{s}\Omega_{\mathbf{\Lambda}}$ is nonempty, then all partitions $\lambda^{(s)}$ are N-symmetric, see also Lemma 3.4.4 below.

The self-dual Grassmannian is related to the Gaudin model in types B and C, see [MV04] and Theorem 3.4.5 below. We show that $\mathrm{sGr}(N, d)$ also has a remarkable stratification structure similar to the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$, governed by representation theory of $\mathfrak{g}_N$, see Theorems 3.4.11 and 3.4.13.

Remark 3.4.2. The self-dual Grassmannian also has a stratification induced from the usual Schubert cell decomposition (3.3.1), (3.3.2). For $z \in \mathbb{P}^1$, and an N-symmetric partition λ with $\lambda_1 \leqslant d-N$, set $\mathrm{s}\Omega_\lambda(\mathcal{F}(z)) = \Omega_\lambda(\mathcal{F}(z)) \cap \mathrm{sGr}(N,d)$. Then it is easy to see that

$$\mathrm{sGr}(N,d) = \bigsqcup_{\substack{N-\text{symmetric } \mu, \\ \mu_1 \leqslant d-N}} \mathrm{s}\Omega_\mu(\mathcal{F}(z)) \quad \text{and} \quad \overline{\mathrm{s}\Omega}_\lambda(\mathcal{F}(z)) = \bigsqcup_{\substack{N-\text{symmetric } \mu, \\ \mu_1 \leqslant d-N, \ \lambda \subseteq \mu}} \mathrm{s}\Omega_\mu(\mathcal{F}(z)).$$

3.4.2 Bethe Algebras of Types B and C and Self-Dual Grassmannian

The Bethe algebra $\mathcal{B}$ (the algebra of higher Gaudin Hamiltonians) for a simple Lie algebras $\mathfrak{g}$ were described in [FFR94]. The Bethe algebra $\mathcal{B}$ is a commutative subalgebra of $\mathcal{U}(\mathfrak{g}[t])$ which commutes with the subalgebra $\mathcal{U}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g}[t])$. An explicit set of generators of the Bethe algebra in Lie algebras of types B, C, and D was given in [Mol13]. Such a description in the case of $\mathfrak{gl}_N$ is given above in Section 3.2.4. For the case of $\mathfrak{g}_N$ we only need the following fact.

Recall our notation $g(x)$ for the current of $g \in \mathfrak{g}$, see (3.2.1).

Proposition 3.4.3 ([FFR94, Mol13])**.** *Let $N > 3$. There exist elements $F_{ij} \in \mathfrak{g}_N$, $i,j = 1,\dots,N$, and polynomials $G_s(x)$ in $d^k F_{ij}(x)/dx^k$, $s = 1,\dots,N$, $k = 0,\dots,N$, such that the Bethe algebra of $\mathfrak{g}_N$ is generated by coefficients of $G_s(x)$ considered as formal power series in x^{-1}.* □

Similar to the $\mathfrak{gl}_N$ case, for a collection of dominant integral $\mathfrak{g}_N$-weights $\mathbf{\Lambda} = (\lambda^{(1)},\dots,\lambda^{(n)})$ and $\boldsymbol{z} = (z_1,\dots,z_n) \in \mathring{\mathbb{P}}_n$, we set $V_{\mathbf{\Lambda},\boldsymbol{z}} = \bigotimes_{s=1}^n V_{\lambda^{(s)}}(z_s)$, considered as a $\mathcal{B}$-module. Namely, if $\boldsymbol{z} \in \mathbb{C}^n$, then $V_{\mathbf{\Lambda},\boldsymbol{z}}$ is a tensor product of evaluation $\mathfrak{g}_N[t]$-modules and therefore a $\mathcal{B}$-module. If, say, $z_n = \infty$, then $\mathcal{B}$ acts trivially on $V_{\lambda^{(n)}}(\infty)$. More precisely, in this case, $b \in \mathcal{B}$ acts by $b \otimes 1$ where the first factor acts on $\bigotimes_{s=1}^{n-1} V_{\lambda^{(s)}}(z_s)$ and 1 acts on $V_{\lambda^{(n)}}(\infty)$.

We also denote $V_{\mathbf{\Lambda}}$ the module $V_{\mathbf{\Lambda},\boldsymbol{z}}$ considered as a $\mathfrak{g}_N$-module.

Let μ be a dominant integral $\mathfrak{g}_N$-weight and $k \in \mathbb{Z}_{\geqslant 0}$. Define an N-symmetric partition $\mu_{A,k}$ with at most N parts by the rule: $(\mu_{A,k})_N = k$ and

$$(\mu_{A,k})_i - (\mu_{A,k})_{i+1} = \begin{cases} \langle \mu, \check{\alpha}_i \rangle, & \text{if } 1 \leqslant i \leqslant \left[\frac{N}{2}\right], \\ \langle \mu, \check{\alpha}_{N-i} \rangle, & \text{if } \left[\frac{N}{2}\right] < i \leqslant N-1. \end{cases} \tag{3.4.2}$$

We call $\mu_{A,k}$ the partition *associated with weight* μ *and integer* k.

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and let $\boldsymbol{k} = (k_1, \dots, k_n)$ be an n-tuple of nonnegative integers. Then denote

$$\mathbf{\Lambda}_{A,\boldsymbol{k}} = (\lambda^{(1)}_{A,k_1}, \dots, \lambda^{(n)}_{A,k_n})$$

the sequence of partitions associated with $\lambda^{(s)}$ and k_s, $s = 1, \dots, n$.

We use notation $\mu_A = \mu_{A,0}$ and $\mathbf{\Lambda}_A = \mathbf{\Lambda}_{A,(0,\dots,0)}$.

Lemma 3.4.4. *If Ξ is a d-nontrivial sequence of partitions with at most N parts and $\mathrm{s}\Omega_{\Xi}$ is nonempty, then Ξ has the form $\Xi = \mathbf{\Lambda}_{A,\boldsymbol{k}}$ for a sequence of dominant integral $\mathfrak{g}_N$-weights $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ and an n-tuple $\boldsymbol{k}$ of nonnegative integers. The pair $(\mathbf{\Lambda}, \boldsymbol{k})$ is uniquely determined by Ξ. Moreover, if $N = 2r$, then $\sum_{s=1}^n \langle \lambda^{(s)}, \check{\alpha}_r \rangle$ is even.*

Proof. The first statement follows from Lemma 3.4.1. If $N = 2r$ is even, the second statement follows from the equality

$$N(d-N) = |\Xi| = \sum_{s=1}^n r\Big(2\sum_{i=1}^{r-1} \langle \lambda^{(s)}, \check{\alpha}_i \rangle + \langle \lambda^{(s)}, \check{\alpha}_r \rangle\Big) + N \sum_{s=1}^n k_s. \qquad \square$$

Therefore the strata are effectively parameterized by sequences of dominant integral $\mathfrak{g}_N$-weights and tuples of nonnegative integers. In what follows we write $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}$ for $\mathrm{s}\Omega_{\mathbf{\Lambda}_{A,\boldsymbol{k}}}$ and $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ for $\mathrm{s}\Omega_{\mathbf{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$.

Define a formal differential operator

$$\mathcal{D}^{\mathcal{B}} = \partial_x^N + \sum_{i=1}^N G_i(x) \partial_x^{N-i}.$$

For a $\mathcal{B}$-eigenvector $v \in V_{\mathbf{\Lambda},\boldsymbol{z}}$, $G_i(x)v = h_i(x)v$, we denote $\mathcal{D}_v = \partial_x^N + \sum_{i=1}^N h_i(x)\partial_x^{N-i}$ the corresponding scalar differential operator.

Theorem 3.4.5. *Let $N > 3$. There exists a choice of generators $G_i(x)$ of the $\mathfrak{g}_N$ Bethe algebra $\mathcal{B}$ (see Proposition 3.4.3), such that for any sequence of dominant integral $\mathfrak{g}_N$-weights $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$, any $\boldsymbol{z} \in \mathring{\mathbb{P}}_n$, and any $\mathcal{B}$-eigenvector $v \in (V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$, we have* $\operatorname{Ker}\left((T_1 \dots T_N)^{1/2} \cdot \mathcal{D}_v \cdot (T_1 \dots T_N)^{-1/2}\right) \in \mathrm{s}\Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}}$, *where $\boldsymbol{T} = (T_1, \dots, T_N)$ is associated with $\boldsymbol{\Lambda}_A, \boldsymbol{z}$.*

Moreover, if $|\boldsymbol{\Lambda}_A| = N(d-N)$, then this defines a bijection between the joint eigenvalues of $\mathcal{B}$ on $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ and $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}} \subset \mathrm{Gr}(N,d)$.

Proof. Theorem 3.4.5 is deduced from [Ryb18] in Section 3.7.2. □

The second part of the theorem also holds for $N = 3$, see Section 3.4.6.

Remark 3.4.6. In particular, Theorem 3.4.5 implies that if $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N} > 0$, then $\dim(V_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}}})^{\mathfrak{sl}_N} > 0$. This statement also follows from Lemma 3.8.2 given in the Appendix.

We also have the following lemma from [Ryb18].

Lemma 3.4.7. *Let $\boldsymbol{z}$ be a generic point in $\mathring{\mathbb{P}}_n$. Then the action of the $\mathfrak{g}_N$ Bethe algebra on $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ is diagonalizable and has simple spectrum. In particular, this statement holds for any sequence $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$.* □

3.4.3 Properties of the Strata

We describe simple properties of the strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$.

Given $\boldsymbol{\Lambda}, \boldsymbol{k}, \boldsymbol{z}$, define $\tilde{\boldsymbol{\Lambda}}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{z}}$ by removing all zero components, that is the ones with both $\lambda^{(s)} = 0$ and $k_s = 0$. Then $\mathrm{s}\Omega_{\tilde{\boldsymbol{\Lambda}},\tilde{\boldsymbol{k}},\tilde{\boldsymbol{z}}} = \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ and $\mathrm{s}\Omega_{\tilde{\boldsymbol{\Lambda}},\tilde{\boldsymbol{k}}} = \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$. Also, by Remark 3.4.6, if $(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N} \neq 0$, then $\dim(V_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}}})^{\mathfrak{sl}_N} > 0$, thus $|\boldsymbol{\Lambda}_{A,\boldsymbol{k}}|$ is divisible by N.

We say that $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is *d-nontrivial* if and only if $(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N} \neq 0$, $|\lambda^{(s)}_{A,k_s}| > 0$, $s = 1, \dots, n$, and $|\boldsymbol{\Lambda}_{A,\boldsymbol{k}}| = N(d-N)$.

If $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial then the corresponding stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ is nonempty, see Proposition 3.4.9 below.

Note that $|\boldsymbol{\Lambda}_{A,\boldsymbol{k}}| = |\boldsymbol{\Lambda}_A| + N|\boldsymbol{k}|$, where $|\boldsymbol{k}| = k_1 + \cdots + k_n$. In particular, if $(\boldsymbol{\Lambda}, \boldsymbol{0})$ is d-nontrivial then $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is $(d + |\boldsymbol{k}|)$-nontrivial. Further, there exists a bijection between $\Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}}$ in $\mathrm{Gr}(N,d)$ and $\Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$ in $\mathrm{Gr}(N, d + |\boldsymbol{k}|)$ given by

$$\Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}} \to \Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}, \quad X \mapsto \prod_{s=1}^{n} (x - z_s)^{k_s} \cdot X. \tag{3.4.3}$$

Moreover, (3.4.3) restricts to a bijection of $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_A,\boldsymbol{z}}$ in $\mathrm{sGr}(N,d)$ and $\mathrm{s}\Omega_{\boldsymbol{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$ in $\mathrm{sGr}(N, d + |\boldsymbol{k}|)$.

If $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial then $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ is d-nontrivial. The converse is not true.

Example 3.4.8. For this example we write the highest weights in terms of fundamental weights, e.g. $(1,0,0,1) = \omega_1 + \omega_4$. We also use $\mathfrak{sl}_N$-modules instead of $\mathfrak{gl}_N$-modules, since the spaces of invariants are the same.

For $N = 4$ and $\mathfrak{g}_4 = \mathfrak{so}_5$ of type B_2, we have

$$\dim(V_{(2,0)} \otimes V_{(1,0)} \otimes V_{(2,0)})^{\mathfrak{g}_4} = 0 \quad \text{and} \quad \dim(V_{(2,0,2)} \otimes V_{(1,0,1)} \otimes V_{(2,0,2)})^{\mathfrak{sl}_4} = 2.$$

Let $\boldsymbol{\Lambda} = ((2,0),(1,0),(2,0))$. Then $\boldsymbol{\Lambda}_A$ is 9-nontrivial, but $(\boldsymbol{\Lambda},(0,0,0))$ is not.

Similarly, for $N = 5$ and $\mathfrak{g}_5 = \mathfrak{sp}_4$ of type C_2, we have

$$\dim(V_{(1,0)} \otimes V_{(0,1)} \otimes V_{(0,1)})^{\mathfrak{g}_5} = 0 \quad \text{and} \quad \dim(V_{(1,0,0,1)} \otimes V_{(0,1,1,0)} \otimes V_{(0,1,1,0)})^{\mathfrak{sl}_5} = 2.$$

Let $\boldsymbol{\Lambda} = ((1,0),(0,1),(1,0))$. Then $\boldsymbol{\Lambda}_A$ is 8-nontrivial, but $(\boldsymbol{\Lambda},(0,0,0))$ is not.

Let $\mu^{(1)}, \dots, \mu^{(a)}$ be all distinct partitions in $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$. Let n_i be the number of occurrences of $\mu^{(i)}$ in $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$, then $\sum_{i=1}^{a} n_i = n$. Denote $\boldsymbol{n} = (n_1, \dots, n_a)$, we shall write $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ in the following order: $\lambda^{(i)}_{A,k_i} = \mu^{(j)}$ for $\sum_{s=1}^{j-1} n_s + 1 \leqslant i \leqslant \sum_{s=1}^{j} n_s$, $j = 1, \dots, a$.

Proposition 3.4.9. *Suppose $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial. The set $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ is a ramified covering of $\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$. Moreover, the degree of the covering is equal to $\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N}$. In particular, $\dim \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} = n$. Over $\mathbb{R}\mathring{\mathbb{P}}_n/S_{\boldsymbol{n}}$, this covering is unramified of the same degree, moreover all points in fibers are real.*

Proof. The proposition follows from Theorem 3.4.5, Lemma 3.4.7, and Theorem 1.1 of [MTV09c]. $\square$

We find strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ of the largest dimension.

Lemma 3.4.10. *If $N = 2r$, then the d-nontrivial stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ with the largest dimension has $(\lambda^{(s)}, k_s) = (\omega_r, 0)$, $s = 1, \dots, 2(d-N)$. In particular, the dimension of this stratum is $2(d-N)$.*

If $N = 2r+1$, the d-nontrivial strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \subset \mathrm{sGr}(N,d)$ with the largest dimension have $(\lambda^{(s)}, k_s)$ equal to either $(\omega_{j_s}, 0)$ with some $j_s \in \{1, \dots, r\}$, or to $(0,1)$, for $s = 1, \dots, d-N$. Each such stratum is either empty or has dimension $d-N$. There is at least one nonempty stratum of this dimension, and if $d > N+1$ then more than one.

Proof. By Proposition 3.4.9, we are going to find the maximal n such that $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial, where $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is a sequence of dominant integral $\mathfrak{g}_N$-weights and $\boldsymbol{k} = (k_1, \dots, k_n)$ is an n-tuple of nonnegative integers. Since $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ is d-nontrivial, it follows that $\lambda^{(s)} \neq 0$ or $\lambda^{(s)} = 0$ and $k_s > 0$, for all $s = 1, \dots, n$.

Suppose $N = 2r$. If $\lambda^{(s)} \neq 0$, we have

$$|\lambda^{(s)}_{A,k_s}| \geqslant |\lambda^{(s)}_{A,0}| = r\Big(2\sum_{i=1}^{r-1}\langle\lambda^{(s)}, \check{\alpha}_i\rangle + \langle\lambda^{(s)}, \check{\alpha}_r\rangle\Big) \geqslant r.$$

If $k_s > 0$, then $|\lambda^{(s)}_{A,k_s}| \geqslant 2rk_s \geqslant 2r$. Therefore, it follows that

$$rn \leqslant \sum_{s=1}^{n} |\lambda^{(s)}_{A,k_s}| = |\boldsymbol{\Lambda}_{A,\boldsymbol{k}}| = (d-N)N.$$

Hence $n \leqslant 2(d-N)$.

If we set $\lambda^{(s)} = w_r$ and $k_s = 0$ for all $s = 1, \dots, 2(d-N)$. Then $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial since

$$\dim(V_{\omega_r} \otimes V_{\omega_r})^{\mathfrak{so}_{2r+1}} = 1.$$

Now let us consider $N = 2r+1$, $r \geqslant 1$. Similarly, if $\lambda^{(s)} \neq 0$, we have

$$|\lambda^{(s)}_{A,k_s}| \geqslant |\lambda^{(s)}_{A,0}| = (2r+1)\sum_{i=1}^{r}\langle\lambda^{(s)}, \check{\alpha}_i\rangle \geqslant 2r+1.$$

If $k_s > 0$, then $|\lambda^{(s)}_{A,k_s}| \geqslant (2r+1)k_s \geqslant 2r+1$. It follows that

$$(2r+1)n \leqslant \sum_{s=1}^{n} |\lambda^{(s)}_{A,k_s}| = |\mathbf{\Lambda}_{A,\boldsymbol{k}}| = (d-N)N.$$

Hence $n \leqslant d-N$. Clearly, the equality is achieved only for the $(\mathbf{\Lambda}, \boldsymbol{k})$ described in the statement of the lemma. Note that if $(\lambda^{(s)}, k_s) = (0,1)$ for all $s = 1, \dots, d-N$, then $(\mathbf{\Lambda}, \boldsymbol{k})$ is d-nontrivial and therefore nonempty. If $d > N+1$ we also have d-nontrivial tuples parameterized by $i = 1, \dots, r$, such that $(\lambda^{(s)}, k_s) = (0,1)$, $s = 3, \dots, d-N$, and $(\lambda^{(s)}, k_s) = (\omega_i, 0)$, $s = 1, 2$. $\square$

3.4.4 The $\mathfrak{g}_N$-Stratification of Self-Dual Grassmannian

The following theorem follows directly from Theorems 3.3.5 and 3.4.5.

Theorem 3.4.11. *We have*

$$\mathrm{sGr}(N,d) = \bigsqcup_{d\text{-nontrivial } (\mathbf{\Lambda}, \boldsymbol{k})} \mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}. \tag{3.4.4}$$

$\square$

Next, for a d-nontrivial $(\mathbf{\Lambda}, \boldsymbol{k})$, we call the closure of $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}$ inside $\mathrm{sGr}(N,d)$, a $\mathfrak{g}_N$-*cycle*. The $\mathfrak{g}_N$-cycles $\overline{\mathrm{s}\Omega}_{\mathbf{\Lambda},\boldsymbol{k}}$ are algebraic sets in $\mathrm{sGr}(N,d)$ and therefore in $\mathrm{Gr}(N,d)$. We describe $\mathfrak{g}_N$-cycles as unions of $\mathfrak{g}_N$-strata similar to (3.3.8).

Define a partial order $\geqslant$ on the set of pairs $\{(\mathbf{\Lambda}, \boldsymbol{k})\}$ as follows. Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$, $\mathbf{\Xi} = (\xi^{(1)}, \dots, \xi^{(m)})$ be two sequences of dominant integral $\mathfrak{g}_N$-weights. Let $\boldsymbol{k} = (k_1, \dots, k_n)$, $\boldsymbol{l} = (l_1, \dots, l_m)$ be two tuples of nonnegative integers. We say that $(\mathbf{\Lambda}, \boldsymbol{k}) \geqslant (\mathbf{\Xi}, \boldsymbol{l})$ if and only if there exists a partition $\{I_1, \dots, I_m\}$ of $\{1, 2, \dots, n\}$ such that

$$\mathrm{Hom}_{\mathfrak{g}_N}(V_{\xi^{(i)}}, \bigotimes_{j\in I_i} V_{\lambda^{(j)}}) \neq 0, \qquad |\xi^{(i)}_{A,l_i}| = \sum_{j\in I_i} |\lambda^{(j)}_{A,k_j}|,$$

for $i = 1, \dots, m$.

If $(\mathbf{\Lambda}, \boldsymbol{k}) \geqslant (\mathbf{\Xi}, \boldsymbol{l})$ are d-nontrivial, we call $\mathrm{s}\Omega_{\mathbf{\Xi},\boldsymbol{l}}$ a *degeneration* of $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}$. If we suppose further that $m = n-1$, we call $\mathrm{s}\Omega_{\mathbf{\Xi},\boldsymbol{l}}$ a *simple degeneration* of $\mathrm{s}\Omega_{\mathbf{\Lambda},\boldsymbol{k}}$.

Theorem 3.4.12. *If* $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$ *is a degeneration of* $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$*, then* $\mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}$ *is contained in the* $\mathfrak{g}_N$*-cycle* $\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}}$.

Theorem 3.4.12 is proved in Section 3.7.2.

Theorem 3.4.13. *For d-nontrivial* $(\boldsymbol{\Lambda},\boldsymbol{k})$*, we have*

$$\overline{\mathrm{s}\Omega}_{\boldsymbol{\Lambda},\boldsymbol{k}} = \bigsqcup_{\substack{(\boldsymbol{\Xi},\boldsymbol{l})\leqslant(\boldsymbol{\Lambda},\boldsymbol{k}),\\ d\text{-nontrivial }(\boldsymbol{\Xi},\boldsymbol{l})}} \mathrm{s}\Omega_{\boldsymbol{\Xi},\boldsymbol{l}}. \tag{3.4.5}$$

Theorem 3.4.13 is proved in Section 3.7.2.

Theorems 3.4.11 and 3.4.13 imply that the subsets $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ with d-nontrivial $(\boldsymbol{\Lambda},\boldsymbol{k})$ give a stratification of $\mathrm{sGr}(N,d)$, similar to the $\mathfrak{gl}_N$-stratification of $\mathrm{Gr}(N,d)$, see (3.3.7) and (3.3.8). We call it the $\mathfrak{g}_N$*-stratification of* $\mathrm{sGr}(N,d)$.

Example 3.4.14. The following picture gives an example for $\mathfrak{so}_5$-stratification of $\mathrm{sGr}(4,6)$. In the following picture, we write $((\lambda^{(1)})_{k_1},\dots,(\lambda^{(n)})_{k_n})$ for $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$. We also simply write $\lambda^{(s)}$ for $(\lambda^{(s)})_0$. For instance, $((0,1)_1,(0,1))$ represents $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ where $\boldsymbol{\Lambda}=((0,1),(0,1))$ and $\boldsymbol{k}=(1,0)$. The solid arrows represent simple degenerations. Unlike the picture in Example 3.3.9 we do not include here the pairs of sequences which are not 6-nontrivial, as there are too many of them.

$$\begin{tikzcd}[column sep=tiny]
 & & & ((0,1),(0,1),(0,1),(0,1)) \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & ((0,2),(0,1),(0,1)) \arrow[dl] \arrow[drrr] & & ((1,0),(0,1),(0,1)) \arrow[dl] \arrow[dr] & & ((0,0)_1,(0,1),(0,1)) \arrow[dl] \arrow[dr] & \\
((0,2),(0,2)) \arrow[drrr] & & ((1,0),(1,0)) \arrow[dr] & & ((0,1)_1,(0,1)) \arrow[dl] & & ((0,0)_1,(0,0)_1) \arrow[dlll] \\
 & & & ((0,0)_2) & & &
\end{tikzcd}$$

In particular, the stratum $\mathrm{s}\Omega_{((0,1),(0,1),(0,1),(0,1))}$ is dense in $\mathrm{sGr}(4,6)$.

Proposition 3.4.15. *If $N = 2r$ is even, then the stratum $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ with $(\lambda^{(s)}, k_s) = (\omega_r, 0)$, where $s = 1, \dots, 2(d-N)$, is dense in $\mathrm{sGr}(N,d)$.*

Proof. For $N = 2r$, one has the $\mathfrak{g}_N$-module decomposition

$$V_{\omega_r} \otimes V_{\omega_r} = V_{2\omega_r} \oplus V_{\omega_1} \oplus \cdots \oplus V_{\omega_{r-1}} \oplus V_{(0,\dots,0)}. \tag{3.4.6}$$

It is clear that $(\boldsymbol{\Lambda}, \boldsymbol{k})$ is d-nontrivial. It also follows from (3.4.6) that if $(\boldsymbol{\Xi}, \boldsymbol{l})$ is d-nontrivial then $(\boldsymbol{\Lambda}, \boldsymbol{k}) \geqslant (\boldsymbol{\Xi}, \boldsymbol{l})$. The proposition follows from Theorems 3.4.11 and 3.4.13. $\square$

Remark 3.4.16. The group of affine translations, see Remark 3.3.11, preserves the self-dual Grassmannian $\mathrm{sGr}(N,d)$ and the strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$.

3.4.5 The $\mathfrak{g}_N$-Stratification of $\mathrm{sGr}(N,d)$ and the Wronski Map

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and let $\boldsymbol{k} = (k_1, \dots, k_n)$ be an n-tuple of nonnegative integers. Let $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$.

Recall that $\lambda_i^{(s)} = \langle \lambda^{(s)}, \check{\alpha}_i \rangle$. If $X \in \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$, one has

$$\mathrm{Wr}(X) = \begin{cases} \Big(\prod_{s=1}^{n} (x - z_s)^{\lambda_1^{(s)} + \cdots + \lambda_r^{(s)} + k_s} \Big)^N, & \text{if } N = 2r+1; \\ \Big(\prod_{s=1}^{n} (x - z_s)^{2\lambda_1^{(s)} + \cdots + 2\lambda_{r-1}^{(s)} + \lambda_r^{(s)} + 2k_s} \Big)^r, & \text{if } N = 2r. \end{cases}$$

We define the *reduced Wronski map* $\overline{\mathrm{Wr}}$ as follows.

If $N = 2r+1$, the reduced Wronski map

$$\overline{\mathrm{Wr}} : \mathrm{sGr}(N,d) \to \mathrm{Gr}(1, d-N+1)$$

is sending $X \in \mathrm{sGr}(N,d)$ to $\mathbb{C}(\mathrm{Wr}(X))^{1/N}$.

If $N = 2r$, the reduced Wronski map

$$\overline{\mathrm{Wr}} : \mathrm{sGr}(N,d) \to \mathrm{Gr}(1, 2(d-N)+1)$$

is sending $X \in \mathrm{sGr}(N,d)$ to $\mathbb{C}(\mathrm{Wr}(X))^{1/r}$.

The reduced Wronski map is also a finite map.

For $N = 2r$, the degree of the reduced Wronski map is given by $\dim(V_{\omega_r}^{\otimes 2(d-N)})^{\mathfrak{g}_N}$. This dimension is given by, see [KLP12],

$$(N-1)!! \prod_{1\leqslant i<j\leqslant r} \big((j-i)(N-i-j+1)\big) \prod_{k=0}^{r-1} \frac{(2(d-N+k))!}{(d-k-1)!(d-N+k)!}. \qquad (3.4.7)$$

Let $\tilde{d} = d-N+1$ if $N = 2r+1$ and $\tilde{d} = 2(d-N)+1$ if $N = 2r$. Let $\boldsymbol{m} = (m_1,\ldots,m_n)$ be an unordered sequence of positive integers such that $|\boldsymbol{m}| = \tilde{d}-1$.

Similar to Section 3.3.4, we have the following proposition.

Proposition 3.4.17. *The preimage of the stratum $\Omega_{\boldsymbol{m}}$ of $\mathrm{Gr}(1,\tilde{d})$ under the reduced Wronski map is a union of all strata $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}$ of $\mathrm{sGr}(N,d)$ such that $|\lambda^{(s)}_{A,k_s}| = Nm_s$, $s = 1,\ldots,n$, if N is odd and such that $|\lambda^{(s)}_{A,k_s}| = rm_s$, $s = 1,\ldots,n$, if $N = 2r$ is even.* □

Let $\boldsymbol{\Lambda} = (\lambda^{(1)},\ldots,\lambda^{(n)})$ be an unordered sequence of dominant integral $\mathfrak{g}_N$-weights and $\boldsymbol{k} = (k_1,\ldots,k_n)$ a sequence of nonnegative integers. Let a be the number of distinct pairs in the set $\{(\lambda^{(s)},k_s),\ s = 1,\ldots,n\}$. We can assume that $(\lambda^{(1)},k_1),\ldots,(\lambda^{(a)},k_a)$ are all distinct, and let $n_1,\ldots,n_a$ be their multiplicities, $n_1+\cdots+n_a = n$.

Consider the unordered set of integers $\boldsymbol{m} = (m_1,\ldots,m_n)$, where $Nm_s = |\lambda^{(s)}_{A,k_s}|$ if N is odd or $rm_s = |\lambda^{(s)}_{A,k_s}|$ if $N = 2r$ is even. Consider the stratum $\Omega_{\boldsymbol{m}}$ in $\mathrm{Gr}(1,\tilde{d})$, corresponding to polynomials with n distinct roots of multiplicities $m_1,\ldots,m_n$.

Proposition 3.4.18. *Let $(\boldsymbol{\Lambda},\boldsymbol{k})$ be d-nontrivial. Then the reduced Wronski map $\overline{\mathrm{Wr}}|_{\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}}} : \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k}} \to \Omega_{\boldsymbol{m}}$ is a ramified covering of degree $b(\boldsymbol{\Lambda}_{A,\boldsymbol{k}})\dim(V_{\boldsymbol{\Lambda}})^{\mathfrak{g}_N}$, where $b(\boldsymbol{\Lambda}_{A,\boldsymbol{k}})$ is given by (3.3.9).*

Proof. The statement follows from Theorem 3.4.5, Lemma 3.4.7, and Proposition 3.4.17. □

In other words, the $\mathfrak{g}_N$-stratification of $\mathrm{sGr}(N,d)$ given by Theorems 3.4.11 and 3.4.13, is adjacent to the swallowtail $\mathfrak{gl}_1$-stratification of $\mathrm{Gr}(1,\tilde{d})$ and the reduced Wronski map.

3.4.6 Self-Dual Grassmannian for $N=3$

Let $N=3$ and $\mathfrak{g}_3=\mathfrak{sl}_2$. We identify the dominant integral $\mathfrak{sl}_2$-weights with nonnegative integers. Let $\boldsymbol{\Lambda}=(\lambda^{(1)},\dots,\lambda^{(n)},\lambda)$ be a sequence of nonnegative integers and $\boldsymbol{z}=(z_1,\dots,z_n,\infty)\in\mathring{\mathbb{P}}_{n+1}$.

Choose d large enough so that $k:=d-3-\sum_{s=1}^{n}\lambda^{(s)}-\lambda\geqslant 0$. Let $\boldsymbol{k}=(0,\dots,0,k)$. Then $\boldsymbol{\Lambda}_{A,\boldsymbol{k}}$ has coordinates

$$\lambda_A^{(s)}=(2\lambda^{(s)},\lambda^{(s)},0),\quad s=1,\dots,n,$$

$$\lambda_{A,k}=\Big(d-3-\sum_{s=1}^{n}\lambda^{(s)}+\lambda,d-3-\sum_{s=1}^{n}\lambda^{(s)},d-3-\sum_{s=1}^{n}\lambda^{(s)}-\lambda\Big).$$

Note that we always have $|\boldsymbol{\Lambda}_{A,\boldsymbol{k}}|=3(d-3)$ and spaces of polynomials in $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ are pure self-dual spaces.

Theorem 3.4.19. *There exists a bijection between the common eigenvectors of the $\mathfrak{gl}_2$ Bethe algebra $\mathcal{B}$ in $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_2}$ and $\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$.*

Proof. Let $X\in\mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$, and let $\boldsymbol{T}=(T_1(x),T_2(x),T_3(x))$ be associated with $\boldsymbol{\Lambda}_{A,\boldsymbol{k}},\boldsymbol{z}$, then

$$T_1(x)=T_2(x)=\prod_{s=1}^{n}(x-z_s)^{\lambda^{(s)}}.$$

Following Section 6 of [MV04], let $\boldsymbol{u}=(u_1,u_2,u_3)$ be a Witt basis of X, one has

$$\mathrm{Wr}(u_1,u_2)=T_1u_1,\quad \mathrm{Wr}(u_1,u_3)=T_1u_2,\quad \mathrm{Wr}(u_2,u_3)=T_1u_3.$$

Let $y(x,c)=u_1+cu_2+\frac{c^2}{2}u_3$, it follows from Lemma 6.15 of [MV04] that

$$\mathrm{Wr}\Big(y(x,c),\frac{\partial y}{\partial c}(x,c)\Big)=T_1y(x,c).$$

Since X has no base points, there must exist $c'\in\mathbb{C}$ such that $y(x,c')$ and $T_1(x)$ do not have common roots. It follows from Lemma 6.16 of [MV04] that $y(x,c')=p^2$ and

$y(x,c) = (p + (c-c')q)^2$ for suitable polynomials $p(x), q(x)$ satisfying $\mathrm{Wr}(p,q) = 2T_1$. In particular, $\{p^2, pq, q^2\}$ is a basis of X. Without loss of generality, we can assume that $\deg p < \deg q$. Then

$$\deg p = \frac{1}{2}\Big(\sum_{s=1}^{n} \lambda^{(s)} - \lambda\Big), \quad \deg q = \frac{1}{2}\Big(\sum_{s=1}^{n} \lambda^{(s)} + \lambda\Big) + 1.$$

Since X has no base points, p and q do not have common roots. Combining with the equality $\mathrm{Wr}(p,q) = 2T_1$, one has that the space spanned by p and q has singular points at $z_1, \dots, z_n$ and ∞ only. Moreover, the exponents at z_s, $s = 1, \dots, n$, are equal to $0, \lambda^{(s)} + 1$, and the exponents at ∞ are equal to $-\deg p, -\deg q$.

By Theorem 3.3.2, the space $\mathrm{span}\{p,q\}$ corresponds to a common eigenvector of the $\mathfrak{gl}_2$ Bethe subalgebra in the subspace

$$\Big(\bigotimes_{s=1}^{n} V_{(\lambda^{(s)},0)}(z_s) \otimes V_{(d-2-\deg p, d-1-\deg q)}(\infty)\Big)^{\mathfrak{sl}_2}.$$

Conversely, given a common eigenvector of the $\mathfrak{gl}_2$ Bethe algebra in $(V_{\boldsymbol{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_2}$, by Theorem 3.3.2, it corresponds to a space $\tilde{X}$ of polynomials in $\mathrm{Gr}(2,d)$ without base points. Let $\{p,q\}$ be a basis of $\tilde{X}$, define a space of polynomials $\mathrm{span}\{p^2, pq, q^2\}$ in $\mathrm{Gr}(3,d)$. It is easy to see that $\mathrm{span}\{p^2, pq, q^2\} \in \mathrm{s}\Omega_{\boldsymbol{\Lambda},\boldsymbol{k},\boldsymbol{z}}$ is a pure self-dual space. $\square$

Let $X \in \mathrm{Gr}(2,d)$, denote by X^2 the space spanned by f^2 for all polynomials $f \in X$. It is clear that $X^2 \in \mathrm{sGr}(3, 2d-1)$. Define

$$\pi : \mathrm{Gr}(2,d) \to \mathrm{sGr}(3, 2d-1) \tag{3.4.8}$$

by sending X to X^2. The map π is an injective algebraic map.

Corollary 3.4.20. *The map π defines a bijection between the subset of spaces of polynomials without base points in* $\mathrm{Gr}(2,d)$ *and the subset of pure self-dual spaces in* $\mathrm{sGr}(3, 2d-1)$. $\square$

Note that not all self-dual spaces in $\mathrm{sGr}(3, 2d-1)$ can be expressed as X^2 for some $X \in \mathrm{Gr}(2,d)$ since the greatest common divisor of a self-dual space does not have to be a square of a polynomial.

3.4.7 Geometry and Topology

It would be very interesting to determine the topology and geometry of the strata and cycles of $\mathrm{Gr}(N,d)$ and of $\mathrm{sGr}(N,d)$. In particular, it would be interesting to understand the geometry and topology of the self-dual Grassmannian $\mathrm{sGr}(N,d)$. Here are some simple examples of small dimension.

Of course, $\mathrm{sGr}(N,N)=\mathrm{Gr}(N,N)$ is just one point. Also, $\mathrm{sGr}(2r+1,2r+2)$ is just $\mathbb{P}^1$.

Consider $\mathrm{sGr}(2r,2r+1)$, $r\geqslant 1$. It has only two strata: $\mathrm{s}\Omega_{(\omega_r,\omega_r),(0,0)}$ and $\mathrm{s}\Omega_{(0),(1)}$. Moreover, the reduced Wronski map has degree 1 and defines a bijection: $\overline{\mathrm{Wr}}$: $\mathrm{sGr}(2r,2r+1)\to\mathrm{Gr}(1,3)$. In particular, the $\mathfrak{so}_{2r+1}$-stratification in this case is identified with the swallowtail $\mathfrak{gl}_1$-stratification of quadratics. There are two strata: polynomials with two distinct roots and polynomials with one double root. Therefore through the reduced Wronski map, the self-dual Grassmannian $\mathrm{sGr}(2r,2r+1)$ can be identified with $\mathbb{P}^2$ with coordinates $(a_0:a_1:a_2)$ and the stratum $\mathrm{s}\Omega_{(0),(1)}$ is a nonsingular curve of degree 2 given by the equation $a_1^2-4a_0a_2=0$.

Consider $\mathrm{sGr}(2r+1,2r+3)$, $r\geqslant 1$. In this case we have $r+2$ strata: $\mathrm{s}\Omega_{(\omega_i,\omega_i),(0,0)}$, $i=1,\dots,r$, $\mathrm{s}\Omega_{(0,0),(1,1)}$, and $\mathrm{s}\Omega_{(0),(2)}$. The reduced Wronski map $\overline{\mathrm{Wr}}$: $\mathrm{sGr}(2r+1,2r+3)\to\mathrm{Gr}(1,3)$ restricted to any strata again has degree 1. Therefore, through the reduced Wronski map, the self-dual Grassmannian $\mathrm{sGr}(2r+1,2r+3)$ can be identified with $r+1$ copies of $\mathbb{P}^2$ all intersecting in the same nonsingular degree 2 curve corresponding to the stratum $\mathrm{s}\Omega_{(0),(2)}$. In particular, every 2-dimensional $\mathfrak{sp}_{2r}$-cycle is just $\mathbb{P}^2$.

Consider $\mathrm{sGr}(2r+1,2r+4)$, $r\geqslant 1$. We have $\dim\mathrm{sGr}(2r+1,2r+4)=3$. This is the last case when for all strata the coverings of Proposition 3.4.9 have degree one. There are already many strata. For example, consider $\mathrm{sGr}(5,8)$, that is $r=2$. There are four strata of dimension 3 corresponding to the following sequences of $\mathfrak{sp}_4$-weights and 3-tuples of nonnegative integers:

$$\boldsymbol{\Lambda}_1=(\omega_1,\omega_1,0),\quad \boldsymbol{k}_1=(0,0,1);\qquad \boldsymbol{\Lambda}_2=(\omega_1,\omega_1,\omega_2),\quad \boldsymbol{k}_2=(0,0,0);$$

$$\mathbf{\Lambda}_3 = (\omega_2, \omega_2, 0), \quad \boldsymbol{k}_3 = (0,0,1); \qquad \mathbf{\Lambda}_4 = (0,0,0), \quad \boldsymbol{k}_4 = (1,1,1).$$

By the reduced Wronski map, the stratum $\Omega_{\mathbf{\Lambda}_4, \boldsymbol{k}_4}$ is identified with the subset of $\mathrm{Gr}(1,4)$ represented by cubic polynomials without multiple roots and the cycle $\overline{\Omega}_{\mathbf{\Lambda}_4, \boldsymbol{k}_4}$ with $\mathrm{Gr}(1,4) = \mathbb{P}^3$. The stratification of $\overline{\Omega}_{\mathbf{\Lambda}_4, \boldsymbol{k}_4}$ is just the swallowtail of cubic polynomials. However, for other three strata the reduced Wronski map has degree 3. Using instead the map in Proposition 3.4.9, we identify each of these strata with $\mathring{\mathbb{P}}_3/(\mathbb{Z}/2\mathbb{Z})$ or with the subset of $\mathrm{Gr}(1,3) \times \mathrm{Gr}(1,2)$ represented by a pair of polynomials (p_1, p_2), such that $\deg(p_1) \leqslant 2$, $\deg(p_2) \leqslant 1$ and such that all three roots (including infinity) of $p_1 p_2$ are distinct. Then the corresponding $\mathfrak{sp}_4$-cycles $\overline{\Omega}_{\mathbf{\Lambda}_i, \boldsymbol{k}_i}$, $i = 1,2,3$, are identified with $\mathrm{Gr}(1,3) \times \mathrm{Gr}(1,2) = \mathbb{P}^2 \times \mathbb{P}^1$.

A similar picture is observed for 3-dimensional strata in the case of $\mathrm{sGr}(2r, 2r+2)$. Consider, for example, $\mathrm{Gr}(2,4)$, see Example 3.3.9. Then the 4-dimensional stratum $\Omega_{(1,0),(1,0),(1,0),(1,0)}$ is dense and (relatively) complicated, as the corresponding covering in Proposition 3.3.4 has degree 2. But for the 3-dimensional strata the degrees are 1. Therefore, $\Omega_{(2,0),(1,0),(1,0)}$ and $\Omega_{(1,1),(1,0),(1,0)}$ are identified with $\mathring{\mathbb{P}}_3/(\mathbb{Z}/2\mathbb{Z})$ and the corresponding cycles are just $\mathrm{Gr}(1,3) \times \mathrm{Gr}(1,2) = \mathbb{P}^2 \times \mathbb{P}^1$.

3.5 More Notation

3.5.1 Lie Algebras

Let $\mathfrak{g}$ and $\mathfrak{h}$ be as in Section 3.2.2. One has the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Introduce also the positive and negative Borel subalgebras $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$.

Let $\mathscr{G}$ be a simple Lie group, $\mathscr{B}$ a Borel subgroup, and $\mathscr{N} = [\mathscr{B}, \mathscr{B}]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n}_+ \subset \mathfrak{b} \subset \mathfrak{g}$. Let $\mathscr{G}$ act on $\mathfrak{g}$ by adjoint action.

Let $E_1, \dots, E_r \in \mathfrak{n}_+$, $\check{\alpha}_1, \dots, \check{\alpha}_r \in \mathfrak{h}$, $F_1, \dots, F_r \in \mathfrak{n}_-$ be the Chevalley generators of $\mathfrak{g}$. Let p_{-1} be the regular nilpotent element $\sum_{i=1}^r F_i$. The set $p_{-1} + \mathfrak{b} = \{p_{-1} + b \mid b \in$

$\mathfrak{b}\}$ is invariant under conjugation by elements of $\mathscr{N}$. Consider the quotient space $(p_{-1}+\mathfrak{b})/\mathscr{N}$ and denote the $\mathscr{N}$-conjugacy class of $g \in p_{-1}+\mathfrak{b}$ by $[g]_{\mathfrak{g}}$.

Let $\check{\mathcal{P}} = \{\check{\lambda} \in \mathfrak{h} | \langle \alpha_i, \check{\lambda}\rangle \in \mathbb{Z},\ i = 1,\dots,r\}$ and $\check{\mathcal{P}}^{+} = \{\check{\lambda} \in \mathfrak{h} | \langle \alpha_i, \check{\lambda}\rangle \in \mathbb{Z}_{\geqslant 0},\ i = 1,\dots,r\}$ be the coweight lattice and the cone of dominant integral coweights. Let $\rho \in \mathfrak{h}^*$ and $\check{\rho} \in \mathfrak{h}$ be the Weyl vector and covector such that $\langle \rho, \check{\alpha}_i\rangle = 1$ and $\langle \alpha_i, \check{\rho}\rangle = 1$, $i = 1,\dots,r$.

The Weyl group $\mathcal{W} \subset \mathrm{Aut}(\mathfrak{h}^*)$ is generated by simple reflections s_i, $i = 1,\dots,r$,

$$s_i(\lambda) = \lambda - \langle \lambda, \check{\alpha}_i\rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The restriction of the bilinear form $(\cdot,\cdot)$ to $\mathfrak{h}$ is nondegenerate and induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. The action of $\mathcal{W}$ on $\mathfrak{h}$ is given by $s_i(\check{\mu}) = \check{\mu} - \langle \alpha_i, \check{\mu}\rangle \check{\alpha}_i$ for $\check{\mu} \in \mathfrak{h}$. We use the notation

$$w \cdot \lambda = w(\lambda+\rho) - \rho, \quad w \cdot \check{\lambda} = w(\check{\lambda}+\check{\rho}) - \check{\rho}, \quad w \in \mathcal{W},\ \lambda \in \mathfrak{h}^*,\ \check{\lambda} \in \mathfrak{h},$$

for the shifted action of the Weyl group on $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively.

Let ${}^t\mathfrak{g} = \mathfrak{g}({}^tA)$ be the Langlands dual Lie algebra of $\mathfrak{g}$, then ${}^t(\mathfrak{so}_{2r+1}) = \mathfrak{sp}_{2r}$ and ${}^t(\mathfrak{sp}_{2r}) = \mathfrak{so}_{2r+1}$. A system of simple roots of ${}^t\mathfrak{g}$ is $\check{\alpha}_1,\dots,\check{\alpha}_r$ with the corresponding coroots $\alpha_1,\dots,\alpha_r$. A coweight $\check{\lambda} \in \mathfrak{h}$ of $\mathfrak{g}$ can be identified with a weight of ${}^t\mathfrak{g}$.

For a vector space X we denote by $\mathcal{M}(X)$ the space of X-valued meromorphic functions on $\mathbb{P}^1$. For a group R we denote by $R(\mathcal{M})$ the group of R-valued meromorphic functions on $\mathbb{P}^1$.

3.5.2 $\mathfrak{sp}_{2r}$ as a Subalgebra of $\mathfrak{sl}_{2r}$

Let $v_1,\dots,v_{2r}$ be a basis of $\mathbb{C}^{2r}$. Define a nondegenerate skew-symmetric form χ on $\mathbb{C}^{2r}$ by

$$\chi(v_i, v_j) = (-1)^{i+1}\delta_{i,2r+1-j}, \quad i,j = 1,\dots,2r.$$

The special symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2r}$ by definition consists of all endomorphisms K of $\mathbb{C}^{2r}$ such that $\chi(Kv,v') + \chi(v,Kv') = 0$ for all $v,v' \in \mathbb{C}^{2r}$. This identifies $\mathfrak{sp}_{2r}$ with a Lie subalgebra of $\mathfrak{sl}_{2r}$.

Denote E_{ij} the matrix with zero entries except 1 at the intersection of the i-th row and j-th column.

The Chevalley generators of $\mathfrak{g} = \mathfrak{sp}_{2r}$ are given by

$$E_i = E_{i,i+1} + E_{2r-i,2r+1-i}, \quad F_i = E_{i+1,i} + E_{2r+1-i,2r-i}, \quad i = 1, \dots, r-1,$$

$$E_r = E_{r,r+1}, \qquad F_r = E_{r+1,r},$$

$$\check{\alpha}_j = E_{jj} - E_{j+1,j+1} + E_{2r-j,2r-j} - E_{2r+1-j,2r+1-j}, \;\; \check{\alpha}_r = E_{rr} - E_{r+1,r+1}, \;\; j = 1, \dots, r-1.$$

Moreover, a coweight $\check{\lambda} \in \mathfrak{h}$ can be written as

$$\check{\lambda} = \sum_{i=1}^{r} \Big(\langle \alpha_i, \check{\lambda} \rangle + \dots + \langle \alpha_{r-1}, \check{\lambda} \rangle + \langle \alpha_r, \check{\lambda} \rangle / 2 \Big) (E_{ii} - E_{2r+1-i,2r+1-i}). \tag{3.5.1}$$

In particular,

$$\check{\rho} = \sum_{i=1}^{r} \frac{2r-2i+1}{2} (E_{ii} - E_{2r+1-i,2r+1-i}).$$

For convenience, we denote the coefficient of E_{ii} in the right hand side of (3.5.1) by $(\check{\lambda})_{ii}$, for $i = 1, \dots, 2r$.

3.5.3 $\mathfrak{so}_{2r+1}$ as a Subalgebra of $\mathfrak{sl}_{2r+1}$

Let $v_1, \dots, v_{2r+1}$ be a basis of $\mathbb{C}^{2r+1}$. Define a nondegenerate symmetric form χ on $\mathbb{C}^{2r+1}$ by

$$\chi(v_i, v_j) = (-1)^{i+1} \delta_{i,2r+2-j}, \quad i, j = 1, \dots, 2r+1.$$

The special orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}_{2r+1}$ by definition consists of all endomorphisms K of $\mathbb{C}^{2r+1}$ such that $\chi(Kv, v') + \chi(v, Kv') = 0$ for all $v, v' \in \mathbb{C}^{2r+1}$. This identifies $\mathfrak{so}_{2r+1}$ with a Lie subalgebra of $\mathfrak{sl}_{2r+1}$.

Denote E_{ij} the matrix with zero entries except 1 at the intersection of the i-th row and j-th column.

The Chevalley generators of $\mathfrak{g} = \mathfrak{so}_{2r+1}$ are given by

$$E_i = E_{i,i+1} + E_{2r+1-i,2r+2-i}, \quad F_i = E_{i+1,i} + E_{2r+2-i,2r+1-i}, \quad i = 1, \dots, r,$$

$$\check{\alpha}_j = E_{jj} - E_{j+1,j+1} + E_{2r+1-j,2r+1-j} - E_{2r+2-j,2r+2-j}, \quad j = 1, \ldots, r.$$

Moreover, a coweight $\check{\lambda} \in \mathfrak{h}$ can be written as

$$\check{\lambda} = \sum_{i=1}^{r} \Big(\langle \alpha_i, \check{\lambda} \rangle + \cdots + \langle \alpha_r, \check{\lambda} \rangle \Big) (E_{ii} - E_{2r+2-i,2r+2-i}). \tag{3.5.2}$$

In particular,

$$\check{\rho} = \sum_{i=1}^{r} (r+1-i)(E_{ii} - E_{2r+2-i,2r+2-i}).$$

For convenience, we denote the coefficient of E_{ii} in the right hand side of (3.5.2) by $(\check{\lambda})_{ii}$, for $i = 1, \ldots, 2r+1$.

3.5.4 Lemmas on Spaces of Polynomials

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)}, \lambda)$ be a sequence of partitions with at most N parts such that $|\mathbf{\Lambda}| = N(d-N)$ and let $\boldsymbol{z} = (z_1, \ldots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$.

Given an N-dimensional space of polynomials X, denote by $\mathcal{D}_X$ the monic scalar differential operator of order N with kernel X. The operator $\mathcal{D}_X$ is a monodromy-free Fuchsian differential operator with rational coefficients.

Lemma 3.5.1. *A subspace $X \subset \mathbb{C}_d[x]$ is a point of $\Omega_{\mathbf{\Lambda},\boldsymbol{z}}$ if and only if the operator $\mathcal{D}_X$ is Fuchsian, regular in $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$, the exponents at z_s, $s = 1, \ldots, n$, being equal to $\lambda_N^{(s)}, \lambda_{N-1}^{(s)} + 1, \ldots, \lambda_1^{(s)} + N - 1$, and the exponents at ∞ being equal to $1 + \lambda_N - d, 2 + \lambda_{N-1} - d, \ldots, N + \lambda_1 - d$.* □

Let $\boldsymbol{T} = (T_1, \ldots, T_N)$ be associated with $\mathbf{\Lambda}, \boldsymbol{z}$, see (3.4.1). Let $\Gamma = \{u_1, \ldots, u_N\}$ be a basis of $X \in \Omega_{\mathbf{\Lambda},\boldsymbol{z}}$, define a sequence of polynomials

$$y_{N-i} = \mathrm{Wr}^{\dagger}(u_1, \ldots, u_i), \quad i = 1, \ldots, N-1. \tag{3.5.3}$$

Denote $(y_1, \ldots, y_{N-1})$ by $\boldsymbol{y}_\Gamma$. We say that $\boldsymbol{y}_\Gamma$ is *constructed from the basis* Γ.

Lemma 3.5.2 ([MV04]). *Suppose $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and let $\Gamma = \{u_1, \dots, u_N\}$ be a basis of X. If $\boldsymbol{y}_\Gamma = (y_1, \dots, y_{N-1})$ is constructed from Γ, then*

$$\mathcal{D}_X = \Big(\partial_x - \ln'\Big(\frac{T_1 \cdots T_N}{y_1}\Big)\Big)\Big(\partial_x - \ln'\Big(\frac{y_1 T_2 \cdots T_N}{y_2}\Big)\Big) \times \dots \\ \times \Big(\partial_x - \ln'\Big(\frac{y_{N-2} T_{N-1} T_N}{y_{N-1}}\Big)\Big)\Big(\partial_x - \ln'(y_{N-1} T_N)\Big).$$

□

Let $\mathcal{D} = \partial_x^N + \sum_{i=1}^N h_i(x)\partial_x^{N-i}$ be a differential operator with meromorphic coefficients. The operator $\mathcal{D}^* = \partial_x^N + \sum_{i=1}^N (-1)^i \partial_x^{N-i} h_i(x)$ is called the *formal conjugate to $\mathcal{D}$.*

Lemma 3.5.3. *Let $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ and let $\{u_1, \dots, u_N\}$ be a basis of X, then*

$$\frac{\mathrm{Wr}(u_1, \dots, \widehat{u_i}, \dots, u_N)}{\mathrm{Wr}(u_1, \dots, u_N)}, \quad i = 1, \dots, N,$$

form a basis of $\mathrm{Ker}((\mathcal{D}_X)^*)$. *The symbol $\widehat{u_i}$ means that u_i is skipped. Moreover, given an arbitrary factorization of $\mathcal{D}_X$ to linear factors, $\mathcal{D}_X = (\partial_x + f_1)(\partial_x + f_2) \dots (\partial_x + f_N)$, we have $(\mathcal{D}_X)^* = (\partial_x - f_N)(\partial_x - f_{N-1}) \dots (\partial_x - f_1)$.*

Proof. The first statement follows from Theorem 3.14 of [MTV08a]. The second statement follows from the first statement and Lemma A.5 of [MV04]. □

Lemma 3.5.4. *Let $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$. Then*

$$\mathcal{D}_{X^\dagger} = (T_1 \cdots T_N) \cdot \big(\mathcal{D}_X\big)^* \cdot (T_1 \cdots T_N)^{-1}.$$

Proof. The statement follows from Lemma 3.5.3 and the definition of $X^\dagger$. □

Lemma 3.5.5. *Suppose $X \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ is a pure self-dual space and z is an arbitrary complex number, then there exists a basis $\Gamma = \{u_1, \dots, u_N\}$ of X such that for $\boldsymbol{y}_\Gamma = (y_1, \dots, y_{N-1})$ given by* (3.5.3)*, we have $y_i = y_{N-i}$ and $y_i(z) \neq 0$ for every $i = 1, \dots, N-1$.*

Proof. The lemma follows from the proofs of Theorem 8.2 and Theorem 8.3 of [MV04].

□

3.6 $\mathfrak{g}$-Oper

We fix N, $N \geqslant 4$, and set $\mathfrak{g}$ to be the Langlands dual of $\mathfrak{g}_N$. Explicitly, $\mathfrak{g} = \mathfrak{sp}_{2r}$ if $N = 2r$ and $\mathfrak{g} = \mathfrak{so}_{2r+1}$ if $N = 2r + 1$.

3.6.1 Miura $\mathfrak{g}$-Oper

Fix a global coordinate x on $\mathbb{C} \subset \mathbb{P}^1$. Consider the following subset of differential operators

$$\mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1) = \{\partial_x + p_{-1} + \boldsymbol{v} \mid \boldsymbol{v} \in \mathcal{M}(\mathfrak{b})\}.$$

This set is stable under the gauge action of the unipotent subgroup $\mathscr{N}(\mathcal{M}) \subset \mathscr{G}(\mathcal{M})$. The space of $\mathfrak{g}$-*opers* is defined as the quotient space $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1) := \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)/\mathscr{N}(\mathcal{M})$. We denote by $[\nabla]$ the class of $\nabla \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)$.

We say that $\nabla = \partial_x + p_{-1} + \boldsymbol{v} \in \mathrm{op}_{\mathfrak{g}}(\mathbb{P}^1)$ is *regular* at $z \in \mathbb{P}^1$ if $\boldsymbol{v}$ has no pole at z. A $\mathfrak{g}$-oper $[\nabla]$ is said to be *regular* at z if there exists $f \in \mathscr{N}(\mathcal{M})$ such that $f^{-1} \cdot \nabla \cdot f$ is regular at z.

Let $\nabla = \partial_x + p_{-1} + \boldsymbol{v}$ be a representative of a $\mathfrak{g}$-oper $[\nabla]$. Consider ∇ as a $\mathscr{G}$-connection on the trivial principal bundle $p : \mathscr{G} \times \mathbb{P}^1 \to \mathbb{P}^1$. The connection has singularities at the set $\mathrm{Sing} \subset \mathbb{C}$ where the function $\boldsymbol{v}$ has poles (and maybe at infinity). Parallel translations with respect to the connection define the monodromy representation $\pi_1(\mathbb{C} \setminus \mathrm{Sing}) \to \mathscr{G}$. Its image is called the *monodromy group* of ∇. If the monodromy group of one of the representatives of $[\nabla]$ is contained in the center of $\mathscr{G}$, we say that $[\nabla]$ is a *monodromy-free* $\mathfrak{g}$-oper.

A *Miura* $\mathfrak{g}$-*oper* is a differential operator of the form $\nabla = \partial_x + p_{-1} + \boldsymbol{v}$, where $\boldsymbol{v} \in \mathcal{M}(\mathfrak{h})$.

A $\mathfrak{g}$-oper $[\nabla]$ has *regular singularity* at $z \in \mathbb{P}^1 \backslash \{\infty\}$, if there exists a representative ∇ of $[\nabla]$ such that

$$(x - z)^{\check{\rho}} \cdot \nabla \cdot (x - z)^{-\check{\rho}} = \partial_x + \frac{p_{-1} + \boldsymbol{w}}{x - z},$$

where $\boldsymbol{w} \in \mathcal{M}(\mathfrak{b})$ is regular at z. The residue of $[\nabla]$ at z is $[p_{-1} + \boldsymbol{w}(z)]_{\mathfrak{g}}$. We denote the residue of $[\nabla]$ at z by $\mathrm{res}_z[\nabla]$.

Similarly, a $\mathfrak{g}$-oper $[\nabla]$ has *regular singularity* at $\infty \in \mathbb{P}^1$, if there exists a representative ∇ of $[\nabla]$ such that

$$x^{\check{\rho}} \cdot \nabla \cdot x^{-\check{\rho}} = \partial_x + \frac{p_{-1} + \tilde{\boldsymbol{w}}}{x},$$

where $\tilde{\boldsymbol{w}} \in \mathcal{M}(\mathfrak{b})$ is regular at ∞. The residue of $[\nabla]$ at ∞ is $-[p_{-1} + \tilde{\boldsymbol{w}}(\infty)]_{\mathfrak{g}}$. We denote the residue of $[\nabla]$ at ∞ by $\mathrm{res}_\infty[\nabla]$.

Lemma 3.6.1. *For any $\check{\lambda}, \check{\mu} \in \mathfrak{h}$, we have $[p_{-1} - \check{\rho} - \check{\lambda}]_{\mathfrak{g}} = [p_{-1} - \check{\rho} - \check{\mu}]_{\mathfrak{g}}$ if and only if there exists $w \in \mathcal{W}$ such that $\check{\lambda} = w \cdot \check{\mu}$.* □

Hence we can write $[\check{\lambda}]_{\mathcal{W}}$ for $[p_{-1} - \check{\rho} - \check{\lambda}]_{\mathfrak{g}}$. In particular, if $[\nabla]$ is regular at z, then $\mathrm{res}_z[\nabla] = [0]_{\mathcal{W}}$.

Let $\check{\boldsymbol{\Lambda}} = (\check{\lambda}^{(1)}, \dots, \check{\lambda}^{(n)}, \check{\lambda})$ be a sequence of $n+1$ dominant integral $\mathfrak{g}$-coweights and let $\boldsymbol{z} = (z_1, \dots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$. Let $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$ denote the set of all $\mathfrak{g}$-opers with at most regular singularities at points z_s and ∞ whose residues are given by

$$\mathrm{res}_{z_s}[\nabla] = [\check{\lambda}^{(s)}]_{\mathcal{W}}, \quad \mathrm{res}_\infty[\nabla] = -[\check{\lambda}]_{\mathcal{W}}, \quad s = 1, \dots, n,$$

and which are regular elsewhere. Let $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}} \subset \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$ denote the subset consisting of those $\mathfrak{g}$-opers which are also monodromy-free.

Lemma 3.6.2 ([Fre05])**.** *For every $\mathfrak{g}$-oper $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$, there exists a Miura $\mathfrak{g}$-oper as one of its representatives.* □

Lemma 3.6.3 ([Fre05])**.** *Let ∇ be a Miura $\mathfrak{g}$-oper, then $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)^{\mathrm{RS}}_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$ if and only if the following conditions hold:*

(i) ∇ is of the form

$$\nabla = \partial_x + p_{-1} - \sum_{s=1}^{n} \frac{w_s \cdot \check{\lambda}^{(s)}}{x - z_s} - \sum_{j=1}^{m} \frac{\tilde{w}_j \cdot 0}{x - t_j} \tag{3.6.1}$$

for some $m \in \mathbb{Z}_{\geqslant 0}$, $w_s \in \mathcal{W}$ for $s = 1, \dots, n$ and $\tilde{w}_j \in \mathcal{W}$, $t_j \in \mathbb{P}^1 \setminus \boldsymbol{z}$ for $j = 1, \dots, m$, *(ii) there exists $w_\infty \in \mathcal{W}$ such that*

$$\sum_{s=1}^{n} w_s \cdot \check{\lambda}^{(s)} + \sum_{j=1}^{m} \tilde{w}_j \cdot 0 = w_\infty \cdot \check{\lambda}, \tag{3.6.2}$$

(iii) $[\nabla]$ is regular at t_j for $j = 1, \ldots, m$.

□

Remark 3.6.4. The condition (3.6.2) implies that $\sum_{s=1}^{n} \langle \alpha_r, \check{\lambda}^{(s)} \rangle + \langle \alpha_r, \check{\lambda} \rangle$ is even if $N = 2r$.

3.6.2 Miura Transformation

Following [DS85], one can associate a linear differential operator L_∇ to each Miura $\mathfrak{g}$-oper $\nabla = \partial_x + p_{-1} + \boldsymbol{v}(x)$, $\boldsymbol{v}(x) \in \mathcal{M}(\mathfrak{h})$.

In the case of $\mathfrak{sl}_{r+1}$, $\boldsymbol{v}(x) \in \mathcal{M}(\mathfrak{h})$ can be viewed as an $(r+1)$-tuple

$$(v_1(x), \ldots, v_{r+1}(x))$$

such that $\sum_{i=1}^{r+1} v_i(x) = 0$. The *Miura transformation* sends $\nabla = \partial_x + p_{-1} + \boldsymbol{v}(x)$ to the operator

$$L_\nabla = (\partial_x + v_1(x)) \ldots (\partial_x + v_{r+1}(x)).$$

Similarly, the Miura transformation takes the form

$$L_\nabla = (\partial_x + v_1(x)) \ldots (\partial_x + v_r(x))(\partial_x - v_r(x)) \ldots (\partial_x - v_1(x))$$

for $\mathfrak{g} = \mathfrak{sp}_{2r}$ and

$$L_\nabla = (\partial_x + v_1(x)) \ldots (\partial_x + v_r(x))\partial_x(\partial_x - v_r(x)) \ldots (\partial_x - v_1(x))$$

for $\mathfrak{g} = \mathfrak{so}_{2r+1}$. The formulas of the corresponding linear differential operators for the cases of $\mathfrak{sp}_{2r}$ and $\mathfrak{so}_{2r+1}$ can be understood with the embeddings described in Sections 3.5.2 and 3.5.3.

It is easy to see that different representatives of $[\nabla]$ give the same differential operator, we can write this map as $[\nabla] \mapsto L_{[\nabla]}$.

Recall the definition of $(\check{\lambda})_{ii}$ for $\check{\lambda} \in \mathfrak{h}$ from Sections 3.5.2 and 3.5.3.

Lemma 3.6.5. *Suppose ∇ is a Miura $\mathfrak{g}$-oper with $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$, then $L_{[\nabla]}$ is a monic Fuchsian differential operator with singularities at points in $\boldsymbol{z}$ only. The exponents of $L_{[\nabla]}$ at z_s, $s = 1, \dots, n$, are $(\check{\lambda}^{(s)})_{ii} + N - i$, and the exponents at ∞ are $-(\check{\lambda})_{ii} - N + i$, $i = 1, \dots, N$.*

Proof. Note that ∇ satisfies the conditions (i)-(iii) in Lemma 3.6.3. By Theorem 5.11 in [Fre05] and Lemma 3.6.1, we can assume $w_s = 1$ for given s. The lemma follows directly. $\square$

Denote by $Z(\mathscr{G})$ the center of $\mathscr{G}$, then

$$Z(\mathscr{G}) = \begin{cases} \{I_{2r+1}\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1}, \\ \{\pm I_{2r}\} & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}. \end{cases}$$

We have the following lemma.

Lemma 3.6.6. *Suppose ∇ is a Miura $\mathfrak{g}$-oper with $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$. If $\mathfrak{g} = \mathfrak{so}_{2r+1}$, then $L_{[\nabla]}$ is a monodromy-free differential operator. If $\mathfrak{g} = \mathfrak{sp}_{2r}$, then the monodromy of $L_{[\nabla]}$ around z_s is $-I_{2r}$ if and only if $\langle \alpha_r, \check{\lambda}^{(s)} \rangle$ is odd for given $s \in \{1, \dots, n\}$.* $\square$

3.6.3 Relations with Pure Self-Dual Spaces

Let $\check{\mathbf{\Lambda}} = (\check{\lambda}^{(1)}, \dots, \check{\lambda}^{(n)}, \check{\lambda})$ be a sequence of $n+1$ dominant integral $\mathfrak{g}$-coweights and let $\boldsymbol{z} = (z_1, \dots, z_n, \infty) \in \mathring{\mathbb{P}}_{n+1}$.

Consider $\check{\mathbf{\Lambda}}$ as a sequence of dominant integral $\mathfrak{g}_N$-weights. Choose d large enough so that $k := d - N - \sum_{s=1}^{n} (\check{\lambda}^{(s)})_{11} - (\check{\lambda})_{11} \geqslant 0$. (We only need to consider the case that $\sum_{s=1}^{n} (\check{\lambda}^{(s)})_{11} + (\check{\lambda})_{11}$ is an integer for $N = 2r$, see Lemma 3.4.4 and Remark 3.6.4.) Let $\boldsymbol{k} = (0, \dots, 0, k)$. Note that we always have $|\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}}| = N(d - N)$ and spaces of polynomials in $\mathrm{s}\Omega_{\check{\mathbf{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$ ($= \mathrm{s}\Omega_{\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}},\boldsymbol{z}}$) are pure self-dual spaces.

Theorem 3.6.7. *There exists a bijection between $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\mathbf{\Lambda}},\boldsymbol{z}}$ and $\mathrm{s}\Omega_{\check{\mathbf{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$ given by the map $[\nabla] \mapsto \mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f)$, where $\boldsymbol{T} = (T_1, \dots, T_N)$ is associated with $\check{\mathbf{\Lambda}}_{A,\boldsymbol{k}}, \boldsymbol{z}$ and $f = (T_1 \dots T_N)^{-1/2}$.*

Proof. We only prove it for the case of $\mathfrak{g} = \mathfrak{sp}_{2r}$. Suppose $[\nabla] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$, by Lemmas 3.6.2 and 3.6.3, we can assume ∇ has the form (3.6.1) satisfying the conditions (i), (ii), and (iii) in Lemma 3.6.3.

Note that if $\langle \alpha_r, \check{\lambda}^{(s)} \rangle$ is odd, f has monodromy $-I_{2r}$ around the point z_s. By Lemma 3.6.6, one has that $f^{-1} \cdot L_{[\nabla]} \cdot f$ is monodromy-free around the point z_s for $s = 1, \dots, n$. Note also that $\sum_{s=1}^{n} \langle \alpha_r, \check{\lambda}^{(s)} \rangle + \langle \alpha_r, \check{\lambda} \rangle$ is even, it follows that $f^{-1} \cdot L_{[\nabla]} \cdot f$ is also monodromy-free around the point ∞. Hence $f^{-1} \cdot L_{[\nabla]} \cdot f$ is a monodromy-free differential operator.

It follows from Lemmas 3.5.1 and 3.6.5 that $\mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f) \in \Omega_{\check{\boldsymbol{\Lambda}}_{A},\boldsymbol{k},\boldsymbol{z}}$. Since $L_{[\nabla]}$ takes the form

$$(\partial_x + v_1(x)) \dots (\partial_x + v_r(x))(\partial_x - v_r(x)) \dots (\partial_x - v_1(x)),$$

it follows that $\mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f)$ is a pure self-dual space by Lemma 3.5.4.

If there exist $[\nabla_1], [\nabla_2] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$ such that $f^{-1} \cdot L_{[\nabla_1]} \cdot f = f^{-1} \cdot L_{[\nabla_2]} \cdot f$, then they are the same differential operator constructed from different bases of $\mathrm{Ker}(f^{-1} \cdot L_{[\nabla]} \cdot f)$ as described in Lemma 3.5.2. Therefore they correspond to the same $\mathfrak{so}_{2r+1}$-population by Theorem 7.5 of [MV04]. It follows from Theorem 4.2 and remarks in Section 4.3 of [MV05a] that $[\nabla_1] = [\nabla_2]$.

Conversely, give a self-dual space $X \in \mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$. By Lemma 3.5.5, there exists a basis Γ of X such that for $\boldsymbol{y}_\Gamma = (y_1, \dots, y_{N-1})$ we have $y_i = y_{N-i}$, $i = 1, \dots, N-1$. Following [MV05a], define $\boldsymbol{v} \in \mathcal{M}(\mathfrak{h})$ by

$$\langle \alpha_i, \boldsymbol{v} \rangle = -\ln' \Big(T_i \prod_{j=1}^{r} y_j^{-a_{i,j}} \Big),$$

then we introduce the Miura $\mathfrak{g}$-oper $\nabla_\Gamma = \partial_x + p_{-1} + \boldsymbol{v}$, which only has regular singularities. It is easy to see from Lemma 3.5.2 that $f^{-1} \cdot L_{[\nabla_\Gamma]} \cdot f = \mathcal{D}_X$. It follows from the same argument as the previous paragraph that $[\nabla_\Gamma] = [\nabla_{\Gamma'}]$ for any other basis Γ' of X and hence $[\nabla_\Gamma]$ is independent of the choice of Γ. Again by Lemma 3.5.5, for any $x_0 \in \mathbb{C} \setminus \boldsymbol{z}$ we can choose Γ such that $y_i(x_0) \neq 0$ for all $i = 1, \dots, N-1$, it follows that $[\nabla_\Gamma]$ is regular at x_0. By exponents reasons, see Lemma 3.6.5, we have

$$\mathrm{res}_{z_s}[\nabla_\Gamma] = [\check{\lambda}^{(s)}]_{\mathcal{W}}, \quad \mathrm{res}_{\infty}[\nabla_\Gamma] = -[\check{\lambda}]_{\mathcal{W}}, \quad s = 1, \dots, n.$$

On the other hand, $[\nabla_\Gamma]$ is monodromy-free by Theorem 4.1 of [MV05a]. It follows that $[\nabla_\Gamma] \in \mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$, which completes the proof. $\square$

3.7 Proof of Main Theorems

3.7.1 Proof of Theorems 3.3.6 and 3.3.8

We prove Theorem 3.3.6 first.

By assumption, $\boldsymbol{\Xi} = (\xi^{(1)}, \dots, \xi^{(n-1)})$ is a simple degeneration of

$$\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)}).$$

Without loss of generality, we assume that $\xi^{(i)} = \lambda^{(i)}$ for $i = 1, \dots, n-2$ and

$$\dim(V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})^{\mathrm{sing}}_{\xi^{(n-1)}} > 0.$$

Recall the strata $\Omega_{\boldsymbol{\Lambda}}$ is a union of intersections of Schubert cells $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$, see (3.3.6). Taking the closure of $\Omega_{\boldsymbol{\Lambda}}$ is equivalent to allowing coordinates of $\boldsymbol{z} \in \mathring{\mathbb{P}}_n$ coincide.

Let $\boldsymbol{z}_0 = (z_1, \dots, z_{n-1}) \in \mathring{\mathbb{P}}_{n-1}$. Let $X \in \Omega_{\boldsymbol{\Xi},\boldsymbol{z}_0}$. By Theorem 3.3.2, there exists a common eigenvector $v \in (V_{\boldsymbol{\Xi},\boldsymbol{z}_0})^{\mathfrak{sl}_N}$ of the Bethe algebra $\mathcal{B}$ such that $\mathcal{D}_v = \mathcal{D}_X$.

Let $\boldsymbol{z}_0' = (z_1, \dots, z_{n-1}, z_{n-1})$. Consider the $\mathcal{B}$-module $V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'}$, then we have

$$\begin{aligned} V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'} =& \Big(\bigotimes_{s=1}^{n-2} V_{\lambda^{(s)}}(z_s)\Big) \otimes (V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})(z_{n-1}) \\ =& \bigoplus_{\mu} c^{\mu}_{\lambda^{(n-1)},\lambda^{(n)}} \Big(\bigotimes_{s=1}^{n-2} V_{\lambda^{(s)}}(z_s)\Big) \otimes V_\mu(z_{n-1}), \end{aligned}$$

where $c^{\mu}_{\lambda^{(n-1)},\lambda^{(n)}} := \dim(V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})^{\mathrm{sing}}_{\mu}$ are the Littlewood-Richardson coefficients. Since $\dim(V_{\lambda^{(n-1)}} \otimes V_{\lambda^{(n)}})^{\mathrm{sing}}_{\xi^{(n-1)}} > 0$, we have $V_{\boldsymbol{\Xi},\boldsymbol{z}_0} \subset V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'}$. In particular, $(V_{\boldsymbol{\Xi},\boldsymbol{z}_0})^{\mathfrak{sl}_N} \subset (V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'})^{\mathfrak{sl}_N}$. Hence v is a common eigenvector of the Bethe algebra $\mathcal{B}$ on $(V_{\boldsymbol{\Lambda},\boldsymbol{z}_0'})^{\mathfrak{sl}_N}$ such that $\mathcal{D}_v = \mathcal{D}_X$.

It follows that X is a limit point of $\Omega_{\boldsymbol{\Lambda},\boldsymbol{z}}$ as z_n approaches z_{n-1}. This completes the proof of Theorem 3.3.6.

Theorem 3.3.8 follows directly from Theorem 3.3.6.

3.7.2 Proof of Theorems 3.4.5, 3.4.12, and 3.4.13

We prove Theorem 3.4.5 first. We follow the convention of Section 3.6.

We can identify the sequence $\check{\boldsymbol{\Lambda}} = (\check{\lambda}^{(1)}, \dots, \check{\lambda}^{(n)}, \check{\lambda})$ of dominant integral $\mathfrak{g}$-coweights as a sequence of dominant integral $\mathfrak{g}_N$-weights. Consider the $\mathfrak{g}_N$-module $V_{\check{\boldsymbol{\Lambda}}} = V_{\check{\lambda}^{(1)}} \otimes \cdots \otimes V_{\check{\lambda}^{(n)}} \otimes V_{\check{\lambda}}$. It follows from Theorem 3.2 and Corollary 3.3 of [Ryb18] that there exists a bijection between the joint eigenvalues of the $\mathfrak{g}_N$ Bethe algebra $\mathcal{B}$ acting on $(V_{\check{\lambda}^{(1)}}(z_1) \otimes \cdots \otimes V_{\check{\lambda}^{(n)}}(z_n))^{\mathrm{sing}}$ and the $\mathfrak{g}$-opers in $\mathrm{Op}_{\mathfrak{g}}(\mathbb{P}^1)_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}}$ for all possible dominant integral $\mathfrak{g}$-coweight $\check{\lambda}$. In fact, one can show that Theorem 3.2 and Corollary 3.3 of [Ryb18] are also true for the subspaces of $(V_{\check{\lambda}^{(1)}}(z_1) \otimes \cdots \otimes V_{\check{\lambda}^{(n)}}(z_n))^{\mathrm{sing}}_{\check{\lambda}}$ with specific $\mathfrak{g}_N$-weight $\check{\lambda}$. Recall that $\boldsymbol{k} = (0, \dots, 0, k)$, where $k = d - N - \sum_{s=1}^{n} (\check{\lambda}^{(s)})_{11} - (\check{\lambda})_{11} \geqslant 0$. Since one has the canonical isomorphism of $\mathcal{B}$-modules

$$(V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N} \cong (V_{\check{\lambda}^{(1)}}(z_1) \otimes \cdots \otimes V_{\check{\lambda}^{(n)}}(z_n))^{\mathrm{sing}}_{\check{\lambda}},$$

by Theorem 3.6.7, we have the following theorem.

Theorem 3.7.1. *There exists a bijection between the joint eigenvalues of the $\mathfrak{g}_N$ Bethe algebra $\mathcal{B}$ acting on $(V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N}$ and $\mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}} \subset \mathrm{sGr}(N,d)$ such that given a joint eigenvalue of $\mathcal{B}$ with a corresponding $\mathcal{B}$-eigenvector v in $(V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N}$ we have* $\mathrm{Ker}\,((T_1 \dots T_N)^{1/2} \cdot \mathcal{D}_v \cdot (T_1 \dots T_N)^{-1/2}) \in \mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$. □

The fact that $\mathrm{Ker}\,((T_1 \dots T_N)^{1/2} \cdot \mathcal{D}_v \cdot (T_1 \dots T_N)^{-1/2}) \in \mathrm{s}\Omega_{\check{\boldsymbol{\Lambda}},\boldsymbol{k},\boldsymbol{z}}$ for the eigenvector $v \in (V_{\check{\boldsymbol{\Lambda}},\boldsymbol{z}})^{\mathfrak{g}_N}$ of the $\mathfrak{g}_N$ Bethe algebra (except for the case of even N when there exists $s \in \{1, 2, \dots, n\}$ such that $\langle \alpha_r, \check{\lambda}^{(s)} \rangle$ is odd) also follows from the results of [LMV16] and [MM17].

Note that by Proposition 2.10 in [Ryb18], the i-th coefficient of the scalar differential operator $L_{[\nabla]}$ in Theorem 3.6.7 is obtained by action of a universal series $G_i(x) \in \mathcal{U}(\mathfrak{g}_N[t][[x^{-1}]])$. Theorem 3.4.5 for the case of $N \geqslant 4$ is a direct corollary of Theorems 3.6.7 and 3.7.1.

Thanks to Theorem 3.4.5, Theorems 3.4.12 and 3.4.13 can be proved in a similar way as Theorems 3.3.6 and 3.3.8.

3.8 Self-dual Spaces and ϖ-Invariant Vectors

3.8.1 Diagram Automorphism ϖ

There is a diagram automorphism $\varpi : \mathfrak{sl}_N \to \mathfrak{sl}_N$ such that

$$\varpi(E_i) = E_{N-i}, \quad \varpi(F_i) = F_{N-i}, \quad \varpi^2 = 1, \quad \varpi(\mathfrak{h}_A) = \mathfrak{h}_A.$$

The automorphism ϖ is extended to the automorphism of $\mathfrak{gl}_N$ by

$$\mathfrak{gl}_N \to \mathfrak{gl}_N, \quad e_{ij} \mapsto (-1)^{i-j-1} e_{N+1-j,N+1-i}, \quad i,j = 1,\dots,N.$$

By abuse of notation, we denote this automorphism of $\mathfrak{gl}_N$ also by ϖ.

The restriction of ϖ to the Cartan subalgebra $\mathfrak{h}_A$ induces a dual map $\varpi^* : \mathfrak{h}_A^* \to \mathfrak{h}_A^*$, $\lambda \mapsto \lambda^\star$, by

$$\lambda^\star(h) = \varpi^*(\lambda)(h) = \lambda(\varpi(h)),$$

for all $\lambda \in \mathfrak{h}_A^*, h \in \mathfrak{h}_A$.

Let $(\mathfrak{h}_A^*)^0 = \{\lambda \in \mathfrak{h}_A^* \mid \lambda^\star = \lambda\} \subset \mathfrak{h}_A^*$. We call elements of $(\mathfrak{h}_A^*)^0$ *symmetric weights*.

Let $\mathfrak{h}_N$ be the Cartan subalgebra of $\mathfrak{g}_N$. Consider the root system of type A_{N-1} with simple roots $\alpha_1^A, \dots, \alpha_{N-1}^A$ and the root system of $\mathfrak{g}_N$ with simple roots $\alpha_1, \dots, \alpha_{[\frac{N}{2}]}$.

There is a linear isomorphism $P_\varpi^* : \mathfrak{h}_N^* \to (\mathfrak{h}_A^*)^0$, $\lambda \mapsto \lambda_A$, where λ_A is defined by

$$\langle \lambda_A, \check{\alpha}_i^A \rangle = \langle \lambda_A, \check{\alpha}_{N-i}^A \rangle = \langle \lambda, \check{\alpha}_i \rangle, \quad i = 1, \dots, \left[\frac{N}{2}\right]. \tag{3.8.1}$$

Let $\lambda \in \mathfrak{h}_A^*$ and fix two nonzero highest weight vectors $v_\lambda \in (V_\lambda)_\lambda, v_{\lambda^\star} \in (V_{\lambda^\star})_{\lambda^\star}$. Then there exists a unique linear isomorphism $\mathcal{I}_\varpi : V_\lambda \to V_{\lambda^\star}$ such that

$$\mathcal{I}_\varpi(v_\lambda) = v_{\lambda^\star}, \quad \mathcal{I}_\varpi(gv) = \varpi(g)\mathcal{I}_\varpi(v), \tag{3.8.2}$$

for all $g \in \mathfrak{sl}_N, v \in V_\lambda$. In particular, if λ is a symmetric weight, $\mathcal{I}_\varpi$ is a linear automorphism of V_λ, where we always assume that $v_\lambda = v_{\lambda^\star}$.

Let M be a finite-dimensional $\mathfrak{sl}_N$-module with a weight space decomposition $M = \bigoplus_{\mu\in\mathfrak{h}_A^*}(M)_\mu$. Let $f : M \to M$ be a linear map such that $f(hv) = \varpi(h)f(v)$ for $h \in \mathfrak{h}_A, v \in M$. Then it follows that $f((M)_\mu) \subset (M)_{\mu^\star}$ for all $\mu \in \mathfrak{h}_A^*$. Define a formal sum

$$\mathrm{Tr}^{\varpi}_M f = \sum_{\mu\in(\mathfrak{h}_A^*)^0} \mathrm{Tr}(f|_{(M)_\mu})e(\mu),$$

where $\mathrm{Tr}(f|_{(M)_\mu})$ for $\mu \in (\mathfrak{h}_A^*)^0$ denotes the trace of the restriction of f to the weight space $(M)_\mu$.

Lemma 3.8.1. *We have* $\mathrm{Tr}^{\varpi}_{M\otimes M'}(f \otimes f') = (\mathrm{Tr}^{\varpi}_M f) \cdot (\mathrm{Tr}^{\varpi}_{M'} f')$. □

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of dominant integral $\mathfrak{g}_N$-weights, then the tuple $\mathbf{\Lambda}^A = (\lambda^{(1)}_A, \dots, \lambda^{(n)}_A)$ is a sequence of symmetric dominant integral $\mathfrak{sl}_N$-weights. Let $V_{\mathbf{\Lambda}^A} = \bigotimes_{s=1}^n V_{\lambda^{(s)}_A}$. The tensor product of maps $\mathcal{I}_\varpi$ in (3.8.2) with respect to $\lambda^{(s)}_A$, $s = 1, \dots, n$, gives a linear isomorphism

$$\mathcal{I}_\varpi : V_{\mathbf{\Lambda}^A} \to V_{\mathbf{\Lambda}^A}, \tag{3.8.3}$$

of $\mathfrak{sl}_N$-modules. Note that the map $\mathcal{I}_\varpi$ preserves the weight spaces with symmetric weights and the corresponding spaces of singular vectors. In particular, $(V_{\mathbf{\Lambda}^A})^{\mathfrak{sl}_N}$ is invariant under $\mathcal{I}_\varpi$.

Lemma 3.8.2. *Let μ be a $\mathfrak{g}_N$-weight. Then we have*

$$\dim(V_{\mathbf{\Lambda}})^{\mathrm{sing}}_\mu = \mathrm{Tr}\big(\mathcal{I}_\varpi|_{(V_{\mathbf{\Lambda}^A})^{\mathrm{sing}}_{\mu^A}}\big), \qquad \dim(V_{\mathbf{\Lambda}})_\mu = \mathrm{Tr}\big(\mathcal{I}_\varpi|_{(V_{\mathbf{\Lambda}^A})_{\mu^A}}\big).$$

In particular, $\dim(V_{\mathbf{\Lambda}})^{\mathfrak{g}_N} = \mathrm{Tr}\big(\mathcal{I}_\varpi|_{(V_{\mathbf{\Lambda}^A})^{\mathfrak{sl}_N}}\big)$.

Proof. The statement follows from Lemma 3.8.1 and Theorem 1 of Section 4.4 of [FSS96]. □

3.8.2 Action of ϖ on the Bethe Algebra

The automorphism ϖ is extended to the automorphism of current algebra $\mathfrak{gl}_N[t]$ by the formula $\varpi(g \otimes t^s) = \varpi(g) \otimes t^s$, where $g \in \mathfrak{gl}_N$ and $s = 0, 1, 2, \dots$. Recall the operator $\mathcal{D}^{\mathcal{B}}$, see (3.2.3).

Proposition 3.8.3. *We have the following identity*

$$\varpi(\mathcal{D}^{\mathcal{B}}) = \partial_x^N + \sum_{i=1}^{N} (-1)^i \partial_x^{N-i} B_i(x).$$

Proof. It follows from the proof of Lemma 3.5 of [BHLW17] that no nonzero elements of $\mathcal{U}(\mathfrak{gl}_N[t])$ kill all $\bigotimes_{s=1}^n L(z_s)$ for all $n \in \mathbb{Z}_{>0}$ and all $z_1, \dots, z_n$. It suffices to show the identity when it evaluates on $\bigotimes_{s=1}^n L(z_s)$.

Following the convention of [MTV10], define the $N \times N$ matrix

$$\mathcal{G}_h = \mathcal{G}_h(N, n, x, p_x, \boldsymbol{z}, \boldsymbol{\lambda}, X, P)$$

by the formula

$$\mathcal{G}_h := \Big((p_x - \lambda_i)\, \delta_{ij} + \sum_{a=1}^{n} (-1)^{i-j} \frac{x_{N+1-i,a} p_{N+1-j,a}}{x - z_a} \Big)_{i,j=1}^{N}.$$

By Theorem 2.1 of [MTV10], it suffices to show that

$$\operatorname{rdet}(\mathcal{G}_h) \prod_{a=1}^{n} (x - z_a) = \sum_{A,B,|A|=|B|} \prod_{b \notin A} (p_x - \lambda_b) \prod_{a \notin B} (x - z_a) \det(x_{ab})_{a\in A}^{b \in B} \det(p_{ab})_{a\in A}^{b\in B}. \quad (3.8.4)$$

The proof of (3.8.4) is similar to the proof of Theorem 2.1 in [MTV10] with the following modifications.

Let m be a product whose factors are of the form $f(x)$, p_x, p_{ij}, x_{ij} where $f(x)$ is a rational function in x. Then the product m will be called *normally ordered* if all factors of the form p_x, x_{ij} are on the left from all factors of the form $f(x)$, p_{ij}.

Correspondingly, in Lemma 2.4 of [MTV10], we put the normal order for the first i factors of each summand. □

We have the following corollary of Proposition 3.8.3.

Corollary 3.8.4. *The $\mathfrak{gl}_N$ Bethe algebra $\mathcal{B}$ is invariant under ϖ, that is $\varpi(\mathcal{B}) = \mathcal{B}$.* □

Let $\boldsymbol{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \dots, z_n) \in \mathring{\mathbb{P}}_n$.

Let $v \in (V_{\mathbf{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ be an eigenvector of the $\mathfrak{gl}_N$ Bethe algebra $\mathcal{B}$. Denote the $\varpi(\mathcal{D}^{\mathcal{B}})_v$ the scalar differential operator obtained by acting by the formal operator $\varpi(\mathcal{D}^{\mathcal{B}})$ on v.

Corollary 3.8.5. *Let $v \in (V_{\mathbf{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ be a common eigenvector of the $\mathfrak{gl}_N$ Bethe algebra; then the identity $\varpi(\mathcal{D}^{\mathcal{B}})_v = \big(\mathcal{D}_v\big)^*$ holds.* □

Let $\Xi = (\xi^{(1)}, \dots, \xi^{(n)})$ be a sequence of N-tuples of integers. Suppose

$$\xi^{(s)} - \lambda^{(s)} = m_s(1, \dots, 1), \quad s = 1, \dots, n.$$

Define the following rational functions depending on m_s, $s = 1, \dots, n$,

$$\varphi(x) = \prod_{s=1}^{n}(x - z_s)^{m_s}, \quad \psi(x) = \ln'(\varphi(x)) = \sum_{s=1}^{n} \frac{m_s}{x - z_s}.$$

Here we use the convention that $1/(x - z_s)$ is considered as the constant function 0 if $z_s = \infty$.

Lemma 3.8.6. *For any formal power series $a(x)$ in x^{-1} with complex coefficients, the linear map obtained by sending $e_{ij}(x)$ to $e_{ij}(x)+\delta_{ij}a(x)$ induces an automorphism of $\mathfrak{gl}_N[t]$.* □

We denote the automorphism in Lemma 3.8.6 by $\eta_{a(x)}$.

Lemma 3.8.7. *The $\mathcal{B}$-module obtained by pulling $V_{\mathbf{\Lambda},\boldsymbol{z}}$ via $\eta_{\psi(x)}$ is isomorphic to $V_{\Xi,\boldsymbol{z}}$.* □

By Lemma 3.8.7, we can identify the $\mathcal{B}$-module $V_{\Xi,\boldsymbol{z}}$ with the $\mathcal{B}$-module $V_{\mathbf{\Lambda},\boldsymbol{z}}$ as vector spaces. This identification is an isomorphism of $\mathfrak{sl}_N$-modules. For $v \in (V_{\mathbf{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ we use $\eta_{\psi(x)}(v)$ to express the same vector in $(V_{\Xi,\boldsymbol{z}})^{\mathfrak{sl}_N}$ under this identification.

Lemma 3.8.8. *The following identity for differential operators holds*

$$\eta_{\psi(x)}(\mathcal{D}^{\mathcal{B}}) = \varphi(x)\mathcal{D}^{\mathcal{B}}(\varphi(x))^{-1}.$$

Proof. The lemma follows from the simple computation:

$$\varphi(x)(\partial_x - e_{ii}(x))(\varphi(x))^{-1} = \partial_x - e_{ii}(x) - \psi(x).$$

$\square$

Proposition 3.8.9. *Let $v \in (V_{\mathbf{\Lambda},\boldsymbol{z}})^{\mathfrak{sl}_N}$ be an eigenvector of the Bethe algebra such that $\mathcal{D}_v = \mathcal{D}_X$ for some $X \in \Omega_{\mathbf{\Lambda},\boldsymbol{z}}$, then $\mathcal{D}_{\eta_{\psi(x)}(v)} = \mathcal{D}_{\varphi(x)\cdot X}$.*

Proof. With the identification between the $\mathcal{B}$-modules $V_{\mathbf{\Xi},\boldsymbol{z}}$ and $V_{\mathbf{\Lambda},\boldsymbol{z}}$, we have

$$\mathcal{D}_{\eta_{\psi(x)}(v)} = \left(\eta_{\psi(x)}(\mathcal{D}^{\mathcal{B}})\right)_v = \varphi(x)\mathcal{D}_v(\varphi(x))^{-1} = \varphi(x)\mathcal{D}_X(\varphi(x))^{-1} = \mathcal{D}_{\varphi(x)\cdot X}.$$

The second equality follows from Lemma 3.8.8. $\square$

3.8.3 $\mathcal{I}_\varpi$-Invariant Bethe Vectors and Self-Dual Spaces

Let $\mathbf{\Lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a tuple of dominant integral $\mathfrak{g}_N$-weights. Recall the map $\mathcal{I}_\varpi : V_{\mathbf{\Lambda}^A} \to V_{\mathbf{\Lambda}^A}$, from (3.8.3).

Note that an $\mathfrak{sl}_N$-weight can be lifted to a $\mathfrak{gl}_N$-weight such that the N-th coordinate of the corresponding $\mathfrak{gl}_N$-weight is zero. From now on, we consider $\lambda_A^{(s)}$ from (3.8.1) as $\mathfrak{gl}_N$-weights obtained from (3.4.2), that is as the partitions with at most $N-1$ parts.

Let $\mathbf{\Xi} = (\xi^{(1)}, \dots, \xi^{(n)})$ be a sequence of N-tuples of integers such that

$$\xi^{(s)} - \lambda_A^{(s)} = -(\lambda_A^{(s)})_1(1, \dots, 1), \quad s = 1, \dots, n.$$

Consider the $\mathfrak{sl}_N$-module $V_{\mathbf{\Lambda}^A}$ as the $\mathfrak{gl}_N$-module $V_{\mathbf{\Lambda}_A}$, the image of $V_{\mathbf{\Lambda}_A}$ under $\mathcal{I}_\varpi$ in (3.8.3), considered as a $\mathfrak{gl}_N$-module, is $V_{\mathbf{\Xi}}$. Furthermore, the image of $(V_{\mathbf{\Lambda}_A})^{\mathfrak{sl}_N}$ under $\mathcal{I}_\varpi$ is $(V_{\mathbf{\Xi}})^{\mathfrak{sl}_N}$.

Let $\boldsymbol{T} = (T_1, \dots, T_N)$ be associated with $\mathbf{\Lambda}_A, \boldsymbol{z}$, we have

$$T_1 \cdots T_N = \prod_{s=1}^{n} (x - z_s)^{(\lambda_A^{(s)})_1}.$$

Let $\varphi(x) = T_1 \cdots T_N$ and let $\psi(x) = \varphi'(x)/\varphi(x)$. Hence by Lemma 3.8.7, the pull-back of $V_{\mathbf{\Xi},\boldsymbol{z}}$ through $\eta_{\psi(x)}$ is isomorphic to $V_{\mathbf{\Lambda}_A,\boldsymbol{z}}$. Furthermore, the pull-back of $(V_{\mathbf{\Xi},\boldsymbol{z}})^{\mathfrak{sl}_N}$ through $\eta_{\psi(x)}$ is isomorphic to $(V_{\mathbf{\Lambda}_A,\boldsymbol{z}})^{\mathfrak{sl}_N}$.

Theorem 3.8.10. *Let $v \in (V_{\mathbf{\Lambda}_A,\boldsymbol{z}})^{\mathfrak{sl}_N}$ be an eigenvector of the $\mathfrak{gl}_N$ Bethe algebra $\mathcal{B}$ such that $\mathcal{D}_v = \mathcal{D}_X$ for some $X \in \Omega_{\mathbf{\Lambda}_A,\boldsymbol{z}}$, then $\mathcal{D}_{\eta_{\psi(x)}\circ\mathcal{I}_\varpi(v)} = \mathcal{D}_{X^\dagger}$. Moreover, X is self-dual if and only if $\mathcal{I}_\varpi(v) = v$.*

Proof. It follows from Proposition 3.8.9, Corollary 3.8.5, and Lemma 3.5.4 that

$$\begin{aligned}\mathcal{D}_{\eta_{\psi(x)}\circ\mathcal{I}_\varpi(v)} =&\varphi(x)\mathcal{D}_{\mathcal{I}_\varpi(v)}(\varphi(x))^{-1} = \varphi(x)\varpi(\mathcal{D}^{\mathcal{B}})_v(\varphi(x))^{-1}\\ =&(T_1\dots T_N)(\mathcal{D}_X)^*(T_1\dots T_N)^{-1} = \mathcal{D}_{X^\dagger}.\end{aligned}$$

Since $(\lambda_A^{(s)})_N = 0$ for all $s = 1, \dots, n$, X has no base points. Therefore X is self-dual if and only if $\mathcal{D}_X = \mathcal{D}_{X^\dagger}$. Suppose X is self-dual, it follows from Theorem 3.3.2 that $\eta_{\psi(x)} \circ \mathcal{I}_\varpi(v)$ is a scalar multiple of v. By our identification, in terms of an $\mathfrak{sl}_N$-module homomorphism, $\eta_{\psi(x)}$ is the identity map. Moreover, since $\mathcal{I}_\varpi$ is an involution, we have $\mathcal{I}_\varpi(v) = \pm v$.

Finally, generically, we have an eigenbasis of the action of $\mathcal{B}$ in $(V_{\mathbf{\Lambda}_A,\boldsymbol{z}})^{\mathfrak{sl}_N}$ (for example for all $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$). In such a case, by the equality of dimensions using Lemma 3.8.2, we have $\mathcal{I}_\varpi(v) = v$. Then the general case is obtained by taking the limit. $\square$

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4. LOWER BOUNDS FOR NUMBERS OF REAL SELF-DUAL SPACES IN PROBLEMS OF SCHUBERT CALCULUS

4.1 Introduction

It is well known that the problem of finding the number of real solutions to algebraic systems is very difficult, and not many results are known. In particular, the counting of real points in problems of Schubert calculus in the Grassmannian has received a lot of attention, see [EG02, HHS13, HS, HSZ16, MT16, SS06, Sot10] for example. In this paper, we give lower bounds for the numbers of real self-dual spaces in intersections of Schubert varieties related to osculating flags in the Grassmannian.

We define the Grassmannian $\mathrm{Gr}(N,d)$ to be the set of all N-dimensional subspaces of the d-dimensional space $\mathbb{C}_d[x]$ of polynomials in x of degree less than d. In other words, we always assume for $X \in \mathrm{Gr}(N,d)$, we have $X \subset \mathbb{C}_d[x]$. Set $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then, for any $z \in \mathbb{P}^1$, we have the osculating flag $\mathcal{F}(z)$, see (3.3.3), (3.3.4). Denote the Schubert cells corresponding to $\mathcal{F}(z)$ by $\Omega_\xi(\mathcal{F}(z))$, where $\xi = (d-N \geqslant \xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_N \geqslant 0)$ are partitions. Then the set $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ consists of spaces $X \in \mathrm{Gr}(N,d)$ such that X belongs to the intersection of Schubert cells $\Omega_{\xi^{(i)}}(\mathcal{F}(z_i))$ for $\boldsymbol{z} = (z_1, \dots, z_n)$ and $\boldsymbol{\xi} = \left(\xi^{(1)}, \dots, \xi^{(n)}\right)$, where all $z_i \in \mathbb{P}^1$ are distinct and $\xi^{(i)}$ are partitions, see (3.3.5). A point $X \in \mathrm{Gr}(N,d)$ is called *real* if it has a basis consisting of polynomials with all coefficients real. A lower bound for the number of real points in $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is given in [MT16].

Let $X \in \mathrm{Gr}(N,d)$ be an N-dimensional subspace of polynomials in x. Let $X^\vee$ be the N-dimensional space of polynomials which are Wronskian determinants of $N-1$ elements of X

$$X^\vee = \left\{ \det \left(d^{i-1}\varphi_j/dx^{i-1}\right)_{i,j=1}^{N-1},\ \varphi_j(x) \in X \right\}.$$

The space X is called *self-dual* if $X^\vee = \psi \cdot X$ for some polynomial $\psi(x)$, see [MV04]. We define $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ the subset of $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ consisting of all self-dual spaces. Our main result of this paper is a lower bound for the number of real self-dual spaces in $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$, see Corollary 4.7.4, i.e., a lower bound for the number of real points in $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$, by following the idea of [MT16].

Let $\mathfrak{g}_N$ be the Lie algebra $\mathfrak{so}_{2r+1}$ if $N = 2r$ or the Lie algebra $\mathfrak{sp}_{2r}$ if $N = 2r+1$. We also set $\mathfrak{g}_3 = \mathfrak{sl}_2$. It is known from [LMV16], see also [MV04, Section 6.1], that if $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is nonempty, then $\xi_i^{(s)} - \xi_{i+1}^{(s)} = \xi_{N-i}^{(s)} - \xi_{N-i+1}^{(s)}$ for $i = 1, \dots, N-1$. Hence the $\mathfrak{sl}_N$-weight corresponding to the partition $\xi^{(s)}$ has certain symmetry and thus induces a $\mathfrak{g}_N$-weight $\lambda^{(s)}$, cf. (4.4.4). Therefore, the sequence of partitions $\boldsymbol{\xi}$ with nonempty $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ can be expressed in terms of a sequence of dominant integral $\mathfrak{g}_N$-weights $\boldsymbol{\lambda} = \left(\lambda^1, \dots, \lambda^{(n)}\right)$ and a sequence of nonnegative integers $\boldsymbol{k} = (k_1, \dots, k_n)$, see Lemma 4.4.1. In particular, $k_i = \xi_N^{(i)}$. We call $\boldsymbol{\xi}, \boldsymbol{z}$ or $\boldsymbol{\lambda}, \boldsymbol{k}, \boldsymbol{z}$ the *ramification data*.

As a subset of $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$, $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ can be empty even if $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is infinite. However, if $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is nonempty, then $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is finite if and only if $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is finite. More precisely, if

$$|\boldsymbol{\xi}| := \sum_{i=1}^{n} \left|\xi^{(i)}\right| = N(d-N),$$

then the number of points in $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ counted with multiplicities equals the multiplicity of the trivial $\mathfrak{g}_N$-module in the tensor product $V_{\lambda^{(1)}} \otimes \dots \otimes V_{\lambda^{(n)}}$ of irreducible $\mathfrak{g}_N$-modules of highest weights $\lambda^{(1)}, \dots, \lambda^{(n)}$. Since we are interested in the counting problem, from now on, we always assume that $|\boldsymbol{\xi}| = N(d-N)$.

For brevity, we consider ∞ to be real. If all $z_1, \dots, z_n$ are real, it follows from [MTV09c, Theorem 1.1] that all points in $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ are real. Hence the number of real points is maximal possible in this case. Moreover, it follows from [MTV09b, Corollary 6.3] that all points in $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ are multiplicity-free.

Then we want to know how many real points we can guarantee in other cases. In general, a necessary condition for the existence of real points is that the set $\{z_1, \ldots, z_n\}$ should be invariant under the complex conjugation and the partitions at the complex conjugate points are the same. In other words, $\left(\lambda^{(i)}, k_i\right) = \left(\lambda^{(j)}, k_j\right)$ provided $z_i = \bar{z}_j$. In this case we say that $\boldsymbol{z}$, $\boldsymbol{\lambda}$, $\boldsymbol{k}$ are *invariant under conjugation.* Moreover, the greatest common divisor of $X \in \mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ in this case is a real polynomial. Hence we reduce the problem to the case that $k_i = 0$, for all $i = 1, \ldots, n$.

The derivation of the lower bounds is based on the identification of the self-dual spaces of polynomials with points of spectrum of higher Gaudin Hamiltonians of types B and C ($\mathfrak{g}_N$, $N \geqslant 4$) built in [LMV16] and [MV04], see Theorem 4.5.2. We show that higher Gaudin Hamiltonians of types B and C have certain symmetry with respect to the Shapovalov form which is positive definite Hermitian, see Proposition 4.6.1. In particular, these operators are self-adjoint with respect to the Shapovalov form for real $z_1, \ldots, z_n$ and hence have real eigenvalues. Therefore, it follows from Theorem 4.5.2 that self-dual spaces with real $z_1, \ldots, z_n$ are real.

If some of $z_1, \ldots, z_n$ are not real, but the data $\boldsymbol{z}$, $\boldsymbol{\lambda}$, $\boldsymbol{k}$ are invariant under the complex conjugation, the higher Gaudin Hamiltonians are self-adjoint with respect to a nondegenerate (indefinite) Hermitian form. One of the key observations for computing the lower bound for the number of real points in $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is the fact that the number of real eigenvalues of such operators is at least the absolute value of the signature of the Hermitian form, see Lemma 4.6.4.

The computation of the signature of the form is reduced to the computation of the character values of products of symmetric groups on products of commuting transpositions. The formula for such character, similar to the Frobenius formula in [Fro00] and [MT16, Proposition 2.1], is given in Proposition 4.3.1. Consequently, we obtain our main result, a lower bound for the number of real points in $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ for $N \geqslant 4$, see Corollary 4.7.4. The case $N = 2$ is the same as that of [MT16] since every 2-dimensional space of polynomials is self-dual. By the proof of [LMV16, Theorem 4.19], the case $N = 3$ is reduced to the case of [MT16], see Section 4.7.2.

Based on the identification of the self-self-dual spaces of polynomials with points of spectrum of higher Gaudin Hamiltonians of type G_2 built in [BM05] and [LM19a], we expect that lower bounds for the numbers of real self-self-dual spaces in $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ with $N=7$ can also be given in a similar way as conducted in this paper.

It is also interesting to find an algorithm to compute all (real) self-dual spaces with prescribed ramification data. The solutions to the Bethe ansatz equations described in [LMV16] can be used to find nontrivial examples of self-dual spaces.

The paper is organized as follows. We start with the standard notation of Lie theory in Section 4.2 and computations of characters of a product of symmetric groups in Section 4.3. Then we recall notation and definitions for osculating Schubert calculus and self-dual spaces in Section 4.4. In Section 4.5 we recall the connections between Gaudin model of types B, C and self-dual spaces of polynomials. The symmetry of higher Gaudin Hamiltonians with respect to Shapovalov form and the key lemma from linear algebra are discussed in Section 4.6. In Section 4.7 we prove our main results, see Theorem 4.7.2 and Corollary 4.7.4. Finally, we display some simple data computed from Corollary 4.7.4 in Section 4.8.

4.2 Simple Lie Algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan matrix $A=(a_{i,j})_{i,j=1}^r$, where r is the rank of $\mathfrak{g}$. Let $D=\mathrm{diag}(d_1,\dots,d_r)$ be the diagonal matrix with positive relatively prime integers d_i such that DA is symmetric.

Let $\mathfrak{h}\subset\mathfrak{g}$ be a Cartan subalgebra with the Cartan decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. Fix simple roots $\alpha_1,\dots,\alpha_r$ in $\mathfrak{h}^*$. Let $\check\alpha_1,\dots,\check\alpha_r\in\mathfrak{h}$ be the corresponding coroots. Fix a nondegenerate invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$ such that $(\check\alpha_i,\check\alpha_j)=a_{i,j}/d_j$. The corresponding bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i,\alpha_j)=d_ia_{i,j}$. We have $\langle\lambda,\check\alpha_i\rangle=2(\lambda,\alpha_i)/(\alpha_i,\alpha_i)$ for $\lambda\in\mathfrak{h}^*$. In particular, $\langle\alpha_j,\check\alpha_i\rangle=a_{i,j}$. Let $\omega_1,\dots,\omega_r\in\mathfrak{h}^*$ be the fundamental weights, $\langle\omega_j,\check\alpha_i\rangle=\delta_{i,j}$.

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* \,|\, \langle \lambda, \check{\alpha}_i \rangle \in \mathbb{Z},\ i = 1, \dots, r\}$ and $\mathcal{P}^+ = \{\lambda \in \mathfrak{h}^* \,|\, \langle \lambda, \check{\alpha}_i \rangle \in \mathbb{Z}_{\geqslant 0},\ i = 1, \dots, r\}$ be the weight lattice and the cone of dominant integral weights.

Let $e_1, \dots, e_r \in \mathfrak{n}_+$, $\check{\alpha}_1, \dots, \check{\alpha}_r \in \mathfrak{h}$, $f_1, \dots, f_r \in \mathfrak{n}_-$ be the Chevalley generators of $\mathfrak{g}$.

Given a $\mathfrak{g}$-module M, denote by $(M)^{\mathfrak{g}}$ the subspace of $\mathfrak{g}$-invariants in M. The subspace $(M)^{\mathfrak{g}}$ is the multiplicity space of the trivial $\mathfrak{g}$-module in M.

A sequence of nonnegative integers $\xi = (\xi_1, \dots, \xi_k)$ such that $\xi_1 \geqslant \xi_2 \geqslant \dots \geqslant \xi_k \geqslant 0$ is called *a partition with at most k parts*. Set $|\xi| = \sum_{i=1}^{k} \xi_i$.

For $\lambda \in \mathfrak{h}^*$, let V_λ be the irreducible $\mathfrak{g}$-module with highest weight λ. For any $\mathfrak{g}$-weights λ and μ, it is well known that $\dim(V_\lambda \otimes V_\mu)^{\mathfrak{g}} = \delta_{\lambda,\mu}$ for $\mathfrak{g} = \mathfrak{so}_{2r+1}, \mathfrak{sp}_{2r}$.

For any Lie algebra $\mathfrak{g}$, denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

4.3 Characters of the Symmetric Groups

Let $\mathfrak{g}_N$ be the Lie algebra $\mathfrak{so}_{2r+1}$ if $N = 2r$ or the Lie algebra $\mathfrak{sp}_{2r}$ if $N = 2r + 1$, $r \geqslant 2$. We also set $\mathfrak{g}_3 = \mathfrak{sl}_2$. Let G_N be the respective classical group with Lie algebra $\mathfrak{g}_N$.

Let $\mathfrak{S}_k$ be the symmetric group permuting a set of k elements. In this section we deduce a formula for characters of a product of the symmetric groups acting on a tensor product of finite-dimensional irreducible $\mathfrak{g}_N$-modules.

For each dominant integral $\mathfrak{g}_N$-weight λ, denote by $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_r)$ the partition with at most r parts such that

$$2\langle \lambda, \check{\alpha}_i \rangle = \bar{\lambda}_i - \bar{\lambda}_{i+1}, \quad i = 1, \dots, r-1, \quad \text{and} \quad \bar{\lambda}_r = \begin{cases} \langle \lambda, \check{\alpha}_r \rangle, & \text{if } N = 2r, \\ 2\langle \lambda, \check{\alpha}_r \rangle, & \text{if } N = 2r+1. \end{cases}$$

Define an anti-symmetric Laurent polynomial Δ_N in $x_1, \dots, x_r$ as follows

$$\Delta_N = \det\left(x_i^{N+1-2j} - x_i^{-(N+1-2j)}\right)_{i,j=1}^{r}. \tag{4.3.1}$$

We call Δ_N the *Vandermonde determinant of* $\mathfrak{g}_N$.

Let λ be a dominant integral $\mathfrak{g}_N$-weight. It is well known that the character of the module V_λ is given by

$$\mathcal{S}_\lambda^N(x_1,\dots,x_r) = \mathrm{tr}_{V_\lambda} X_N = \frac{\det\left(x_i^{\bar\lambda_j+N+1-2j} - x_i^{-(\bar\lambda_j+N+1-2j)}\right)_{i,j=1}^r}{\Delta_N}, \tag{4.3.2}$$

where $X_N \in \mathrm{G}_N$ is given by

$$X_N = \begin{cases} \mathrm{diag}\left(x_1^2,\dots,x_r^2,1,x_r^{-2},\dots,x_1^{-2}\right), & \text{if } N=2r, \\ \mathrm{diag}\left(x_1^2,\dots,x_r^2,x_r^{-2},\dots,x_1^{-2}\right), & \text{if } N=2r+1. \end{cases}$$

We call $\mathcal{S}_\lambda^N$ the *Schur function of* $\mathfrak{g}_N$ *associated with the weight* λ.

Note that $\mathcal{S}_\lambda^N$ are symmetric Laurent polynomials in $x_1,\dots,x_r$, namely

$$\mathcal{S}_\lambda^N \in \left(\mathbb{C}\left[x_1^{\pm1},\dots,x_r^{\pm1}\right]\right)^{\mathfrak{S}_r}.$$

Let $\lambda^{(1)},\dots,\lambda^{(s)}$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and $k_1,\dots,k_s$ a sequence of positive integers. Consider the tensor product of $\mathfrak{g}_N$-modules

$$V_{\boldsymbol{\lambda}} = V_{\lambda^{(1)}}^{\otimes k_1} \otimes V_{\lambda^{(2)}}^{\otimes k_2} \otimes \cdots \otimes V_{\lambda^{(s)}}^{\otimes k_s}$$

and its decomposition into irreducible $\mathfrak{g}_N$-submodules

$$V_{\boldsymbol{\lambda}} = \bigoplus_{\mu} V_\mu \otimes M_{\boldsymbol{\lambda},\mu}.$$

By permuting the corresponding tensor factors of $V_{\boldsymbol{\lambda}}$, the product of symmetric groups $\mathfrak{S}_{\boldsymbol{k}} = \mathfrak{S}_{k_1} \times \mathfrak{S}_{k_2} \times \cdots \times \mathfrak{S}_{k_s}$ acts naturally on $V_{\boldsymbol{\lambda}}$. Note that the $\mathfrak{S}_{\boldsymbol{k}}$-action commutes with the $\mathfrak{g}_N$-action, therefore the group $\mathfrak{S}_{\boldsymbol{k}}$ acts on the multiplicity space $M_{\boldsymbol{\lambda},\mu}$ for all μ.

For $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_s \in \mathfrak{S}_{\boldsymbol{k}}$, $\sigma_i \in \mathfrak{S}_{k_i}$. Suppose all σ_i are written as a product of disjoint cycles. Denote by c_i the number of cycles in the product representing σ_i and l_{ij}, $j=1,\dots,c_i$, the lengths of cycles. Note that $l_{i,1}+\cdots+l_{i,c_i} = k_i$.

We then consider the value of the character of $\mathfrak{S}_{\boldsymbol{k}}$ corresponding to the representation $M_{\boldsymbol{\lambda},\mu}$ on σ. Let $\chi_{\boldsymbol{\lambda},\mu} = \mathrm{tr}_{M_{\boldsymbol{\lambda},\mu}}$.

Proposition 4.3.1. *The character value* $\chi_{\boldsymbol{\lambda},\mu}(\sigma)$ *equals the coefficient of the monomial*

$$x_1^{\bar{\mu}_1+N-1}x_2^{\bar{\mu}_2+N-3}\cdots x_r^{\bar{\mu}_r+N+1-2r}$$

in the Laurent polynomial

$$\Delta_N\cdot\prod_{i=1}^{s}\prod_{j=1}^{c_i}\mathcal{S}^N_{\lambda^{(i)}}\big(x_1^{l_{ij}},\dots,x_r^{l_{ij}}\big).$$

Proof. The proof of the statement is similar to that of [MT16, Proposition 2.1]. $\square$

4.4 Osculating Schubert Calculus and Self-Dual Spaces

Let N, $d\in\mathbb{Z}_{>0}$ be such that $N\leqslant d$. Consider $\mathbb{P}^1:=\mathbb{C}\cup\{\infty\}$. Set

$$\mathring{\mathbb{P}}_n:=\big\{\boldsymbol{z}=(z_1,\dots,z_n)\in(\mathbb{P}^1)^n\,|\,z_i\neq z_j\text{ for }1\leqslant i<j\leqslant n\big\},$$
$$\mathbb{R}\mathring{\mathbb{P}}_n:=\big\{\boldsymbol{z}=(z_1,\dots,z_n)\in\mathring{\mathbb{P}}_n\,|\,z_i\in\mathbb{R}\text{ or }z_i=\infty,\ \text{for }1\leqslant i\leqslant n\big\}.$$

4.4.1 Osculating Schubert Calculus

Let $\mathbb{C}_d[x]$ be the space of polynomials in x with complex coefficients of degree less than d. We have $\dim\mathbb{C}_d[x]=d$. Let $\mathrm{Gr}(N,d)$ be the Grassmannian of all N-dimensional subspaces in $\mathbb{C}_d[x]$. The Grassmannian $\mathrm{Gr}(N,d)$ is a smooth projective complex variety of dimension $N(d-N)$.

Let $\mathbb{R}_d[x]\subset\mathbb{C}_d[x]$ be the set of polynomials in x with real coefficients of degree less than d. Let $\mathrm{Gr}^{\mathbb{R}}(N,d)\subset\mathrm{Gr}(N,d)$ be the set of subspaces which have a basis consisting of polynomials with all coefficients real. For $X\in\mathrm{Gr}(N,d)$ we have $X\in\mathrm{Gr}^{\mathbb{R}}(N,d)$ if and only if $\dim_{\mathbb{R}}(X\cap\mathbb{R}_d[x])=N$. We call such points X *real*.

For a complete flag $\mathcal{F} = \{0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_d = \mathbb{C}_d[x]\}$ and a partition $\xi = (\xi_1, \ldots, \xi_N)$ such that $\xi_1 \leqslant d - N$, the Schubert cell $\Omega_\xi(\mathcal{F}) \subset \mathrm{Gr}(N, d)$ is given by

$$\Omega_\xi(\mathcal{F}) = \big\{X \in \mathrm{Gr}(N, d) \mid \dim(X \cap \mathscr{F}_{d-j-\xi_{N-j}}) = N - j,$$
$$\dim(X \cap \mathscr{F}_{d-j-\xi_{N-j}-1}) = N - j - 1\big\}.$$

Note that $\operatorname{codim} \Omega_\xi(\mathcal{F}) = |\xi|$.

Let $\mathcal{F}(\infty)$ be the complete flag given by

$$\mathcal{F}(\infty) = \big\{0 \subset \mathbb{C}_1[x] \subset \mathbb{C}_2[x] \subset \cdots \subset \mathbb{C}_d[x]\big\}. \tag{4.4.1}$$

The subspace X is a point of $\Omega_\xi(\mathcal{F}(\infty))$ if and only if for every $i = 1, \ldots, N$, it contains a polynomial of degree $d - i - \xi_{N+1-i}$.

For $z \in \mathbb{C}$, consider the complete flag

$$\mathcal{F}(z) = \big\{0 \subset (x - z)^{d-1}\mathbb{C}_1[x] \subset (x - z)^{d-2}\mathbb{C}_2[x] \subset \cdots \subset \mathbb{C}_d[x]\big\}. \tag{4.4.2}$$

The subspace X is a point of $\Omega_\xi(\mathcal{F}(z))$ if and only if for every $i = 1, \ldots, N$, it contains a polynomial with a root at z of order exactly $\xi_i + N - i$.

A point $z \in \mathbb{C}$ is called a *base point* for a subspace $X \subset \mathbb{C}_d[x]$ if $\varphi(z) = 0$ for every $\varphi \in X$.

Let $\boldsymbol{\xi} = \big(\xi^{(1)}, \ldots, \xi^{(n)}\big)$ be a sequence of partitions with at most N parts and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathring{\mathbb{P}}_n$. Set $|\boldsymbol{\xi}| = \sum_{s=1}^{n} \big|\xi^{(s)}\big|$.

Assuming $|\boldsymbol{\xi}| = N(d - N)$, denote by $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ the intersection of the Schubert cells

$$\Omega_{\boldsymbol{\xi},\boldsymbol{z}} = \bigcap_{s=1}^{n} \Omega_{\xi^{(s)}}(\mathcal{F}(z_s)). \tag{4.4.3}$$

Note that due to our assumption, $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is a finite subset of $\mathrm{Gr}(N, d)$.

Define a sequence of polynomials $\boldsymbol{T} = (T_1, \ldots, T_N)$ by the formulas

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{\xi_i^{(s)} - \xi_{i+1}^{(s)}}, \qquad i = 1, \ldots, N,$$

where $\xi_{N+1}^{(s)} = 0$. Here and in what follows we use the convention that $x - z_s$ is considered as the constant function 1 if $z_s = \infty$. We say that $\boldsymbol{T}$ is *associated with* $\boldsymbol{\xi}$, $\boldsymbol{z}$.

4.4.2 Self-Dual Spaces

Let $X \in \mathrm{Gr}(N,d)$ be an N-dimensional subspace of polynomials in x. Given a polynomial ψ in x, denote by $\psi \cdot X$ the space of polynomials of the form $\psi \cdot \varphi$ for all $\varphi \in X$.

Let $X^\vee$ be the N-dimensional space of polynomials which are Wronskian determinants of $N-1$ elements of X

$$X^\vee = \left\{ \det \left(d^{i-1}\varphi_j/dx^{i-1} \right)_{i,j=1}^{N-1},\ \varphi_j(x) \in X \right\}.$$

The space X is called *self-dual* if $X^\vee = \psi \cdot X$ for some polynomial $\psi(x)$, see [MV04].

Let $\mathrm{sGr}(N,d)$ be the set of all self-dual spaces in $\mathrm{Gr}(N,d)$. We call $\mathrm{sGr}(N,d)$ the *self-dual Grassmannian.* The self-dual Grassmannian $\mathrm{sGr}(N,d)$ is an algebraic subset of $\mathrm{Gr}(N,d)$.

Denote by $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ the set of all self-dual spaces in $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$

$$\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}} = \Omega_{\boldsymbol{\xi},\boldsymbol{z}} \bigcap \mathrm{sGr}(N,d).$$

Let μ be a dominant integral $\mathfrak{g}_N$-weight and $k \in \mathbb{Z}_{\geqslant 0}$. Define a partition $\mu_{A,k}$ with at most N parts by the rule: $(\mu_{A,k})_N = k$ and

$$(\mu_{A,k})_i - (\mu_{A,k})_{i+1} = \begin{cases} \langle \mu, \check{\alpha}_i \rangle, & \text{if } 1 \leqslant i \leqslant \left[\frac{N}{2}\right], \\ \langle \mu, \check{\alpha}_{N-i} \rangle, & \text{if } \left[\frac{N}{2}\right] < i \leqslant N-1. \end{cases} \tag{4.4.4}$$

We call $\mu_{A,k}$ the partition *associated with weight μ and integer k.*

Let $\boldsymbol{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right)$ be a sequence of dominant integral $\mathfrak{g}_N$-weights and let $\boldsymbol{k} = (k_1, \ldots, k_n)$ be an n-tuple of nonnegative integers. Then denote

$$\boldsymbol{\lambda}_{A,\boldsymbol{k}} = \left(\lambda^{(1)}_{A,k_1}, \ldots, \lambda^{(n)}_{A,k_n}\right)$$

the sequence of partitions associated with $\lambda^{(s)}$ and k_s, $s = 1, \ldots, n$.

We use the notation $\mu_A = \mu_{A,0}$ and $\boldsymbol{\lambda}_A = \boldsymbol{\lambda}_{A,(0,\ldots,0)}$.

Lemma 4.4.1 ([LMV16])**.** *If $\boldsymbol{\xi}$ is a sequence of partitions with at most N parts such that $|\boldsymbol{\xi}| = N(d-N)$ and $\mathrm{s}\Omega_{\boldsymbol{\xi},\boldsymbol{z}}$ is nonempty, then $\boldsymbol{\xi}$ has the form $\boldsymbol{\xi} = \boldsymbol{\lambda}_{A,\boldsymbol{k}}$ for* *a sequence of dominant integral $\mathfrak{g}_N$-weights $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \dots, \lambda^{(n)}\big)$ and a sequence of nonnegative integers $\boldsymbol{k} = (k_1, \dots, k_n)$. The pair $(\boldsymbol{\lambda}, \boldsymbol{k})$ is uniquely determined by $\boldsymbol{\xi}$.*

In what follows we write $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$, $\Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}}$, $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$, $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}}$ for $\Omega_{\boldsymbol{\lambda}_A,\boldsymbol{z}}$, $\Omega_{\boldsymbol{\lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$, $\mathrm{s}\Omega_{\boldsymbol{\lambda}_A,\boldsymbol{z}}$, $\mathrm{s}\Omega_{\boldsymbol{\lambda}_{A,\boldsymbol{k}},\boldsymbol{z}}$, respectively.

Note that $|\boldsymbol{\lambda}_{A,\boldsymbol{k}}| = |\boldsymbol{\lambda}_A| + N|\boldsymbol{k}|$, where $|\boldsymbol{k}| = k_1 + \cdots + k_n$. Suppose $|\boldsymbol{\lambda}_A| = N(d-N)$, there exists a bijection between $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ in $\mathrm{Gr}(N,d)$ and $\Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}}$ in $\mathrm{Gr}(N, d+|\boldsymbol{k}|)$ given by

$$\Omega_{\boldsymbol{\lambda},\boldsymbol{z}} \to \Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}}, \qquad X \mapsto \prod_{s=1}^{n} (x - z_s)^{k_s} \cdot X. \tag{4.4.5}$$

Moreover, (3.4.3) restricts to a bijection between $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ in $\mathrm{sGr}(N,d)$ and $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}}$ in $\mathrm{sGr}(N, d+|\boldsymbol{k}|)$.

4.5 Gaudin Model

Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{g}$-valued polynomials with the pointwise commutator. We call it the *current algebra* of $\mathfrak{g}$. We identify the Lie algebra $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1$ of constant polynomials in $\mathfrak{g}[t]$.

It is convenient to collect elements of $\mathfrak{g}[t]$ in generating series of a formal variable x. For $g \in \mathfrak{g}$, set

$$g(x) = \sum_{k=0}^{\infty} \big(g \otimes t^k\big) x^{-k-1}. \tag{4.5.1}$$

For each $a \in \mathbb{C}$, we have the evaluation homomorphism $\mathrm{ev}_a \colon \mathfrak{g}[t] \to \mathfrak{g}$ where ev_a sends $g \otimes t^s$ to $a^s g$ for all $g \in \mathfrak{g}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Its restriction to the subalgebra $\mathfrak{g} \subset \mathfrak{g}[t]$ is the identity map. For any $\mathfrak{g}$-module M, we denote by $M(a)$ the $\mathfrak{g}[t]$-module, obtained by pulling M back through the evaluation homomorphism ev_a. The $\mathfrak{g}[t]$-module $M(a)$ is called an *evaluation module.* The generating series $g(x)$ acts on the evaluation module $M(a)$ by $g/(x-a)$.

The Bethe algebra $\mathcal{B}$ (the algebra of higher Gaudin Hamiltonians) for a simple Lie algebra $\mathfrak{g}$ was described in [FFR94]. The Bethe algebra $\mathcal{B}$ is a commutative subalgebra of $\mathcal{U}(\mathfrak{g}[t])$ which commutes with the subalgebra $\mathcal{U}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g}[t])$. An explicit set of generators of the Bethe algebra in Lie algebras of types B, C, and D was given in [Mol13].

Proposition 4.5.1 ([FFR94, Mol13])**.** *Let $N > 3$. There exist elements $F_{ij} \in \mathfrak{g}_N$, $i, j = 1, \ldots, N$, and polynomials $B_s(x)$ in $d^k F_{ij}(x)/dx^k$, $s = 1, \ldots, N$, $k = 0, \ldots, N$, such that the Bethe algebra $\mathcal{B}$ of $\mathfrak{g}_N$ is generated by the coefficients of $B_s(x)$ considered as formal power series in x^{-1}.*

We denote $M(\infty)$ the $\mathfrak{g}_N$-module M with the trivial action of the Bethe algebra $\mathcal{B}$, see [LMV16] for more detail.

For a collection of $\mathfrak{g}_N$-weights $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \ldots, \lambda^{(n)}\big)$ and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathring{\mathbb{P}}_n$, we set

$$V_{\boldsymbol{\lambda},\boldsymbol{z}} = \bigotimes_{s=1}^{n} V_{\lambda^{(s)}}(z_s),$$

considered as a $\mathcal{B}$-module. We also denote $V_{\boldsymbol{\lambda}}$ the module $V_{\boldsymbol{\lambda},\boldsymbol{z}}$ considered as a $\mathfrak{g}_N$-module.

Let ∂_x be the differentiation with respect to x. Define a formal differential operator

$$\mathcal{D}^{\mathcal{B}} = \partial_x^N + \sum_{i=1}^{N} B_i(x)\partial_x^{N-i},$$

where

$$B_i(x) = \sum_{j=i}^{\infty} B_{ij} x^{-j} \tag{4.5.2}$$

and $B_{ij} \in \mathcal{U}(\mathfrak{g}_N[t])$, $j \in \mathbb{Z}_{\geqslant i}$, $i = 1, \ldots, N$. The operator $\mathcal{D}^{\mathcal{B}}$ is called *the universal operator.*

Let $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathring{\mathbb{P}}_n$ and let $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \ldots, \lambda^{(n)}\big)$ be a sequence of dominant integral $\mathfrak{g}_N$-weights. For every $g \in \mathfrak{g}_N$, the series $g(x)$ acts on $V_{\boldsymbol{\lambda},\boldsymbol{z}}$ as a rational function of x.

Since the Bethe algebra $\mathcal{B}$ commutes with $\mathfrak{g}_N$, $\mathcal{B}$ acts on the invariant space $(V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$. For $b \in \mathcal{B}$, denote by $b(\boldsymbol{\lambda}, \boldsymbol{z}) \in \mathrm{End}((V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N})$ the corresponding linear operator.

Given a common eigenvector $v \in (V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ of the operators $b(\boldsymbol{\lambda},\boldsymbol{z})$, denote by $b(\boldsymbol{\lambda},\boldsymbol{z};v)$ the corresponding eigenvalues, and define the scalar differential operator

$$\mathcal{D}_v = \partial_x^N + \sum_{i=1}^{N}\sum_{j=i}^{\infty} B_{ij}(\boldsymbol{\lambda},\boldsymbol{z};v)x^{-j}\partial_x^{N-i}.$$

The following theorem connects self-dual spaces in the Grassmannian $\mathrm{Gr}(N,d)$ with the Gaudin model associated to $\mathfrak{g}_N$.

Theorem 4.5.2 ([LMV16]). *Let $N > 3$. There exists a choice of generators $B_i(x)$ of the Bethe algebra $\mathcal{B}$, such that for any sequence of dominant integral $\mathfrak{g}_N$-weights $\boldsymbol{\lambda} = \left(\lambda^{(1)},\ldots,\lambda^{(n)}\right)$, any $\boldsymbol{z} \in \mathring{\mathbb{P}}_n$, and any $\mathcal{B}$-eigenvector $v \in (V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$, we have*

$$\operatorname{Ker}\left((T_1\cdots T_N)^{1/2}\cdot\mathcal{D}_v\cdot(T_1\cdots T_N)^{-1/2}\right) \in \mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}},$$

where $\boldsymbol{T} = (T_1,\ldots,T_N)$ is associated with $\boldsymbol{\lambda}_A$, $\boldsymbol{z}$.

Moreover, if $|\boldsymbol{\lambda}_A| = N(d-N)$, then this defines a bijection between the joint eigenvalues of $\mathcal{B}$ on $(V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ and $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}} \subset \mathrm{Gr}(N,d)$.

4.6 Shapovalov Form and the Key Lemma

4.6.1 Shapovalov Form

Define the anti-involution $\varpi\colon \mathfrak{g}_N \to \mathfrak{g}_N$ sending $e_1,\ldots,e_r$, $\check{\alpha}_1,\ldots,\check{\alpha}_r$, $f_1,\ldots,f_r$ to $f_1,\ldots,f_r$, $\check{\alpha}_1,\ldots,\check{\alpha}_r$, $e_1,\ldots,e_r$, respectively.

For any dominant integral $\mathfrak{g}_N$-weight λ, the irreducible $\mathfrak{g}_N$-module V_λ admits a positive definite Hermitian form $(\cdot,\cdot)_\lambda$ such that $(gv,w)_\lambda = (v,\overline{\varpi(g)}w)_\lambda$ for any $v,w \in V_\lambda$ and $g \in \mathfrak{g}_N$. Such a form is unique up to multiplication by a positive real number. We call this form the *Shapovalov form.*

Let $\boldsymbol{\lambda} = \left(\lambda^{(1)},\ldots,\lambda^{(n)}\right)$ be a sequence of dominant integral $\mathfrak{g}_N$-weights. We define the positive definite Hermitian form $(\cdot,\cdot)_{\boldsymbol{\lambda}}$ on the tensor product $V_{\boldsymbol{\lambda}}$ as the product of Shapovalov forms on the tensor factors. The form $(\cdot,\cdot)_{\boldsymbol{\lambda}}$ induces a positive definite Hermitian form $(\cdot|\cdot)_{\boldsymbol{\lambda}}$ on $(V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$.

Proposition 4.6.1. *For any $i = 1, \dots, N$, $j \in \mathbb{Z}_{\geqslant i}$, and any $v, w \in (V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$, we have*

$$\big(B_{ij}(\boldsymbol{\lambda}, \boldsymbol{z})v|w\big)_{\boldsymbol{\lambda}} = \big(v|B_{ij}(\boldsymbol{\lambda}, \bar{\boldsymbol{z}})w\big)_{\boldsymbol{\lambda}},$$

where B_{ij} are given by (4.5.2)*, $\bar{\boldsymbol{z}} = (\bar{z}_1, \dots, \bar{z}_n)$ and the bar stands for the complex conjugation.*

Proof. We prove the proposition in Section 4.6.3. □

If $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$, then $B_{ij}(\boldsymbol{\lambda}, \boldsymbol{z})$ are self-adjoint with respect to the Shapovalov form. Therefore all $B_{ij}(\boldsymbol{\lambda}, \boldsymbol{z})$ are simultaneously diagonalizable. Moreover, all eigenvalues of $B_{ij}(\boldsymbol{\lambda}, \boldsymbol{z})$ are real.

The following statement is also known.

Theorem 4.6.2 ([Ryb18])**.** *For generic $\boldsymbol{z} \in \mathring{\mathbb{P}}_n$, the action of the Bethe algebra $\mathcal{B}$ on $(V_{\boldsymbol{\lambda},\boldsymbol{z}})^{\mathfrak{g}_N}$ is diagonalizable and has simple spectrum. In particular, this statement holds for any sequence $\boldsymbol{z} \in \mathbb{R}\mathring{\mathbb{P}}_n$.*

If some of the partitions $\lambda^{(1)}, \dots, \lambda^{(n)}$ coincide, the operators $b(\boldsymbol{\lambda}, \boldsymbol{z})$ admit additional symmetry. Assume that $\lambda^{(i)} = \lambda^{(i+1)}$ for some i. Let $P_i \in \mathrm{End}(V_{\boldsymbol{\lambda}})$ be the flip of the i-th and $(i+1)$-st tensor factors and $\tilde{\boldsymbol{z}}^{(i)} = (z_1, \dots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \dots, z_n)$.

Lemma 4.6.3. *For any $b \in \mathcal{B}$, we have $P_i b(\boldsymbol{\lambda}, \boldsymbol{z}) P_i = b\big(\boldsymbol{\lambda}, \tilde{\boldsymbol{z}}^{(i)}\big)$.*

4.6.2 Self-Adjoint Operators with Respect to Indefinite Hermitian Form

In this section we recall the key lemma from linear algebra, see [Pon44].

Given a finite-dimensional vector space M, a linear operator $\mathfrak{T} \in \mathrm{End}(M)$, and a number $\alpha \in \mathbb{C}$, let $M_{\mathfrak{T}}(\alpha) = \ker(\mathfrak{T} - \alpha)^{\dim M}$. When $M_{\mathfrak{T}}(\alpha)$ is not trivial, it is the subspace of generalized eigenvectors of $\mathfrak{T}$ with eigenvalue α.

Lemma 4.6.4 ([Pon44])**.** *Let M be a complex finite-dimensional vector space with a nondegenerate Hermitian form of signature κ, and let $\mathcal{A} \subset \mathrm{End}(M)$ be a commutative subalgebra over $\mathbb{R}$, whose elements are self-adjoint operators. Let $R = \bigcap_{\mathfrak{T}\in\mathcal{A}} \bigoplus_{\alpha\in\mathbb{R}} M_{\mathfrak{T}}(\alpha)$. Then the restriction of the Hermitian form on R is nondegenerate and has signature κ. In particular, $\dim R \geqslant |\kappa|$.*

4.6.3 Proof of Proposition 4.6.1

In this section, we give the proof of Proposition 4.6.1. We follow the convention of [MM17]. We only introduce the necessary notation and refer the reader to [Mol13, Section 5] and [MM17, Section 3] for more detail.

Proof of Proposition 4.6.1. We prove it for the case $N = 2r$ first.

Let E_{ij} with $i, j = 1, \dots, 2r+1$ be the standard basis of $\mathfrak{gl}_{2r+1}$. The Lie subalgebra of $\mathfrak{gl}_{2r+1}$ generated by the elements $F_{ij} = E_{ij} - E_{2r+2-j,2r+2-i}$ is isomorphic to the Lie algebra $\mathfrak{so}_{2r+1} = \mathfrak{g}_N$. With this isomorphism, the anti-involution $\varpi\colon \mathfrak{g}_N \to \mathfrak{g}_N$ is realized by taking transposition, $F_{ij} \mapsto F_{ji}$. To be consistent with the notation in [MM17], we write $\mathfrak{g}$ for $\mathfrak{g}_N$. The number N in [MM17] is $2r + 1$ in our notation.

We write $F_{ij}[s]$ for $F_{ij} \otimes t^s$ in the loop algebra $\mathfrak{g}\left[t, t^{-1}\right]$. Consider the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}\left[t, t^{-1}\right] \oplus \mathbb{C}K$, which is the central extension of the loop algebra $\mathfrak{g}\left[t, t^{-1}\right]$, where the element K is central in $\widehat{\mathfrak{g}}$ and

$$[g_1[k], g_2[l]] = [g_1, g_2][k + l] + k\delta_{k,-l}(g_1, g_2)K, \qquad g_1, g_2 \in \mathfrak{g}, \qquad k, l \in \mathbb{Z}.$$

Consider the extended affine Lie algebra $\widehat{\mathfrak{g}} \oplus \mathbb{C}\tau = \mathfrak{g}\left[t, t^{-1}\right] \oplus \mathbb{C}K \oplus \mathbb{C}\tau$, where τ satisfies

$$[\tau, F_{ij}[s]] = -sF_{ij}[s - 1], \qquad [\tau, K] = 0, \qquad s \in \mathbb{Z}.$$

Set $\mathcal{U} = \mathcal{U}(\widehat{\mathfrak{g}} \oplus \mathbb{C}\tau)$ and fix $m \in \{1, \dots, N\}$. Introduce the element $F[s]_a$ of the algebra $\left(\operatorname{End}\left(\mathbb{C}^{2r+1}\right)\right)^{\otimes m} \otimes \mathcal{U}$, see [MM17, equation (3.5)], by

$$F[s]_a = \sum_{i,j=1}^{2r+1} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes F_{ij}[s],$$

where $e_{ij} \in \operatorname{End}\left(\mathbb{C}^{2r+1}\right)$ denote the standard matrix units. The map ϖ induces an anti-involution

$$\varpi\colon\ \mathcal{U}\left(t^{-1}\mathfrak{g}\left[t^{-1}\right]\right) \to \mathcal{U}\left(t^{-1}\mathfrak{g}\left[t^{-1}\right]\right), \qquad F_{ij}[s] \mapsto F_{ji}[s], \qquad s \in \mathbb{Z}_{\leqslant -1}.$$

For $1 \leqslant a < b \leqslant m$, consider the operators P_{ab} and Q_{ab} in $\left(\operatorname{End}\left(\mathbb{C}^{2r+1}\right)\right)^{\otimes m}$ defined as follows

$$P_{ab} = \sum_{i,j=1}^{2r+1} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(m-b)},$$

$$Q_{ab} = \sum_{i,j=1}^{2r+1} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{2r+2-i,2r+2-j} \otimes 1^{\otimes(m-b)}.$$

Set

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 + \frac{P_{ab}}{b-a} - \frac{2Q_{ab}}{2r+2b-2a-1}\right),$$

where the product is taken in the lexicographic order on the pairs (a, b). The element $S^{(m)}$ is the symmetrizer of the Brauer algebra acting on $\left(\mathbb{C}^{2r+1}\right)^{\otimes m}$. In particular, for any $1 \leqslant a < b \leqslant m$, the operator $S^{(m)}$ satisfies

$$S^{(m)} Q_{ab} = Q_{ab} S^{(m)} = 0, \qquad S^{(m)} P_{ab} = P_{ab} S^{(m)} = S^{(m)}.$$

Replacing τ with ∂_x and $F_{ij}[-\ell-1]$ with $-\partial_x^\ell F_{ij}(x)/\ell!$, where $F_{ij}(x)$ is defined in (4.5.1), for $\ell \in \mathbb{Z}_{\geqslant 0}$, in the element

$$\frac{2r+m-1}{2r+2m-1} \operatorname{tr} S^{(m)} (\tau + F[-1]_1) \cdots (\tau + F[-1]_m),$$

see [MM17, formula (3.26)], where the trace is taken on all m copies of $\operatorname{End}\left(\mathbb{C}^{2r+1}\right)$, we get a differential operator

$$\vartheta_{m0}(x) \partial_x^m + \vartheta_{m1}(x) \partial_x^{m-1} + \cdots + \vartheta_{mm}(x),$$

where $\vartheta_{mi}(x)$ is a formal power series in x^{-1} with coefficients in $\mathcal{U}(\mathfrak{g}[t])$. The Bethe subalgebra $\mathcal{B}$ of $\mathcal{U}(\mathfrak{g}[t])$ is generated by the coefficients of $\vartheta_{mi}(x)$, $m = 1, \ldots, N$, $i = 0, \ldots, m$, see [Mol13, Section 5].

Therefore, to prove the proposition, it suffices to show that the element

$$\frac{2r+m-1}{2r+2m-1} \operatorname{tr} S^{(m)} (\tau + F[-1]_1) \cdots (\tau + F[-1]_m), \tag{4.6.1}$$

is stable under the anti-involution ϖ. Here ϖ maps τ to τ.

Applying transposition on a-th and b-th components to the commutator relation

$$F[k]_a F[l]_b - F[l]_b F[k]_a = (P_{ab} - Q_{ab}) F[k+l]_b - F[k+l]_b (P_{ab} - Q_{ab}),$$

see the proof of [MM17, Lemma 3.6], we get

$$F^\top[k]_a F^\top[l]_b - F^\top[l]_b F^\top[k]_a = F^\top[k+l]_b (P_{ab} - Q_{ab}) - (P_{ab} - Q_{ab}) F^\top[k+l]_b,$$

for all $1 \leqslant a < b \leqslant m$. Here $\top$ stands for transpose, explicitly,

$$F^\top[s]_a = \sum_{i,j=1}^{2r+1} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes F_{ji}[s].$$

Thus one can use the same argument as in the proof of [MM17, Lemma 3.2] to show that the image of (4.6.1) under the anti-involution ϖ equals

$$\frac{2r+m-1}{2r+2m-1} \operatorname{tr} S^{(m)} \big(\tau + F^\top[-1]_1\big) \cdots \big(\tau + F^\top[-1]_m\big). \tag{4.6.2}$$

By applying the simultaneous transposition $e_{ij} \to e_{ji}$ to all m copies of $\operatorname{End}\big(\mathbb{C}^{2r+1}\big)$ we conclude that (4.6.2) coincides with (4.6.1) because the transformation takes each factor $\tau + F^\top[-1]_a$ to $\tau + F[-1]_a$ whereas the symmetrizer $S^{(m)}$ stays invariant. Hence we complete the proof of Proposition 4.6.1 for the case $N = 2r$.

The case $N = 2r + 1$ is proved similarly, see for example [MM17, Lemma 3.9]. $\square$

4.7 The Lower Bound

In this section we prove our main results – the lower bound for the number of real self-dual spaces in $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$, see Theorem 4.7.2 and Corollary 4.7.4.

Recall the notation from Section 4.4. For positive integers N, d such that $d \geqslant N$ we consider the Grassmannian $\mathrm{Gr}(N,d)$ of N-dimensional planes in the space $\mathbb{C}_d[x]$ of polynomials of degree less than d. A point $X \in \mathrm{Gr}(N,d)$ is called real if it has a basis consisting of polynomials with all coefficients real.

4.7.1 The general case $N \geqslant 4$

Let us first consider the case $N \geqslant 4$.

Let $\boldsymbol{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right)$ be a sequence of dominant integral $\mathfrak{g}_N$-weights, $\boldsymbol{k} = (k_1, \ldots, k_n)$ an n-tuple of nonnegative integers, and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathring{\mathbb{P}}_n$. Suppose that $|\boldsymbol{\lambda}_{A,\boldsymbol{k}}| = N(d-N)$. Denote by $d(\boldsymbol{\lambda}, \boldsymbol{k}, \boldsymbol{z})$ the number of real points counted with multiplicities in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}} \subset \mathrm{Gr}(N, d)$.

Clearly, $d(\boldsymbol{\lambda}, \boldsymbol{k}, \boldsymbol{z}) = 0$ unless the set $\{z_1, \ldots, z_n\}$ is invariant under the complex conjugation and $\left(\lambda^{(i)}, k_i\right) = \left(\lambda^{(j)}, k_j\right)$ whenever $z_i = \bar{z}_j$. In particular, the polynomial $\prod_{s=1}^{n}(x - z_s)^{k_s}$ has only real coefficients. It follows from (3.4.3) that the number of real points in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{z}} \subset \mathrm{Gr}(N, d)$ is equal to that of $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}} \subset \mathrm{Gr}(N, d - |\boldsymbol{k}|)$. From now on, we shall only consider the case that $\boldsymbol{k} = (0, \ldots, 0)$. We simply write $d(\boldsymbol{\lambda}, \boldsymbol{z})$ for $d(\boldsymbol{\lambda}, \boldsymbol{k}, \boldsymbol{z})$ if $\boldsymbol{k} = (0, \ldots, 0)$.

Let $\boldsymbol{T} = (T_1, \ldots, T_N)$ be associated with $\boldsymbol{\lambda}_{A,\boldsymbol{k}}$, $\boldsymbol{z}$. Note that if $\boldsymbol{z}$, $\boldsymbol{\lambda}$, $\boldsymbol{k}$ is invariant under conjugation, then the polynomial $T_1 \cdots T_N$ also has only real coefficients.

In what follows we denote by c the number of complex conjugate pairs in the set $\{z_1, \ldots, z_n\}$ and without loss of generality assume that $z_1 = \bar{z}_2, \ldots, z_{2c-1} = \bar{z}_{2c}$ while $z_{2c+1}, \ldots, z_n$ are real (one of them can be infinity). We will also always assume that $\lambda^{(1)} = \lambda^{(2)}, \ldots, \lambda^{(2c-1)} = \lambda^{(2c)}$.

Recall that for any $\boldsymbol{\lambda}$ and generic $\boldsymbol{z} \in \mathring{\mathbb{P}}_n$, all points of $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ are multiplicity-free. The same also holds true with $\boldsymbol{\lambda}$ imposed above for any c.

Consider the decomposition of $V_{\boldsymbol{\lambda}}$ into irreducible $\mathfrak{g}_N$-submodules

$$V_{\boldsymbol{\lambda}} = \bigoplus_{\mu} V_\mu \otimes M_{\boldsymbol{\lambda},\mu}.$$

Then $M_{\boldsymbol{\lambda},0} = (V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$. Since $\lambda^{(2i-1)} = \lambda^{(2i)}$ for $i = 1, \ldots, c$, the flip P_{2i-1} of the $(2i-1)$-st and $2i$-th tensor factors of $V_{\boldsymbol{\lambda}}$ commutes with the $\mathfrak{g}_N$-action and thus acts on $(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$. Denote by $P_{\boldsymbol{\lambda},c} \in \mathrm{End}((V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N})$ the action of the product $P_1 P_3 \cdots P_{2c-1}$ on $(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$.

The operator $P_{\boldsymbol{\lambda},c}$ is self-adjoint relative to the Hermitian form $(\cdot|\cdot)_{\boldsymbol{\lambda}}$ on $(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$ given in Section 4.6. Define a new Hermitian form $(\cdot,\cdot)_{\boldsymbol{\lambda},c}$ on $(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$ by the rule: for any $v, w \in (V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$

$$(v, w)_{\boldsymbol{\lambda},c} = (P_{\boldsymbol{\lambda},c} v | w)_{\boldsymbol{\lambda}}.$$

Denote by $q(\boldsymbol{\lambda},c)$ the signature of the form $(\cdot,\cdot)_{\boldsymbol{\lambda},c}$.

Proposition 4.7.1. *The signature $q(\boldsymbol{\lambda},c)$ equals the coefficient of the monomial*

$$x_1^{N-1}x_2^{N-3}\cdots x_r^{N+1-2r},$$

in the Laurent polynomial

$$\Delta_N\cdot\prod_{i=1}^{c}\mathcal{S}^N_{\lambda^{(2i)}}\big(x_1^2,\dots,x_r^2\big)\prod_{j=2c+1}^{n}\mathcal{S}^N_{\lambda^{(j)}}(x_1,\dots,x_r).$$

Here Δ_N and $\mathcal{S}^N_{\lambda^{(s)}}$ are given by (4.3.1) *and* (4.3.2)*, respectively.*

Proof. Since $P^2_{\boldsymbol{\lambda},c}=1$ and $M_{\boldsymbol{\lambda},0}=(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$, we have $q(\boldsymbol{\lambda},c)=\mathrm{tr}_{M_{\boldsymbol{\lambda},0}}P_{\boldsymbol{\lambda},c}$, and the claim follows from Proposition 4.3.1. □

Theorem 4.7.2. *The number $d(\boldsymbol{\lambda},\boldsymbol{z})$ of real self-dual spaces in $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ is no less than $|q(\boldsymbol{\lambda},c)|$.*

Proof. Our proof is parallel to that of [MT16, Theorem 7.2].

By Proposition 4.6.1 and Lemma 4.6.3, the operators $B_{ij}(\boldsymbol{\lambda},\boldsymbol{z})\in\mathrm{End}((V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N})$ are self-adjoint relative to the form $(\cdot,\cdot)_{\boldsymbol{\lambda},c}$. By Lemma 4.6.4,

$$\dim\bigg(\bigcap_{i,j}\bigoplus_{\alpha\in\mathbb{R}}\big((V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}\big)_{B_{ij}(\boldsymbol{\lambda},\boldsymbol{z})}(\alpha)\bigg)\geqslant|q(\boldsymbol{\lambda},c)|.$$

By Theorem 4.6.2, for any $\boldsymbol{\lambda}$ and generic $\boldsymbol{z}\in\mathring{\mathbb{P}}_n$ the operators $B_{ij}(\boldsymbol{\lambda},\boldsymbol{z})$ are diagonalizable and the action of the Bethe algebra $\mathcal{B}$ on $(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$ has simple spectrum. The same also holds true with $\boldsymbol{\lambda}$ imposed above for any c. Thus for generic $\boldsymbol{z}$, the operators $B_{ij}(\boldsymbol{\lambda},\boldsymbol{z})$ have at least $|q(\boldsymbol{\lambda},c)|$ common eigenvectors with distinct real eigenvalues, which provides $|q(\boldsymbol{\lambda},c)|$ distinct real points in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ by Theorem 4.5.2. Hence, $d(\boldsymbol{\lambda},\boldsymbol{z})\geqslant|q(\boldsymbol{\lambda},c)|$ for generic $\boldsymbol{z}$, and therefore, for any $\boldsymbol{z}$, due to counting with multiplicities. □

Remark 4.7.3. If $\dim(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}$ is odd, it follows from Theorem 4.7.2 by counting parity that

$$d(\boldsymbol{\lambda},\boldsymbol{z})\geqslant|q(\boldsymbol{\lambda},c)|\geqslant 1.$$

Therefore, there exists at least one real point in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$. In particular, if $\dim(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N} = 1$, then the only point in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ is always real.

The following corollary of Proposition 4.7.1 and Theorem 4.7.2 is our main result.

Corollary 4.7.4. *The number $d(\boldsymbol{\lambda},\boldsymbol{z})$ of real self-dual spaces in $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ (real points in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$) is no less than $|a(\boldsymbol{\lambda},c)|$, where $a(\boldsymbol{\lambda},c)$ is the coefficient of the monomial $x_1^{N-1}x_2^{N-3}\cdots x_r^{N+1-2r}$ in the Laurent polynomial*

$$\Delta_N\cdot\prod_{i=1}^{c}\mathcal{S}^N_{\lambda^{(2i)}}\left(x_1^2,\dots,x_r^2\right)\prod_{j=2c+1}^{n}\mathcal{S}^N_{\lambda^{(j)}}(x_1,\dots,x_r).$$

Here Δ_N is the Vandermonde determinant of $\mathfrak{g}_N$ and $\mathcal{S}^N_{\lambda^{(s)}}$ is the Schur function of $\mathfrak{g}_N$ associated with $\lambda^{(s)}$, $s=1,\dots,n$, see (4.3.1) *and* (4.3.2).

Remark 4.7.5. Recall that the total number of points (counted with multiplicities) in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ equals $\dim(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N} = q(\boldsymbol{\lambda},0)$. Hence if $\boldsymbol{z}\in\mathbb{R}\mathring{\mathbb{P}}_n$, Theorem 4.7.2 claims that all points in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ are real. It is proved in [MTV09b, Corollary 6.3] that for $\boldsymbol{z}\in\mathbb{R}\mathring{\mathbb{P}}_n$ all points in $\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ are real and multiplicity-free, so are the points in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$.

4.7.2 The case $N = 2, 3$

Now let us consider the case $N = 2,3$. Note that $\mathrm{sGr}(2,d) = \mathrm{Gr}(2,d)$, this case is the usual Grassmannian, which has already been discussed in [MT16].

Let $N = 3$ and $\mathfrak{g}_3 = \mathfrak{sl}_2$. It suffices for us to consider the case that points in $\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ have no base points, see the beginning of Section 4.7.1 for more detail. We shall consider $\mathrm{sGr}(3,2d-1)$ instead of $\mathrm{sGr}(3,d)$, see [LMV17, Section 4.6]. We identify the dominant integral $\mathfrak{sl}_2$-weights with nonnegative integers. Let $\boldsymbol{\lambda} = \left(\lambda^{(1)},\dots,\lambda^{(n)}\right)$ be a sequence of nonnegative integers and $\boldsymbol{z} = (z_1,\dots,z_n)\in\mathring{\mathbb{P}}_n$. Then $\boldsymbol{\lambda}_A$ has coordinates

$$\lambda_A^{(s)} = \left(2\lambda^{(s)},\lambda^{(s)},0\right),\qquad s = 1,\dots,n.$$

We also assume $|\boldsymbol{\lambda}_A| = 6(d-2)$.

Recall from [LMV17, Theorem 4.19], if $X\in\mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$, then there exist monic polynomials φ and ψ such that φ^2, $\varphi\psi$, ψ^2 form a basis of X. Denote by $\sqrt{X}$ the space of polynomials spanned by φ and ψ. Let $\xi^{(i)}$ be the partitions with at most two parts defined by $\big(\lambda^{(i)},0\big)$, $i=1,\dots,n$. Set $\boldsymbol{\xi}=\big(\xi^{(1)},\dots,\xi^{(n)}\big)$, then $|\boldsymbol{\xi}|=2(d-2)$. It follows from the proof of [LMV17, Theorem 4.19] that $\sqrt{X}\in\Omega_{\boldsymbol{\xi},\boldsymbol{z}}\subset\mathrm{Gr}(2,d)$. The map $\Omega_{\boldsymbol{\xi},\boldsymbol{z}}\to \mathrm{s}\Omega_{\boldsymbol{\lambda},\boldsymbol{z}}$ given by $\sqrt{X}\mapsto X$ is bijective.

Lemma 4.7.6. *The self-dual space X is real if and only if $\sqrt{X}$ is real.*

Proof. It is obvious that X is real if $\sqrt{X}$ is real.

Conversely, if X is real, then there exist complex numbers a_i, b_i, c_i, $i=1,2,3$, such that

$$a_i\varphi^2+b_i\varphi\psi+c_i\psi^2,\qquad i=1,2,3,$$

are real polynomials and form a basis of X. Without loss of generality, we assume $\deg\varphi<\deg\psi$. Since $\deg\varphi<\deg\psi$, we have $c_i\in\mathbb{R}$, $i=1,2,3$. At least one of c_i is nonzero. We assume $c_3\neq 0$. By subtracting a proper real multiple of $a_3\varphi^2+b_3\varphi\psi+c_3\psi^2$, we assume further $c_1=c_2=0$. Continuing with the previous step, we assume that $b_1=0$, $b_2\neq 0$, $a_1\neq 0$ and hence obtain that $a_1,b_2,c_3\in\mathbb{R}$. Then $a_1\varphi^2$ is a real polynomial, so is φ. Therefore, $a_2\varphi+b_2\psi$ is also a real polynomial, which implies that the space of polynomials $\sqrt{X}$ is also real. $\square$

Because of Lemma 4.7.6, the case $N=3$ is reduced to the lower bound for the number of real solutions to osculating Schubert problems of $\mathrm{Gr}(2,d)$, see [MT16]. Moreover, Corollary 4.7.4 also applies for this case by putting $N=3$, $r=1$, and $\mathfrak{g}_N=\mathfrak{sl}_2$.

4.8 Some data for small N

In this section, we give some data obtained from Corollary 4.7.4 when N is small. Since the cases $N=2,3$ reduce to the cases of [MT16], we start with $N=4$.

We always assume that $\boldsymbol{\lambda}$, $\boldsymbol{k}$, $\boldsymbol{z}$ are invariant under conjugation. By Remark 4.7.3, we shall only consider the cases that $\dim(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_N}\geqslant 2$. We also exclude the cases that $\boldsymbol{z}\in\mathbb{R}\mathring{\mathbb{P}}_n$. In particular, the cases that all pairs $\big(\lambda^{(s)},k_s\big)$, $s=1,\dots,n$, are different.

We write the highest weights in terms of fundamental weights, for example

$$(1,0,0,1) = \omega_1 + \omega_4.$$

We also write $\left(\lambda^{(1)}\right)_{k_1}, \ldots, \left(\lambda^{(n)}\right)_{k_n}$ for $(\boldsymbol{\lambda}, \boldsymbol{k})$ and simply write $\lambda^{(s)}$ for $(\lambda^{(s)})_0$. We use $\left(\lambda_1^{(s)}, \lambda_2^{(s)}\right)_{k_s}^{\otimes m}$ to indicate that the pair $\left(\left(\lambda_1^{(s)}, \lambda_2^{(s)}\right), k_s\right)$ appears in $(\boldsymbol{\lambda}, \boldsymbol{k})$ exactly m times. For instance, $(0,1)_1, (0,1)^{\otimes 3}$ represents the pair $(\boldsymbol{\lambda}, \boldsymbol{k})$ where $\boldsymbol{\lambda} = ((0,1),(0,1),(0,1),(0,1))$ and $\boldsymbol{k} = (1,0,0,0)$.

4.8.1 The case $N = 4, 5$

For each $\mathfrak{g}_4$-weight $\lambda = (\lambda_1, \lambda_2)$, denote by λ_C the $\mathfrak{g}_5$-weight (λ_2, λ_1). Note that $\mathfrak{g}_4 = \mathfrak{so}_5$ is isomorphic to $\mathfrak{g}_5 = \mathfrak{sp}_4$, the lower bound obtained from the ramification data $\boldsymbol{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right)$ and $\boldsymbol{k} = (k_1, \ldots, k_n)$ of $\mathfrak{g}_4$ is the same as that obtained from the ramification data $\boldsymbol{\lambda_C} = \left(\lambda_C^{(1)}, \ldots, \lambda_C^{(n)}\right)$ and $\boldsymbol{k} = (k_1, \ldots, k_n)$ of $\mathfrak{g}_5$.

In Table 4.1, we give lower bounds for the cases from $\mathrm{Gr}(4,7)$ and $\mathrm{Gr}(5,10)$. By the observation above, we transform the case from $\mathrm{Gr}(5,10)$ to its counter part in $\mathrm{Gr}(4,d)$ for some d depending on the ramification data. The number in the column of dimension is equal to $\dim(V_{\boldsymbol{\lambda}})^{\mathfrak{g}_4}$ for the corresponding ramification data $\boldsymbol{\lambda}$ in each row. The numbers in the column of $c = i$ equal the lower bounds computed from Corollary 4.7.4 with the corresponding c.

For a given c, there may exist several choices of complex conjugate pair corresponding to different pairs of $\mathfrak{g}_N$-weights. If the corresponding lower bounds are the same, we just write one number. For example, in the case of $(0,2)^{\otimes 2}$, $(0,1)^{\otimes 2}$ and $c = 1$ of Table 4.1, the complex conjugate pair may correspond to the weights $(0,2)^{\otimes 2}$ or $(0,1)^{\otimes 2}$. However, they give the same lower bound 1. Hence we just write 1 for $c = 1$. If the bounds are different, we write the lower bound with the conjugate pairs corresponding to the leftmost $2c$ weights first while the one with the conjugate pairs corresponding to the rightmost $2c$ weights last, in terms of the order of the ramification data displayed on each row. Since we have at most 3 cases, the possible remaining case is clear. For instance, in the case $(0,1,0)^{\otimes 4}$, $(0,0,1)^{\otimes 4}$ and $c = 2$ of

Table 4.1.
The case $N = 4, 5$.

ramification data	dimension	$c = 1$	$c = 2$	$c = 3$
$(0,1)^{\otimes 6}$	14	2	2	6
$(1,0)^{\otimes 3}, (0,1)^{\otimes 2}$	4	0,2	2	
$(1,0)^{\otimes 3}, (1,0)_1$	3	1		
$(1,0)^{\otimes 4}, (0,0)_1$	3	1	3	
$(0,2), (0,1)^{\otimes 4}$	6	0	2	
$(0,0)_1, (0,1)^{\otimes 4}$	3	1	3	
$(1,0), (0,1)^{\otimes 4}$	5	1	1	
$(1,1), (0,1)^{\otimes 3}$	2	0		
$(0,1)_1, (0,1)^{\otimes 3}$	3	1		
$(0,2)^{\otimes 2}, (0,1)^{\otimes 2}$	3	1	3	
$(1,0)^{\otimes 2}, (0,1)^{\otimes 2}$	2	0	2	
$(0,2), (1,0), (0,1)^{\otimes 2}$	2	0		
$(1,0)_1, (1,0), (0,1)^{\otimes 2}$	2	0		
$(1,0)^{\otimes 2}, (0,1)_1, (0,1)$	2	0		
$(1,1), (1,0)^{\otimes 2}, (0,1)$	2	0		

Table 4.2, the two complex conjugate pairs corresponding to $(0,1,0)^{\otimes 4}$ give the lower bound 12 while the two complex conjugate pairs corresponding to $(0,0,1)^{\otimes 4}$ give the lower bound 24. The remaining case, where the two conjugate pairs corresponding to $(0,1,0)^{\otimes 2}$ and $(0,0,1)^{\otimes 2}$, gives the lower bound 2.

4.8.2 The case $N = 6$

In what follows, we give lower bounds for ramification data consisting of fundamental weights when $N = 6$. We follow the same convention as in Section 4.8.1.

Table 4.2.
The case $N = 6$.

ramification data	dimension	$c = 1$	$c = 2$	$c = 3$	$c = 4$
$(0,0,1)^{\otimes 4}$	4	0	4		
$(0,1,0)^{\otimes 4}$	6	2	6		
$(1,0,0)^{\otimes 4}$	3	1	3		
$(0,0,1)^{\otimes 2}, (0,1,0)^{\otimes 2}$	3	1	3		
$(0,0,1)^{\otimes 2}, (1,0,0)^{\otimes 2}$	2	0	2		
$(0,1,0)^{\otimes 2}, (1,0,0)^{\otimes 2}$	3	1	3		
$(0,0,1)^{\otimes 6}$	30	2	2	10	
$(0,1,0)^{\otimes 6}$	130	8	14	36	
$(1,0,0)^{\otimes 6}$	15	3	3	7	
$(0,1,0)^{\otimes 2}, (0,0,1)^{\otimes 4}$	34	4,2	0,6	16	
$(0,1,0)^{\otimes 4}, (0,0,1)^{\otimes 2}$	55	3,1	3,7	19	
$(1,0,0)^{\otimes 2}, (0,0,1)^{\otimes 4}$	16	2	0,4	10	
$(1,0,0)^{\otimes 4}, (0,0,1)^{\otimes 2}$	10	0,2	2,0	6	
$(1,0,0)^{\otimes 2}, (0,1,0)^{\otimes 4}$	46	2	6	18	
$(1,0,0)^{\otimes 4}, (0,1,0)^{\otimes 2}$	21	1,3	5,3	11	
$(1,0,0)^{\otimes 2}, (0,1,0)^{\otimes 2}, (0,0,1)^{\otimes 2}$	20	2	0,4,0	10	
$(0,0,1)^{\otimes 8}$	330	20	6	0	50
$(0,1,0)^{\otimes 8}$	6111	69	59	113	311
$(1,0,0)^{\otimes 8}$	105	15	9	7	25
$(0,1,0)^{\otimes 4}, (0,0,1)^{\otimes 4}$	984	22,28	12,2,24	0,38	108
$(1,0,0)^{\otimes 4}, (0,0,1)^{\otimes 4}$	116	6,12	8,2,12	0,10	32
$(1,0,0)^{\otimes 4}, (0,1,0)^{\otimes 4}$	510	6,12	22,4,18	28,18	74

REFERENCES

REFERENCES

[AGZV85] V. Arnold, S. Gusein-Zade, and A. Varchenko. *Singularities of differentiable map Volume I.* Birkh́ause, Boston, 1985.

[Bet31] H. Bethe. Zur Theorie der Metalle I. Eigenwerte und Eigenfunktionen der linearen Atomkette. *Z. Physik*, 71(3):205–226, 1931.

[BHLW17] A. Beliakova, K. Habiro, A. Lauda, and B. Webster. Current algebras and categorified quantum groups. *J. London Math. Soc.*, 95(1):248–276, 2017.

[BK06] J. Brundan and A. Kleshchev. Shifted Yangians and finite W-algebras. *Adv. Math.*, 200:136–195, 2006.

[BM05] L. Borisov and E. Mukhin. Self-self-dual spaces of polynomials. *Adv. Math.*, 190(1):77–118, 2005.

[CP91] V. Chari and A. Pressley. Quantum affine algebras. *Comm. Math. Phys.*, 142(2):261–283, 1991.

[CP94] V. Chari and A. Pressley. *A guide to quantum groups.* Cambridge University Press, Cambridge, 1994.

[CP95] V. Chari and A. Pressley. Quantum affine algebras and their representations. In A co-publication of the AMS and Canadian Mathematical Society, editors, *Amer. Math. Soc.*, volume 16, pages 1–58. Birkhäuser, Providence, 1995.

[CT06] A. Chervov and D. Talalaev. Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence. *arXiv*, pages arXiv:hep–th/0604128, 2006.

[Dri85] V. Drinfeld. Hopf algebras and the quantum Yang–Baxter equation. *Sov. Math. Dokl.*, 32(1):254–258, 1985.

[DS85] V. Drinfeld and V. Sokolov. Lie algebras and KdV type equations. *J. Sov. Math*, 30:1975–2036, 1985.

[EG02] A. Eremenko and A. Gabrielov. Degrees of real Wronski maps. *Discrete Comput. Geom.*, 28:331–347, 2002.

[FF92] B. Feigin and E. Frenkel. Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebra. *Int. J. Mod. Phys.*, 7(Suppl. 1A):197–215, 1992.

[FFR94] B. Feigin, E. Frenkel, and N. Reshetikhin. Gaudin model, Bethe ansatz and critical level. *Comm. Math. Phys.*, 166(1):27–62, 1994.

[FFR10] B. Feigin, E. Frenkel, and L. Rybnikov. Opers with irregular singularity and spectra of the shift of argument subalgebra. *Comm. Math. Phys.*, 155(2):337–363, 2010.

[FFTL10] B. Feigin, E. Frenkel, and V. Toledano-Laredo. Gaudin models with irregular singularities. *Adv. Math.*, 223(3):873–948, 2010.

[FH15] E. Frenkel and D. Hernandez. Baxter's Relations and Spectra of Quantum Integrable Models. *Duke Math. J.*, 164(12):2407–2460, 2015.

[FJMM17] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin. Finite type modules and Bethe ansatz equations. *Ann. Henri Poincaré*, 18(8):2543–2579, 2017.

[FJMR16] L. Frappat, N. Jing, A. Molev, and E. Ragoucy. Higher Sugawara Operators for the Quantum Affine Algebras of Type A. *Commun. Math. Phys.*, 345:631–657, 2016.

[FR99] E. Frenkel and N. Reshetikhin. The q-characters of representations of quantum affine agebras and deformations of $\mathcal{W}$-algebras. In N. Jing and K. Misra, editors, *Recent Developments in Quantum Affine Algebras and Related Topics*, volume 248, pages 163–205. Amer. Math. Soc., Providence, 1999.

[Fre05] E. Frenkel. Gaudin model and opers. In P. Kulish, N. Manojlovic, and H. Samtleben, editors, *Infinite dimensional algebras and quantum integrable systems*, volume 237, pages 1–58. Birkhäuser, Basel, 2005.

[Fre07] E. Frenkel. *Langlands Correspondence for Loop Groups*. Cambridge University Press, Cambridge, 2007.

[Fro00] F. Frobenius. Über die Charaktere der symmetrischen Gruppe. *König. Preuss. Akad. Wiss. Berlin*, pages 516–534, 1900.

[FRT88] L. Faddeev, N. Reshetikhin, and L. Takhtajan. Quantization of Lie Groups and Lie Algebras. In Masaki Kashiwara and Takahiro Kawai, editors, *Algebraic Analysis*, volume 1, pages 129–139. Academic Press, San Diego, 1988.

[FSS96] J. Fuchs, B. Schellekens, and C. Schweigert. From Dynkin diagram symmetries to fixed point structures. *Comm. Math. Phys.*, 180(1):39–97, 1996.

[Gau76] M. Gaudin. Diagonalisation d'une classe d'hamiltoniens de spin. *J. de Physique*, 37(10):1087–1098, 1976.

[Gau83] M. Gaudin. *La fonction d'onde de Bethe*. Masson, Paris, 1983.

[GH94] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Addison-Wesley, Reading Massachusetts, 1994.

[GRTV12] V. Gorbounov, R. Rimányi, V. Tarasov, and A. Varchenko. Cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra. *J. Geom. Phys.*, 74(37):56–86, 2012.

[HHS13] N. Hein, C.J. Hillar, and F. Sottile. Lower bounds in real Schubert calculus. *São Paulo J. Math. Sci.*, 7:33–58, 2013.

[HS] N. Hein and F. Sottile. Beyond the Shapiro conjecture and Eremenko–Gabrielov lower bounds. *Lower Bounds Experimental Project*, page http://www.math.tamu.edu/ secant/lowerBounds/lowerBounds.php.

[HSZ16] N. Hein, F. Sottile, and I. Zelenko. A congruence modulo four in real Schubert calculus. *J. Reine Angew. Math.*, 714:151–174, 2016.

[Hum78] J. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York, 2 edition, 1978.

[JKMY18] N. Jing, S. Kozic, A. Molev, and F. Yang. Center of the quantum affine vertex algebra in type A. *J. Algebra*, 496:138–186, 2018.

[KLP12] P. Kulish, V. Lyakhovsky, and O. Postnova. Tensor power decomposition. B_n case. *J. Phys.: Conf. Ser.*, 343(1):012095, 2012.

[Kni95] H. Knight. Spectra of tensor products of finite-dimensional representations of Yangians. *J. Algebra*, 174(1):187–196, 1995.

[KR86] A. Kirillov and N. Reshetikhin. The Bethe Ansatz and the Combinatorics of Young Tableaux. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 166:65–115, 1986.

[Kup96] G. Kuperberg. Another proof of the alternating sign matrix conjecture. *Int. Math. Res. Not. IMRN*, 1996(3):139–150, 1996.

[LM19a] Kang Lu and E. Mukhin. On the Gaudin model of type G_2. *Commun. Contemp. Math.*, 21(3):1–31, 2019.

[LM19b] Kang Lu and E. Mukhin. On the supersymmetric XXX spin chains associated to $\mathfrak{gl}_{1|1}$. *Preprint*, pages 1–32, 2019.

[LMV16] Kang Lu, E. Mukhin, and A. Varchenko. On the Gaudin model associated to Lie algebras of classical types. *J. Math. Phys.*, 57(10):1–23, 2016.

[LMV17] Kang Lu, E. Mukhin, and A. Varchenko. Self-dual Grassmannian, Wronski map, and representations of $\mathfrak{gl}_N$, $\mathfrak{sp}_{2r}$, $\mathfrak{so}_{2r+1}$. *Pure Appl. Math. Q.*, 13(2):291–335, 2017.

[Lu18] Kang Lu. Lower bounds for numbers of real self-dual spaces in problems of Schubert calculus. *Symmetry Integrability Geom. Methods Appl.*, 14:1–15, 2018.

[Lu20] Kang Lu. Perfect integrability and Gaudin model. *preprint*, pages 1–8, 2020.

[MM17] A. Molev and E. Mukhin. Eigenvalues of Bethe vectors in the Gaudin model. *Theor. Math. Phys.*, 192(3):1258–1281, 2017.

[Mol13] A. Molev. Feigin—Frenkel center in types B, C and D. *Invent. Math.*, 191(1):1–34, 2013.

[Mol18] A. Molev. *Sugawara Operators for Classical Lie Algebras.* American Mathematical Society, Providence, 2018.

[MR14] A. Molev and E. Ragoucy. The MacMahon Master Theorem for right quantum superalgebras and higher Sugawara operators for $\widehat{\mathfrak{gl}}(m|n)$. *Moscow Math. J.*, 14(1):83–119, 2014.

[MRR16] A. Molev, E. Ragoucy, and N. Rozhkovskaya. Segal–Sugawara vectors for the Lie algebra of type G_2. *J. Algebra*, 455:386–401, 2016.

[MT16] E. Mukhin and V. Tarasov. Lower bounds for numbers of real solutions in problems of Schubert calculus. *Acta Math.*, 217(1):177–193, 2016.

[MTV06] E. Mukhin, V. Tarasov, and A. Varchenko. Bethe eigenvectors of higher transfer matrices. *J. Stat. Mech. Theor. Exp.*, 2006(P08002):1–44, 2006.

[MTV08a] E. Mukhin, V. Tarasov, and A. Varchenko. Bispectral and $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ dualities, discrete versus differential. *Adv. Math.*, 218(1):216–265, 2008.

[MTV08b] E. Mukhin, V. Tarasov, and A. Varchenko. Spaces of quasi-exponentials and representations of $\mathfrak{gl}_N$. *J. Phys. A: Math. Theor.*, 41(19):1–28, 2008.

[MTV09a] E. Mukhin, V. Tarasov, and A. Varchenko. On reality property of wronski maps. *Confluentes Math.*, 1(2):225–247, 2009.

[MTV09b] E. Mukhin, V. Tarasov, and A. Varchenko. Schubert calculus and representations of general linear group. *J. Amer. Math. Soc.*, 22(4):909–940, 2009.

[MTV09c] E. Mukhin, V. Tarasov, and A. Varchenko. The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz. *Ann. Math.*, 170(2):863–881, 2009.

[MTV10] E. Mukhin, V. Tarasov, and A. Varchenko. A generalization of the Capelli identity. In Yuri Tschinkel and Yuri Zarhin, editors, *Algebra, Arithmetic, and Geometry*, volume 2, pages 383–398. Birkhäuser, Boston, 2010.

[MTV14] E. Mukhin, V. Tarasov, and A. Varchenko. Spaces of quasi-exponentials and representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. *Transform. Groups*, 19(3):861–885, 2014.

[MV00] E. Mukhin and A. Varchenko. Remarks on critical points of phase functions and norms of Bethe vectors. In M. Falk and H. Terao, editors, *Arrangements–Tokyo 1998*, pages 239–246. Mathematical Society of Japan, Tokyo, 2000.

[MV04] E. Mukhin and A. Varchenko. Critical points of master functions and flag varieties. *Commun. Contemp. Math.*, 6(1):111–163, 2004.

[MV05a] E. Mukhin and A. Varchenko. Miura opers and critical points of master functions. *Cent. Eur. J. Math*, 3(2):155–182, 2005.

[MV05b] E. Mukhin and A. Varchenko. Norm of a Bethe vector and the Hessian of the master function. *Compos. Math.*, 141(4):1012–1028, 2005.

[MV07] E. Mukhin and A. Varchenko. Multiple orthogonal polynomials and a counterexample to the Gaudin Bethe ansatz conjecture. *Trans. Amer. Math. Soc.*, 359(11):5383–5418, 2007.

[MV08] E. Mukhin and A. Varchenko. Quasi-polynomials and the Bethe ansatz. *Geom. Topol. Monogr.*, 13:385–420, 2008.

[MVY15] E. Mukhin, B. Vicedo, and C. Young. Gaudin Model for $\mathfrak{gl}(m|n)$. *J. Math. Phys.*, 56(5):1–30, 2015.

[Pon44] L. Pontrjagin. Hermitian operators in spaces with indefinite metric. *Izvestia Akad. Nauk SSSR*, 8:243–280, 1944.

[RS01] A. Razumov and Y. Stroganov. Spin chains and combinatorics. *J. Phys. A: Math. Gen*, 34(14):3185–3190, 2001.

[RSTV11] R. Rimányi, V. Schechtman, V. Tarasov, and A. Varchenko. Cohomology of a flag variety as a Bethe algebra. *Func. Anal. Appl.*, 45(4):252–264, 2011.

[RSZJ07] A. Razumov, Y. Stroganov, and P. Zinn-Justin. Polynomial solutions of qKZ equation and ground state of XXZ spin chain at $\Delta = -1/2$. *J. Phys. A: Math. Gen*, 40(39):11827–11847, 2007.

[RTV15] R. Rimányi, V. Tarasov, and A. Varchenko. Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety. *J. Geom. Phys.*, 94:81–119, 2015.

[RV95] N. Reshetikhin and A. Varchenko. Quasiclassical asymptotics of solutions to the KZ equations. In Shing-Tung Yau, editor, *Geometry, Topology and Physics for R. Bott*, pages 293–322. Intern. Press, Boston, 1995.

[Ryb06] L. Rybnikov. The argument shift method and the Gaudin model. *Funct. Anal. Appl.*, 40(3):188–199, 2006.

[Ryb18] L. Rybnikov. A Proof of the Gaudin Bethe Ansatz Conjecture. *Int. Math. Res. Not. IMRN*, page rny245, 2018.

[Sch86] H. Schubert. Anzahl-Bestimmungen für lineare Räume beliebiger Dimension. *Acta. Math.*, 8:97–118, 1886.

[Sot10] F. Sottile. Frontiers of reality in Schubert calculus. *Bull. Amer. Math. Soc.*, 47(1):31–71, 2010.

[SS06] E. Soprunova and F. Sottile. Lower bounds for real solutions to sparse polynomial systems. *Adv. Math.*, 204(1):116–151, 2006.

[SV91] V. Schechtman and A. Varchenko. Arrangements of hyperplanes and Lie algebra homology. *Invent. Math.*, 106:139–194, 1991.

[Tal06] D. Talalaev. The quantum Gaudin system. *Funct. Anal. Appl.*, 40(1):73–77, 2006.

[Var95] A. Varchenko. Critical points of the product of powers of linear functions and families of bases of singular vectors. *Compos. Math.*, 97(3):385–401, 1995.

[Var03] A. Varchenko. *Special Functions, KZ Type Equations, and Representation Theory.* American Mathematical Society, Providence, 2003.

[Var11] A. Varchenko. Quantum integrable model of an arrangement of hyperplanes. *Symmetry Integrability Geom. Methods Appl.*, 7(032):1–55, 2011.