BOUNDING THE DEGREES OF THE DEFINING EQUATIONS

OF REES RINGS FOR CERTAIN DETERMINANTAL AND PFAFFIAN IDEALS

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For my wife, Paula.

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ABSTRACT

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We consider ideals of minors of a matrix, ideals of minors of a symmetric matrix, and ideals of Pfaffians of an alternating matrix. Assuming these ideals are of generic height, we characterize the condition G_s for these ideals in terms of the heights of smaller ideals of minors or Pfaffians of the same matrix. We additionally obtain bounds on the generation and concentration degrees of the defining equations of Rees rings for a subclass of such ideals via specialization of the Rees rings in the generic case. We do this by proving that, given sufficient height conditions on ideals of minors or Pfaffians of the matrix, the specialization of a resolution of a graded component of the Rees ring in the generic case is an approximate resolution of the same component of the Rees ring in question. We end the paper by giving some examples of explicit generation and concentration degree bounds.

1. INTRODUCTION

Let R be a Noetherian commutative ring, and consider an R-ideal $I = (a_1, \ldots, a_n)$. Our main object of study is the Rees algebra of I which is defined to be the subring

$$\mathcal{R}(I) := R[It] = R[a_1 t, \ldots, a_n t] \subset R[t]$$

where t is an indeterminate. The Rees algebra of I plays a fundamental role in studying the powers of I and this connection is made apparent by the natural isomorphism $\mathcal{R}(I) \cong \bigoplus_{i \geq 0} I^i$.

One approach to studying $\mathcal{R}(I)$ is to give a presentation for $\mathcal{R}(I)$ as an R-algebra. We can define a R-algebra homomorphism

$$\rho : R[T_1, \ldots, T_n] \twoheadrightarrow \mathcal{R}(I)$$

induced by $\rho(T_i) = a_i t$. We let $\mathcal{J} = \ker \rho$ (called the Rees ideal of I) and then we have a description of $\mathcal{R}(I)$ as a quotient of a polynomial ring over R via $\mathcal{R}(I) \cong R[T_1, \ldots, T_n]/\mathcal{J}$.

Much work has been done in studying the defining equations of $\mathcal{R}(I)$ in a few cases. Considerable work has been done in the case that I is perfect of grade two [22] [40] [41] [16] [24] [12] [35] [13] [14] [42] [39] [5] [36] [45] [33]. Some work has also been done in the case that I is a perfect Gorenstein ideal of grade three [40] [30] [37]. More generally, there has been some work in the cases that I is the ideal of minors of a matrix; in particular, [7] and [25] study this problem for ideals of minors of a generic matrix, and [8] studies this problem for matrices with linear entries.

From a geometric perspective $\mathcal{R}(I)$ is the coordinate ring of the blowup of $\mathrm{Spec}(R)$ along the subvariety $V(I)$. In this way Rees algebras are involved in the study of resolving singularities. In addition, there is a special interpretation for $\mathcal{R}(I)$ provided $R = K[x_1, \ldots, x_d]$ is a polynomial ring in d variables over the field K and I is a

homogeneous ideal with homogeneous generators $a_1, \ldots, a_n$ of the same degree. In this setting each polynomial a_i is well defined on $\mathbb{P}_K^{d-1} \setminus V(a_i)$, and so we are able to use these polynomials as coordinates parameterized by the variables $x_1, \ldots, x_d$ which gives rise to a rational map

$$\Phi : \mathbb{P}_K^{d-1} \dashrightarrow \mathbb{P}_K^{n-1}$$

with

$$\Phi(p_0 : \cdots : p_d) = (a_1(p_0 : \cdots : p_d) : \cdots : a_n(p_0 : \cdots : p_d)).$$

This map is well defined for points in $\mathbb{P}_K^{d-1}$ except at points in the set of common zeros of the generators of I, i.e. on $\mathbb{P}_K^{d-1} \setminus V(I)$. Conversely, if we are given a rational map, then we can obtain such an ideal I. One can show $\mathcal{R}(I)$ is the bihomogeneous coordinate ring of the graph of Φ.

Another application of the Rees algebra appears in the study of geometric modeling where we set $R = K[s, t]$ and $x, y, z \in R$ are homogeneous polynomials defining a parameterized curve $\mathbb{P}_K^1 \dashrightarrow \mathbb{P}_K^2$ given by

$$(p : q) \mapsto (x(p, q) : y(p, q) : z(p, q)).$$

Following the discussion in the introduction of [17] we see that this rational map can truly be viewed as a curve by setting $q = 1$ and noting that

$$(x(p, 1) : y(p, 1) : z(p, 1)) = \left(\frac{x(p, 1)}{z(p, 1)} : \frac{y(p, 1)}{z(p, 1)} : 1 \right)$$

from which it follows that these choices afford a rational map $\mathbb{A}_K^1 \dashrightarrow \mathbb{A}_K^2$ with $p \mapsto \left(\frac{x(p,1)}{z(p,1)}, \frac{y(p,1)}{z(p,1)} \right)$ which is defined except at the finitely many zeroes of the single variable polynomial given by $h(X) := z(X, 1)$. Next, one considers homogeneous polynomials $f \in S = R[X, Y, Z]$ with

$$f(x(p, q), y(p, q), z(p, q)) = 0. \tag{1.1}$$

Graphically, once we have fixed $s = p$ and $t = q$, $f = 0$ corresponds to a curve in $\mathbb{A}_K^2$, and condition (1.1) means that the points from the rational map given by $x, y,$ and z all lie on this curve. In particular we note that a linear form in S corresponds

graphically to a line that "moves with" the parameterized curve, but (1.1) also shows that such a form describes an R-linear relation on the generators of the ideal $I = (x, y, z) \subset R$. One then sees that the ideal generated by these forms in S coincides with the Rees ideal of I.

The aim of this thesis is to study the properties of Rees algebras associated to classes of determinantal ideals. In particular we shall obtain results related to ideals of minors for ordinary and symmetric matrices, and for ideals of Pfaffians in the case of alternating matrices. We use the process of specialization to transfer properties that hold for generic versions of these matrices to matrices satisfying height conditions on some of their ideals of lower minors or lower Pfaffians.

In chapter two we define the technical terms and results we will need throughout the thesis. In chapter three we characterize the condition G_s for matrices of generic height and then classify G_∞ for the generic versions of the various matrices.

Chapter four explores the topic of specialization of Rees algebras. Specifically, given a matrix A and the ideal $I = I_t(A)$, we start with a generic matrix X having the same size as A and let $J = I_t(X)$ be the analogous ideal. We use the term "specialization" to refer to a more formal algebraic way to describe the process of substituting the entries of A for the variables in X. We find that, given some restriction on the heights of $I_j(A)$ for j between 1 and t, we are able to carry some properties from the generic case to the specialized case. Of particular interest to us are the behaviours of resolutions and what we call "approximate resolutions" for the ideals J^k under such specialization.

The degree bounds for the defining equations of the Rees algebra we obtain are the subject of our final chapter. We use height restrictions together with the criterion of Buchsbaum-Eisenbud to describe sufficient conditions to obtain an approximate resolution for I^k. Finally, given an approximate resolution for I^k, we determine the resulting bounds for the defining equations of the Rees algebra by combining work of Kustin and Ulrich with information about Castelnuovo-Mumford regularity in various cases.

2. PRELIMINARIES

We use this chapter to set forth the definitions and theorems upon which this thesis shall rely. We assume, unless otherwise noted, that R is a Noetherian ring.

2.1 Determinantal Ideals

Let A be an $m \times n$ matrix with entries in R. Throughout this thesis, we use the convention that $m \leq n$, though this shall be restated throughout for emphasis. We use $I_t(A)$ to denote the ideal generated by the $t \times t$ minors of A when $1 \leq t \leq m$. By convention, if $t \leq 0$, we set $I_t(A) = R$, and if $t > m$, we set $I_t(A) = 0$. Notice that, by virtue of Laplace expansion for determinants, these ideals form a nested sequence

$$I_1(A) \supset I_2(A) \supset \cdots \supset I_{m-1}(A) \supset I_m(A),$$

and the conventions set for other indices extend this nesting in both directions.

By a *generic matrix* X over R, we mean a matrix whose entries are distinct variables over R.

Definition 2.1.1 It is well-known from Eagon-Northcott [18, Theorem 3] that if $I_t(A)$ is a proper ideal, then $\operatorname{ht} I_t(A) \leq (m - t + 1)(n - t + 1)$, and that equality is achieved if A is a generic matrix [19, Theorem 2]. As such, an ideal $I_t(A)$ is said to be *of generic height* if it achieves this maximal possible height, i.e., if $\operatorname{ht} I_t(A) = (m - t + 1)(n - t + 1)$.

For any $m \times n$ matrix A, the notation A^T refers to the *transpose* of A. We say is A *symmetric* if $A^T = A$. We say A is *alternating* if $A^T = -A$ and if the diagonal entries of A are 0. We note that symmetric and alternating matrices are square, i.e., that $m = n$.

Definition 2.1.2 By a *generic symmetric matrix* X over R, we mean a symmetric matrix whose upper triangle consists of distinct variables over R. It is known that if A is an $n \times n$ symmetric matrix and if $I_t(A)$ is a proper ideal, then ht $I_t(A) \leq \binom{n-t+2}{2}$ [31, Theorem 2.1], and equality is achieved if A is a generic symmetric matrix via [38, Proposition 6.2]. As such, if A is an $n \times n$ symmetric matrix, then the ideal $I_t(A)$ is said to be *of generic symmetric height* if it achieves this maximal possible height, i.e., if ht $I_t(A) = \binom{n-t+2}{2}$.

Observe that a generic symmetric matrix has one variable in the first column of its upper triangle, then two variables in the second column and so on. Therefore there are $1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$ distinct variables in the entries of a generic symmetric matrix.

Example 2.1.3 Let $X_{11}, X_{12}, X_{13}, X_{22}, X_{23}$, and X_{33} be distinct variables over R, then

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{33} \end{bmatrix}$$

is a generic symmetric 3×3 matrix with $\binom{4}{2} = 6$ distinct variables. We can check directly that ht $I_1(X) = 6$, ht $I_2(X) = 3$, ht $I_3(X) = 1$.

Definition 2.1.4 Suppose U, V, W, and Z are $p \times p, p \times q, q \times p$, and $q \times q$ matrices with entries in a commutative ring R and suppose U is invertible. Let

$$A = \left[\begin{array}{c|c} U & V \\ \hline W & Z \end{array} \right]$$

so that A is a $(p + q) \times (p + q)$ square matrix. The *Schur complement* of A by U is

$$A/U := Z - WU^{-1}V.$$

The sizes of these matrices are suitable for block matrix multiplication and carrying out such computations confirms that

$$\left[\begin{array}{c|c} I & 0 \\ \hline -WU^{-1} & I \end{array} \right] \left[\begin{array}{c|c} U & V \\ \hline W & Z \end{array} \right] \left[\begin{array}{c|c} I & -U^{-1}V \\ \hline 0 & I \end{array} \right] = \left[\begin{array}{c|c} U & 0 \\ \hline 0 & Z - WU^{-1}V \end{array} \right].$$

Hence

$$\det A = \det U \cdot \det A/U, \qquad (2.1.4.a)$$

which serves to motivate the notation used for the Schur complement.

Lemma 2.1.5 *Let X be a $2n \times 2n$ alternating matrix whose distinct entries are the variables X_{ij} over $\mathbb{Z}$ for $1 \leq i < j \leq 2n$, then $\det X$ is the square of an element $\rho \in \mathbb{Z}[X]$.*

Proof We proceed by induction, noting that when $n = 1$ we have

$$X = \begin{bmatrix} 0 & X_{12} \\ -X_{12} & 0 \end{bmatrix} \implies \det X = X_{12}^2 \implies \rho = X_{12}.$$

Now, for the inductive step, suppose we have shown the result for matrices of size up to $2n \times 2n$ and let X be a $2(n+1) \times 2(n+1)$ alternating matrix whose distinct entries are the varialbes X_{ij} over $\mathbb{Z}$ for $1 \leq i < j \leq 2(n+1)$. Now write X in block form as in Definition 2.1.4 where U is 2×2, V is $2 \times 2n$, W is $2n \times 2$, and Z is $2n \times 2n$. If we view X as a matrix in $\mathbb{Q}(X)$, then U is invertible. Notice

$$U^{-1} = \begin{bmatrix} 0 & -\frac{1}{X_{12}} \\ \frac{1}{X_{12}} & 0 \end{bmatrix}$$

and $-V^T = W$, so the product $WU^{-1}V$ is an alternating matrix with entries in $\mathbb{Q}(X)$ where every entry can be taken to have denominator equal to X_{11}. Since Z is a $2n \times 2n$ alternating matrix with entries in $\mathbb{Z}[X]$ it follows that $A/U = Z - WU^{-1}V$ is an alternating matrix whose entries, after finding a common denominator for the entries of Z, can be taken to have denominator equal to X_{12}. Hence, $X_{12} \cdot (A/U)$ is an alternating $2n \times 2n$ matrix with entries in $\mathbb{Z}[X]$ and our inductive hypothesis shows that $\det (X_{12} \cdot (A/U)) = \rho_1^2$ for some $\rho_1 \in \mathbb{Z}[X]$. Now from (2.1.4.a) we know

$$\det X = \det U \cdot \det A/U = X_{12}^2 \cdot \det A/U$$

$$\implies X_{12}^{2n-2} \det X = X_{12}^{2n} \cdot \det A/U = \det (X_{12} \cdot (A/U)) = \rho_1^2.$$

$$\implies \det X = \left(\frac{\rho_1}{X_{11}^{n-1}}\right)^2.$$

Now we let $\rho_2 = \frac{\rho_1}{X_{12}^{n-1}} \in \mathbb{Q}(X)$ and see that ρ_2 is a root of the monic polynomial $Y^2 - \det X \in (\mathbb{Z}[X])[Y]$. Therefore, since $\mathbb{Z}[X]$ is a UFD–and is, therefore, integrally closed in its field of fractions–ρ_2 is an element of $\mathbb{Z}[X]$ as required. ■

Definition 2.1.6 By a *generic alternating matrix* X over R, we mean an alternating matrix whose non-diagonal upper triangular entries are distinct variables over R.

Now it follows from Lemma 2.1.5 that if A is an $n \times n$ alternating matrix, then $\det A$ is a perfect square in R. As such, one considers the *Pfaffian* of an alternating matrix A, denoted $\mathrm{Pf}(A)$, as a polynomial in the entries of A whose square is $\det A$. For a definition and a more detailed discussion of the properties of Pfaffians, see [21, Appendix D] or [2, pp. 140-142]. When n is odd, $\det A$ must be zero, so we instead focus on submatrices of A with even size. Hence, for any size matrix, we shall consider submatrices of size $2t \times 2t$ instead of $t \times t$ as in the case of ideals of minors.

Given an $n \times n$ alternating matrix A, by $\mathrm{Pf}_{2t}(A)$, we denote the ideal generated by the Pfaffians of the $2t \times 2t$ principal submatrices of A in the case that $2 \leq 2t \leq n$. Where a *principal submatrix* of A is a submatrix whose diagonal coincides with the diagonal of A. This is equivalent to using the same rows as columns when designating a particular submatrix (e.g., the submatrix of a 5×5 matrix A determined by rows $2, 3, 5$, and columns $2, 3, 5$ is a principal submatrix of A.) Such submatrices give all of the alternating submatrices, and these are needed to make sense of the notion of Pfaffian. By convention, if $2t \leq 0$, then $\mathrm{Pf}_{2t}(A) = R$, and if $2t > n$, then $\mathrm{Pf}_{2t}(A) = 0$. For simplicity, we also refer to the Pfaffians of the $2t \times 2t$ principal submatrices of A as the $2t \times 2t$ Pfaffians of A.

Definition 2.1.7 It is a known that if A is an $n \times n$ alternating matrix and if $\mathrm{Pf}_{2t}(A)$ is a proper ideal, then $\mathrm{ht}\,\mathrm{Pf}_{2t}(A) \leq \binom{n-2t+2}{2}$, and equality is achieved if A is a generic alternating matrix [32, Theorem 2.1, and Theorem 2.3]. As such, if A is an $n \times n$ alternating matrix, then the ideal $\mathrm{Pf}_{2t}(A)$ is said to be *of generic alternating height* if it achieves this maximal possible height, i.e., if $\mathrm{ht}\,\mathrm{Pf}_{2t}(A) = \binom{n-2t+2}{2}$.

Example 2.1.8 Consider the following matrices whose entries are distinct variables over R:

$$
X = \begin{bmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & X_{23} & X_{24} \\ -X_{13} & -X_{23} & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{bmatrix}, \qquad
Y = \begin{bmatrix} 0 & Y_{12} & Y_{13} & Y_{14} & Y_{15} \\ -Y_{12} & 0 & Y_{23} & Y_{24} & Y_{25} \\ -Y_{13} & -Y_{23} & 0 & Y_{34} & Y_{35} \\ -Y_{14} & -Y_{24} & -Y_{34} & 0 & Y_{45} \\ -Y_{15} & -Y_{25} & -Y_{35} & -Y_{45} & 0 \end{bmatrix}.
$$

These matrices are both generic alternating matrices and, as we can see, there are $\binom{4}{2} = 6$ distinct variables in X and $\binom{5}{2} = 10$ distinct variables in Y.

Since X is 4×4 we can obtain the Pfaffian using a Laplace-type expansion as described in Definition 2.1.9 below:

$$
\mathrm{Pf}(X) = -X_{12}\,\mathrm{Pf}^{12}(X) + X_{13}\,\mathrm{Pf}^{13}(X) - X_{14}\,\mathrm{Pf}^{14}(X)
$$

$$
= -X_{12}\,\mathrm{Pf}\begin{pmatrix} 0 & X_{34} \\ -X_{34} & 0 \end{pmatrix} + X_{13}\,\mathrm{Pf}\begin{pmatrix} 0 & X_{24} \\ -X_{24} & 0 \end{pmatrix} - X_{14}\,\mathrm{Pf}\begin{pmatrix} 0 & X_{23} \\ -X_{23} & 0 \end{pmatrix}
$$

$$
= -X_{12}X_{34} + X_{13}X_{24} - X_{14}X_{23}.
$$

The only other non-trivial Pfaffians associated to X are those of size 2×2 which generate $\mathrm{Pf}_2(X)$. Considering all the distinct alternating submatrices of X we see $\mathrm{Pf}_2(X)$ is generated by the distinct variables appearing in the entries of X. As such we confirm $\mathrm{ht}\,\mathrm{Pf}_2(X) = \binom{4-2+2}{2} = 6$, and $\mathrm{ht}\,\mathrm{Pf}_4(X) = \binom{4-4+2}{2} = 1$.

On the other hand, since Y is 5×5, we know its determinant and hence Pfaffian are zero. For such a matrix we would consider $\mathrm{Pf}_4(Y)$ and $\mathrm{Pf}_2(Y)$ as the Pfaffian ideals associated to Y. Notice the 4×4 Pfaffians of Y correspond to alternating 4×4 submatrices of Y which then gives us one distinct Pfaffian for each entry on the diagonal of Y via deletion of the row and column containing that entry. One can follow the computation of $\mathrm{Pf}(X)$ above to see how to obtain each of the 5 distinct 4×4 Pfaffians of Y and note that these will be quadratic in polynomials in the variables Y_{ij}.

Definition 2.1.9 Let A be an $n \times n$ matrix with entries in a ring R. It is a classical result that there exists an $n \times n$ matrix $\mathrm{Adj}(A)$, called the *classical adjoint* of A, with entries in R satisfying the property $\mathrm{Adj}(A)\, A = \det(A)\, I_{n \times n}$. This follows from the Laplace expansion of $\det A$.

Analogously, let n be even, and let A be an $n \times n$ alternating matrix with entries in R. Then there exists an $n \times n$ alternating matrix $\mathrm{PfAdj}(A)$, called the *Pfaffian adjoint* of A, with entries in R satisfying the property $\mathrm{PfAdj}(A)\, A = \mathrm{Pf}(A)\, I_{n \times n}$.

As in the non-alternating case, this result follows from a Laplace expansion of the Pfaffian. In particular, if A is an $n \times n$ alternating matrix with n even, then for a fixed integer $j, 1 \leq j \leq n$, one has

$$\mathrm{Pf}(A) = \sum_{i<j} (-1)^{i+j-1}\, A_{ij}\mathrm{Pf}^{ij}(A) + \sum_{i>j} (-1)^{i+j}\, A_{ij}\mathrm{Pf}^{ij}(A)\,,$$

where $\mathrm{Pf}^{ij}(A)$ denotes the Pfaffian of the alternating matrix obtained from A by deleting rows and columns i and j (see, for instance, [21, Appendix D]). We saw a small application of this expansion in Example 2.1.8 above.

Define $\mathrm{PfAdj}(A)$ to be the $n \times n$ alternating matrix where, for $i < j$,

$$[\mathrm{PfAdj}(A)]_{ij} = (-1)^{i+j}\,\mathrm{Pf}^{ij}(A)\,.$$

Using the Laplace expansion above, one verifies that $\mathrm{PfAdj}(A)\, A = \mathrm{Pf}(A)\, I_{n \times n}$. Notice that the off-diagonal entries of this product are zero since the square of a Pfaffian is the determinant of the given matrix, and off-diagonal entries here correspond to Pfaffians (and therefore to determinants) of submatrices with repeated rows and columns. This is true without requiring R to be a reduced ring since we can use the approach found in Lemma 2.1.5 to demonstrate this as a "universal" property of Pfaffian adjoints.

2.2 Resolutions, and Approximate Resolutions

Definition 2.2.1 Given a graded Noetherian ring R and a graded R-Module M we define

$$b_0(M) = \inf \left\{ p \,\middle|\, R\left(\bigoplus_{j \leq p} [M]_j\right) = M \right\},$$

$$\text{topdeg}\, M = \sup \left\{ j \,\middle|\, [M]_j \neq 0 \right\}.$$

Note that if $M = 0$, then we have

$$\text{topdeg}(M) = b_0(M) = -\infty.$$

We think of $b_0(M)$ as the maximum generator degree since it describes the largest graded piece of M needed in a minimal generating set for M. We also refer to topdeg M as the concentration degree for M or say "M is concentrated in degrees up to topdeg M". This describes the fact that if $q > \text{topdeg}\, M$, then the entire component $[M]_q$ is the zero module.

Given a standard graded polynomial ring R over a field K with maximal homogeneous ideal $\mathfrak{m}$, M a finitely generated graded R-module, and let $(\mathbf{C_\bullet}, \partial_\bullet)$ be a minimal homogeneous free resolution of M, then the Castelnuovo-Mumford *regularity* of M is

$$\text{reg}\, M = \sup \left\{ b_0(\mathbf{C_j}) - j \right\}. \tag{2.2.1.a}$$

This invariant is a measure of the maximal degree of the generators of the modules $\mathbf{C_j}$ and, in particular, provides a way to place a bound on these generators given reg M.

Definition 2.2.2 Let R be a Noetherian ring, given a complex

$$\mathbf{F_\bullet}: 0 \to F_s \xrightarrow{\varphi_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

of finite free R-modules we set $r_i = \sum_{j=i}^{s} (-1)^{j-i} \text{rank}(F_j)$ and we use $I(\varphi_i)$ to denote $I_{r_i}(\varphi_i)$.

Theorem 2.2.3 (Buchsbaum-Eisenbud Criterion [9, Theorem 1.4.13])
Let R be a Noetherian ring, and

$$\mathbf{F_\bullet}: 0 \to F_s \xrightarrow{\varphi_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

a complex of finite free R-modules. Set $r_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank}(F_j)$, then the following are equivalent:

(a) $\mathbf{F_\bullet}$ *is acyclic;*

(b) $\operatorname{grade} I_{r_i}(\varphi_i) \geq i$ *for $i = 1, \ldots, s$.*

As a consequence of this theorem we have the following result which will also be of use.

Corollary 2.2.4 (Buchsbaum-Eisenbud) *Let R be a Noetherian ring and M an R-module with a finite free resolution*

$$0 \to R^{b_n} \xrightarrow{\varphi_n} R^{b_{n-1}} \to \cdots \to R^{b_1} \xrightarrow{\varphi_1} R^{b_0}.$$

Then

(i) $\sqrt{I(\varphi_i)} \subset \sqrt{I(\varphi_{i+1})}$ *for all $1 \leq i \leq n-1$.*

(ii) *If $\operatorname{ann}(M) \neq 0$, then $\operatorname{rank}(\varphi_1) = b_0$, $\operatorname{grade} I(\varphi_1) = \operatorname{grade} M = g$, and*

$$\sqrt{I(\varphi_1)} = \cdots = \sqrt{I(\varphi_g)} \subsetneq \sqrt{I(\varphi_{g+1})}$$

if $M \neq 0$.

We now introduce the notion of an approximate resolution which we shall combine with the results above when dealing with localizations of particular complexes.

Definition 2.2.5 Let R be a Noetherian positively graded ring of dimension $d > 0$ with R_0 local and with unique maximal homogeneous ideal $\mathfrak{m}$, and suppose M is a graded R-module. For any homogeneous complex of finitely generated graded modules $\mathbf{D_\bullet}$ with $M \cong \operatorname{H}_0(\mathbf{D_\bullet})$ we say $\mathbf{D_\bullet}$ is a *approximate resolution* of M if both of the following conditions hold:

(i) $\dim \mathrm{H}_j(\mathbf{D}_\bullet) \leq j$ whenever $1 \leq j \leq d-1$, and

(ii) $\min\{d, j+2\} \leq \operatorname{depth} \mathbf{D}_j$ whenever $0 \leq j \leq d-1$.

While we define this in terms of graded rings and modules, we note that this definition also applies to local rings by giving the trivial grading. Indeed, if $(R, \mathfrak{m})$ is a local ring, then we can take R to be a graded ring having $R_0 = R$ and $R_j = 0$ for $j > 0$.

We will focus on complexes of free modules; hence, condition (ii) will be automatically satisfied if R is Cohen-Macaulay. Indeed, in this case we have

$$\operatorname{depth} \mathbf{D}_j = \operatorname{depth} R = \dim R = d \geq \min\{d, j+2\}.$$

The name "approximate resolution" is a reference to the way homological invariants for M may still be obtained–or at least bounded–from information about $\mathbf{D}_\bullet$ without possessing an explicit resolution for M.

2.3 Invariants and the condition G_s for Ideals

Definition 2.3.1 When R is a Noetherian ring, I is a proper R-ideal, and M is a nonzero finitely generated R-module, it is always the case that $\operatorname{grade}(\operatorname{ann}_R(M)) \leq \operatorname{pd}_R M$, where $\operatorname{pd}_R M$ denotes the projective dimension of M as an R-module. If this inequality is an equality for a particular module M, then we say that M is a *perfect* R-module. In the case of an ideal–despite having the structure of an R-module–we say that I is a *perfect ideal* when the R-module R/I is perfect.

Given a perfect R-ideal I with $\operatorname{grade} I = g$ we say I is a *Gorenstein ideal* if $\operatorname{Ext}_R^g(R/I, R)$ is a cyclic R-module. Ideals of this kind are studied directly in [37], and appear in the study of Pfaffian ideals since a structure theorem for Gorenstein ideals of height three tells us that such ideals are naturally Pfaffian ideals.

For a ring R and an R-module M we use the notation $\tau_R(M)$ to denote the R-torsion of M. We use the notation $\mu_R(M)$ to refer to the minimal number of generators of a finitely generated module M over the local ring R.

Definition 2.3.2 (The Condition G_s) Given an ideal I in a Noetherian ring R we say that I satisfies the condition G_s (or I satisfies G_s) if for all $\mathfrak{p} \in V(I)$

$$\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \leq \operatorname{ht}\mathfrak{p} \text{ when } \operatorname{ht}\mathfrak{p} < s.$$

Notice that this condition can be trivially extended to primes in $\operatorname{Spec}(R)$ instead of primes in $V(I)$ when $\operatorname{ht} I > 0$, since if $\mathfrak{q} \in \operatorname{Spec}(R) \setminus V(I)$, then $I_\mathfrak{q} = R_\mathfrak{q} \implies \mu_{R_\mathfrak{q}}(I_\mathfrak{q}) = 1 \leq \operatorname{ht}\mathfrak{q}$. We say the ideal I satisfies G_∞ when I satisfies G_s for all s. In particular, when $\dim R = d$, the condition G_∞ is equivalent to G_{d+1}.

We make use of an equivalent description of the condition G_s in terms of heights of Fitting ideals. We first give a definition of Fitting ideals and identify the key property needed for making the connection with G_s.

Definition 2.3.3 Let ψ be an $m \times n$ matrix with entries in R and $M = \operatorname{Coker}\psi$. The ith *Fitting ideal* of M is defined to be the ideal $I_{m-i}(\psi)$.

Notice that Fitting ideals are defined for any index by following the convention for determinantal ideals. It is a useful exercise to show that these ideals only depend on the module M, and not on the presentation for M. In particular we are interested in a characterization of the variety defined by Fitting ideals.

Proposition 2.3.4 ([10, Proposition 16.3]) *Let R be a commutative ring and let M be a finitely presented R-module, then*

$$V(\operatorname{Fitt}_i(M)) = \left\{ \mathfrak{p} \in \operatorname{Spec}(R) \,\middle|\, \mu_{R_\mathfrak{p}}(M_\mathfrak{p}) > i \right\}$$

We can now restate G_s in terms of Fitting ideals.

Proposition 2.3.5 *Let R be a Noetherian ring and let I be an R-ideal with $\operatorname{ht} I > 0$, then I satisfies G_s if and only if $\operatorname{ht}\operatorname{Fitt}_i(I) > i$ for $0 < i < s$.*

Proof Suppose I satisfies G_s, and fix i in the range $0 < i < s$. For $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht}\mathfrak{p} \leq i < s$, since I satisfies G_s we must have $\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \leq \operatorname{ht}\mathfrak{p} \leq i$, and this is

true (by Proposition 2.3.4) if and only if $\mathfrak{p} \notin V(\text{Fitt}_i(I))$. Since no prime ideals of height at most i contain $\text{Fitt}_i(I)$ we can conclude $\text{ht}\,\text{Fitt}_i(I) > i$.

Conversely, if $\text{ht}\,\text{Fitt}_i(I) > i$ for each i in the range $0 < i < s$, then we know that no prime of height at most i contains $\text{Fitt}_i(I)$. Hence primes $\mathfrak{p} \in V(I)$ of height i must satisfy $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) \leq i = \text{ht}\,\mathfrak{p}$. Therefore $\text{ht}\,\text{Fitt}_i(I) > i$ for $0 < i < s$ is sufficient to show I satisfies G_s. ∎

The above argument works because primes containing $\text{Fitt}_i(I)$ are precisely those for which G_s would fail. We avoid this failure by forcing the Fitting ideals to be "too tall" for such a failure to occur via the condition

$$\text{ht}\,\text{Fitt}_i(I) > i \text{ for } 0 < i < s.$$

We would like to make use of a more compact statement of this condition. We can modify the statement of Proposition 2.3.5 to the following:

$$I \text{ satisfies } G_s \text{ if and only if } \text{ht}\,\text{Fitt}_i(I) \geq \min\{i+1, s\} \text{ for all } i \geq 1. \qquad (2.3.5.\text{a})$$

We observe that the height of the unit ideal is taken to be infinite, so this inequality is trivially satisfied for values of i such that $\text{Fitt}_i(I) = R$.

2.4 The Rees Ring and Other Related Algebras

Definition 2.4.1 Let R be a Noetherian ring, and suppose I is an R-ideal. The *Rees ring* of I is defined to be the R-subalgebra $R[It] \subseteq R[t]$, where t is an indeterminate over R, and is denoted by $\mathcal{R}(I)$. We note that $\mathcal{R}(I)$ is a standard graded R-algebra whose grading is induced by the standard grading on $R[t]$.

If $R = K[x_1, \ldots, x_d]$ is a standard graded polynomial ring over a field K and I is an R-ideal generated by homogeneous forms $f_1, \ldots, f_n$ of the same degree D, then we define $S = R[T_1, \ldots, T_n]$ as a standard bigraded K-algebra where $\deg x_i = (1, 0)$ and $\deg T_i = (0, 1)$. We give $\mathcal{R}(I)$ the bigrading with $\deg x_i = (1, 0)$ and $\deg t = (-D, 1)$. This grading makes sense since we will want each variable T_i to correspond to $f_i t \in$

$\mathcal{R}(I)$. Indeed, since f_i has degree D in the variables of R, the product $f_i t$ will have bidegree $(D, 0) + (-D, 1) = (0, 1)$. With this choice the R-algebra homomorphism $\pi \colon S \to \mathcal{R}(I)$ given by $T_i \mapsto f_i t$ is a bihomogeneous K-algebra epimorphism. We use $\mathcal{J}$ to denote the bihomogeneous ideal $\ker \pi$ and call this the *defining ideal* of $\mathcal{R}(I)$. An element of a minimal bihomogeneous generating set of $\mathcal{J}$ is referred to as a *defining equation* of $\mathcal{R}(I)$ and any element of $\mathcal{J}$ is an *equation* of $\mathcal{R}(I)$.

In the rest of this section we shall fix $R = K[x_1, \ldots, x_d]$ as a standard graded polynomial ring over a field K, and $I = (f_1, \ldots, f_n)$ an ideal generated by homogeneous forms $f_1, \ldots, f_n$ of the same degree D.

Definition 2.4.2 Another algebra of importance we associate to the ideal I is called the *symmetric algebra* of I and is denoted by $\mathcal{S}(I)$. We define the symmetric algebra as follows:

$$\mathcal{S}(I) = \frac{\mathcal{T}(I)}{\mathcal{H}}$$

where $\mathcal{T}(I)$ is the tensor algebra of I (see, e.g., [47, Section 1.1]), and $\mathcal{H}$ is the two sided ideal of $\mathcal{T}(I)$ generated by $x \otimes y - y \otimes x$ for $x, y \in I$. Given a presentation

$$R^m \xrightarrow{\varphi} R^n \to I \to 0$$

of I we can describe $\mathcal{S}(I)$ as an R-algebra with defining ideal

$$I_1\left(\begin{bmatrix} T_1 & \cdots & T_n \end{bmatrix} \cdot \varphi\right).$$

This algebra is closely connected with the Rees algebra. Indeed it is possible to show that $\mathcal{R}(I) \cong \mathcal{S}(I) / \tau_R(\mathcal{S}(I))$. It follows that the defining equations of $\mathcal{S}(I(D))$ are a subset of those found in $\mathcal{J}$.

To ensure the gradings are compatible we always consider the R-module $I(D)$ which is the ideal I shifted by D. Since I is generated in degree D we see $I(D)$ is generated in degree zero, thus $\mathcal{S}(I(D))$ is a standard bigraded K-algebra. Indeed, assuming $I = (g_1, \ldots, g_n)$ with $\deg g_i = D$, we can describe a presentation for $\mathcal{S}(I)$ by defining an algebra map

$$\sigma \colon R[T_1, \ldots, T_n] \twoheadrightarrow \mathcal{S}(I)$$

with

$$T_i \mapsto g_i \in \mathcal{S}_1(I) \text{ for } i, 1 \leq i \leq n.$$

We see the generators of I sit in the degree one part of the symmetric algebra by definition and their degree in R gives the other component of the bigrading. Hence, without a shift, $\deg g_i = (D, 1)$ as an element of $\mathcal{S}(I)$ and σ cannot be bihomogeneous since $\deg T_i = (0, 1)$. Viewed as an element in $\mathcal{S}(I(D))$, however, $\deg g_i = (0, 1)$ and σ is bihomogeneous. We shall use $\mathcal{L}$ to denote the bihomogeneous defining ideal of $\mathcal{S}(I(D))$.

This approach to describing these algebras via presentations gives a precise way of seeing that the symmetric algebra for I is an R-algebra with one algebra generator for each generator of I, and where these algebra generators are related only by the first syzygies of I, i.e. R-linear combinations of the generators adding to zero. This stands in contrast with the Rees algebra which is also an R-algebra with the same number of algebra generators, but with relations among the generators obtained by considering R-linear combinations involving the *products* of the generators equal to zero. This distinction can be observed through a simple example:

Example 2.4.3 Let $R = K[x, y]$, and let $I = (x, y)^2 = (x^2, xy, y^2)$. We have three generators for I, so we have three algebra generators T_1, T_2, and T_3 for $\mathcal{S}(I)$ and $\mathcal{R}(I)$. To obtain $\mathcal{S}(I)$ we note that the first syzygies of I are generated by the Koszul relations obtained from $(y)x^2 + (-x)xy = 0$, and $(y)xy + (-x)y^2 = 0$ which can be directly translated into relations on T_1, T_2, and T_3 via the forms $yT_1 - xT_2$, and $yT_2 - xT_3$. This is to say $\mathcal{L} = (yT_1 - xT_2, yT_2 - xT_3)$ (we abuse notation by using $\mathcal{L}$ without adjusting the grading of I in this case) so that

$$\mathcal{S}(I) = \frac{R[T_1, T_2, T_3]}{\mathcal{L}} = \frac{K[x, y][T_1, T_2, T_3]}{(yT_1 - xT_2, yT_2 - xT_3)}.$$

There is one additional relation found among the generators of I that cannot be expressed as an R-linear combination of the generators of $\mathcal{L}$. It is clear that we have $(x^2)(y^2) - (xy)^2 = 0$, but this corresponds to the equation $T_1T_3 - T_2^2 = 0$. Notice that this equation does not involve the variables x, and y; we call such an equation a

"fiber" equation for reasons that will become clear when we define the special fiber ring of I below. This means $(yT_1 - xT_2, yT_2 - xT_3, T_1T_3 - T_2^2) \subset \mathcal{J}$. It turns out that this containment is, in fact, equality so

$$\mathcal{R}(I) = \frac{R[T_1, T_2, T_3]}{\mathcal{J}} = \frac{K[x, y][T_1, T_2, T_3]}{(yT_1 - xT_2, yT_2 - xT_3, T_1T_3 - T_2^2)}.$$

It is worth pointing out here that this implies $\tau_R(\mathcal{S}(I))$ is generated by the image of $T_1T_3 - T_2^2$ in $\mathcal{S}(I)$. It is, at least, clear that this element is R-torsion since

$$x \cdot \overline{T_1T_3 - T_2^2} = \overline{T_1} \cdot \overline{xT_3} - \overline{xT_2} \cdot \overline{T_2} = \overline{T_1} \cdot \overline{yT_2} - \overline{yT_1} \cdot \overline{T_2} = 0.$$

Since $\mathcal{S}(I(D))$ is a standard bigraded K-algebra, we know the natural R-algebra homomorphism $\alpha \colon \mathcal{S}(I(D)) \to \mathcal{R}(I)$ given by $f_i \mapsto f_i t$ is a bihomogeneous K-algebra epimorphism. We use $\mathcal{A}$, or $\mathcal{A}(I)$, to denote the bihomogeneous ideal $\ker \alpha$. Our results involve comparisons of $\mathcal{A}$ for multiple ideals, hence the usage of $\mathcal{A}(I)$ to distinguish the corresponding kernels.

It is important to recognize that the R-algebra homomorphism $f \colon S \to \mathcal{S}(I(D))$ with $T_i \mapsto f_i$ is also a bihomogeneous K-algebra epimorphism. As noted above we use $\mathcal{L}$ to denote the bihomogeneous ideal $\ker f$. We then obtain the following commutative diagram with exact rows in which all maps are bihomogeneous.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L} & \longrightarrow & S & \overset{f}{\longrightarrow} & \mathcal{S}(I(D)) & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & S & \overset{\pi}{\longrightarrow} & \mathcal{R}(I) & \longrightarrow & 0
\end{array}
$$

As such, with an application of the Snake Lemma, one sees that $\mathcal{A} \cong \mathcal{J}/\mathcal{L}$. Moreover, it is known that $S[\mathcal{J}]_{(*,1)} := S\left(\bigoplus_{k=0}^\infty [\mathcal{J}]_{(k,1)}\right) = \mathcal{L}$. As such, $[\mathcal{A}]_{(*,1)} = 0$. Moreover, since we have seen $\mathcal{L}$ is easy to find given a presentation matrix of I, we devote our attention to $\mathcal{A}$.

Notation 2.4.4 We adopt the notation that $\mathcal{S}_k(I(D)) = [\mathcal{S}(I(D))]_{(*,k)}$. In the same vein, we use the notation $\mathcal{A}_k(I) = [\mathcal{A}(I)]_{(*,k)}$. As such, when we write $[\mathcal{A}_k(I)]_p$, we mean $[\mathcal{A}(I)]_{(p,k)}$. This notation is convenient for us since we often fix the second component in the bidegree.
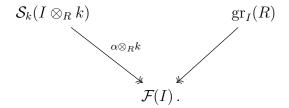
Definition 2.4.5 An ideal I is said to be of *linear type* if $\mathcal{A}(I) = 0$ and is said to be of *fiber type* if $\mathcal{A}_k(I)$ is generated in degree zero for all k with $\mathcal{A}_k(I) \neq 0$.

Remark 2.4.6 *Suppose $(R, \mathfrak{m})$ is local then I is of linear type on the punctured spectrum of R (i.e., $I_{\mathfrak{p}}$ is of linear type for all primes in $\mathrm{Spec}(R) \setminus \mathfrak{m}$) if and only if each component $\mathcal{A}_k(I)$ is supported only at $\mathfrak{m}$. Hence, in either case, $\mathcal{A}_k(I)$ has finite top degree, and since $\mathcal{A}(I)$ is finitely generated, there is a uniform bound for the top degree of $\mathcal{A}_k(I)$ for each k so that $\mathcal{A}(I)$ also has finite top degree.*

Definition 2.4.7 Suppose $(R, \mathfrak{m})$ is a Noetherian standard graded ring with unique maximal homogeneous ideal $\mathfrak{m}$, and I is a proper R-ideal generated by forms of the same degree. The *special fiber ring* of I, denoted $\mathcal{F}(I)$, is defined as $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$.

The Krull dimension of the special fiber ring is denoted by $\ell(I)$ and serves as a rather useful invariant we call the analytic spread of I. One surprising application of analytic spread comes from Burch's inequality which relates the analytic spread of I to depths of the powers of I. Since depth and projective dimension are related via the Auslander-Buchsbaum formula we are then able to take information about the analytic spread of I and make conclusions about maximal projective dimension for the powers of I.

We can make a first observation about bounds on analytic spread by considering the surjection from the associated graded ring onto $\mathcal{F}(I)$ along with the surjection obtained by taking the tensor product of the natural map $\alpha \colon \mathcal{S}(I) \to \mathcal{R}(I)$ with the residue field $k = R/\mathfrak{m}$. These give

$$
\mathcal{S}_k(I \otimes_R k) \xrightarrow{\ \alpha \otimes_R k\ } \mathcal{F}(I) \xleftarrow{\hspace{1.5em}} \mathrm{gr}_I(R)
$$

Now, since the symmetric algebra of a free module is a polynomial ring over the base ring in a number of variables equal to the free rank of the module, we observe

$I \otimes_R k \cong k^{\mu_R(I)}$ so $\mathcal{S}(I \otimes_R k) \cong k[T_1, \ldots, T_{\mu_R(I)}]$. Thus $\dim \mathcal{S}_k(I \otimes_R k) = \mu_R(I)$. Since we know $\dim \mathrm{gr}_I(R) = \dim R$ we can use these epimorphisms to conclude $\ell(I) \leq \min\{\mu_R(I), \dim R\}$.

We endeavor to use analytic spread as a way to study the projective dimension of the modules R/I^k. Hence we provide the following result which gives us a way to connect analytic spread to the depths of R/I^k.

Theorem 2.4.8 (Burch, [11, Corollary p. 373]) *Suppose $(R, \mathfrak{m})$ is a local ring, $I \subset \mathfrak{m}$ is an R-ideal, then*

$$\ell(I) + \inf\left\{\mathrm{depth}_R R/I^k\right\} \leq \dim R$$

The case where equality is obtained in this theorem is also of interest to us, so we state the required theorem here.

Theorem 2.4.9 (Eisenbud-Huneke, [20, Theorem 3.3]) *Let $(R, \mathfrak{m})$ be a local ring and let I be an ideal of R of height at least one. Suppose R and $\mathcal{R}(I)$ are Cohen-Macaulay. Then*

$$\ell(I) = \dim R - \inf\left\{\mathrm{depth}\, R/I^k\right\},$$

and if $\mathrm{depth}\, R/I^r = \inf\left\{\mathrm{depth}\, R/I^k\right\}$, then $\mathrm{depth}\, R/I^{r+1} = \mathrm{depth}\, R/I^r$.

We remark that the assumptions provided in Theorem 2.4.9 are sufficient for the hypothesis found in [20, Theorem 3.3] where it is only required that $\mathrm{gr}_I(R)$ be Cohen-Macaulay.

The following formula provides a connection between projective dimension and depth.

Theorem 2.4.10 (Auslander-Buchsbaum) *Let R be a Noetherian local ring and M a finite nonzero R-module of finite projective dimension. Then*

$$\mathrm{pd}_R M + \mathrm{depth}\, M = \mathrm{depth}\, R$$

When R is a regular local ring (to ensure projective dimensions remain finite) we can use Theorem 2.4.10 to obtain

$$\text{depth } R/I^k = \dim R - \text{pd}_R R/I^k \implies \inf\left\{\text{depth } R/I^k\right\} = \dim R - \sup\left\{\text{pd}_R R/I^k\right\}$$

In this case Theorem 2.4.8 becomes

$$\ell(I) \leq \sup\left\{\text{pd}_R R/I^k\right\}.$$

Finally Theorem 2.4.9 tells us this is an equality when, in addition, $\mathcal{R}(I)$ is Cohen-Macaulay. Further we know that this supremum of projective dimensions is realized by R/I^k for some k, so we can replace the supremum by a maximum. We collect these results here for future reference:

Proposition 2.4.11 *Let $(R, \mathfrak{m})$ be a regular local ring and let I be an R-ideal, then*

$$\ell(I) \leq \sup\left\{\text{pd}_R R/I^k\right\}$$

If, in addition, $\mathcal{R}(I)$ is Cohen-Macaulay, then

$$\ell(I) = \max\left\{\text{pd}_R R/I^k\right\}.$$

Observation 2.4.12 This thesis will be primarily concerned with graded (in particular polynomial) rings for its core results. Therefore we note Theorem 2.4.8, Theorem 2.4.9, Theorem 2.4.10, and Proposition 2.4.11 hold in the graded case via [9, Section 1.5].

We highlight the results from this reference which are particularly germane to our task.

Definition 2.4.13 ([9, Definition 1.5.13]) Let R be a graded ring. A homogeneous ideal $\mathfrak{m}$ of R is called **maximal* (read "star maximal"), if every homogeneous ideal properly containing $\mathfrak{m}$ is equal to R. The ring R is called **local* if it has a unique *maximal ideal $\mathfrak{m}$. A *local ring R with *maximal ideal $\mathfrak{m}$ will be denoted by $(R, \mathfrak{m})$.

With this definition in hand we can make sense of analytic spread in the *local case by using the *maximal ideal $\mathfrak{m}$ in the quotient $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Observe that if R is non-negatively graded and R_0 is local with maximal ideal $\mathfrak{m}_0$, then R has a unique maximal homogeneous ideal $\mathfrak{M} = \mathfrak{m}_0 + R_+$ which is also a maximal ideal of R, so R is naturally *local. We can now state a proposition which shows how the required theorems naturally hold when we use (in particular) standard graded polynomial rings.

Proposition 2.4.14 ([9, Proposition 1.5.15]) *Let $(R, \mathfrak{m})$ be a Noetherian *local ring, M a finite graded R-module, and I a homogeneous R-ideal. Then*

(a) *every minimal homogeneous system of generators of M has exactly $\mu_{R_\mathfrak{m}}(M_\mathfrak{m})$ elements,*

(b) *if $\mathbf{F}_\bullet$ is a minimal graded free resolution of M, then $\mathbf{F}_\bullet \otimes R_\mathfrak{m}$ is a minimal free resolution of $M_\mathfrak{m}$,*

(c) *the functor $- \otimes R_\mathfrak{m}$ is faithfully exact on the category $\mathcal{M}_0(R)$ (the category of graded R-modules whose objects are graded R-modules and whose morphisms are homogeneous R-linear maps),*

(d) *M is projective if and only if it is free,*

(e) *one has*

$$\mathrm{pd}_R M = \mathrm{pd}_{R_\mathfrak{m}} M_\mathfrak{m}, \qquad \mathrm{grade}\,(\mathfrak{m}, M) = \mathrm{depth}\, M_\mathfrak{m},$$
$$\mathrm{grade}\, M = \mathrm{grade}\, M_\mathfrak{m}, \qquad \mathrm{grade}\,(I, M) = \mathrm{grade}\,(I_\mathfrak{m}, M_\mathfrak{m}).$$

3. THE CONDITION G_s

In much of the work that has been done on studying Rees rings of an ideal I, the ideal I has been assumed to satisfy the condition G_s for some s. One finds that G_s is often important for the study of Rees rings because of the relation of the condition to the dimension of the symmetric algebra of I, as seen in [26, Theorem 2.6] which states that if R is a Noetherian ring and I is an R-ideal, then $\dim \mathcal{S}(I) = \sup\{\dim R/\mathfrak{p} + \mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$.

Suppose R is local, equidimensional, catenary, and I satisfies G_∞. If has positive height, then we know $\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) = 1$ for $\mathfrak{p} \in \operatorname{Spec}(R) \setminus V(I)$, and $\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p}$ for $\mathfrak{p} \in V(I)$. Hence, for primes $\mathfrak{p} \in V(I)$ we observe

$$\dim R/\mathfrak{p} + \mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \leq \dim R/\mathfrak{p} + \operatorname{ht} \mathfrak{p} \leq \dim R$$

and, since $\operatorname{ht} I > 0$, the minimal primes of R are found in $\operatorname{Spec}(R) \setminus V(I)$, so

$$\dim \mathcal{S}(I) = \sup \{\dim R/\mathfrak{p} + 1 \mid \mathfrak{p} \in \operatorname{Min}(R)\} = \dim R + 1 = \dim \mathcal{R}(I).$$

One can also consider the same setting when I satisfies G_d and $\dim R = d$ wherein G_d simply tells us not to consider primes of height less than d in computing the supremum. This leaves only maximal ideals containing I so, in the local setting, G_d implies $\dim \mathcal{S}_R(I) = \max\{\dim R + 1, \mu_R(I)\}$.

Another reason G_d is important comes from the utility found in working with zeroth local cohomology of the symmetric algebra of I. In particular, to get $\mathcal{A}(I) \cong \operatorname{H}^0_\mathfrak{m}(M)$, I must be of linear type on the punctured spectrum of R. Since G_∞ is necessary for an ideal to be of linear type it is necessary that I satisfy G_∞ locally except at the maximal ideal of R, and this is equivalent to the condition G_d.

Typically one checks whether an ideal I satisfies G_s for some s by computing the heights of Fitting ideals of I as described in Proposition 2.3.5. To compute Fitting

ideals, one needs a presentation matrix φ of I and to compute ideals of minors of φ (or Pfaffians of φ). In this thesis we work with determinantal ideals, so we always start with a matrix in tandem with our ideal. Unfortunately this matrix will often not serve as a presentation matrix for $I = I_t(A)$, but it is natural to ask whether the ideals of minors of A (or the Pfaffians of A) can be used to check G_s. We answer this question in the affirmative assuming I has the appropriate "generic height".

3.1 The Condition G_s for Ideals of Minors and Pfaffians

First we establish a technical lemma allowing us to restate G_s in a way that will be compatible with the minors or Pfaffians of A. The key insight is that, up to radical, distinct Fitting ideals only occur for indices equal to the distinct minimal number of generators of I locally at the primes containing I. Thus, if we consider the collection of integers corresponding to $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}})$ for all primes $\mathfrak{p} \in V(I)$, then for integers k between two (consecutive) of these values $\sqrt{\mathrm{Fitt}_k(I)}$ "collapses" to the radical of the Fitting ideal with index the closest value less than or equal to k.

Lemma 3.1.1 *Let R be a Noetherian ring and let I be a proper ideal with $\operatorname{ht} I > 0$. Let $(\mu_j)_{j=0}^q$ be a strictly decreasing sequence of positive integers so that*

   (i) *I can be generated by μ_0 elements,*

   (ii) *$\operatorname{ht} I = \mu_q$, and*

   (iii) *$\left\{ \mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) \,\middle|\, \mathfrak{p} \in V(I) \right\} \subseteq \left\{ \mu_j \,\middle|\, 0 \leq j \leq q \right\}.$*

Then I satisfies G_s if and only if

$$\operatorname{ht} \mathrm{Fitt}_{\mu_j}(I) \geq \min\{\mu_{j-1}, s\} \text{ for each } 1 \leq j \leq q.$$

Proof First we observe how the Fitting ideals "collapse" to the values in $\mathcal{M} = \{\mu_j\}_{j=0}^q$. If $k \in \mathbb{N}$ with $\mu_{j-1} > k \geq \mu_j$ for some j, then for each $\mathfrak{p} \in V\left(\mathrm{Fitt}_{\mu_j}(I)\right)$ one

has $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) > \mu_j$ by Proposition 2.3.4, but since $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) \in \mathcal{M}$ and μ_{j-1} is the next largest element of $\mathcal{M}$, we must have $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) \geq \mu_{j-1}$. Hence

$$\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) \geq \mu_{j-1} > k \implies \mathfrak{p} \in V(\operatorname{Fitt}_k(I)).$$

This shows $\sqrt{\operatorname{Fitt}_k(I)} \subset \sqrt{\operatorname{Fitt}_{\mu_j}(I)}$. The reverse containment is always true since Fitting ideals are nested, i.e., $\operatorname{Fitt}_{\mu_j}(I) \subset \operatorname{Fitt}_k(I)$ since $\mu_j \leq k$. Thus we have

$$\mu_{j-1} > k \geq \mu_j \implies \sqrt{\operatorname{Fitt}_{\mu_j}(I)} = \sqrt{\operatorname{Fitt}_k(I)}. \tag{3.1.1.a}$$

We need to know about the height of $\operatorname{Fitt}_k(I)$ for all values of k, so we must determine what happens when $k < \mu_q$ and when $k \geq \mu_0$. When $1 \leq k < \mu_q = \operatorname{ht} I$ a prime $\mathfrak{p}$ in $\operatorname{Fitt}_k(I)$ has $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) > k$, but since $\operatorname{ht} I = \operatorname{ht} I_{\mathfrak{p}} > k$, this is true for every element of $V(I)$ by Krull's Altitude Theorem. Hence $\sqrt{\operatorname{Fitt}_k(I)} = \sqrt{I}$.

Now when $k \geq \mu_0$ primes $\mathfrak{p}$ containing $\operatorname{Fitt}_k(I)$ require a *minimal* number of local generators larger than a *global* generating set. Since no primes can satisfy this condition we can conclude $\operatorname{Fitt}_k(I) = R$.

Recall from (2.3.5.a) if $\operatorname{ht} I > 0$, then I satisfies G_s if and only if

$$\operatorname{ht} \operatorname{Fitt}_k(I) \geq \min\{k+1, s\} \text{ for all } k \geq 1.$$

Now suppose I satisfies G_s: fix j with $1 \leq j \leq q$, and consider a prime $\mathfrak{p} \in V(\operatorname{Fitt}_{\mu_j}(I))$. We immediately know $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) > \mu_j$ and so $\mu_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) \geq \mu_{j-1}$. Now choose $k_0 = \mu_{j-1} - 1$, whence $\mu_{j-1} > k_0 \geq \mu_j$ and by (3.1.1.a) we have $\sqrt{\operatorname{Fitt}_{\mu_j}(I)} = \sqrt{\operatorname{Fitt}_{k_0}(I)}$.

We now apply G_s to obtain

$$\operatorname{ht} \operatorname{Fitt}_{\mu_j}(I) = \operatorname{ht} \operatorname{Fitt}_{k_0}(I) \overset{G_s}{\geq} \min\{k_0 + 1, s\} = \min\{\mu_{j-1}, s\}.$$

which is the desired inequality.

Conversely, suppose $\operatorname{ht} \operatorname{Fitt}_{\mu_j}(I) \geq \min\{\mu_{j-1}, s\}$ for each $1 \leq j \leq q$. To check G_s we fix $k \geq 1$. If $1 \leq k < \mu_q$ we have seen above $\sqrt{\operatorname{Fitt}_k(I)} = \sqrt{I}$ so by (ii) $\operatorname{ht} \operatorname{Fitt}_k(I) = \operatorname{ht} I \geq \mu_q > k$ as needed. On the other hand if $\mu_0 \leq k$, then $\operatorname{Fitt}_k(I) = R$

which has infinite height and so G_s is satisfied. This leaves the situation where, for some j, $\mu_{j-1} > k \geq \mu_j$. By (3.1.1.a) we then have $\sqrt{\text{Fitt}_k(I)} = \sqrt{\text{Fitt}_{\mu_j}(I)}$. Now we have

$$\text{ht Fitt}_k(I) = \text{ht Fitt}_{\mu_j}(I) \geq \min\{\mu_{j-1}, s\} \geq \min\{k+1, s\}$$

as required. ∎

Using this lemma we can restate the condition G_s for determinantal and Pfaffian ideals of generic height. This is made possible by the fact that we can compute the minimal number of generators of these ideals by rearranging the matrix in the local setting and ending up again with a determinantal (or Pfaffian) ideal of generic height.

Proposition 3.1.2 *Let R be a Noetherian ring.*

(a) *Suppose $1 \leq t \leq m \leq n$, and let A be an $m \times n$ matrix with entries in R. Suppose $I = I_t(A)$ is of generic height. Then I satisfies G_s if and only if*

$$\text{ht } I_j(A) \geq \min\left\{ \binom{m-j+1}{m-t}\binom{n-j+1}{n-t}, s \right\} \quad \text{for all } 1 \leq j \leq t-1.$$

(b) *Suppose $1 \leq t \leq n$, and A is a symmetric $n \times n$ matrix. Let $I = I_t(A)$ be of generic symmetric height. Then I satisfies G_s if and only if*

$$\text{ht } I_j(A) \geq \min\left\{ \frac{1}{n-j+2}\binom{n-j+2}{n-t}\binom{n-j+2}{n-t+1}, s \right\} \quad \text{for all } 1 \leq j \leq t-1.$$

(c) *Suppose $2 \leq 2t \leq n$, and A is an alternating $n \times n$ matrix. Let $I = \text{Pf}_{2t}(A)$ be of generic alternating height. Then I satisfies G_s if and only if*

$$\text{ht } \text{Pf}_{2j}(A) \geq \min\left\{ \binom{n-2j+2}{n-2t}, s \right\} \quad \text{for all } 1 \leq j \leq t-1.$$

Proof We prove part (b). We define the sequence $\mu_j = \frac{1}{n-j+1}\binom{n-j+1}{n-t}\binom{n-j+1}{n-t+1}$ for $0 \leq j \leq t-1$. One can compute that μ_j is strictly decreasing. Since I is of generic symmetric height, $\mu_{t-1} = \text{ht } I$. Moreover, I can be generated by μ_0 elements, since [46, 3.1.5] proves $I_t(I)$ may be generated by $\frac{1}{n+1}\binom{n+1}{t+1}\binom{n+1}{t}$ many elements.

Thus, by Lemma 3.1.1 it suffices to show that $\{\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \mid \mathfrak{p} \in V(I)\} \subseteq \{\mu_j\}$ and $\sqrt{I_j(A)} = \sqrt{\mathrm{Fitt}_{\mu_j}(I)}$.

Let $\mathfrak{p} \in V(I)$. Suppose that in $R_\mathfrak{p}$, the matrix $A_\mathfrak{p}$ is equivalent to

$$
\left(
\begin{array}{c|c|c}
I_{h \times h} & 0 & 0 \\
\hline
0 & J_{k \times k} & 0 \\
\hline
0 & 0 & C
\end{array}
\right),
$$

where $I_{h \times h}$ is the $h \times h$ identity matrix, $J_{k \times k}$ is a $k \times k$ block diagonal matrix with each block of the form $\begin{pmatrix} a & u \\ u & b \end{pmatrix}$ so that u is a unit and a and b are non-units, and C is symmetric with no unit entries. We refer to this as the *standard form* of $A_\mathfrak{p}$. Note that, for any $\mathfrak{p} \in V(I)$, the matrix $A_\mathfrak{p}$ can be placed into standard form without altering the ideal $I_t(A_\mathfrak{p})$.

Observe that if $A_\mathfrak{p}$ is in standard form and $j = h + k$, then $\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) = \mu_j$. Indeed, note that $I_\mathfrak{p} = I_t(A_\mathfrak{p}) = I_{t-j}(C)$, and then apply Lemma 3.1.3 to $I_{t-j}(C)$ to see $\{\mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \mid \mathfrak{p} \in V(I)\} \subseteq \{\mu_j \mid 0 \leq j \leq q\}$.

Finally, we show the smaller ideals of minors agree (up to radical) with the Fitting ideals. Suppose $\mathfrak{p} \in V(I)$. Transform $A_\mathfrak{p}$ into standard form, and let $\ell = h + k$. Then, as above, $\mu_\ell = \mu_{R_\mathfrak{p}}(I_\mathfrak{p})$.

$$
\begin{aligned}
\mathfrak{p} \notin V(I_j(A)) &\iff A_\mathfrak{p} \text{ has a unit } j \times j \text{ minor} \\
&\iff j \leq \ell \\
&\iff \mu_j \geq \mu_\ell = \mu_{R_\mathfrak{p}}(I_\mathfrak{p}) \\
&\iff \mathfrak{p} \notin V\left(\mathrm{Fitt}_{\mu_j}(I)\right)
\end{aligned}
$$

Therefore, the result obtains as soon as we have proven Lemma 3.1.3. The proofs for parts (a) and (c) are similar using the sequences $\mu_j = \binom{m-j}{m-t}\binom{n-j}{n-t}$ and $\mu_j = \binom{n-2j}{n-2t}$, respectively. ∎

The next lemma supplies the final piece to the proof of Proposition 3.1.2 by computing the minimal number of generators for the ideals of interest to us. If we

start with an ideal of generic height, then any localization at a prime containing this ideal will again be of generic height. We can then compute the minimal number of generators locally at such primes as follows.

Lemma 3.1.3 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $1 \leq t \leq m \leq n$.*

(a) *Let A be an $m \times n$ matrix with entries in $\mathfrak{m}$. If $I = I_t(A)$ is of generic height, then $\mu_R(I) = \binom{m}{t}\binom{n}{t}$.*

(b) *Let A be an $n \times n$ symmetric matrix with entries in $\mathfrak{m}$. If $I = I_t(A)$ is of generic symmetric height, then $\mu_R(I) = \frac{1}{n+1}\binom{n+1}{t+1}\binom{n+1}{t}$.*

(c) *Let A be an $n \times n$ alternating matrix with entries in $\mathfrak{m}$. If $I = \mathrm{Pf}_{2t}(A)$ is of generic alternating height, then $\mu_R(I) = \binom{n}{2t}$.*

Proof We begin with a reduction to the generic case. We prove this reduction for (b). The reduction is similar for (a) and (c). Let $X = (X_{ij})$ be an $n \times n$ generic symmetric matrix over R, and define $B = R[X]$. Let $\mathfrak{n} = \mathfrak{m} + (X)$ be the unique maximal homogeneous ideal of B. Let N be the B-ideal generated by the entries of the matrix $X - A$. We give R the B-algebra structure induced by $R \cong B/N$. We note $R/\mathfrak{m} \cong (B/N)/((\mathfrak{m}+N)/N) \cong B/\mathfrak{n}$ as B-algebras since $\mathfrak{m} + N = \mathfrak{n}$. Since N is generated by a regular sequence modulo $I_t(X)$, we also have that $I_t(X) \otimes_B R \cong I_t(A)$. Hence,

$$I_t(A) \otimes_R R/\mathfrak{m} \cong I_t(X) \otimes_B R \otimes_R R/\mathfrak{m} \cong I_t(X) \otimes_B B/\mathfrak{n}.$$

Therefore, $I_t(A) \otimes_R R/\mathfrak{m} \cong I_t(X) \otimes_B B/\mathfrak{n}$ as $B/\mathfrak{n}$-vector spaces. Hence, by Nakayama's lemma, $\mu_R(I_t(A)) = \mu_B(I_t(X))$.

For (a), we note that B is a free R-module with free basis given by the monomials in the variables which are the entries of X. Thus, it suffices to show that the distinct minors of X are R-linearly independent. By the Laplace expansion of determinants, every monomial in the support of a $t \times t$ minor is a squarefree monomial of degree t. Given a monomial α in the support of a fixed $t \times t$ minor M, if the variable X_{ij} divides α, then any submatrix of X having determinant M must use row i and column

j. Thus, since there are t distinct variables dividing α, there is only one submatrix of X for which α is in the support of the determinant. Therefore, α determines the corresponding minor M. Hence, the distinct $t \times t$ minors of X are R-linearly independent, and thus form a minimal generating set of $I_t(X)$. There are $\binom{m}{t}\binom{n}{t}$ distinct $t \times t$ minors of X.

Similarly, for (c), we note that B is a free R-module with free basis given by the monomials in the variables which are the entries above the diagonal of X. Thus, it suffices to show that the distinct Pfaffians of X are R-linearly independent. By the Laplace expansion of Pfaffians, every monomial in the support of a $2t \times 2t$ Pfaffian is a squarefree monomial of degree t. However, given a monomial α in the monomial support of a fixed $2t \times 2t$ Pfaffian P, if the variable X_{ij} divides α, then any principal submatrix of X having Pfaffian P must use row i and column j. Thus, since there are t distinct variables dividing α, there is only one principal submatrix of X for which α is in the support of the Pfaffian. Therefore, α determines the corresponding Pfaffian P. Hence, the distinct $2t \times 2t$ Pfaffians of X are R-linearly independent, and thus form a minimal generating set of $\operatorname{Pf}_{2t}(X)$. There are $\binom{n}{2t}$ distinct principal submatrices X.

In case (b), we refer to [46, Lemma 3.1.5] to obtain a count for the number of linearly independent $t \times t$ minors. In particular we obtain $\mu_B(I_t(X)) = \frac{1}{n+1}\binom{n+1}{t+1}\binom{n+1}{t}$. ∎

There are special combinations of $t, m,$ and n of interest, so we record some of these situations below. These results are a simple consequence of substituting these special combinations into the inequalities found in Proposition 3.1.2.

Observation 3.1.4 *Let R be a Noetherian ring.*

(a) *Suppose $1 \leq t = m \leq n$, and A is an $m \times n$ matrix with entries in R. If $I = I_m(A)$ is of generic height, then I satisfies G_s if and only if*

$$\operatorname{ht} I_j(A) \geq \min\left\{\binom{n-j+1}{n-m}, s\right\} \text{ for all } 1 \leq j \leq m-1.$$

In particular if $n = m + 1$, then I satisfies G_s if and only if

$$\operatorname{ht} I_j(A) \geq \min\{m - j + 2, s\} \ \text{ for all } 1 \leq j \leq m - 1.$$

(b) *Suppose $2 \leq m = n = t + 1$, and A is a square matrix with entries in R. If $I = I_{m-1}(A)$ is of generic height, then I satisfies G_s if and only if*

$$\operatorname{ht} I_j(A) \geq \min\left\{(m - j + 1)^2, s\right\} \ \text{ for all } 1 \leq j \leq t - 1.$$

(c) *Suppose $2 \leq n = t + 1$, and A is a symmetric $n \times n$ matrix with entries in R. If $I = I_{n-1}(A)$ is of generic symmetric height, then I satisfies G_s if and only if*

$$\operatorname{ht} I_j(A) \geq \min\left\{\frac{1}{2}(n - j + 2)(n - j + 1), s\right\} \ \text{ for all } 1 \leq j \leq t - 1.$$

(d) *Suppose $2 \leq n = 2t + 1$, and A is an alternating $n \times n$ matrix. Let $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height. The I satisfies G_s if and only if*

$$\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min\{n - 2j + 2, s\} \ \text{ for all } 1 \leq j \leq t - 1.$$

(e) *Suppose $2 \leq n = 2t + 2$, and A is an alternating $n \times n$ matrix. Let $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height. Then I satisfies G_s if and only if*

$$\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min\left\{\frac{1}{2}(n - 2j + 2)(n - 2j + 1), s\right\} \ \text{ for all } 1 \leq j \leq 2t - 2.$$

We will now use Proposition 3.1.2 to describe all of situations in which the ideal $I_t(X)$ for a generic matrix X satisfies G_∞. For those matrices where G_∞ does not hold we indicate the largest value of s for which $I_t(X)$ satisfies G_s.

Corollary 3.1.5 *Let X be a generic $m \times n$ matrix, $1 \leq t \leq m \leq n$, and $I = I_t(X)$. Then I satisfies G_∞ if and only if one of the following conditions holds:*

(i) $t = 1$.

(ii) $t = m = n$.

(iii) $m = n$ and $t = n - 1$.

(iv) $n = m + 1$ and $t = m$.

(v) $n = m + 2$ and $t = m$.

(vi) $m = 2$, $n = 5$, and $t = 2$.

If $3 \leq t = m$ and $n = m + 3$, then the maximal s for which I satisfies G_s is $s = 18$. In all other cases, the maximal s for which I satisfies G_s is $s = (m - t + 2)(n - t + 2)$.

Proof From Proposition 3.1.2.a, in order to satisfy G_∞ we must have

$$(m - j + 1)(n - j + 1) = \operatorname{ht} I_j(X) \geq \binom{m - j + 1}{m - t}\binom{n - j + 1}{n - t} \qquad (3.1.4.\mathrm{a})$$

for all $1 \leq j \leq t - 1$. Cases (i),(ii), and (iii) lead directly to equality in (3.1.4.a) while in (iv) we find

$$(m - j + 1)(m - j + 2) \geq \binom{m - j + 2}{1}$$

which is clearly true for all values of j in the given range. For (v) we have

$$(m - j + 1)(m - j + 3) \geq \binom{m - j + 3}{2} \iff 2(m - j + 1) \geq (m - j + 2)$$

which reduces to $j \leq m$. Since $j < t = m$ we again have G_∞. Finally, for (vi) we try the substitution $t = m$ and $n = m + 3$. This gives

$$(m - j + 1)(m - j + 4) \geq \binom{m - j + 4}{3} \iff$$

$$6(m - j + 1) \geq (m - j + 3)(m - j + 2).$$

We view this as a quadratic equation in $\lambda := (m - j)$ to find

$$6\lambda \geq \lambda^2 + 5\lambda \iff \lambda(\lambda - 1) \leq 0 \iff \lambda \leq 1.$$

This means we have G_∞ if and only if $m - j \leq 1 \iff j \geq m - 1$ for all $1 \leq j \leq t - 1$. This only works for $t = m = 2$ which forces $n = 5$ and establishes G_∞ for case (vi).

In order to obtain an upper bound on s for the other cases, we show that (3.1.4.a) cannot be satisfied when substituting $j = t - 1$, except in the cases (i)-(vi) where G_∞ holds and for one exceptional case. We focus on $j = t - 1$ because failure for this value of j implies failure for each of the smaller values of j due to the growth of the functions under comparison as j decreases. In this way satisfying (3.1.4.a) $j = t - 1$ represents a minimum necessary numerical requirement for G_∞.

When substituting $j = t - 1$ into (3.1.4.a), we obtain the inequality

$$4 \geq (m - t + 1)(n - t + 1).$$

The following table lists the cases where this inequality is satisfied.

$(m-t+1)(n-t+1)$	Result		
1	$t = m$	and	$m = n$
2	$t = m$	and	$n = m + 1$
3	$t = m$	and	$n = m + 2$
4	$t = m$	and	$n = m + 3$
	or		
	$t = m - 1$	and	$n = m$

These results correspond to cases (i)-(vi) and the exceptional case $3 \leq t = m$ and $n = m + 3$.

For the exceptional case $3 \leq t = m$ and $n = m + 3$, we check the next lower value for j and so we substitute $j = t - 2 = m - 2$, to obtain $\binom{m-j+1}{m-t}\binom{n-j+1}{n-t} = 20 > 18 = (m - j + 1)(n - j + 1)$ which fails (3.1.4.a). So what is the largest value of s for which G_s is satisfied in this case? Since ht $I_{m-2}(X) = 18$ the ideal $I = I_m(X)$ will satisfy G_{18}, and $s > 18$ is not possible.

Apart from the aforementioned cases, since ht $I_{t-1}(X) < \binom{m-j+1}{m-t}\binom{n-j+1}{n-t}$ when $j = t - 1$, the only way that G_s could be satisfied is if

$$s \leq (m - t + 2)(n - t + 2) = \text{ht } I_{t-1}(X).$$

Since ht $I_j(X) \geq$ ht $I_{t-1}(X)$ for all $1 \leq j \leq t - 2$, it follows that the maximal s for which G_s is satisfied is when $s = (m - t + 2)(n - t + 2)$. ∎

We now repeat our analysis for the case of generic symmetric matrices.

Corollary 3.1.6 *Let X be a generic symmetric $n \times n$ matrix, $1 \leq t \leq n$, and $I = I_t(X)$. Then I satisfies G_∞ if and only if $t = 1$, $t = n$, or $t = n - 1$. Otherwise, the maximal s for which I satisfies G_s is $s = \binom{n-t+3}{2}$.*

Proof From Proposition 3.1.2.b, in order for G_∞ to be satisfied, we must have

$$\binom{n - j + 2}{2} = \text{ht } I_j(X) \geq \frac{1}{n - j + 2}\binom{n - j + 2}{n - t}\binom{n - j + 2}{n - t + 1} \qquad (3.1.5.\text{a})$$

for all $1 \leq j \leq t - 1$.

For $t = 1$ there is nothing to check, so we proceed to the case $t = n$ where we have

$$\binom{n - j + 2}{2} \geq \frac{1}{n - j + 2}\binom{n - j + 2}{1} = 1$$

which is clearly true for $1 \leq j \leq n - 1$.

In the case $t = n - 1$ we find

$$\binom{n - j + 2}{2} \geq \frac{1}{n - j + 2}\binom{n - j + 2}{1}\binom{n - j + 2}{2} = \binom{n - j + 2}{2}$$

which is again true for all values of j.

In order to obtain an upper bound on s for the other cases, we show that (3.1.5.a) cannot be satisfied when substituting $j = t - 1$, except in the cases where G_∞ holds.

When substituting $j = t - 1$ into (3.1.5.a), we obtain the inequality

$$6 \geq (n - t + 2)(n - t + 1).$$

Restricting to $t \leq n$, the inequality is satisfied for $n - 1 \leq t \leq n$. This means, apart from $t = 1, n$, and $n - 1$, it must be the case that

$$\text{ht } I_{t-1}(X) < \frac{1}{n - j + 2}\binom{n - j + 2}{n - t}\binom{n - j + 2}{n - t + 1}.$$

Hence, the only way that G_s could be satisfied is if ht $I_{t-1}(X) = \binom{n-t+3}{2} \geq s$. Since ht $I_j(X) \geq$ ht $I_{t-1}(X)$ for all $1 \leq j \leq t - 2$, it follows that the maximal s for which G_s is satisfied is when $s = \binom{n-t+3}{2}$. ∎

We now apply this same analysis to the case of generic alternating matrices.

Corollary 3.1.7 *Let X be a generic alternating $n \times n$ matrix, $2 \leq 2t \leq n$, and $I = \mathrm{Pf}_{2t}(X)$. Then I satisfies G_∞ if and only if one of the following conditions hold:*

(i) $2t = 2$,

(ii) $2t = n$,

(iii) $2t = n - 2$, *or*

(iv) $2t = n - 1$.

Otherwise, the maximal s for which I satisfies G_s is $s = \binom{n-2t+4}{2}$.

Proof From Proposition 3.1.2.c, in order for G_∞ to be satisfied, we must have

$$\binom{n - 2j + 2}{2} = \mathrm{ht}\,\mathrm{Pf}_{2j}(X) \geq \binom{n - 2j + 2}{n - 2t} \qquad (3.1.6.\mathrm{a})$$

for all $1 \leq j \leq t - 1$.

There is nothing to check in case (i), so we set $2t = n$ for case (ii) to find

$$\binom{n - 2j + 2}{2} \geq 1$$

which is clearly true for $1 \leq j \leq t - 1$.

In case (iii) we have $2t = n - 2$ which gives

$$\binom{n - 2j + 2}{2} \geq \binom{n - 2j + 2}{2}$$

which is again true for all values of j.

In case (iv) we have $2t = n - 1$ and this gives

$$\binom{n - 2j + 2}{2} \geq \binom{n - 2j + 2}{1}$$

which also holds for all values of j.

In order to obtain an upper bound on s for the other cases, we show that (3.1.6.a) cannot be satisfied when substituting $j = t - 1$, except in the cases where G_∞ holds.

When substituting $j = t - 1$ into (3.1.6.a) and restricting to $2 \leq 2t \leq n$, we obtain the inequality $n \geq 2t \geq n - 2$. This happens only for cases (ii)-(iv) where G_∞ is satisfied.

Apart from the aforementioned cases, since $\operatorname{ht} I_{t-1}(X) < \binom{n-2j+2}{n-2t}$ when $j = t - 1$, the only way that G_s could be satisfied is if $\binom{n-2t+4}{2} = \operatorname{ht} \operatorname{Pf}_{2t-2}(X) \geq s$. Since $\operatorname{ht} \operatorname{Pf}_{2j}(X) \geq \operatorname{ht} \operatorname{Pf}_{2t-2}(X)$ for all $1 \leq j \leq t - 2$, it follows that the maximal s for which G_s is satisfied is when $s = \binom{n-2t+4}{2}$. $\blacksquare$

3.2 Analytic Spread and G_ℓ

We conclude this chapter by describing the analytic spread of the ideals $I_t(X)$ and then combining those computations with Proposition 3.1.2 to indicate the required height bounds on the ideals $I_j(X)$ (for $1 \leq j \leq t - 1$) required to satisfy G_ℓ.

The following result is a reworking of the argument found in [15] and is the main tool we use to compute $\ell(I_t(X))$.

Lemma 3.2.1 *Let X be a generic $n \times n$ matrix over the field K. Let $\Delta_1, \ldots, \Delta_s$ be the $(n-1) \times (n-1)$ minors of X, and let $F = K(\Delta_1, \ldots, \Delta_s)$ be the fraction field of $\mathcal{F}(I_{n-1}(X)) = K[\Delta_1, \ldots, \Delta_s]$, then $F \subset K(X)$ is an algebraic extension of fields.*

In particular, we find

$$\ell(I_{n-1}(X)) = \dim \mathcal{F}(I_{n-1}(X)) = \operatorname{trdeg}_K F = \dim K[X] = n^2.$$

Proof (Cowsik-Nori) Our first step is to make use of the classical adjoint of X which gives us the following matrix equation:

$$\operatorname{Adj}(X) \cdot X = (\det X) \cdot I_{n \times n}. \tag{3.2.1.a}$$

Computing determinants of both sides of this equality shows

$$\det\left(\operatorname{Adj}(X)\right) \det X = (\det X)^n.$$

Hence, $\det\left(\mathrm{Adj}(X)\right) = \left(\det X\right)^{n-1}$. Now since the entries of $\mathrm{Adj}(X)$ are elements of $K[\Delta_1, \ldots, \Delta_s]$ we know $\det\left(\mathrm{Adj}(X)\right) \in K[\Delta_1, \ldots, \Delta_s]$. Thus $\det X$ is a root of the polynomial

$$Y^{n-1} - \det\left(\mathrm{Adj}(X)\right) \in (K[\Delta_1, \ldots, \Delta_s])[Y]$$

and so $F \subset F(\det X)$ is an algebraic extension of fields.

We can now complete the proof by showing $X_{ij} \in F(\det X)$. We return to equation (3.2.1.a) and view it as a system of equalities: one for each column of X multiplied by $\mathrm{Adj}(X)$ equal to the corresponding column of $(\det X) \cdot I_{n \times n}$. Using Cramer's rule on these equations allows us to "solve" for the variables X_{ij}. To be precise, we find

$$X_{ij} = \frac{\det\left(Z_{ij}\right)}{\det\left(\mathrm{Adj}(X)\right)} \in F(\det X)$$

where Z_{ij} is the matrix obtained from $\mathrm{Adj}(X)$ by replacing column j with column i from $(\det X) \cdot I_{n \times n}$.

Thus we have shown $K(X) \subset F(\det X)$ and therefore $K(X) = F(\det X)$ is an algebraic extension of F, as needed. ∎

We can adapt this lemma to easily compute the analytic spread in more general settings. Our first generalization is to compute the analytic spread in the case of an ordinary generic $m \times n$ matrix.

Proposition 3.2.2 *Let X be a generic $m \times n$ matrix over a field K with $1 \leq t \leq m \leq n$. Let $J = I_t(X)$.*

(i) *If $t = m$, then $\ell(J) = m\,(n - m) + 1$.*

(ii) *If $t < m$, then $\ell(J) = nm$.*

Proof The proof of (i) comes from realizing that the analytic spread for the case of maximal minors is the same as the dimension of a corresponding Grassmanian. It follows that the analytic spread is one plus the dimension of the Grassman variety in its Plücker embedding, hence $\ell(J) = m(n - m) + 1$. See [10, Theorem 1.3] for background on this result.

To prove (ii) we let X' be a $(t+1) \times (t+1)$ submatrix of X so that an application of Lemma 3.2.1 to X' shows the variables of X contained in X' are algebraic over $\mathcal{F}(J)$. Letting X' vary over all of the distinct $(t+1) \times (t+1)$ submatrices of X then shows that every variable X_{ij} is algebraic over $\mathcal{F}(J)$, so $\ell(J) = \dim \mathcal{F}(J) = \dim K[X] = nm$. ∎

Observe that the application of Lemma 3.2.1 simply shows that each distinct variable among the entries of X will be algebraically independent. We can, therefore, extend this result to obtain analytic spread in the case of generic symmetric matrices and generic alternating matrices.

Since these cases correspond to square matrices a maximal submatrix simply corresponds to the full matrix and, therefore, corresponds to an ideal generated by a regular element. Hence the analytic spread in such cases is one. In all cases where $t < m$ we use Lemma 3.2.1 to count the number of distinct variables among the entries of X to obtain the analytic spread of generic symmetric matrices and generic alternating matrices. We summarize these results here.

Remark 3.2.3 *Let K be a field.*

(i) *Suppose X is a symmetric $n \times n$ generic matrix, then for $t < n$*

$$\ell(I_n(X)) = 1, \text{ and } \ell(I_t(X)) = \binom{n+1}{2}.$$

(ii) *Suppose X is an alternating $2n \times 2n$ generic matrix, then for $t < n$*

$$\ell(\mathrm{Pf}_{2n}(X)) = 1, \text{ and } \ell(\mathrm{Pf}_{2t}(X)) = \binom{n}{2}.$$

(iii) *Suppose X is an alternating $(2n+1) \times (2n+1)$ generic matrix, then for $t = n$*

$$\ell(\mathrm{Pf}_{2n}(X)) = 2n.$$

Proof The discussion preceding this remark together with the fact that a generic symmetric matrix has $\binom{n+1}{2}$ distinct variables among its entries and a generic alternating matrix has $\binom{n}{2}$ distinct variables among its entries is enough to show (i) and (ii).

Finally (iii) follows from [27, 2.2] where it is shown that $\mathrm{Pf}_{2n}(X)$ is of linear type. In this case we can compute the special fiber ring directly as

$$\mathcal{F}(\mathrm{Pf}_{2n}(X)) = K \otimes \mathcal{R}(\mathrm{Pf}_{2n}(X)) \cong K \otimes \mathcal{S}(\mathrm{Pf}_{2n}(X)) \cong \mathcal{S}(K \otimes \mathrm{Pf}_{2n}(X))$$

$$\cong \mathcal{S}\big(K^{\mu(\mathrm{Pf}_{2n}(X))}\big) \cong K[T_1, \ldots, T_{2n}].$$

This completes the proof. ■

With the analytic spread now determined for the family of generic matrices of interest to us we combine these values with Corollary 3.1.5. As a result of this combination we can determine when $I_t(X)$ or $\mathrm{Pf}_{2t}(X)$ satisfy G_ℓ. In particular, for ideals which do not satisfy G_∞ (since these will trivially satisfy G_ℓ) Corollary 3.1.5, Corollary 3.1.6, and Corollary 3.1.7 give the maximum s for which the ideal satisfies G_s. Thus we can compare $\ell(J)$ with this maximum s to classify those situations where G_ℓ holds.

Corollary 3.2.4 *Let K be a field.*

(i) *Let $1 \leq t = m \leq n$ and let X be a generic $m \times n$ matrix. Let $J = I_t(X)$ and suppose J does not satisfy G_∞, then J satisfies G_ℓ if and only if one of the following hold*

    (a) $m = 2$ *and* $n \geq 6$, *or*

    (b) $m = 3$ *and* $n = 6$.

(ii) *Let $1 \leq t \leq n$ and let X be a generic symmetric $n \times n$ matrix. Let $J = I_t(X)$ and suppose J does not satisfy G_∞, then J satisfies G_ℓ if and only if $t = 2$ and $n \geq 4$.*

(iii) *Let $1 \leq 2t \leq n$ and let X be a generic alternating $n \times n$ matrix. Let $J = \mathrm{Pf}_{2t}(X)$ and suppose J does not satisfy G_∞, then J satisfies G_ℓ if and only if $t = 2$ and $n \geq 7$.*

Proof Using Corollary 3.1.5 in case (i) we identify the cases where G_∞ already holds. For the remaining cases we have the maximum s for which J satisfies G_s given by

$$s = 2\,(n - m + 2)\,.$$

On the other hand, by Proposition 3.2.2, we know the analytic spread of J. Thus we find J satisfies G_ℓ if and only if

$$2\,(n - m + 2) \geq m(n - m) + 1. \tag{3.2.4.a}$$

If $m = 2$, as in part (a) this becomes $2n \geq 2n - 3$ which is satisfied for all n. To demonstrate (b) we rearrange (3.2.4.a) to find J satisfies G_ℓ if and only if

$$m^2 - 2m + 3 \geq n\,(m - 2) \overset{m \geq 2}{\Longleftrightarrow} n \leq m + \frac{3}{m - 2}.$$

Thus, for $m > 5$ only $n = m$ is possible, and, for $m = 4$ and 5, n is at most $m + 1$. Therefore only $m = 3$ and $n = 6$ gives a solution to this inequality such that J is not also G_∞.

Using Corollary 3.1.6, Corollary 3.1.7 for the maximal s for which J satisfies G_s, and Remark 3.2.3 for $\ell(J)$, we can quickly see these quantities are equal when $t = 2$. Since the maximal s for which J satisfies G_s is a strictly decreasing function of t, and $\ell(J)$ does not depend on t, we can conclude J satisfies G_ℓ if $t = 2$. To avoid the cases where we already have G_∞ we simply require $t < n - 1 \implies n \geq 4$ in (ii), and $2t < n - 2 \implies n \geq 7$ in (iii) as required. ∎

4. APPROXIMATION OF REES RINGS AND OF RESOLUTIONS VIA SPECIALIZATION

We now discuss the techniques needed to compare data obtained from resolutions of the ideals $J^k = I_t(X)^k$ and $J^k = \mathrm{Pf}_{2t}(X)^k$ to their specialized counterparts.

4.1 Introduction to Specialization

Let R be a Cohen-Macaulay ring and let X be a generic $m \times n$ matrix over R. For any $m \times n$ matrix A whose entries are in R, there exists a surjective R-algebra homomorphism $R[X] \to R$ given by $X_{ij} \mapsto A_{ij}$. This is the specialization map we shall be implicitly using to pass from the generic case to the setting of the matrix A. In particular we can use this surjective map to view R as an $R[X]$-module according to the isomorphism

$$R \cong \frac{R[X]}{\mathcal{N}}$$

where $\mathcal{N} = (\{X_{ij} - A_{ij} \mid 1 \le i \le m,\, 1 \le j \le n\})$, and we will see

$$I_t(X) \otimes_{R[X]} R \cong \frac{I_t(X)}{\mathcal{N} I_t(X)} \cong \frac{I_t(X) + \mathcal{N}}{\mathcal{N}} \cong I_t(A)\,.$$

The middle isomorphism between the two quotient ideals comes from the short exact sequence

$$0 \to \frac{\mathcal{N} \cap I_t(X)}{\mathcal{N} I_t(X)} \to \frac{I_t(X)}{\mathcal{N} I_t(X)} \to \frac{I_t(X) + \mathcal{N}}{\mathcal{N}} \to 0.$$

One can check that the left most module in this sequence is the kernel obtained from applying $R \otimes_{R[X]} -$ to the inclusion $I_t(X) \hookrightarrow R[X]$. Thus we can characterize this kernel as a Tor module via

$$\frac{\mathcal{N} \cap I_t(X)}{\mathcal{N} I_t(X)} \cong \mathrm{Tor}_1^{R[X]}\left(\frac{R[X]}{\mathcal{N}}, \frac{R[X]}{I_t(X)} \right).$$

We can show this module is zero as soon as we know the sequence given by the entries of $X - A$ is a weak $R[X]/I_t(X)$-sequence. To see this we observe $R[X]/I_t(X)$ is Cohen-Macaulay since $I_t(X)$ is a perfect $R[X]$-ideal by [23]. Now we compute

$$\dim \left(\frac{R[X]/I_t(X)}{(\mathcal{N} + I_t(X))/I_t(X)} \right) = \dim \frac{R[X]}{\mathcal{N} + I_t(X)} = \dim \frac{R}{I_t(A)} = \dim R - \operatorname{ht} I_t(A).$$

We also have

$$\dim \frac{R[X]}{I_t(X)} = \dim R[X] - \operatorname{ht} I_t(X) = \dim R + nm - \operatorname{ht} I_t(A).$$

Hence we are able to show $\mathcal{N}$ is weakly $R[X]/I_t(X)$-regular via

$$\dim \frac{R[X]}{I_t(X)} - \dim \left(\frac{R[X]/I_t(X)}{(\mathcal{N} + I_t(X))/I_t(X)} \right)$$

$$= (\dim R + nm - \operatorname{ht} I_t(A)) - (\dim R - \operatorname{ht} I_t(A)) = nm.$$

Since nm is the number of elements in $X - A$ which corresponds to the sequence generating $\mathcal{N}$ we are able to conclude this sequence is, in fact, $R[X]/I_t(X)$-regular.

We can then conclude the vanishing of the Tor module with the following useful result which we record here along with a proof for future reference.

Lemma 4.1.1 *Suppose R is Noetherian and M is an R-module. If $\mathfrak{a} = (a_1, \ldots, a_s)$, where $a_1, \ldots, a_s$ is a weak M-sequence, then $\operatorname{Tor}_1^R(R/\mathfrak{a}, M) = 0$.*

Proof We proceed by induction on s. When $s = 1$ we have $\mathfrak{a} = (a)$, and there is an exact sequence

$$0 \to \frac{R}{\left(0 \underset{R}{:} a \right)} \xrightarrow{\cdot a} R \to \frac{R}{(a)} \to 0$$

to which we can associate the long exact sequence for Tor to obtain the exact sequence

$$\operatorname{Tor}_1^R(R, M) \to \operatorname{Tor}_1^R\left(\frac{R}{(a)}, M \right) \to \frac{M}{\left(0 \underset{R}{:} a \right) M} \xrightarrow{\cdot a} M.$$

Since R is a flat R-module we see $\operatorname{Tor}_1^R(R, M) = 0$ so we must have

$$\operatorname{Tor}_1^R\left(\frac{R}{(a)}, M \right) \cong \ker \left(\frac{M}{\left(0 \underset{R}{:} a \right) M} \xrightarrow{\cdot a} M \right) = \frac{\left(0 \underset{M}{:} a \right)}{\left(0 \underset{R}{:} a \right) M}.$$

Now, since a is a non-zerodivisor on M, we have $\left(0 \underset{M}{:} a\right) = 0$ as needed.

For the induction step simply follow the same steps as in the $s = 1$ case but let $\mathfrak{a}' = (a_1, \ldots, a_{s-1})$ and instead consider the short exact sequence

$$0 \to \frac{R}{\left(\mathfrak{a}' \underset{R}{:} a_s\right)} \xrightarrow{\cdot a_s} \frac{R}{\mathfrak{a}'} \to \frac{R}{\mathfrak{a}} \to 0.$$

We then arrive at

$$\mathrm{Tor}_1^R\left(\frac{R}{\mathfrak{a}}, M\right) \cong \frac{\left(\mathfrak{a}'M \underset{M}{:} a_s\right)}{\left(\mathfrak{a}' \underset{R}{:} a_s\right) M}.$$

We conclude the argument by observing that, since $a_1, \ldots, a_s$ is a weak M-sequence, we have $\left(\mathfrak{a}'M \underset{M}{:} a_s\right) = 0$. ∎

4.2   Specializing Resolutions and Approximate Resolutions

More generally, if B is a Cohen-Macaulay ring, J is some B-ideal, and $\mathcal{N} = (Y_1, \ldots, Y_D)$ where $Y_1, \ldots, Y_D$ is a regular sequence on B and B/J, then $B/\mathcal{N}$ is said to be a specialization of B and $J(B/\mathcal{N})$ is a specialization of J following Lemma 4.1.1. The primary goal of this section is to investigate how properties of the specialization of an ideal J can be obtained from properties of the ideal J.

In particular, one hopes that understanding $\mathcal{R}(I_t(X))$ would give enough information to understand $\mathcal{R}(I_t(A))$ under suitable assumptions on A.

We now prove a technical lemma that will be critical in our results on specialization and degree bounds.

Lemma 4.2.1 *Let R be a Cohen-Macaulay ring, and suppose M is a finitely generated R-module with finite projective dimension. Let L be an R-module. Suppose $f \colon M \to L$ is an R-linear map and that $\ker f = \tau_R(M)$.*

(i) *Then f is injective if and only if $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \mathrm{ht}\,\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}\,\mathfrak{p} > 0$.*

(ii) *Suppose R is positively graded with R_0 local, $\dim R > 0$, M and L are graded R-modules, and f is homogeneous. Denote the unique maximal homogeneous ideal of R as $\mathfrak{m}$. Then $\ker f = \mathrm{H}^0_{\mathfrak{m}}(M)$ if and only if $\mathrm{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \mathrm{ht}\,\mathfrak{p}$ for all $\mathfrak{p}$ in $\mathrm{Proj}(R)$ with $\mathrm{ht}\,\mathfrak{p} > 0$.*

Proof Let $\mathcal{K} = \ker f$, and let $\mathbf{X}_0 = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}\,\mathfrak{p} > 0\}$.

To prove (i), note that $\mathcal{K} = 0$ if and only if $\mathcal{K}_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Thus, we just have to show that

$$f \text{ is injective locally at all } \mathfrak{p} \in \mathrm{Spec}(R) \iff \mathrm{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \mathrm{ht}\,\mathfrak{p} \text{ for all } \mathfrak{p} \in \mathbf{X}_0.$$

We establish this equivalence by working through a sequence of equivalences connecting the two conditions.

First we show a more direct equivalence:

$$f \text{ is injective locally at all } \mathfrak{p} \in \mathrm{Spec}(R) \iff M_{\mathfrak{p}} \text{ is torsion-free for all } \mathfrak{p} \in \mathrm{Spec}(R).$$

Since $\mathcal{K} = \tau_R(M)$ and R is Cohen-Macaulay, we see the torsion of M localizes. Indeed, since $\tau_R(M) = \ker(M \to \mathrm{Quot}(R) \otimes_R M)$ we can see $\tau_R(M)_{\mathfrak{p}} \subset \tau_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ and this inclusion can be proper when R has embedded primes. Since R is Cohen-Macaulay we know all of the associated primes of R are minimal primes, so the inclusion becomes equality. Thus, using the local-to-global principle, it becomes clear that these conditions are equivalent.

The next equivalence we demonstrate is

$$M_{\mathfrak{p}} \text{ is torsion-free for all } \mathfrak{p} \in \mathrm{Spec}(R) \iff \mathrm{depth}\,M_{\mathfrak{p}} > 0 \text{ for all } \mathfrak{p} \in \mathbf{X}_0.$$

First suppose $M_{\mathfrak{p}}$ is torsion free for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Recall that an S-module N is torsion free if and only if zerodivisors on N are also zerodivisors on S which is true if and only if every associated prime of N is contained in an associated prime of S. So suppose we have $\mathfrak{p} \in \mathbf{X}_0$, then by virtue of having positive height in the Cohen-Macaulay ring R, we know $\mathrm{depth}\,R_{\mathfrak{p}} > 0$. Now positive depth for a module is equivalent to the existence of regular elements in the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$, hence is

equivalent to $\mathfrak{p}R_\mathfrak{p} \notin \mathrm{Ass}(R_\mathfrak{p})$. Now since $M_\mathfrak{p}$ is torsion-free we can conclude $\mathfrak{p}R_\mathfrak{p} \notin \mathrm{Ass}(M_\mathfrak{p})$, since, if $\mathfrak{p}R_\mathfrak{p}$ were associated to $M_\mathfrak{p}$, then it would not be contained in an associated prime of $R_\mathfrak{p}$. Thus depth $M_\mathfrak{p} > 0$.

Conversely suppose depth $M_\mathfrak{p} > 0$ whenever $\mathfrak{p} \in \mathbf{X}_0$. So let $\mathfrak{q} \in \mathrm{Ass}_R(M_\mathfrak{p})$. We would like to show $\mathfrak{q}$ is contained an associated prime of $R_\mathfrak{p}$. By virtue of being an associated prime of $M_\mathfrak{p}$, $\mathfrak{q} \subset \mathfrak{p}$ and so $(M_\mathfrak{p})_\mathfrak{q} \cong M_\mathfrak{q}$. As associated primes consist of zerodivisors on the module $M_\mathfrak{p}$ we have depth $(M_\mathfrak{p})_\mathfrak{q} = 0$ hence depth $M_\mathfrak{q} = 0$. Hence $\mathfrak{q}$ cannot be in $\mathbf{X}_0$ according to our assumption. This means $\mathrm{ht}\,\mathfrak{q} = 0$, thus $\mathfrak{q} \in \mathrm{Ass}(R) \implies \mathfrak{q}R_\mathfrak{p} \in \mathrm{Ass}(R_\mathfrak{p})$ showing that every associated prime of $M_\mathfrak{p}$ is contained in an associated prime of $R_\mathfrak{p}$, as needed.

Our last equivalence is

$$\text{depth } M_\mathfrak{p} > 0 \text{ for all } \mathfrak{p} \in \mathbf{X}_0 \iff \mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \mathrm{ht}\,\mathfrak{p} \text{ for all } \mathfrak{p} \in \mathbf{X}_0.$$

So fix $\mathfrak{p} \in \mathbf{X}_0$, then we use Theorem 2.4.10 to observe

$$\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} + \text{depth } M_\mathfrak{p} = \text{depth } R_\mathfrak{p} \overset{R\,\mathrm{CM}}{=} \dim R_\mathfrak{p} = \mathrm{ht}\,\mathfrak{p}.$$

Therefore depth $M_\mathfrak{p} > 0 \iff \mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} = \mathrm{ht}\,\mathfrak{p} - \text{depth } M_\mathfrak{p} < \mathrm{ht}\,\mathfrak{p}$.

We now prove (ii). Since $\mathrm{H}^0_\mathfrak{m}(M)$ consists of all elements of M which are annihilated by a power of $\mathfrak{m}$ and depth $R = \dim R > 0$, it follows that $\mathrm{H}^0_\mathfrak{m}(M) \subseteq \tau_R(M) = \mathcal{K} \subseteq M$. Under these circumstances, $\mathcal{K} = \mathrm{H}^0_\mathfrak{m}(M)$ if and only if $\mathrm{Supp}(\mathcal{K}) \subseteq \{\mathfrak{m}\}$. This is because $\mathrm{H}^0_\mathfrak{m}(M)$ is the largest module supported only on $\mathfrak{m}$. Thus, since f is homogeneous, we get $\mathcal{K} = \mathrm{H}^0_\mathfrak{m}(M)$ if and only if $f_\mathfrak{p}$ is injective for all $\mathfrak{p} \in \mathrm{Proj}(R)$. The string of equivalences in the proof of (i) is still valid when replacing $\mathrm{Spec}(R)$ with $\mathrm{Proj}(R)$. As such, we obtain $\mathcal{K} = \mathrm{H}^0_\mathfrak{m}(M)$ if and only if $f_\mathfrak{p}$ is injective for all $\mathfrak{p} \in \mathrm{Proj}(R)$ if and only if $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \mathrm{ht}\,\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Proj}(R)$ with $\mathrm{ht}\,\mathfrak{p} > 0$. ∎

The next lemma is a main tool in our results which provides key insight into the specialization of powers of an ideal. While part (i) provides a resolution of the specialization of powers of an ideal, parts (ii) and (iii) are critical tools in providing degree bounds on $\mathcal{A}(I)$ via specialization.

Lemma 4.2.2 (The Specialization Lemma) *Let B be a Cohen-Macaulay positively graded ring of dimension d with B_0 local, and use $\mathfrak{m}$ to denote the unique maximal homogeneous ideal of B. Let J be a homogeneous B-ideal generated by forms of the same degree q. Suppose $Y_1, \ldots, Y_D$ is a homogeneous sequence in B which is weakly regular on B/J and on B. Let $\mathcal{N} = (Y_1, \ldots, Y_D)$, $R = B/\mathcal{N}$, and $I = JR$. For each positive integer k, let $\left(\mathbf{D}^{\mathbf{k}}_{\bullet}, \varphi^{\mathbf{k}}_{\bullet}\right)$ be a homogeneous finite free B-resolution of J^k where each $\mathbf{D}^{\mathbf{k}}_i$ is finitely generated.*

Suppose $\{K_i\}$ is a family of B-ideals so that

$$K_i \subseteq \sqrt{I\left(\varphi^{\mathbf{k}}_i\right)}$$

for all i and for all k.

(i) *If $\operatorname{ht} K_i R \geq i$ for all i, then $\mathbf{D}^{\mathbf{k}}_{\bullet} \otimes_B R$ is a homogeneous free R-resolution of $J^k \otimes_B R$ for each k.*

(ii) *If $\operatorname{ht} K_i R \geq \min\{i, d-1\}$ for all i, then $\mathbf{D}^{\mathbf{k}}_{\bullet} \otimes_B R$ is an approximate R-resolution of $J^k \otimes_B R$ for each k.*

(iii) *Let $\psi_k \colon J^k \otimes_B R \to I^k$ be the natural surjection. For each k, there is a homogeneous exact sequence of R-modules*

$$\mathcal{A}_k(J) \otimes_B R \to \mathcal{A}_k(I) \to (\ker \psi_k)(kq) \to 0.$$

Proof To show (i) we first observe $\mathbf{D}^{\mathbf{k}}_{\bullet} \otimes_B R$ is a finite complex of free R-modules with $\operatorname{H}_0\left(\mathbf{D}^{\mathbf{k}}_{\bullet} \otimes_B R\right) \cong J^k \otimes_B R$. Notice that $K_i \subset \sqrt{I(\varphi^{\mathbf{k}}_i)}$ implies

$$K_i \otimes_B R \cong K_i R \subset \sqrt{I(\varphi^{\mathbf{k}}_i) \otimes_B R} = \sqrt{I(\varphi^{\mathbf{k}}_i \otimes_B R)}.$$

Hence, for each i,

$$i \leq \operatorname{ht} K_i R \leq \operatorname{ht} \sqrt{I(\varphi^{\mathbf{k}}_i) \otimes_B R} = \operatorname{ht} I\left(\varphi^{\mathbf{k}}_i\right) \otimes_B R = \operatorname{grade} I\left(\varphi^{\mathbf{k}}_i \otimes_B R\right)$$

since R is Cohen-Macaulay. Thus, $\mathbf{D}^{\mathbf{k}}_{\bullet} \otimes_B R$ is a resolution by the Buchsbaum-Eisenbud criterion Theorem 2.2.3.

For (ii) we recall the notion of an approximate resolution from Definition 2.2.5. Fix $i_0 \geq 1$. Our approach will be to show that no prime $\mathfrak{p}$ of height less than $d - i_0$ is in the support of the R-module $\mathrm{H}_{i_0}\big(\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R\big)$, whence

$$\dim \mathrm{H}_{i_0}\big(\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R\big) = \dim R/\operatorname{ann}_R\big(\mathrm{H}_{i_0}\big(\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R\big)\big)$$

$$\leq \dim R - \dim V\big(\operatorname{ann}_R\big(\mathrm{H}_{i_0}\big(\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R\big)\big)\big) \leq d - (d - i_0) = i_0.$$

Thus let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht}\mathfrak{p} \leq d - i_0 - 1$. Since $K_i R \subset \sqrt{I(\varphi^{\mathbf{k}}_i \otimes_B R)}$ for all i, we also have

$$K_i R_\mathfrak{p} \subset \sqrt{I(\varphi^{\mathbf{k}}_i \otimes_B R)}_\mathfrak{p} \cong \sqrt{I(\varphi^{\mathbf{k}}_i \otimes_B R_\mathfrak{p})}$$

for all i. Now since $\operatorname{ht} K_i R \geq \min\{i, d-1\}$ when we localize at a prime of height $d - i_0 - 1 \leq d - 2$ we find $\operatorname{ht} K_i R_\mathfrak{p} \geq i$ for all i. Indeed if $\min\{i, d-1\} = i$, then there is nothing to show, while if $\min\{i, d-1\} = d-1$, then $K_i R_\mathfrak{p} \cong R_\mathfrak{p}$ is an ideal of infinite height trivially satisfying $\operatorname{ht} K_i R_\mathfrak{p} \geq i$. This allows us to use Corollary 2.2.4 once more to conclude $\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R_\mathfrak{p}$ is a resolution, and so its associated homology modules vanish. In particular we find $\mathrm{H}_{i_0}\big(\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R\big)_\mathfrak{p} = 0$ as desired.

To prove (iii) we first show ψ_1 is an isomorphism. To do this, note that $\ker \psi_1 = (\mathcal{N} \cap J)/\mathcal{N}J = \operatorname{Tor}_1^B(B/\mathcal{N}, B/J) = 0$ by Lemma 4.1.1 since $Y_1, \ldots, Y_D$ is weakly regular on B and on B/J. In particular,

$$\mathcal{S}_k(J) \otimes_B R \cong \mathcal{S}_k(J \otimes_B R) \cong \mathcal{S}_k(I)$$

for all k.

Thus, for each k, we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccc}
\mathcal{A}_k(J) \otimes_B R & \longrightarrow & \mathcal{S}_k(J) \otimes_B R & \longrightarrow & J^k \otimes_B R & \longrightarrow & 0 \\
\downarrow & & \downarrow{\wr} & & \downarrow{\psi_k} & & \\
0 \longrightarrow \mathcal{A}_k(I) & \longrightarrow & \mathcal{S}_k(I) & \longrightarrow & I^k & \longrightarrow & 0
\end{array}
$$

Therefore, by the Snake Lemma, we can see

$$\operatorname{Coker}\big(\mathcal{A}_k(J) \otimes_B R \to \mathcal{A}_k(I)\big) \cong \ker \psi_k,$$

hence the sequence

$$\mathcal{A}_k(J) \otimes_B R \to \mathcal{A}_k(I) \to \ker \psi_k \to 0$$

is exact.

Keeping in mind the gradings on the symmetric algebra and the Rees ring as in Definition 2.4.1 and Definition 2.4.2, the sequence is homogeneous provided that we use $(\ker \psi_k)(kq)$. ∎

We will use this lemma in the case of ideals of minors of generic matrices (or Pfaffian ideals of generic alternating matrices) where the family of ideals K_i will be a subcollection of the family of ideals $\sqrt{I_j(X)}$ (or $\mathrm{Pf}_{2j}(X)$), which will allow us to use the heights of the ideals $I_j(A)$ (or $\mathrm{Pf}_{2j}(A)$) to control $\mathcal{A}(I)$. This will be shown in Theorem 4.2.5; however, we must further supply some well-known technical lemmas.

Lemma 4.2.3 *Let R be a polynomial ring in d variables over the field K.*

(i) *Let X be an $m \times n$ generic matrix over R with $1 \le t = m \le n$, $B = R[X]$, and $J = I_m(X)$. Then $\max_k\{\mathrm{pd}_B J^k\} = m(n-m)$.*

(ii) *Let X be an $m \times n$ generic matrix over R with $1 \le t < m \le n$, $B = R[X]$, and $J = I_t(X)$. Then $\max_k\{\mathrm{pd}_B J^k\} = mn - 1$.*

(iii) *Let X be an $n \times n$ generic symmetric matrix over R, $B = R[X]$, $1 \le t < n$, and $J = I_t(X)$. Then $\max_k\{\mathrm{pd}_B J^k\} = \binom{n+1}{2} - 1$.*

(iv) *Let X be an $n \times n$ generic alternating matrix over R, $2 \le 2t = n-1$, $B = R[X]$, and $J = \mathrm{Pf}_{n-1}(X)$. Then $\max_k\{\mathrm{pd}_B J^k\} = n - 1$.*

(v) *Let X be an $n \times n$ generic alternating matrix over R, $B = R[X]$, $2 \le 2t < n-1$, and $J = \mathrm{Pf}_{2t}(X)$. Then $\max_k\{\mathrm{pd}_B J^k\} = \binom{n}{2} - 1$.*

Proof Since $B = R[X]$ is a polynomial ring over $K[X]$ and since the generators of J^k are elements of $K[X]$, $\mathrm{pd}_B J^k = \mathrm{pd}_{K[X]} J^k$, so we may reduce to the case that $B = K[X]$. We remark here that Burch's inequality requires K to be an infinite field,

but we may reduce to this case by replacing B by $B[Y]_{\mathfrak{m}B[Y]}$, where $\mathfrak{m} = (X)$ and Y is an indeterminate.

To prove (i) and (iv), we note that $\mathcal{R}(J)$ is Cohen-Macaulay (see [20, Theorem 3.5] for (i) and [27, Theorem 2.2] for (iv)). Hence, $\ell(J) = \max_k\{\mathrm{pd}_B B/J^k\}$ by Proposition 2.4.11. In particular, $\max_k\{\mathrm{pd}_B J^k\} = \ell(J) - 1$. In (i) we refer to Proposition 3.2.2 to see we have $\ell(J) = m\,(n-m) + 1$. In (iv) we use Remark 3.2.3 to see $\ell(J) = \mu_B(J) = n$.

For (ii), (iii), and (v), by Hilbert's Syzygy Theorem, we have $\mathrm{pd}_B B/J^k \leq \dim B$ for all k; thus, $\max_k\{\mathrm{pd}_B J^k\} \leq \dim B - 1$. On the other hand, by the inequality in Proposition 2.4.11 we have

$$\ell(J) - 1 \leq \max_k\{\mathrm{pd}_B J^k\} \implies \ell(J) \leq \mathrm{pd}_B B/J^k \leq \dim B.$$

Thus, we are done once we have shown $\ell(J) = \dim B$. This is indeed the case as is shown in Proposition 3.2.2 and Remark 3.2.3 where the analytic spread in these cases simply coincides with the number of distinct variables. Since this number of variables is the same as the dimension of B we are done. ∎

In the above proof, we use the fact that $\mathcal{R}(J)$ is Cohen-Macaulay only in (i) and (iv), despite the fact that $\mathcal{R}(J)$ is known to be Cohen-Macaulay in (ii) and (v) when K has characteristic zero or sufficiently large characteristic (see [6, Theorem 3.7] and [3, Theorem 3.4]). We are unaware of results concerning whether $\mathcal{R}(J)$ is Cohen-Macaulay in (iii) for $t < n-1$. Despite these results, we still obtain $\max_k\{\mathrm{pd}_B J^k\}$ in cases (ii), (iii), and (v) without using that $\mathcal{R}(J)$ is Cohen-Macaulay. Thus, we note that, unless otherwise stated, the rest of the results in this thesis only make use of the fact that $\mathcal{R}(J)$ is Cohen-Macaulay in cases (i) and (iv).

The next lemma allows us to work with localizations of our matrix X. In particular we see how creating unit block submatrices in X locally at some prime $\mathfrak{p}$ allows us to reduce to a smaller matrix which can also be assumed generic. The argument use here is an adaptation of a similar argument found in [10, Proposition 2.4].

Lemma 4.2.4 *Let B be a Noetherian ring, $1 \leq j \leq t \leq m \leq n$,*

$$X = \left[\begin{array}{c|c} U & V \\ \hline W & Z \end{array} \right]$$

be an $m \times n$ generic matrix over B with U a $j \times j$ matrix, and $Y = (Y_{rs})$ be an $(m-j) \times (n-j)$ generic matrix over B. Write $\Delta = \det U$ and $C = B[U, V, W, \Delta^{-1}]$. There is an C-algebra isomorphism

$$\varphi \colon C[Z] \to C[Y]$$

given by

$$Z_{rs} \mapsto \left(Y + WU^{-1}V \right)_{rs}.$$

Moreover, the extension of $I_t(X) C[Z]$ along φ is $I_{t-j}(Y) C[Y]$.

Proof First notice $I_t(X) C[Z] = I_{t-j}\left(\widetilde{X} \right) C[Z]$, where $\widetilde{X} = Z - WU^{-1}V$ is the Schur complement X/U defined in [Definition 2.1.4](#) and is a $(m-j) \times (n-j)$ matrix with entries in $C[Z]$. Therefore, the image of $I_t(X) C[Z]$ via φ is $I_{t-j}(Y) C[Y]$.

To see that φ is an isomorphism simply consider the C-algebra homomorphism $\psi \colon C[Y] \to C[Z]$ defined by $Y_{rs} \mapsto (Z - WU^{-1}V)_{rs}$. Computing the composition of these maps element-wise reveals that φ and ψ are inverse C-algebra homomorphisms, as required. ∎

We note that the above proof also holds assuming X and Y are generic symmetric matrices. In this case, U and Z are symmetric and $W = V^T$; hence, the maps φ and ψ are well-defined and send corresponding entries of $\widetilde{X}$ and Y to each other. The above proof also holds assuming X and Y are generic alternating matrices and assuming we replace ideals of minors with ideals of Pfaffians, provided that we assume t and j are even. Indeed, in this case, U and Z are alternating and $W = -V^T$; hence, the maps φ and ψ are well-defined and send corresponding entries of $\widetilde{X}$ and Y to each other.

Lemma 4.2.5 *Let K be a field, and suppose R is a polynomial ring in d variables over K.*

(i) *Let X be an $m \times n$ generic matrix over R with $1 \leq t = m \leq n$, $B = R[X]$, and $J = I_m(X)$. Further, for each k, let $\left(\mathbf{D}^\mathbf{k}_\bullet, \varphi^\mathbf{k}_\bullet\right)$ be a finite free B-resolution of J^k where each $\mathbf{D}^\mathbf{k}_i$ is finitely generated. Then for each $1 \leq j \leq m - 1$, and for all k*

$$i \geq (m - j)(n - m) + 1 \implies \sqrt{I_j(X)} \subseteq \sqrt{I(\varphi^\mathbf{k}_i)}.$$

(ii) *Let X be an $m \times n$ generic matrix over R with $1 \leq t < m \leq n$, $B = R[X]$, and $J = I_t(X)$. Further, for each k, let $\left(\mathbf{D}^\mathbf{k}_\bullet, \varphi^\mathbf{k}_\bullet\right)$ be a finite free B-resolution of J^k where each $\mathbf{D}^\mathbf{k}_i$ is finitely generated. Then for each $1 \leq j \leq t - 1$, and for all k*

$$i \geq (m - j)(n - j) \implies \sqrt{I_j(X)} \subseteq \sqrt{I(\varphi^\mathbf{k}_i)}.$$

(iii) *Let X be an $n \times n$ generic symmetric matrix over R, $B = R[X]$, $1 \leq t \leq n$, and $J = I_t(X)$. Further, for each k, let $\left(\mathbf{D}^\mathbf{k}_\bullet, \varphi^\mathbf{k}_\bullet\right)$ be a finite free B-resolution of J^k where each $\mathbf{D}^\mathbf{k}_i$ is finitely generated. Then for each $1 \leq j \leq t - 1$, and for all k*

$$i \geq \binom{n - j + 1}{2} \implies \sqrt{I_j(X)} \subseteq \sqrt{I(\varphi^\mathbf{k}_i)}.$$

(iv) *Let X be an $n \times n$ generic alternating matrix over R, $2 \leq 2t = n - 1$, $B = R[X]$, and $J = \mathrm{Pf}_{n-1}(X)$. Further, for each k, let $\left(\mathbf{D}^\mathbf{k}_\bullet, \varphi^\mathbf{k}_\bullet\right)$ be a finite free B-resolution of J^k where each $\mathbf{D}^\mathbf{k}_i$ is finitely generated. Then for each $2 \leq 2j \leq n - 3$, and for all k*

$$i \geq n - 2j \implies \sqrt{\mathrm{Pf}_{2j}(X)} \subseteq \sqrt{I(\varphi^\mathbf{k}_i)}.$$

(v) *Let X be an $n \times n$ generic alternating matrix over R, $B = R[X]$, $2 \leq 2t < n - 1$, and $J = \mathrm{Pf}_{2t}(X)$. Further, for each k, let $\left(\mathbf{D}^\mathbf{k}_\bullet, \varphi^\mathbf{k}_\bullet\right)$ be a finite free B-resolution of J^k where each $\mathbf{D}^\mathbf{k}_i$ is finitely generated. Then for each $1 \leq j \leq t - 1$, and for all k*

$$i \geq \binom{n - 2j}{2} \implies \sqrt{\mathrm{Pf}_{2j}(X)} \subseteq \sqrt{I(\varphi^\mathbf{k}_i)}.$$

Proof We prove parts (i) and (ii). Parts (iii)-(v) are similar.

Fix $1 \leq j \leq t - 1$. We will show that if $\mathfrak{p} \in \mathrm{Spec}(B) \setminus V(I_j(X))$, then $\mathfrak{p} \in \mathrm{Spec}(B) \setminus V\big(I\big(\varphi^{\mathbf{k}}_i\big)\big)$ for all k when i satisfies the inequality in the hypothesis. To do this we notice that if a prime does not contain an ideal $I_j(X)$, then X has a unit $j \times j$ minor over the ring $B_\mathfrak{p}$. We then use row and column operations to change the shape of the matrix and use this new matrix to bound the projective dimension of $J^k_\mathfrak{p}$. If the projective dimension is less than i, then $\varphi^{\mathbf{k}}_{i\,\mathfrak{p}}$ is a differential in the resolution of $J^k_\mathfrak{p}$ past the projective dimension, so $I\big(\varphi^{\mathbf{k}}_i\big)_\mathfrak{p} = B_\mathfrak{p}$ and we conclude $\mathfrak{p}$ does not contain $I\big(\varphi^{\mathbf{k}}_i\big)$. The contrapositive of this implication gives the desired containment of radical ideals.

So let $\mathfrak{p} \in \mathrm{Spec}(B) \setminus V(I_j(X))$. Our goal is to show that $\mathrm{pd}_{B_\mathfrak{p}} J^k_\mathfrak{p} < i$ whenever i satisfies the given inequality, and for all k. We use Lemma 4.2.4 and we recall, in this setting, X is written in block matrix form as

$$X = \left[\begin{array}{c|c} U & V \\ \hline W & Z \end{array} \right]$$

where we may assume U is a $j \times j$ matrix whose determinant we shall assume is a unit when viewed as an element in $B_\mathfrak{p}$. We also recall that, with these blocks specified, we define $\Delta = \det U$ and then define $C = B[U, V, W, \Delta^{-1}]$. We see that, since Δ is not in $\mathfrak{p}$, $B_\mathfrak{p}$ is a localization of $C[Z]$. Therefore there is a $(n - j) \times (m - j)$ generic matrix Y with

$$\mathrm{pd}_{B_\mathfrak{p}} J^k_\mathfrak{p} \leq \mathrm{pd}_{C[Z]} I_t(X)^k = \mathrm{pd}_{C[Y]} I_{t-j}(Y)^k = \mathrm{pd}_{R[U, \Delta^{-1}, Y]} I_{t-j}(Y)^k$$

$$= \mathrm{pd}_{R[U, Y]_\Delta} I_{t-j}(Y)^k \leq \mathrm{pd}_{R[U, Y]} I_{t-j}(Y)^k = \mathrm{pd}_{K[Y]} I_{t-j}(Y)^k$$

for all k. The above string of equalities and inequalities follows from two main facts. First, if A is a Noetherian ring and $\mathfrak{a}$ is an A-ideal, then $\mathrm{pd}_{A_\mathfrak{p}} \mathfrak{a}_\mathfrak{p} \leq \mathrm{pd}_A \mathfrak{a}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Second, if A is a Noetherian ring and $\mathfrak{a}$ is an $A[x]$-ideal whose generators are elements of A, then $\mathrm{pd}_{A[x]} \mathfrak{a} = \mathrm{pd}_A(\mathfrak{a} \cap A)$. Now apply Lemma 4.2.3 which shows in (i) that

$$\mathrm{pd}_{K[Y]} I_{m-j}(Y)^k \leq (m - j)(n - m) < (m - j)(n - m) + 1.$$

Thus, if $i \geq (m-j)(n-m)+1$, then the combination of these inequalities gives us the desired containment of radical ideals. Applying the remaining parts of Lemma 4.2.3 together with reductions analogous to Lemma 4.2.4 for symmetric and alternating generic matrices provides the required bound on $\mathrm{pd}_{K[Y]} I_{t-j}(Y)^k$, and hence, on $\mathrm{pd}_{B_{\mathfrak{p}}} J_{\mathfrak{p}}^k$. ∎

4.3 Specialization of the Rees Algebra

The next proposition addresses cases where the Rees ring specializes. Much of the proposition is a direct consequence of [20, Theorem 1.1], and (i) is even given in [20, Thorem 3.5]. However, our method of proof, at least concerning cases (ii), (iii), and (v), do not use that $\mathcal{R}(J)$ is Cohen-Macaualy, which is a hypothesis of [20, Theorem 1.1]. We will also make use of this result to guarantee that certain determinantal and Pfaffian ideals are of linear type or of fiber type.

Proposition 4.3.1 *Let K be a field. Suppose R is a polynomial ring in d variables over K.*

(i) *Let A be an $m \times n$ matrix over R with $1 \leq m \leq n$, $I = I_m(A)$ be of generic height, X be a generic $m \times n$ matrix over R, $B = R[X]$, and $J = I_m(X)$. If*

$$\mathrm{ht}\, I_j(A) \geq (m-j+1)(n-m)+1 \ \ for\ all\ \ 1 \leq j \leq m-1,$$

then $\mathcal{R}(J) \otimes_B R \cong \mathcal{R}(I)$ via the natural map. Moreover, $\mathcal{R}(I)$ is Cohen-Macaulay.

(ii) *Let A be an $m \times n$ matrix over R, $1 \leq t < m \leq n$, $I = I_t(A)$ be of generic height, X be a generic $m \times n$ matrix over R, $B = R[X]$, and $J = I_t(X)$. If*

$$\mathrm{ht}\, I_j(A) \geq (m-j+1)(n-j+1) \ \ for\ all\ \ 1 \leq j \leq t-1,$$

then $\mathcal{R}(J) \otimes_B R \cong \mathcal{R}(I)$ via the natural map.

If, in addition, $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K > \min\{t, m-t\}$, then $\mathcal{R}(I)$ is Cohen-Macaulay.

(iii) *Let A be an $n \times n$ symmetric matrix over R, $1 \le t \le n$, $I = I_t(A)$ be of generic symmetric height, X be an $n \times n$ generic symmetric matrix over R, $B = R[X]$, and $J = I_t(X)$. If*

$$\operatorname{ht} I_j(A) \ge \binom{n - j + 2}{2} \quad \text{for all } 1 \le j \le t - 1,$$

then $\mathcal{R}(J) \otimes_B R \cong \mathcal{R}(I)$ via the natural map.

(iv) *Let A be an $n \times n$ alternating matrix over R with n odd, $3 \le n$, $I = \operatorname{Pf}_{n-1}(A)$ be of generic alternating height, X be an $n \times n$ generic alternating matrix over R, $B = R[X]$, and $J = \operatorname{Pf}_{n-1}(X)$. If*

$$\operatorname{ht} \operatorname{Pf}_{2j}(A) \ge n - 2j + 2 \quad \text{for all } 2 \le 2j \le n - 3,$$

then $\mathcal{R}(J) \otimes_B R \cong \mathcal{R}(I)$ via the natural map. Moreover, $\mathcal{R}(I)$ is Cohen-Macaulay.

(v) *Let A be an $n \times n$ alternating matrix over R, $2 \le 2t < n - 1$, $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height, X be an $n \times n$ generic alternating matrix over R, $B = R[X]$, and $J = \operatorname{Pf}_{2t}(X)$. If*

$$\operatorname{ht} \operatorname{Pf}_{2j}(A) \ge \binom{n - 2j + 2}{2} \quad \text{for all } 2 \le 2j \le 2t - 2,$$

then $\mathcal{R}(J) \otimes_B R \cong \mathcal{R}(I)$ via the natural map.

If, in addition, $\operatorname{char} K = 0$ or $\operatorname{char} K > \min\{2t, n - 2t\}$, then $\mathcal{R}(I)$ is Cohen-Macaulay.

Proof Let $\mathcal{N}$ be the B-ideal generated by the entries of $X - A$, and give R the B-algebra structure induced by the isomorphism $R \cong B/\mathcal{N}$. Let $\psi_k \colon J^k \otimes_B R \to I^k$ be the natural surjection. Then $\bigoplus_{k=0}^{\infty} \psi_k \colon \mathcal{R}(J) \otimes_B R \to \mathcal{R}(I)$ is the natural surjection. Hence, to show the specialization of the Rees ring, it suffices to prove that $\ker \psi_k = 0$ for all k.

We see that $\ker \psi_k = \tau_R(J^k \otimes_B R)$. Indeed, since $JR = (J + \mathcal{N})/\mathcal{N} = I \ne 0$, J is not contained in $\mathcal{N}$, hence

$$\left(J^k \otimes_B R\right) \otimes_R \operatorname{Quot}(R) \cong J^k \otimes_B R \otimes_R \frac{B_{\mathcal{N}}}{\mathcal{N} B_{\mathcal{N}}} \cong J^k \otimes_B \frac{B_{\mathcal{N}}}{\mathcal{N} B_{\mathcal{N}}} \cong \frac{B_{\mathcal{N}}}{\mathcal{N} B_{\mathcal{N}}} \cong \operatorname{Quot}(R).$$

In addition, $I^k \otimes_R \mathrm{Quot}(R) \cong \mathrm{Quot}(R)$. Hence,

$$\psi_k \otimes_B R \colon \mathrm{Quot}(R) \to \mathrm{Quot}(R)$$

is an isomorphism, whence $\ker \psi_k \otimes_B \mathrm{Quot}(R) = 0$ implies $\ker \psi_k$ is a torsion R-module. To see the other containment we notice that the R-torsion of $J^k \otimes_B R$ must be contained in $\ker \psi_k$ since I^k is R-torsion-free.

Therefore, by Lemma 4.2.1.i, it suffices to show that $\mathrm{pd}_{R_{\mathfrak{p}}}\bigl(J^k \otimes_B R\bigr)_{\mathfrak{p}} < \mathrm{ht}\,\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}\,\mathfrak{p} > 0$. This, in turn, will follow once we show the differentials in the resolutions of $J^k \otimes R$ have height greater than their homological index. Indeed, if ∂_i is the ith differential in a resolution for $J^k \otimes_B R$ and $\mathrm{ht}\,I(\partial_i) > i$, then $I(\partial_i)_{\mathfrak{p}} = R_{\mathfrak{p}}$ when $i = \mathrm{ht}\,\mathfrak{p}$. Hence the projective dimension of $\bigl(J^k \otimes_B R\bigr)_{\mathfrak{p}}$ is less than $\mathrm{ht}\,\mathfrak{p}$.

In order to show this, we first prove that resolutions of J^k specialize to resolutions of $J^k \otimes_B R$. Specifically, for each k, let $\bigl(\mathbf{D}^{\mathbf{k}}_{\bullet}, \varphi^{\mathbf{k}}_{\bullet}\bigr)$ be a finite free B-resolution of J^k where each $\mathbf{D}^{\mathbf{k}}_i$ is finitely generated. We prove that $\mathbf{D}^{\mathbf{k}}_{\bullet} \otimes_B R$ is a free R-resolution. By Lemma 4.2.2.i, it suffices to find a family $\{K_i\}$ of B-ideals with $\mathrm{ht}\,K_i R \geq i$ for each i and with $K_i \subseteq \sqrt{I(\varphi^{\mathbf{k}}_i)}$ for each i and for all k. We handle all cases simultaneously by establishing the following notation:

(i) Let $I_j = I_j(X)$ and $\sigma(j) = (m - j)(n - m) + 1$.

(ii) Let $I_j = I_j(X)$ and $\sigma(j) = (m - j)(n - j)$.

(iii) Let $I_j = I_j(X)$ and $\sigma(j) = \binom{n - j + 1}{2}$.

(iv) Let $I_j = \mathrm{Pf}_{2j}(X)$ and $\sigma(j) = n - 2j$.

(v) Let $I_j = \mathrm{Pf}_{2j}(X)$ and $\sigma(j) = \binom{n - 2j}{2}$.

We now define the members of the family $\{K_i\}$ and we notice that we need members of this family for $i = 1$ up to the maximum possible projective dimension for the powers J^k. We need to divide this range of integers into two portions because we intend to make use of Corollary 2.2.4 where, extending our resolution of J^k to be

one of B/J^k, we have $\sqrt{I(\varphi^{\mathbf{k}}_{i-1})} = \sqrt{I(\varphi^{\mathbf{k}}_i)}$ for $1 \leq 1 \leq \operatorname{ht} I - 1$. This means we must choose the same radical ideal for K_i in this range and, to obtain a resolution after applying the tensor product, this ideal should have height at least $\operatorname{ht} I - 1$. For $i \geq \operatorname{ht} I$ we select K_i in order to meet the criteria of Theorem 4.2.5.

Now for each $1 \leq i \leq \operatorname{ht} I - 1$, we set $K_i = \sqrt{J}$. On the other hand, when $\operatorname{ht} I \leq i \leq \max_k\{\operatorname{pd} J^k\}$, we make use of the fact that σ is strictly decreasing and let j_0 be the smallest j satisfying $i \geq \sigma(j)$. We recall that when this inequality is satisfied Theorem 4.2.5 implies $\sqrt{I_{j_0}} \subset \sqrt{I(\varphi^{\mathbf{k}}_i)}$, so we set $K_i = \sqrt{I_{j_0}}$.

We now check that our choices for K_i indeed give us values of j_0 in the range $1 \leq j_0 \leq t - 1$ as is required for this result. Since I is of generic (symmetric, alternating) height, $\sigma(t - 1) = \operatorname{ht} I \leq i$. For example, in (i), we have $t = m \leq n$, hence

$$\operatorname{ht} I = (m - t + 1)(n - t + 1) = n - m + 1 = \sigma(m - 1).$$

One can check $\sigma(t - 1)$ matches the height of I similarly for each case. Thus, since $j_0 = \min\{j \mid \sigma(j) \leq i\}$ and since $\operatorname{ht} I \leq i$ in this range, we have $j_0 \leq t - 1$. Moreover, by Lemma 4.2.3, we see that $\sigma(0) = \max_k\{\operatorname{pd} J^k\} + 1 > i$. Checking this again for case (i) we see

$$\sigma(0) = m(n - m) + 1 = \max_k\{\operatorname{pd}_B J^k\} + 1.$$

Since $j = 0$ does not satisfy $\sigma(j) \leq i$ we can conclude that $j_0 \geq 1$. Thus, for every j_0, we indeed have $1 \leq j_0 \leq t - 1$.

Next, we wish to show the heights of our K_i are bounded below by i. We first fix i with $1 \leq i \leq \operatorname{ht} I - 1$. Note that we can extend $(\mathbf{D}^{\mathbf{k}}_{\bullet}, \varphi^{\mathbf{k}}_{\bullet})$ to a resolution of B/J^k by taking $0 \to \mathbf{D}^{\mathbf{k}}_{\bullet} \overset{\varphi^{\mathbf{k}}_0}{\to} B$. Then, since $\operatorname{ann}(B/J^k) = J^k \neq 0$ and B is Cohen-Macaulay, we apply Corollary 2.2.4 to see $\sqrt{J^k} = \sqrt{I(\varphi^{\mathbf{k}}_0)} = \cdots = \sqrt{I(\varphi^{\mathbf{k}}_{\operatorname{ht} J^k - 1})}$. Since I is of generic height, $\operatorname{ht} J^k = \operatorname{ht} J = \operatorname{ht} I$. In particular, we have $K_i = \sqrt{J} = \sqrt{J^k} \subseteq \sqrt{I(\varphi^{\mathbf{k}}_i)}$, and so

$$\operatorname{ht} K_i R = \operatorname{ht} I > \operatorname{ht} I - 1 \geq i.$$

For the remaining range we fix i with $\operatorname{ht} I \leq i \leq \max_k\{\operatorname{pd} J^k\}$. Recall that j_0 satisfies $\sigma(j_0 - 1) > i \geq \sigma(j_0)$ and $1 \leq j_0 \leq t - 1$. Since $i \geq \sigma(j_0)$, by The-

orem 4.2.5, we have $K_i = \sqrt{I_{j_0}} \subseteq \sqrt{I(\varphi^{\mathbf{k}}_i)}$. Notice that the height bounds assumed for each case exactly coincide with $\sigma(j-1)$. For example, in case (i), we see $\sigma(j) = (m-j)(n-m) + 1$, hence $\sigma(j-1) = (m-j+1)(n-m) + 1$. Notice also, in each case, the specialized ideal $I_{j_0} R$ is isomorphic to $I_{j_0}(A)$ for ordinary and symmetric matrices or $\mathrm{Pf}_{2j_0}(A)$ for alternating matrices. Putting these facts together yields

$$\mathrm{ht}\, K_i R = \mathrm{ht}\, I_{j_0} R \geq \sigma(j_0 - 1) > i.$$

Therefore, $\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R$ is a free R-resolution of $J^k \otimes_B R$. To finish the proof of specialization, we note that we have actually proven more than $\mathbf{D}^{\mathbf{k}}_\bullet \otimes_B R$ being a free R-resolution of $J^k \otimes_B R$. Indeed, we only needed $\mathrm{ht}\, K_i R \geq i$ for all i, yet we proved a strict inequality for all i. Hence, for any positive integer i, we get $\mathrm{ht}\, I(\varphi^{\mathbf{k}}_i \otimes_B R) > i$. As noted above, this is enough to obtain $\mathrm{pd}_{R_{\mathfrak{p}}}(J^k \otimes_B R)_{\mathfrak{p}} < \mathrm{ht}\, \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}\, \mathfrak{p} > 0$, hence Lemma 4.2.1 shows $\ker \psi_k = 0$ for all k. Hence, $\mathcal{R}(I) \cong \mathcal{R}(J) \otimes_B R$.

All that remains to prove are the statements about $\mathcal{R}(I)$ being Cohen-Macaulay. By [20, Theorem 1.1], $\mathcal{R}(J) \otimes_B R$ is Cohen-Macaulay provided that $\mathcal{R}(J)$ is Cohen-Macaulay. For (i), $\mathcal{R}(J)$ is Cohen-Macaulay [20, Theorem 3.5]. For (ii), $\mathcal{R}(J)$ is Cohen-Macaulay if $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K > \min\{t, m-t\}$ by [6, Theorem 3.7]. For (iv), $\mathcal{R}(J)$ is Cohen-Macaulay by [29, Theorem 2.2]. For (v), $\mathcal{R}(J)$ is Cohen-Macaulay if $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K > \min\{2t, n-2t\}$ by [3, Theorem 3.4]. ∎

Using the characterization of G_∞ from Proposition 3.1.2, one can show that I satisfies G_∞ if and only if the conditions in (ii)-(v) are met or if $n \leq m+1$ and the conditions of (i) are met. However, the conditions in (i) are weaker than G_∞ provided that $n > m+1$.

Based on a number of results for the generic case and combined with the above proposition, we obtain the following results.

Corollary 4.3.2 *Let K be a field. Suppose R is a polynomial ring in d variables over K.*

(i) *Suppose A is an $m \times (m+1)$ matrix with $1 \leq m$, $I = I_m(A)$ is of generic height, and I satisfies G_∞ (i.e., $\operatorname{ht} I_j(A) \geq m - j + 2$ for all $1 \leq j \leq m - 1$). Then I is of linear type.*

(ii) *Suppose A is an $n \times n$ matrix with $2 \leq n$, $I = I_{n-1}(A)$ is of generic height, and I satisfies G_∞ (i.e., $\operatorname{ht} I_j(A) \geq (n - j + 1)^2$ for all $1 \leq j \leq n - 2$). Then I is of linear type.*

(iii) *Suppose A is an $n \times n$ symmetric matrix with $2 \leq n$, $I = I_{n-1}(A)$ is of generic height, and I satisfies G_∞ (i.e., $\operatorname{ht} I_j(A) \geq \binom{n-j+2}{2}$ for all $1 \leq j \leq n-2$). Then I is of linear type.*

(iv) *Suppose A is an $n \times n$ alternating matrix with n odd and $3 \leq n$, $I = \operatorname{Pf}_{n-1}(A)$ is of generic height, and I satisfies G_∞ (i.e., $\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq n - 2j + 2$ for all $2 \leq 2j \leq n - 3$). Then I is of linear type.*

(v) *Suppose $\operatorname{char} K \neq 2$, A is an $n \times n$ alternating matrix with n even and $4 \leq n$, $I = \operatorname{Pf}_{n-2}(A)$ is of generic height, and I satisfies G_∞ (i.e., $\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \binom{n-2j+2}{2}$ for all $2 \leq 2j \leq n - 4$). Then I is of linear type.*

(vi) *Suppose A is an $m \times n$ matrix with $1 \leq m \leq n$, the entries of A are homogeneous of the same degree, $I = I_m(A)$ is of generic height, and $\operatorname{ht} I_j(A) \geq (m - j + 1)(n - m) + 1$ for all $1 \leq j \leq m - 1$. Then I is of fiber type.*

(vii) *Suppose $\operatorname{char} K = 0$, A is a $3 \times n$ matrix with $n \geq 3$, the entries of A are homogeneous of the same degree, $I = I_2(A)$ is of generic height, and I satisfies G_∞ (i.e., $\operatorname{ht} I_1(A) \geq 3n$). Then I is of fiber type.*

Proof By Theorem 4.3.1, we have $\mathcal{R}(J) \otimes_B R \cong \mathcal{R}(I)$ via the natural map for all cases. In particular, by Lemma 4.2.2.iii, the sequence

$$\mathcal{A}_k(J) \otimes_B R \to \mathcal{A}_k(I) \to 0$$

is exact and homogeneous for each k.

If J is of linear type, then $\mathcal{A}_k(J) = 0$ for all k. Thus, $\mathcal{A}_k(I) = 0$ for all k. Hence I is of linear type.

By the following references, we know that J is of linear type. For (i), [28, Theorem 1.1]. For (ii), [29, Theorem 2.6]. For (iii), [34, Proposition 2.10]. For (iv), [27, Theorem 2.2]. For (v), when 2 is invertible, [4, Proposition 2.1].

To prove the fiber type results, we note that being generated in degree 0 is preserved under homogeneous surjections. Therefore, I is of fiber type if J is of fiber type.

For (vi), it is known that J is of fiber type by [8, Theorem 3.7]. For (vii), when char $K = 0$, it is known that J is of fiber type from [25, Corollary 7.3]. ∎

5. DEGREE BOUNDS FOR THE DEFINING EQUATIONS OF THE REES ALGEBRA

In this chapter, we are concerned with bounding the generation and concentration degrees of $\mathcal{A}_k(I)$ for a determinantal or Pfaffian ideal I of generic (symmetric, alternating) height. Our main tool is the homogeneous exact sequence of Lemma 4.2.2.iii and the results of [37, Corollary 5.4, and Theorem 5.6] which relate degree shifts in approximate resolutions to bounds on generation and concentration degrees. In particular, we use [37, Corollary 5.4, and Theorem 5.6] to obtain degree bounds for the kernel of the natural surjection $\psi_k \colon J^k \otimes_B R \to I^k$ using the approximate resolution in Lemma 4.2.2.ii.

We introduce the following data which will be referenced throughout this section.

Data 5.0.1 *Let R be a standard graded polynomial ring in d variables over the field K, and suppose $1 \le t \le m \le n$.*

(a) ***Ordinary matrix:*** *Let A be an $m \times n$ matrix over R with all entries homogeneous of the same degree δ and $I = I_t(A)$ be of generic height. Let X be a generic $m \times n$ matrix over R, $B = R[X]$ where $\deg X_{ij} = 1$ for all X_{ij}, and $J = I_t(X)$. Further, for each k, let $\left(\mathbf{D}^{\mathbf{k}}_{\bullet}, \varphi^{\mathbf{k}}_{\bullet}\right)$ be the minimal homogeneous free B-resolution of $J^k(tk)$.*

(b) ***Symmetric matrix:*** *Let A be an $n \times n$ symmetric matrix over R with all entries homogeneous of the same degree δ and $I = I_t(A)$ be of generic symmetric height. Let X be a generic symmetric $n \times n$ matrix over R, $B = R[X]$ where $\deg X_{ij} = 1$ for all X_{ij}, and $J = I_t(X)$. Further, for each k, let $\left(\mathbf{D}^{\mathbf{k}}_{\bullet}, \varphi^{\mathbf{k}}_{\bullet}\right)$ be the minimal homogeneous free B-resolution of $J^k(tk)$.*

(c) **Alternating matrix**: *Let A be an $n \times n$ alternating matrix over R with all entries homogeneous of the same degree δ and $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height. Let X be a generic alternating $n \times n$ matrix over R, $B = R[X]$ where $\deg X_{ij} = 1$ for all X_{ij}, and $J = \operatorname{Pf}_{2t}(X)$. Further, for each k, let $(\mathbf{D^k_\bullet}, \varphi^{\mathbf{k}}_\bullet)$ be the minimal homogeneous free B-resolution of $J^k(tk)$.*

5.1 Bounds from Approximate Resolutions

The following result is the main argument used to obtain degree bounds on $\mathcal{A}(I)$ from degree bounds on $\mathcal{A}(J)$ and from free resolutions of J^k.

Proposition 5.1.1 *Suppose R is a standard graded polynomial ring in $d > 0$ variables over the field K, and let $A_1, \ldots, A_D$ be a sequence in R where each A_i is homogeneous of the same degree δ. Suppose that $B = R[X_1, \ldots, X_D]$ where $\deg X_i = \delta$ for each i. Let J be a homogeneous B-ideal generated by forms of the same degree q. Define $Y_i = X_i - A_i$, assume that $Y_1, \ldots, Y_D$ is regular on B and B/J, and let $\mathcal{N} = (Y_1, \ldots, Y_D)$.*

We give R the B-algebra structure induced by the homogeneous isomorphism $R \cong B/\mathcal{N}$. Let $I = JR$. For each k, let $(\mathbf{E^k_\bullet}, \tau^{\mathbf{k}}_\bullet)$ be a homogeneous finite free B-resolution of $J^k(kq)$ where each $\mathbf{E^k}_i$ is finitely generated. Suppose that $\{K_i\}$ is a family of B-ideals so that $K_i \subseteq \sqrt{I(\varphi^{\mathbf{k}}_i)}$ for all i and for all k. If

$$\operatorname{ht} K_i R \geq \min\{i+1, d\} \ \text{ for all } i,$$

then

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{b_0(\mathcal{A}_k(J)), b_0(\mathbf{E^k}_{d-1}) - d + 1\right\}, \ \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\operatorname{topdeg}(\mathcal{A}_k(J)), b_0(\mathbf{E^k}_d) - d\right\}.$$

Proof Let $\psi_k \colon J^k \otimes_B R \to I^k$ be the natural surjection.

By Lemma 4.2.2.iii, the homogeneous sequence of R-modules

$$\mathcal{A}_k(J) \otimes_B R \to \mathcal{A}_k(I) \to (\ker \psi_k)(kq) \to 0$$

is exact. Therefore, we have

$$b_0(\mathcal{A}_k(I)) \le \max\left\{b_0(\mathcal{A}_k(J)), b_0((\ker \psi_k)(kq))\right\}, \text{ and}$$

$$\mathrm{topdeg}(\mathcal{A}_k(I)) \le \max\left\{\mathrm{topdeg}(\mathcal{A}_k(J)), \mathrm{topdeg}((\ker \psi_k)(kq))\right\}.$$

It remains to show

$$b_0((\ker \psi_k)(kq)) \le b_0\left(\mathbf{E^k}_{d-1}\right) - d + 1$$

and

$$\mathrm{topdeg}((\ker \psi_k)(kq)) \le b_0\left(\mathbf{E^k}_d\right) - d.$$

We begin by showing that $\ker \psi_k = \mathrm{H}^0_{\mathfrak{m}}\left(J^k \otimes_B R\right)$ for each k. As in the proof of Theorem 4.3.1, $\ker \psi_k$ is the R-torsion of $J^k \otimes_B R$. By Lemma 4.2.1, we have $\ker \psi_k = \mathrm{H}^0_{\mathfrak{m}}\left(J^k \otimes_B R\right)$ if and only if $\mathrm{pd}_{R_{\mathfrak{p}}}\left(J^k \otimes_B R\right)_{\mathfrak{p}} < \mathrm{ht}\,\mathfrak{p}$ for all primes $\mathfrak{p} \in \mathrm{Proj}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_+ \not\subset \mathfrak{p} \text{ and } \mathfrak{p} \text{ is homogeneous}\}$ with $\mathrm{ht}\,\mathfrak{p} > 0$.

By our assumption that $\mathrm{ht}\,K_i R \ge \min\{i+1, d\}$ for all i and since $\mathrm{ht}\,K_i R \le \mathrm{ht}\,I\left(\tau^{\mathbf{k}}_i \otimes_B R\right)$ for all i, we get $\mathrm{ht}\,I\left(\tau^{\mathbf{k}}_i \otimes_B R\right) \ge \min\{i+1, d\}$ for all i. Now, let $\mathfrak{p} \in \mathrm{Proj}(R)$ with $\mathrm{ht}\,\mathfrak{p} > 0$. Then

$$\mathrm{ht}\left(I\left(\tau^{\mathbf{k}}_{\mathrm{ht}\,\mathfrak{p}} \otimes_B R\right)_{\mathfrak{p}}\right) \ge \min\{\mathrm{ht}\,\mathfrak{p} + 1, d\} > \dim R_{\mathfrak{p}}.$$

In particular, since the differential $\tau^{\mathbf{k}}_{\mathrm{ht}\,\mathfrak{p}} \otimes_B R$ of $\mathbf{E^k}_{\bullet} \otimes_B R$ in homological position $\mathrm{ht}\,\mathfrak{p}$ has unit determinantal ideal we can conclude $\mathrm{pd}_{R_{\mathfrak{p}}}\left(J^k \otimes_B R\right)_{\mathfrak{p}} < \mathrm{ht}\,\mathfrak{p}$ for all k. Therefore, $\ker \psi_k = \mathrm{H}^0_{\mathfrak{m}}\left(J^k \otimes_B R\right)$ for each k.

Now since $\mathbf{E^k}_{\bullet} \otimes_B R$ is a homogeneous approximate resolution of $J^k \otimes_B R$ (see Lemma 4.2.2.ii), by [37, Corollary 5.4, and Theorem 5.6], we have

$$b_0((\ker \psi_k)(kq)) \le b_0\left(\mathbf{E^k}_{d-1} \otimes_B R\right) - d + 1 \le b_0\left(\mathbf{E^k}_{d-1}\right) - d + 1.$$

Likewise, we have

$$\mathrm{topdeg}((\ker \psi_k)(kq)) \le b_0\left(\mathbf{E^k}_d \otimes_B R\right) - d \le b_0\left(\mathbf{E^k}_d\right) - d.$$

∎

We wish to apply Proposition 5.1.1 to the setting of Data 5.0.1, where the variables X_i are the algebraically independent entries of the generic (symmetric, alternating) matrix X. However, the hypotheses in Proposition 5.1.1 state that $\deg X_{ij} = \delta$, whereas Data 5.0.1 specifies $\deg X_{ij} = 1$, which is the grading one would typically use to calculate quantities involving the ideals J^k. Since the generators of J^k can be expressed entirely in the entries of the matrix X, transitioning from the grading in Proposition 5.1.1 to the grading in Data 5.0.1 corresponds to multiplying homogeneous polynomials in the entries of X by δ. Therefore, we obtain the following corollary.

Corollary 5.1.2 *Adopt Data 5.0.1.a, Data 5.0.1.b, or Data 5.0.1.c, and suppose that $\{K_i\}$ is a family of B-ideals so that $K_i \subseteq \sqrt{I\left(\varphi^{\mathbf{k}}_i\right)}$ for all i and for all k. If $\operatorname{ht} K_i R \geq \min\{i+1, d\}$ for all i, then, for each k, one has*

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{D^k}_{d-1}\right) - d + 1\right\}, \quad and$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{D^k}_d\right) - d\right\}.$$

To this point in the chapter our results have been more general than the case of determinantal and Pfaffian ideals of generic height. We now place these results in the context of Corollary 5.1.2 where combination with an argument in parallel with the proof of Theorem 4.3.1 converts the hypothesis about a family of B-ideals into concrete assumptions about the heights of smaller minors and Pfaffians.

Corollary 5.1.3 *Suppose one of the following sets of hypotheses is satisfied.*

(i) *Adopt Data 5.0.1.a, and let $t = m$. Suppose*

$$\operatorname{ht} I_j(A) \geq \min\left\{(m-j+1)(n-m)+1, d\right\}$$

for all $1 \leq j \leq m-1$.

(ii) *Adopt Data 5.0.1.a, and let $1 \leq t < m$. Suppose I satisfies G_d, i.e., suppose*

$$\operatorname{ht} I_j(A) \geq \min\left\{(m-j+1)(n-j+1), d\right\}$$

for all $1 \leq j \leq m-1$.

(iii) *Adopt Data 5.0.1.b. Suppose I satisfies G_d, i.e., suppose*

$$\operatorname{ht} I_j(A) \geq \min \left\{ \binom{n-j+2}{2}, d \right\}$$

for all $1 \leq j \leq t-1$.

(iv) *Adopt Data 5.0.1.c, and let $2t = n-1$. Suppose I satisfies G_d, i.e., suppose*

$$\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min \left\{ n - 2j + 2, d \right\}$$

for all $1 \leq j \leq t-1$.

(v) *Adopt Data 5.0.1.c, and let $2 \leq 2t < n-1$. Suppose I satisfies G_d, i.e., suppose*

$$\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min \left\{ \binom{n-2j+2}{2}, d \right\}$$

for all $1 \leq j \leq t-1$.

Then, for each k, one has

$$b_0(\mathcal{A}_k(I)) \leq \max \left\{ \delta b_0(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{D}^{\mathbf{k}}_{d-1}\right) - d + 1 \right\}, \ \ and$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max \left\{ \delta \operatorname{topdeg}(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{D}^{\mathbf{k}}_{d}\right) - d \right\}.$$

Proof By Corollary 5.1.2, it suffices to find a family $\{K_i\}$ of B-ideals so that $K_i \subseteq \sqrt{I\left(\varphi^{\mathbf{k}}_i\right)}$ for all i and for all k and so that $\operatorname{ht} K_i R \geq \min \{i+1, d\}$ for all i. Notice that this proof closely follows the argument establishing the properties of the family $\{K_i\}$ in the proof of Theorem 4.3.1, but our assumptions are weaker. To handle all cases simultaneously, we establish the following notation.

(i) Let $I_j = I_j(X)$ and $\sigma(j) = (m-j)(n-m) + 1$.

(ii) Let $I_j = I_j(X)$ and $\sigma(j) = (m-j)(n-j)$.

(iii) Let $I_j = I_j(X)$ and $\sigma(j) = \binom{n-j+1}{2}$.

(iv) Let $I_j = \operatorname{Pf}_{2j}(X)$ and $\sigma(j) = n - 2j$.

(v) Let $I_j = \mathrm{Pf}_{2j}(X)$ and $\sigma(j) = \binom{n-2j}{2}$.

For each $1 \le i \le \mathrm{ht}\, I - 1$, we set $K_i = \sqrt{J}$. For each $\mathrm{ht}\, I \le i \le \max_k\{\mathrm{pd}\, J^k\}$, we let j_0 be the smallest j satisfying $i \ge \sigma(j)$, and set $K_i = \sqrt{I_{j_0}}$. Since I is of generic height, $\sigma(t-1) = \mathrm{ht}\, I \le i$. Thus, by the definition of j_0, we have $j_0 \le t-1$. By Lemma 4.2.3, we see that $\sigma(0) = \max_k\{\mathrm{pd}\, J^k\} + 1 > i$. Thus, by the definition of j_0, we have $1 \le j_0$.

Fix i with $1 \le i \le \mathrm{ht}\, I - 1$. Note that we can extend $\left(\mathbf{D^k_\bullet}, \varphi^{\mathbf{k}}_\bullet\right)$ to a resolution of B/J^k by taking $0 \to \mathbf{D^k_\bullet} \overset{\varphi^{\mathbf{k}}_0}{\to} B$. Then, since $\mathrm{ann}\left(B/J^k\right) \ne 0$ and B is Cohen-Macaulay, we have $\sqrt{J^k} = \sqrt{I(\varphi^{\mathbf{k}}_0)} = \cdots = \sqrt{I(\varphi^{\mathbf{k}}_{\mathrm{ht}\, J^k - 1})}$. Since I is of generic height, $\mathrm{ht}\, J^k = \mathrm{ht}\, J = \mathrm{ht}\, I$. In particular, we have $K_i = \sqrt{J} = \sqrt{J^k} \subseteq \sqrt{I(\varphi^{\mathbf{k}}_i)}$, and $\mathrm{ht}\, K_i R > i$. Hence, $\mathrm{ht}\, K_i R \ge \min\{i+1, d\}$.

Now, fix i with $\mathrm{ht}\, I \le i \le \max_k\{\mathrm{pd}\, J^k\}$, and recall that j_0 satisfies $\sigma(j_0 - 1) > i \ge \sigma(j_0)$ and $1 \le j_0 \le t-1$. Since $i \ge \sigma(j_0)$, by Theorem 4.2.5, we have $K_i = \sqrt{I_{j_0}} \subseteq \sqrt{I(\varphi^{\mathbf{k}}_i)}$. Moreover, by assumption, $\mathrm{ht}\, K_i R = \mathrm{ht}\, I_{j_0} R = \mathrm{ht}\, I_{j_0}(A) \ge \min\{\sigma(j_0 - 1), d\} \ge \min\{i+1, d\}$. $\blacksquare$

5.2   Main Results from Castelnuovo-Mumford Regularity

In general, free resolutions of J^k are rarely known. As such, we do not know the exact values of $b_0\left(\mathbf{D^k}_{d-1}\right)$ or $b_0\left(\mathbf{D^k}_d\right)$. However, there has been work to study the Castelnuovo-Mumford regularity of J^k for determinantal ideals and Pfaffian ideals for sufficiently large k. Using these results, we obtain degree bounds on $\mathcal{A}_k(I)$ for sufficiently large k. It is important to point out that $\mathrm{topdeg}(\mathcal{A}_k(J))$ is infinite for some values of k unless J is of linear type on the punctured spectrum of R. Indeed, we recall Remark 2.4.6, so to say J is of linear type on the punctured spectrum means $\mathcal{A}_k(J)$ can be supported only at the homogeneous maximal ideal of R. This, in turn, shows the components $\mathcal{A}_k(J)$ have finite length which guarantees concentration in only finitely many x-degrees. Since $\mathcal{A}(J)$ is finitely generated we can obtain a

uniform bound on the top x-degrees for each k, so that $\mathcal{A}(J)$ itself has finite top degree.

Since we are considering generic matrices to obtain bounds one can sometimes predict whether the top degree bounds is non-trivial as we illustrate with an example.

Example 5.2.1 Let X be a 3×6 generic matrix and let $J = I_3(X)$. Will the top degree of $\mathcal{A}_k(J)$ be finite for all k? To answer this question we can take a prime $\mathfrak{p}$ on the punctured spectrum of R. If we choose, say $\mathfrak{p} = (\{X_{ij}\} \setminus \{X_{11}\})$, then viewed in $B_{\mathfrak{p}}$ the entry X_{11} becomes a unit and we may apply Lemma 4.2.4 to see we are in the case of 2×2 minors of a generic 2×5 matrix.

We know the ideal of these minors in this situation is not of linear type since, for example, there are Plücker relations obtained in this situation which will correspond to fiber equations. Therefore J is not of linear type on the punctured spectrum and we must conclude the top degree of $\mathcal{A}_k(J)$ is infinite for some k.

We now turn to a particular case where we do know resolutions for the powers of $J = I_t(X)$: maximal minors. When $t = m$ we can use [1, Theorem 5.4] to learn the top generation degree for each module in the resolution $\mathbf{D^k}_\bullet$. The next theorem is demonstrated implicitly in [1, Theorem 5.4], and we make use of [8, Theorem 3.1, Proposition 3.6] to state the theorem in its present form.

Theorem 5.2.2 (Akin-Buchsbaum-Weyman [1, Theorem 5.4]) *Adopt Data 5.0.1.a and let $t = m$. Then $\mathbf{D^k}_\bullet$ is a linear resolution and its length is $\min\{k, m\}\,(n - m)$. In particular*

$$b_0\left(\mathbf{D^k}_i\right) = \begin{cases} i & 1 \leq i \leq \min\{k, m\}\,(n - m) \\ -\infty & i > \min\{k, m\}\,(n - m). \end{cases}$$

We now give our main result.

Theorem 5.2.3 *Adopt Data 5.0.1.a. Recall the grading information mentioned in Definition 2.4.1, Definition 2.4.2, and Notation 2.4.4, i.e., $\mathcal{A}(I)$ is the kernel of the map $\mathcal{S}(I(D)) \twoheadrightarrow \mathcal{R}(I)$.*

(i) [37, Theorem 6.1.a] *Suppose* $t = m$, $n = m + 1$, *and* I *satisfies* G_d, *i.e.,* ht $I_j(A) \geq \min\{m - j + 2, d\}$ *for all* $1 \leq j \leq m - 1$. *Then for all* k,

$$b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1), \quad and$$

$$\mathrm{topdeg}(\mathcal{A}_k(I)) \leq (\delta - 1)d.$$

In particular, if $d > \min\{k, m\}(n - m)$, *then* $b_0(\mathcal{A}_k(I)) = \mathrm{topdeg}(\mathcal{A}_k(I)) = -\infty$.

(ii) *Suppose* $t = m$, $n \geq m + 2$, *and* ht $I_j(A) \geq \min\{(m - j + 1)(n - m) + 1, d\}$ *for all* $1 \leq j \leq m - 1$. *Then for all* k,

$$b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1), \quad and$$

$$\mathrm{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta\,\mathrm{topdeg}(\mathcal{A}_k(J)), (\delta - 1)d\}.$$

In particular, if $d - 1 > \min\{k, m\}(n - m)$, *then* $b_0(\mathcal{A}_k(I)) \leq 0$.

(iii) *Let* char $K = 0$ *and* $t = 2$.

Suppose I *satisfies* ht $I_1(A) \geq \min\{mn, d\}$. *Then for* $2 \leq k \leq m - 2$,

$$b_0(\mathcal{A}_k(I)) \leq \max\{\delta b_0(\mathcal{A}_k(J)), (\delta - 1)(d - 1) + \delta(m - k - 1)\}, \quad and$$

$$\mathrm{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta\,\mathrm{topdeg}(\mathcal{A}_k(J)), (\delta - 1)d + \delta(m - k - 1)\}.$$

For $k \geq m - 1$,

$$b_0(\mathcal{A}_k(I)) \leq \max\{\delta b_0(\mathcal{A}_k(J)), (\delta - 1)(d - 1)\}, \quad and$$

$$\mathrm{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta\,\mathrm{topdeg}(\mathcal{A}_k(J)), (\delta - 1)d\}.$$

(iv) *Let* char $K = 0$ *and* $1 < t < m$.

Suppose I *satisfies* ht $I_j(A) \geq \min\{(m - j + 1)(n - j + 1), d\}$ *for all* $1 \leq j \leq t - 1$. *Then for* $k \geq m - 1$,

$$b_0(\mathcal{A}_k(I)) \leq \max\{\delta b_0(\mathcal{A}_k(J)), (\delta - 1)(d - 1) + \delta N(t)\}, \quad and$$

$$\text{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta\, \text{topdeg}(\mathcal{A}_k(J)), (\delta - 1)\, d + \delta N(t)\},$$

$$\text{where } N(t) = \begin{cases} \left(\frac{t-1}{2}\right)^2, & t \text{ is odd} \\ \frac{(t-2)t}{4}, & t \text{ is even} \end{cases}.$$

(v) *Let* $\text{char } K = 0$, $n = m$, *and* $t = n - 1$.

Suppose I *satisfies* $\text{ht } I_j(A) \geq \min\left\{(m - j + 1)^2, d\right\}$ *for all* $1 \leq j \leq t$. *Then for* $k \geq n - 1$,

$$b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1) + \delta N(t), \text{ and}$$

$$\text{topdeg}(\mathcal{A}_k(I)) \leq (\delta - 1)\, d + \delta N(t),$$

$$\text{where } N(t) = \begin{cases} \left(\frac{t-1}{2}\right)^2, & t \text{ is odd} \\ \frac{(t-2)t}{4}, & t \text{ is even} \end{cases}.$$

Proof We apply Corollary 5.1.3, which states

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{D^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\text{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta\, \text{topdeg}(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{D^k}_d\right) - d\right\}.$$

We begin by finding upper bounds for $\delta b_0\left(\mathbf{D^k}_{d-1}\right) - d + 1$ and $\delta b_0\left(\mathbf{D^k}_d\right) - d$ in terms of the Castelnuovo-Mumford regularity of J^k. In view of (2.2.1.a), for a fixed homological degree j, we see

$$\text{reg}\left(J^k(tk)\right) \geq b_0\left(\mathbf{D^k}_j\right) - j.$$

Hence we can use this to bound $b_0\left(\mathbf{D^k}_j\right)$. Since $\mathbf{D^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(tk)$, we have

$$b_0\left(\mathbf{D^k}_{d-1}\right) \leq \text{reg}\left(J^k(tk)\right) + d - 1.$$

Hence, $\delta b_0\left(\mathbf{D^k}_{d-1}\right) \leq \delta\, \text{reg}\left(J^k(tk)\right) + \delta(d - 1)$. As such, we get

$$\delta b_0\left(\mathbf{D^k}_{d-1}\right) - d + 1 \leq \delta\, \text{reg}\left(J^k(tk)\right) + \delta(d - 1) - (d - 1)$$

$$= \delta \operatorname{reg}\big(J^k(tk)\big) + (\delta - 1)(d - 1). \tag{5.2.3.a}$$

Likewise, $b_0\big(\mathbf{D^k}_d\big) \leq \operatorname{reg}\big(J^k(tk)\big) + d$. Hence, $\delta b_0\big(\mathbf{D^k}_d\big) \leq \delta \operatorname{reg}\big(J^k(tk)\big) + \delta d$. Hence,

$$\delta b_0\big(\mathbf{D^k}_d\big) - d \leq \delta \operatorname{reg}\big(J^k(tk)\big) + \delta d - d = \delta \operatorname{reg}\big(J^k(tk)\big) + (\delta - 1)d. \tag{5.2.3.b}$$

Since we have now bounded these generation degrees in terms of Castelnuovo-Mumford regularity we turn to the literature where we can make use of some known bounds in each case. But first we point out, for each k, $\operatorname{reg}\big(J^k(tk)\big) = \operatorname{reg}\big(J^k\big) - tk$. Indeed, the shift by tk gives the modules $J^k(tk)$ generated in degree zero, so J^k may be resolved by $\mathbf{D^k}_\bullet(-tk)$, whence $b_0\big(\mathbf{D^k}_j(-tk)\big) = b_0\big(\mathbf{D^k}_j\big) + tk$.

For (i) and (ii), it is known that $\operatorname{reg}\big(J^k\big) = mk$ for all k (see [8, Proposition 3.6]). We can then compute

$$\operatorname{reg}\big(J^k(tk)\big) = (mk - tk).$$

Since $t = m$ in (i) and (ii) we obtain the desired inequality after combining this result with (5.2.3.a). We then use Theorem 5.2.2 to see how the generator degrees depend on the dimension d. We see for $d > \min\{k, m\}(n - m)$ in (i) and $d - 1 > \min\{k, m\}(n - m)$ in (ii) that $\mathbf{D^k}_d$, and $\mathbf{D^k}_{d-1}$ are the zero modules (respectively).

For (iii), [44, Theorem on Regularity] states that when $\operatorname{char} K = 0$,

$$\operatorname{reg}\big(J^k\big) = \begin{cases} k + m - 1, & 1 \leq k \leq m - 2 \\ 2k, & k \geq m - 1. \end{cases}$$

Thus, when $1 \leq k \leq m - 2$, we compute (here $t = 2$)

$$\operatorname{reg}\big(J^k(tk)\big) = k + m - 1 - 2k = m - k - 1$$

and combining this with (5.2.3.a) give the desired inequality. On the other hand, when $m - 1 \leq k$, we then find $\operatorname{reg}\big(J^k(tk)\big) = 2k - 2k = 0$, as needed.

For (iv) and (v), [44, Theorem on Regularity] states that if $\operatorname{char} K = 0$ and $2 < t < m$, then for $k \geq m - 1$, $\operatorname{reg}\big(J^k\big) = tk + N(t) \implies \operatorname{reg}\big(J^k(tk)\big) = N(t)$. Hence we obtain the desired inequality.

We end the proof by showing when we have linear type or fiber type. If J is of linear type, then $b_0(\mathcal{A}_k(J)) = \text{topdeg}(\mathcal{A}_k(J)) = -\infty$. If J is of fiber type, then $b_0(\mathcal{A}_k(J)) \leq 0$ for all k. Therefore, for the cases where J is of linear type or fiber type we can simplify the maximum coming from Corollary 5.1.3.

For (i) and (v), it is known that J is of linear type ([28, Proposition 1.1] and [34, Proposition 2.10], respectively). Hence, $b_0(\mathcal{A}_k(J)) = -\infty$ and $\text{topdeg}(\mathcal{A}_k(J)) = -\infty$ for all k in the setting of (i) and (v). Moreover, in (ii), it is known that J is of fiber type ([8, 3.7]). Hence, $b_0(\mathcal{A}_k(J)) \leq 0$ for all k in the setting of (ii). ∎

We did not mention the condition G_d for part (ii) above. While the condition in (ii) is the same as G_d when $n \leq m + 1$ (see Observation 3.1.4), when $n > m + 1$, the condition in (ii) is weaker than G_d. Thus, (ii) applies to a broader class of ideals than just those satisfying G_d when $n > m + 1$.

Additionally, in (ii), for the case that $\delta = 1$, we see that I is of fiber type since $b_0(\mathcal{A}_k(I)) = 0$ for all k. This has already been proven in [8, Theorem 3.7]; however, our proof above uses a different technique.

In [25, Corollary 7.3], it is proven that if X is $3 \times n$ (for $n \geq 3$) and $t = 2$, then $J = I_2(X)$ is of fiber type. Applying this result to Theorem 5.2.3.iii, we obtain the following result.

Corollary 5.2.4 (2×2 **minors of a** $3 \times n$ **matrix**) *Adopt Data 5.0.1.a, and suppose that* char $K = 0$, $t = 2$, *and* $m = 3$. *Suppose that* I *satisfies* ht $I_1(A) \geq \min\{3n, d\}$. *Then, for all* k, $b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1)$ *and* $\text{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta \, \text{topdeg}(\mathcal{A}_k(J)), (\delta - 1) d\}$.

There has also been work done on obtaining the Castelnuovo-Mumford regularity for powers of Pfaffian ideals of generic alternating matrices. Using these results, we obtain the following bounds on generation and concentration degree.

Theorem 5.2.5 *Adopt Data 5.0.1.c, and assume* char $K = 0$.

(i) *Let $2t = n - 1$. Suppose I satisfies G_d, i.e., $\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min\{n - 2j + 2, d\}$ for all $1 \leq j \leq t - 1$. Then, if k is odd and $2 \leq k \leq n - 2$,*

$$b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1) + \frac{\delta(n - k - 4)}{2}, \quad and$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq (\delta - 1)d + \frac{\delta(n - k - 4)}{2}.$$

If k is even or if k is odd and $k \geq n - 1$,

$$b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1), \quad and$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq (\delta - 1)d.$$

(ii) *Let $2t = n - 2$. Suppose I satisfies G_d, i.e., $\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min\left\{\binom{n-2j+2}{2}, d\right\}$ for all $1 \leq j \leq t - 1$. Then, if $k \geq n - 2$,*

$$b_0(\mathcal{A}_k(I)) \leq (\delta - 1)(d - 1) + \delta N(t), \quad and$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq (\delta - 1)d + \delta N(t),$$

where $N(t) = \begin{cases} \frac{1}{2}(t - 1)^2, & t \text{ is odd} \\ t\left(\frac{t}{2} - 1\right), & t \text{ is even.} \end{cases}$

(iii) *Let $1 < 2t < n - 2$. Suppose I satisfies G_d, i.e., $\operatorname{ht} \operatorname{Pf}_{2j}(A) \geq \min\left\{\binom{n-2j+2}{2}, d\right\}$ for all $1 \leq j \leq t - 1$. Then, if n is even and $k \geq n - 2$ or if n is odd and $k \geq n - 3$,*

$$b_0(\mathcal{A}_k(I)) \leq \max\{\delta b_0(\mathcal{A}_k(J)), (\delta - 1)(d - 1) + \delta N(t)\}, \quad and$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta \operatorname{topdeg}(\mathcal{A}_k(J)), (\delta - 1)d + \delta N(t)\},$$

where $N(t) = \begin{cases} \frac{1}{2}(t - 1)^2, & t \text{ is odd} \\ t\left(\frac{t}{2} - 1\right), & t \text{ is even.} \end{cases}$

Proof Repeat the proof of Theorem 5.2.3 using [43, Theorem 5.9, Theorem A] which states that for char $K = 0$,

if $2t = n-1$, then $\operatorname{reg}\left(J^k\right) = \begin{cases} tk + \frac{1}{2}\left(n-k-4\right), & k \text{ is odd and } d < n-2 \\ tk, & \text{otherwise} \end{cases}$, and

if $1 < 2t < n$, and if either n is even and $k \geq n-2$ or n is odd and $k \geq n-3$, we have

$$\operatorname{reg}\left(J^k\right) = tk + \begin{cases} \frac{1}{2}\left(t-1\right)^2, & t \text{ is odd} \\ t\left(\frac{t}{2}-1\right), & t \text{ is even.} \end{cases}$$

Note that $\operatorname{reg}\left(J^k(tk)\right) = \operatorname{reg}\left(J^k\right) - tk$.

Also note that in (i) and (ii), J is of linear type ([27, 2.2] and [4, 2.1], respectively). Hence, $b_0(\mathcal{A}_k(J)) = -\infty$ and $\operatorname{topdeg}(\mathcal{A}_k(J)) = 0$ for all k in these settings. ∎

We are not aware if work has been done to obtain the Castelnuovo-Mumford regularity of J^k in the symmetric case (as in Data 5.0.1.b), so we only provide the following statement regarding degree bounds for this case, using the fact that J is of linear type in the setting below (see [34, Proposition 2.10]).

Corollary 5.2.6 *Adopt Data 5.0.1.b. Let $t = n-1$.*

Suppose I satisfies G_d, i.e., $\operatorname{ht} I_j(A) \geq \min\left\{\binom{n-j+2}{2}, d\right\}$ for all $1 \leq j \leq t-1$, then

$$b_0(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{D^k}_{d-1}\right) - d + 1 \ \text{ and } \ \operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{D^k}_d\right) - d.$$

Additionally, we note that the degree bounds given in Theorem 5.2.3 and Theorem 5.2.5 likely are not sharp. We expect that if one had access to explicit resolutions of J^k, one would likely find better degree bounds.

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