QUASIDIAGONAL EXTENSIONS OF C^*-ALGEBRAS AND OBSTRUCTIONS IN K-THEORY

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Jacob R. Desmond

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2020

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. Marius Dadarlat

Department of Mathematics

Dr. Thomas Sinclair

Department of Mathematics

Dr. Andrew Toms

Department of Mathematics

Dr. Gregery Buzzard

Department of Mathematics

Approved by:

Dr. Plamen Stefanov

Associate Head for Graduate Studies

ACKNOWLEDGMENTS

I would first and foremost like to thank my advisor Professor Marius Dadarlat for his abundant support, guidance, and patience during my time at Purdue. He taught me the importance of dedication, motivation, and having a genuine passion for research. Additionally, I would like to thank him for his generosity and friendship, for which I am unable to explain how grateful I am.

I would like to thank Professor Thomas Sinclair for being so friendly to graduate students and to me personally as a student in Operator Algebras. I would also like to thank Professor Gregery Buzzard for his compassion.

I am forever indebted to my mother, my father, and my sister for their unwavering support through my six years at Purdue. I would not have been able to complete this degree without their help.

Lastly, I would like to thank my dear friends Stephen, Madeline, Nathan, Daniel, Roberto, and Carmen for their friendship, support, and kindness.

TABLE OF CONTENTS

Page

ABSTRACT

Desmond, Jacob R. Ph.D., Purdue University, August 2020. Quasidiagonal Extensions of C^*-algebras and Obstructions in K-theory. Major Professor: Marius Dadarlat.

Quasidiagonality is a matricial approximation property which asymptotically captures the multiplicative structure of C^*-algebras. Quasidiagonal C^*-algebras must be stably finite. It has been conjectured by Blackadar and Kirchberg that stably finiteness implies quasidiagonality for the class of separable nuclear C^*-algebras. It has also been conjectured that separable exact quasidiagonal C^*-algebras are AF embeddable. In this thesis, we study the behavior of these conjectures in the context of extensions $0 \to I \to E \to B \to 0$. Specifically, we show that if I is exact and connective and B is separable, nuclear, and quasidiagonal (AF embeddable), then E is quasidiagonal (AF embeddable). Additionally, we show that if I is of the form $C(X) \otimes \mathcal{K}$ for a compact metrizable space X and B is separable, nuclear, quasidiagonal (AF embeddable), and satisfies the UCT, then E is quasidiagonal (AF embeddable) if and only if E is stably finite.

1. INTRODUCTION

Quasidiagonality is a matricial approximation property that asymptotically captures the multiplicative structure of C^*-algebras. Fundamental to the study of operator algebras, this notion appears in Elliott's classification program, as a structural property of group C^*-algebras associated to amenable groups, and in the structure theory of nuclear C^*-algebras. Voiculescu's celebrated theorem [1] led to a new characterization of quasidiagonality and was instrumental in proving his theorem on the homotopy invariance of quasidiagonality [2].

Quasidiagonal C^*-algebras must be stably finite. The reduced group C^*-algebra of nonamenable groups is neither quasidiagonal nor nuclear, despite being stably finite. Motivated by this class of examples, Blackadar and Kirchberg conjectured in [3] that all separable nuclear C^*-algebras which are stably finite must be quasidiagonal. As a stunning accomplishment, Tikuisis-Winter-White proved that any separable nuclear C^*-algebra A which satisfies the Universal Coefficient Theorem (UCT) and has a faithful trace is quasidiagonal [4]. In particular, separable unital nuclear simple stably finite C^*-algebras that satisfy the UCT are quasidiagonal.

This thesis confirms the Blackadar-Kirchberg conjecture for certain classes of nonsimple C^*-algebras in the context of extensions. Given a quasidiagonal ideal I and quasidiagonal quotient B, it is natural to ask whether the extension given by the short exact sequence $0 \to I \to E \to B \to 0$ will also be quasidiagonal. The Toeplitz algebra $\mathcal{T}$, an extension of $C(S^1)$ by the compact operators $\mathcal{K}$, is a well-known counterexample. Indeed, since the Toeplitz algebra is generated by a nonunitary isometry, it is not stably finite and therefore is not quasidiagonal. In [5], Spielberg observed a connection between stably finite extensions and the behavior of the boundary map that appears in the 6-term exact sequence. Given that I and B are stably finite, E is stably finite if and only if $\partial(K_1(B)) \cap K_0(I)^+ = \{0\}$.

Constructing embeddings with prescribed K-theory was originally explored by Voiculescu and Pimsner [6] in order to embed irrational rotation algebras into AF algebras. Expanding on this notion and Spielberg's result, N. Brown and Dadarlat developed a method for proving when stably finite extensions are quasidiagonal without having to assume the UCT [7]. Under the additional assumption that the quotient B satisfies the UCT, N. Brown and Dadarlat proved that stably finite extensions are quasidiagonal if and only if there exist embeddings of the ideal I into quasidiagonal C^*-algebras with a particular effect on the K-theory. We use these results to show that if I is exact and connective and B is separable, nuclear, and quasidiagonal, then E is quasidiagonal. In addition, when the ideal is of the form $C(X) \otimes \mathcal{K}$, where X is a compact metrizable space, and B is separable, nuclear, quasidiagonal, and satisfies the UCT, then any extension E is stably finite if and only if it is quasidiagonal.

Aside from quasidiagonality, AF embeddability has also been of great interest in the field. Indeed, every C^*-subalgebra of an AF algebra is separable, exact, and quasidiagonal. It is still open whether the converse holds. One advancement is that cones over separable exact C^*-algebras, which are quasidiagonal by Voiculescu's homotopy invariance theorem, were shown to be AF embeddable by Ozawa [8]. In fact, due to work of Dadarlat-Pennig [9] [10], Gabe [11], and Rørdam [12], we know that all exact connective C^*-algebras are AF embeddable. L. Brown showed in [13] that extensions of AF algebras are AF. Expanding on this result, Spielberg proved that if either the quotient or ideal is AF embeddable and the other is an AF algebra, then any nuclear extension which is stably finite must be AF embeddable [5]. In a similar vein to these results, we show that if I is exact and connective and B is nuclear and AF embeddable, then E is AF embeddable.. With the additional assumption that the B satisfies the UCT, we show that stably finite extensions by ideals of the form $C(X) \otimes \mathcal{K}$, where X is a compact metrizable space, are AF embeddable.

2. QUASIDIAGONALITY

While quasidiagonality is the central focus of this chapter, there are many tools required to understand this notion and its consequences. For this reason, we will give an overview of these concepts to establish the prerequisite theory needed for this thesis.

2.1 PRELIMINARIES

A *-algebra is an algebra A over $\mathbb{C}$ with an involution $^* : A \to A$ that satisfies the following properties:

(i) $(a^*)^* = a$

(ii) $(a+b)^* = a^* + b^*$

(iii) $(\lambda a)^* = \bar{\lambda} a^*$

(iv) $(ab)^* = b^* a^*$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. A *Banach *-algebra* is a *-algebra endowed with a complete submultiplicative norm $\|\cdot\|$ that satisfies $\|a^*\| = \|a\|$ for all $a \in A$. If the Banach *-algebra additionally satisfies the C^*-identity

$$\|a^*a\| = \|a\|^2$$

for all $a \in A$, then it is a *C^*-algebra*. A map $\varphi : A \to B$ between C^*-algebras A and B is called **-preserving* if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. An algebra homomorphism from A to B that is also *-preserving is called a **-homomorphism* and are the fundamental class of morphisms used to study C^*-algebras. A C^*-algebra with a unit is called a unital C^*-algebra. The quintessential example of a C^*-algebra is the set of bounded linear operators $B(H)$ on a Hilbert space H. Any closed self-adjoint subalgebra of $B(H)$, called a C^*-subalgebra, is also a C^*-algebra. In fact, every C^*-algebra has this form.

Theorem 2.1.1 (Gelfand-Naimark). *Every C^*-algebra A is isometrically *-isomorphic to a C^*-algebra contained in $B(H)$ for some Hilbert space H.*

Given a locally compact Hausdorff space X, a continuous function $f \in C(X, \mathbb{C})$ is said to *vanish at infinity* if for every $\epsilon > 0$ the set $\{x \in X : f(x) \geq \epsilon\}$ is compact. The collection of all functions that vanish at infinity is denoted as $C_0(X)$. Equipped with the involution $f \mapsto \bar{f}$ and sup norm, it is clear that $C_0(X)$ is a commutative C^*-algebra. The class of commutative C^*-algebras happens to be one of the most deeply studied due to a structure theorem of Gelfand and Naimark.

Theorem 2.1.2 (Gelfand-Naimark). *Every commutative C^*-algebra A is isometrically *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.*

For this reason, all topological spaces will be assumed to be locally compact Hausdorff. It is well-known that these spaces allow for the existence of partitions of unity. Given an open cover $\{V_i\}_{i\in\mathcal{I}}$, a *partition of unity of X subordinate to* $\{V_i\}_{i\in\mathcal{I}}$ is a collection of continuous functions $\{f_i : X \to [0,1]\}_{i\in\mathcal{I}}$ such that $\mathrm{supp}(f_i) \subset V_i$ and for every $x \in X$

(i) $\sum_{i\in\mathcal{I}} f_i(x) = 1$.

(ii) There exists a neighborhood of x on which only finitely many elements in $\{f_i\}$ are nonzero.

For any compact subset K of a locally compact Hausdorff space X and a finite subcover, there exists a partition of unity of K subordinate to that cover. This notion generalizes to abstract C^*-algebras and is called nuclearity. In order to define it, we must first introduce *contractive completely positive (ccp)* maps. A linear map $\varphi : A \to B$ between C*-algebras A and B is called *positive* if the image of the positive cone $A^+ = \{a^*a : a \in A\}$ of A is contained in the positive cone of B. Positive maps are automatically *-preserving, i.e. $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. Let $M_n(A)$ denote the C^*-algebra which consists of all $n \times n$ matrices with entries in A. $\mathbb{M}_n$ will always stand for $M_n(\mathbb{C})$. Given a positive map $\varphi : A \to B$, if the extensions $\varphi_n : M_n(A) \to M_n(B)$ given by $a_{ij} \mapsto [\varphi(a_{ij})]_{ij}$ are positive for all $n \in \mathbb{N}$, then φ is said to be *completely positive*. Furthermore, if a completely positive map is a contraction, i.e. $\|\varphi\| \le 1$, then φ is contractive and completely positive (ccp). It should be noted that *-homomorphisms are always ccp maps. These maps were popularized by Arveson in an extension theorem analogous to Hahn-Banach for C^*-algebras.

Theorem 2.1.3 (Arveson)**.** *Let A be a unital C^*-algebra and $E \subset A$ a C^*-subalgebra of A. Then every ccp map $\varphi : E \to B(H)$ extends to a ccp map $\tilde{\varphi} : A \to B(H)$.*

With this notion, we are able to define nuclearity, which is a matricial approximation property.

Definition 2.1.4. A C^*-algebra A is *nuclear* if for every finite subset $\mathcal{F} \subset A$ and $\epsilon > 0$, there exist ccp maps $\varphi : A \to \mathbb{M}_n$ and $\psi : \mathbb{M}_n \to A$ such that $\|(\psi \circ \varphi)(a) - a\| < \epsilon$ for all $a \in \mathcal{F}$.

The existence ccp maps which induce the diagrams

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, dashed, "\varphi_n"'] & & A \\ & \mathbb{M}_{k(n)} \arrow[ur, dashed, "\psi_n"'] & \end{tikzcd}$$

is referred to as the ccp map $\mathrm{id}_A : A \to A$ being *nuclear*. A C^*-algebra A is said to be *exact* if there exists a faithful representation $\pi : A \to B(H)$ such that π is nuclear. Examples of nuclear C^*-algebras include commutative C^*-algebrasand AF algebras. The class of nuclear C^*-algebras is closed under quotients, extensions, inductive limits, and other constructions. One of the landmark theorems that utilize this notion is the Choi-Effros Lifting theorem, which will be used repeatedly throughout this thesis.

Theorem 2.1.5 (Choi-Effros Lifting Theorem, [14])**.** *Let A be a separable C^*-algebra. If $\Phi : A \to B/I$ is a nuclear ccp map into a quotient of C^*-algebras, there exists a ccp lift $\varphi : A \to B$ that makes the following diagram commute*

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "\varphi"] \arrow[r, "\Phi"'] & B/I \end{tikzcd}$$

Corollary 2.1.6. *Let A be a separable nuclear C^*-algebra. If $\Phi : A \to B/I$ is a *-homomorphism into a quotient of C^*-algebras, then Φ lifts to a ccp map $\varphi : A \to B$.*

In addition to the Choi-Effros lifting theorem, nuclearity is fundamental to the structure of tensor products. In general, for two C^*-algebras A and B there are many possible tensor product constructions. The two most widely used are the minimal tensor $A \otimes B$ and the maximal tensor $A \otimes_{max} B$. If A is nuclear, then $A \otimes B$ is canonically isomorphic to $A \otimes_{max} B$ for all B, and furthermore implies there is a unique cross-norm on the algebraic tensor of A and B that allows it to become a C*-algebra. In this way, nuclearity of A implies there is a unique norm on the algebraic tensor product of A and B that allows its completion to be a C^*-algebra. See [15] for more information.

2.2 DEFINITIONS AND EXAMPLES

Quasidiagonality was first introduced by Halmos in [16] in 1970 as a means to generalize the notion of a block diagonal operator. Given a separable Hilbert space H, an operator in $B(H)$ is block diagonal if there exists a sequence of finite dimensional projections $P_n \leq P_{n+1}$ which converge to 1 in the strong operator topology and commute with the operator itself. By relaxing the assumption on the projections and allowing them to instead commute asymptotically, we arrive at Halmos's definition of quasidiagonal.

Definition 2.2.1. A set of bounded operators S on a separable Hilbert space is *quasidiagonal* if there exists an increasing sequence of projections $P_n \leq P_{n+1}$ that converge to 1 in the strong operator topology and $\lim_{n\to\infty}[[T,P_n]]=0$ for all n and all $T\in S$.

Let $\mathcal{K}(H)$ denote the compact operators in $B(H)$. When there is no confusion regarding the underlying Hilbert space, we will write $\mathcal{K}$ for brevity. One can show that a single operator T is quasidiagonal if and only if it is a compact perturbation of a block diagonal operator. This is done by using the associated sequence of projections P_n to create a block diagonal matrix B whose blocks are given by $P_{n+1}-P_n$ and showing that the difference $T-B$ is compact. In fact, this holds for quasidiagonal sets as well.

Theorem 2.2.2 (Thm. 5.2 [17])**.** *Let H be a separable Hilbert space. Then S is a quasidiagonal set of operators if and only if for every finite subset $\mathcal{F}\subset S$ and $\epsilon>0$ there exists a block diagonal algebra $B\subset B(\mathcal{H})$ such that $S+\mathcal{K}=B+\mathcal{K}$ and for every $T\in\mathcal{F}$ there exists an element $C\in B$ such that $\|T-C\|<\epsilon$.*

To connect the notion of quasidiagonality with abstract C^*-algebras, we will need to briefly introduce representations. Let H be a separable Hilbert space. A *representation* of a C^*-algebra A on H is a *-homomorphism $\pi: A\to B(H)$. π is called *faithful* if it is injective and *essential* if $\pi(A)\cap\mathcal{K}=\{0\}$. If A is unital, then a representation is called *unital* if $\pi(1)=1_H\in B(H)$. Let $\pi_i: A\to B(H_i)$ be unital representations on separable Hilbert spaces H_i for $i=1,2$. If there exists a sequence of unitaries $u_n: H_1\to H_2$ such that $\|u_n\pi_1(a)u_n^*-\pi_2(a)\|\to 0$ for all $a\in A$, then π_1 and π_2 are said to be *approximately unitarily equivalent*. If the difference $u_n\pi_1(a)u_n^*-\pi_2(a)$ is always compact, then the two representations are said to be *approximately unitarily equivalent modulo the compacts*. Below is one version of Voiculescu's theorem, which is not only essential to the discussion of quasidiagonality but also a fundamental tool within operator algebras in its own right.

Theorem 2.2.3 (Voiculescu, [1])**.** *Let H be a separable Hilbert space and let $A\subset B(H)$ be a C^*-algebra such that $1_H\in B(H)$. Let $\iota: A\hookrightarrow B(H)$ denote the canonical inclusion and $\pi: A\to B(K)$ be a unital representation of A on another Hilbert space*

K such that $\pi(A \cap \mathcal{K}(H)) = 0$. Then $\iota \oplus \pi$ is approximately unitarily equivalent modulo the compact operators to ι.

Corollary 2.2.4. *Let A be a separable unital C^*-algebra and $\pi_i : A \to B(H_i)$, $i = 1, 2$, be faithful unital essential representations of A onto separable Hilbert spaces H_i. Then π_1 and π_2 are approximately unitarily equivalent modulo the compacts.*

Let A be a unital C^*-algebra and let $\pi : A \to B(H)$ be a unital faithful representation of A on H. Suppose that $\pi(A)$ is a quasidiagonal set of operators. While $\pi(A)$ may intersect $\mathcal{K}$ nontrivially, its infinite multiple $\pi_\infty : A \to B(\oplus H)$ given by $a \mapsto \operatorname{diag}(\pi(a), \pi(a), \dots)$ will be a faithful unital essential representation of A. By Voiculescu's theorem, any faithful unital essential representation ρ will be approximately unitarily equivalent modulo the compacts and thus $\rho(A)$ will also be a quasidiagonal set of operators. This motivates the following definition.

Definition 2.2.5. A separable C*-algebra A is said to be *quasidiagonal* if there exists a faithful representation $\pi : A \to B(H)$ on a separable Hilbert space H such that $\pi(A)$ is a quasidiagonal set of operators.

It should be noted, however, that there exist examples of C*-algebras where A is quasidiagonal and π is a faithful representation but $\pi(A)$ is not a quasidiagonal set. To obtain a more usable characterization of quasidiagonality, we again turn to Voiculescu.

Theorem 2.2.6 (Voiculescu, [2])**.** *A C*-algebra A is quasidiagonal if and only if for every finite subset $\mathcal{F} \subset A$ and $\epsilon > 0$ there exists a ccp map $\varphi_n : A \to \mathbb{M}_n$ such that*

(i) $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| < \epsilon$ *for all* $a, b \in \mathcal{F}$

(ii) $\|a\| - \epsilon < \|\varphi_n(a)\|$ *for all* $a \in \mathcal{F}$.

This local definition can be altered to a global characterization which uses nets to account for the possibility that A maybe not be separable. However, since this thesis will only be concerned with separable quasidiagonal C^*-algebras, we will only need the sequential characterization. Let A be a separable quasidiagonal C^*-algebra. By taking a countably dense subset D of A, one may iteratively use the above definition on an increasing subset of D to construct a sequence of ccp maps $\varphi_n : A \to \mathbb{M}_{k(n)}$ that satisfy Properties (i) and (ii) from above. Property (i) implies that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ for all $a, b \in A$ and we say that the sequence φ_n is *asymptotically multiplicative*. Property (ii) implies that $\|\varphi_n(a)\| \to \|a\|$ for all $a \in A$ and we say that the sequence φ_n is *asymptotically isometric*.

Recall that $\prod_n \mathbb{M}_{k(n)}$ is the infinite product of $\mathbb{M}_{k(n)}$, where the norm of an element $m = (m_1, m_2, ...)$ is given by $\|m\| = \sup_n \|m_n\|$. Contained inside this C^*-algebra is the closed two-sided ideal $\bigoplus_n \mathbb{M}_{k(n)}$, which consists of all elements m that satisfy $\|m_n\| \to 0$. If A is quasidiagonal, one can use the sequence of asymptotically multiplicative and asymptotically isometric ccp maps φ_n to embed A into the quotient $\prod \mathbb{M}_{k(n)} / \bigoplus \mathbb{M}_{k(n)}$. Indeed, defining $\psi : A \to \prod \mathbb{M}_{k(n)}$ to map $a \mapsto (\varphi_1(a), \varphi_2(a), ...)$ and composing ψ with the canonical quotient map $\pi : \prod \mathbb{M}_{k(n)} \to \prod \mathbb{M}_{k(n)} / \bigoplus \mathbb{M}_{k(n)}$ yields the desired *-monomorphism. Observe that since the maps φ_n are asymptotically multiplicative and asymptotically isometric, $\pi \circ \psi$ will be multiplicative and injective, respectively.

Proposition 2.2.7. *Let A be a separable C^*-algebra. A is quasidiagonal if and only if there exists a *-monomorphism*

$$\Phi : A \hookrightarrow \prod_n \mathbb{M}_{k(n)} \Big/ \bigoplus_n \mathbb{M}_{k(n)}$$

that has a ccp lift $\psi : A \to \prod_n \mathbb{M}_{k(n)}$.

Proof. We have already described how to construct Φ from an asymptotically multiplicative and asymptotically isometric sequence of ccp maps. Conversely, let Φ be an embedding into the quotient algebra and ψ the corresponding ccp lift. For every n, define $\varphi_n : A \to \mathbb{M}_{k(n)}$ to map an element $a \in A$ to the n^{th} component of the element $\psi(a) = (a_1, a_2, \dots) \in \prod \mathbb{M}_{k(n)}$. Fix a finite subset $\mathcal{F} \subset A$ and $\epsilon > 0$. Let $a, b \in \mathcal{F}$. Since $\Phi(ab) = \Phi(a)\Phi(b)$, there must exist an element

$m \in \bigoplus \mathbb{M}_{k(n)}$ such that $\psi(ab) - \psi(a)\psi(b) = m$. This implies there exists an N such that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| < \epsilon$ for $n \geq N$. Since $\mathcal{F}$ is finite, there are a finite number of pairs of elements in $\mathcal{F}$ and each pair yields an N in this manner. Selecting the maximum shows that the sequence φ_n is asymptotically multiplicative. Recall that the norm of any element $a \in \prod \mathbb{M}_{k(n)}$ is given by

$$\|\pi(a)\| = \limsup_n \|a_n\|$$

Since $\Phi = \pi \circ \psi$, this immediately implies that the sequence φ_n is asymptotically isometric. ■

The existence of a ccp lift is essential to quasidiagonality. C*-algebras that embed into the quotient algebra $\prod \mathbb{M}_{k(n)} / \bigoplus \mathbb{M}_{k(n)}$ are MF-algebras, a strictly weaker property than quasidiagonality. Note that if A is nuclear and an MF-algebra, then it is also quasidiagonal by the Choi-Effros lifting theorem.

We now introduce some permanence properties and examples.

Proposition 2.2.8. *C*-algebras enjoy the following permanence properties:*

(i) Every C^*-subalgebra of a quasidiagonal C-algebra is quasidiagonal.*

(ii) The unitization of any quasidiagonal C-algebra remains quasidiagonal.*

(iii) If A and B are quasidiagonal, then so is their minimal tensor product $A \otimes B$.

(iv) If A_n are quasidiagonal C-algebras, then the inductive limit $\varinjlim A_n$ is quasidiagonal if the connecting maps are injective.*

(v) If A_n are quasidiagonal C-algebras, then so is the infinite product $\prod A_n$.*

One permanence property which shows that quasidiagonality has a topological nature is due to Voiculescu.

Theorem 2.2.9 (Voiculescu [2])**.** *If A is quasidiagonal and homotopic to a C^*-algebra B, then B is also quasidiagonal.*

Example 2.2.10. The matrices $\mathbb{M}_n$ are trivially quasidiagonal, and by the above proposition, so must all the finite dimensional C*-algebras. Since AF algebras can be realized as inductive limits of finite dimensional algebras with injective connecting maps, (iv) from Prop. 2.2.8 implies that all AF algebras are quasidiagonal. In particular, the universal UHF C^*-algebra $\mathcal{U}$ is quasidiagonal and is given by the inductive limit

$$\mathbb{C} \xrightarrow{\varphi_1} \mathbb{M}_{2!} \xrightarrow{\varphi_2} \mathbb{M}_{3!} \xrightarrow{\varphi_3} \mathbb{M}_{4!} \xrightarrow{\varphi_4} \cdots$$

whose connecting maps φ_n map an element $a \in \mathbb{M}_{n!}$ to $n+1$ copies of a along the diagonal within $\mathbb{M}_{(n+1)!}$.

Example 2.2.11. Commutative C*-algebras $C_0(X)$ are always quasidiagonal. Let $\{x_n\}$ be a dense sequence in X. Using point evaluations $x_n \mapsto f(x_n)$ for $f \in C_0(X)$, we can construct ccp maps to be $\varphi_n(f) = \text{diag}(f(x_1), \ldots, f(x_n)$. These maps are clearly multiplicative and since the sequence $\{x_n\}$ is dense in X, $\|\text{diag}(f(x_1), \ldots, f(x_n))\|$ will converge to $\|f\|$. A similar line of reasoning can be used to show nonseparable $C_0(X)$ are quasidiagonal that the residually finite dimensional (RFD) C*-algebras are also quasidiagonal.

Example 2.2.12. Both the cone $CA = C_0(0,1] \otimes A$ and suspension $SA = C_0(0,1) \otimes A$ are quasidiagonal for any C*-algebra A. Indeed since $C_0[0,1) = C\mathbb{C}$ is homotopic to the zero C*-algebra, Voiculescu's homotopy invariance yields $C_0[0,1)$ is quasidiagonal. This homotopy can be passed through after tensoring with A to yield that CA and its subalgebra SA are quasidiagonal. Interestingly, this also shows that quasidiagonality does not behave well under quotients since every C*-algebra is the quotient of quasidiagonal C^*-algebras $A \simeq CA/SA$.

Example 2.2.13. Group C^*-algebras are fundamental examples within the field of operator algebras. It is well-known that if the reduced group C^*-algebra $C^*_\lambda(G)$ is quasidiagonal for a discrete group G, then G is amenable. In a remarkable accomplishment, Tikuisis-Winter-White proved in [4] that the converse holds: all discrete amenable groups G generate quasidiagonal group C^*-algebras.

2.3 STABLY FINITE C^*-ALGEBRAS

In this section we will introduce stably finiteness and its connection to quasidiagonality. We will provide a class of examples that show quasidiagonality is a stronger property and how this class of examples relates to the Blackadar-Kirchberg conjecture. To motivate the notion of stably finite C^*-algebras, we recall a particular characterization of finite sets: a set X is finite if and only if every injective function $f : X \to X$ is a bijection. Replacing sets with Hilbert spaces and injective functions with isometries, we can generalize this notion to C^*-algebras since every C^*-algebra can be concretely represented as operators on some Hilbert space. We are thus led to the following definition.

Definition 2.3.1. A unital C^*-algebra A is said to be *finite* if every isometry is a unitary, i.e. for every $u \in A$, $u^*u = 1$ implies that $uu^* = 1$. A is said to be *stably finite* if $M_n(A)$ is finite for every $n \in \mathbb{N}$. A non-unital C^*-algebra is said to be finite if its unitization is.

Example 2.3.2. For every natural number n, the C^*-algebras $\mathbb{M}_n$ are stably finite. This follows from the fact that $\mathbb{M}_n$ has a faithful trace. Furthermore, the finite dimensional C^*-algebras stably finite since they are direct sums of matrices. This implies that AF algebras are stably finite.

Example 2.3.3. Let X be a locally compact Hausdorff space. Then the commutative C*-algebra $C_0(X)$ is stably finite. Indeed, since $C_0(X)$ is quasidiagonal, it must be stably finite by Prop. 2.3.6.

Example 2.3.4. Let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical orthonormal basis of the infinite dimensional Hilbert space $l^2(\mathbb{N})$. The unilateral shift operator S defined by $Se_i = e_{i+1}$ is a nonunitary isometry. Indeed, we have that

$$S^*S = 1$$

$$SS^* < 1$$

and hence the C^*-algebra it generates is **not** (stably) finite. There are even examples of C^*-algebra which are finite yet not stably finite, as shown by Clarke in [18].

The notion of finiteness can be phrased in terms of projections instead of unitaries. The predominant way to compare the size of projections is using Murray-von Neumann equivalence, which is the backbone of the construction of $K_0(A)$. Two projections $p, q \in P_\infty(A)$ are said to be Murray-von Neumann equivalent, denoted as $p \sim q$, if there exists a $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. A projection $p \in A$ is called *infinite* if there exists a a projection $q \in A$ such that $p \sim q < p$. If p is not infinite, then it is finite.

Proposition 2.3.5. *The following are equivalent for a unital C^*-algebra.*

(i) A is finite.

(ii) All projections in A are finite.

(iii) $1 \in A$ is a finite projection.

Proof. $(i) \implies (ii)$. Suppose $p, q \in A$ are projections such that $p \sim q \leq p$. Let v be a partial isometry such that $v^*v = p$ and $vv^* = q$. Using the well-known identity $v = qv = pv = qvp$ and the observation that $q \leq p$ implies that $qp = pq = q$, we can write $s = v + (1 - p)$ and calculate that $s^*s = 1$ and $ss^* = 1 - (p - q)$. Since A is finite, $p - q = 0$ and hence all projections are finite.
$(ii) \implies (iii)$. Trivial.
$(iii) \implies (i)$. Suppose $s \in A$ satisfies $s^*s = 1$. Then $1 = s^*s \sim ss^* \leq 1$. But 1 is finite, so $ss^* \sim 1$ and hence $ss^* = 1$.

■

We now present one fundamental relationship between stably finiteness and quasidiagonality.

Proposition 2.3.6. *Every quasidiagonal C^*-algebra A is stably finite.*

Proof. It suffices to show that A is finite since the quasidiagonality and nuclearity of $\mathbb{M}_n$ implies that $\mathbb{M}_n(A) \simeq \mathbb{M}_n \otimes A$ is quasidiagonal. Suppose for the sake of contradiction that A contain a nonunitary isometry s. Let $\varphi_n : A \to \mathbb{M}_{k(n)}$ be a set of ccp maps that are asymptotically multiplicative and asymptotically isometric. By assumption, we must have that $\|\varphi_n(s)\varphi_n(s)^* - 1\|$ is bounded away from 0. Note that $\|T_n^* T_n - 1\| \to 0$ as $n \to \infty$ implies $\|T_n T_n^* - 1\| \to 0$ for any sequence of operators $T_n \in \mathbb{M}_{k(n)}$. Since $s^*s = 1$ implies that $\|\varphi_n(s)^*\varphi(s) - 1\| \to 0$ as $n \to \infty$, we arrive at a contradiction. ■

This gives us many ways to construct C^*-algebras which are not quasidiagonal. Simply take any operator $S \in B(H)$ which is a nonunitary isometry and the C^*-algebra it generates will not be quasidiagonal.

The converse to Prop. 2.3.6 not true. $C^*_\lambda(\mathbb{F}_2)$, the reduced C^*-algebra generated by the free group on 2 generators $\mathbb{F}_2$, is stably finite but not quasidiagonal. Stably finiteness follows immediately from the fact that reduced group C^*-algebras have a faithful trace τ. Indeed, if $s^*s = 1$, then $\tau(ss^*) = \tau(s^*s) = 1$, and hence $ss^* - 1 = 0$. Those accustomed to working with operator algebras will notice that $\mathbb{F}_2$ is not amenable. As discussed in Example 2.2.13, this implies that $C^*_\lambda(\mathbb{F}_2)$ is not quasidiagonal. Furthermore, the lack of amenability of $\mathbb{F}_2$ also implies that $C^*_\lambda(\mathbb{F}_2)$ is not nuclear. Nuclearity, like quasidiagonality, is a matricial approximation property. This leads one to question whether the additional assumption of nuclearity combined with being stably finite is strong enough to capture the matricial approximations induced by quasidiagonality. Indeed, this is exactly what Blackadar and Kirchberg conjectured in [3].

Conjecture 2.3.7 (Blackadar-Kirchberg)**.** *Every separable, stably finite, nuclear C^*-algebra is quasidiagonal.*

Even though we have not yet introduced the Universal Coefficient Theorem (UCT), there has been major progress towards answering this question using the UCT. Recall that a C^*-algebra is *simple* if it contains no nontrivial ideals.

Theorem 2.3.8 (Tikuisis-Winter-White, [4])**.** *Let A be a separable, stably finite, nuclear, simple, unital C^*-algebra that satisfies the UCT. Then A is quasidiagonal.*

This major result leads one to draw their focus to the structure of non-simple C^*-algebras and the non-UCT case. Since extensions of C^*-algebras are a method for creating C^*-algebras that contain nontrivial ideals, it is a natural setting in which to test further examples.

3. EXTENSIONS OF C^*-ALGEBRAS

To examine whether or not all nuclear stably finite C*-algebras are quasidiagonal, we will test the Blackadar-Kirchberg conjecture in the setting of extensions. Given a short exact sequence of C*-algebras $0 \to I \to E \to B \to 0$ such that I and B are nuclear and quasidiagonal, we wish to answer two questions. The first is whether the necessary condition that E is stably finite holds. In general, the answer is no. Recall the unilateral shift operator S from Example 2.3.4. The Toeplitz algebra $\mathcal{T}$ is $C^*(S)$, the C^*-algebra generated by S. Let $H = \ell^2(\mathbb{N})$ be a separable, infinite dimensional Hilbert space with canonical orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Let P_n be the projection onto the subspace spanned by e_n. We will first show that the compact operators $\mathcal{K}$ belong to $\mathcal{T}$. Indeed, it suffices to show that $\mathcal{T}$ contains all the finite rank operators. Let $F_n = S^n(S^*)^n$. Then each P_n can be seen to belong to $\mathcal{T}$ using an iterative process:

$$P_n = (1 - F_n) - \sum_{i=1}^{n-1} P_i$$

The projections P_n can be used to construct any finite rank operator and hence the compact operators sit inside of $\mathcal{T}$ as a closed, two-sided ideal. Observe that $\pi(S)$ is a unitary in $\mathcal{T}/\mathcal{K}$ since $1 - SS^* = P_1$ is compact. One can show that $\sigma(\pi(S)) = \mathbb{T}$ and hence $T/\mathcal{K} = C^*(\pi(S)) = C(\mathbb{T})$.

This yields the following extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

We have already shown that both $\mathcal{K}$ and $C(S^1)$ are quasidiagonal and nuclear. Since extensions of nuclear quotients by nuclear ideals are nuclear, $\mathcal{T}$ is nuclear. On the other hand, $\mathcal{T}$ is not finite. This example demonstrates that we will needed an additional assumption on the center term of a short exact sequence to ensure that it is stably finite. Spielberg observed that this condition can be rephrase in terms of K-theory, which we will examined in Section 3.2. The second question is one of the central focuses of this thesis.

Question 3.0.1. *Let $0 \to I \to E \to B \to 0$ be a short exact sequence, where I and B are separable nuclear quasidiagonal C^*-algebras. If E is stably finite, must it be quasidiagonal?*

In a similar manner, we can also examine the AF embeddability conjecture concerning separable exact quasidiagonal C^*-algebras. Phrased in the context of extensions, we present another question this thesis addresses.

Question 3.0.2. *Let $0 \to I \to E \to B \to 0$ be a short exact sequence, where I and B are separable exact AF embeddable C^*-algebras. If E is quasidiagonal, must it be AF embeddable?*

In order to properly discuss this, we start with the theory of extensions.

3.1 EXTENSIONS AND BUSBY INVARIANTS

Within topology, extensions and quotients are natural constructions that allow one to combine topological spaces in order to create new ones. In a similar vein, extensions of C^*-algebras can be seen as a noncommutative analogue to these methods. Suppose X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous map. It is well-known that f induces a *-homomorphism $f^* : C(Y) \to C(X)$ given by $g \mapsto g \circ f$. Let $g, h \in C(Y)$ be two continuous functions such that $g(f(x)) = h(f(x))$ for all $x \in X$. Should f happen to be surjective, then $g = h$ on all of Y and hence f^* is injective. Let us assume that f is injective. Since X is compact and Y is Hausdorff, X is a closed subspace of Y after identifying it with its image $f(X)$. The Tietze extension theorem allows one to show that for any function $h \in C(X)$ there exists a $g \in C(Y)$

such that $g|_X = h$. But since $g|_X = f^*(g)$, we have that f^* is injective. These two observations show that any short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Y/X \longrightarrow 0$$

of compact Hausdorff spaces induces an extension of C^*-algebras

$$0 \longrightarrow C(Y/X) \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow 0.$$

We generalize this construction to arbitrary C^*-algebras with the following definition.

Definition 3.1.1. Given two C^*-algebras I and B, an *extension of B by I* is any C^*-algebra E that satisfies the following short exact sequence

$$0 \longrightarrow I \xrightarrow{\ \iota\ } E \xrightarrow{\ \pi\ } B \longrightarrow 0$$

where I is a closed, two-sided ideal in E, ι is the inclusion of I into E, and π is the surjective quotient map that induces $B \simeq E/I$.

Example 3.1.2. A (trivial) extension of B by I is given by

$$0 \longrightarrow I \longrightarrow I \oplus B \longrightarrow B \longrightarrow 0$$

Example 3.1.3. Extensions of $\mathbb{C}$ by $C_0(\mathbb{R})$:

$$0 \longrightarrow C_0(\mathbb{R}) \longrightarrow E \longrightarrow \mathbb{C} \longrightarrow 0$$

There are several possibilities for E, namely $C(\mathbb{T}), C_0([0,1)), C_0((0,1])$, and the trivial extension $C_0(\mathbb{R}) \oplus \mathbb{C}$.

Example 3.1.4. Recall the Toeplitz algebra $\mathcal{T}$, the C*-algebra generated by the unilateral shift operator is an extension of $C(\mathbb{T})$ by the compact operators $\mathcal{K}$.

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

In order to properly discuss extensions of C^*-algebras, we will use the framework Busby introduced in [19], which is centered around multiplier algebras. Multiplier algebras, similar to unitizations, are a way of adding a unit to any C^*-algebra. While the unitization can be thought of the minimal way to do this, multiplier algebras can be thought of as the maximal way to add a unit.

Definition 3.1.5. An ideal I of a C^*-algebra A is said to be *essential* if for every nontrivial ideal J in A, $I \cap J \neq \{0\}$.

Essential ideals capture information about an ideal that truly relates to the ambient C^*-algebra. Given two arbitrary C^*-algebras A and B, A is always an ideal of $A \oplus B$, but it is certainly not an essential ideal. For arbitrary ideals I and J in A such that $I \cap J = \{0\}$, the product of any element $a \in I$ with $b \in J$ must equal 0. Conversely, if $T \in I \cap J$, then $IJ = 0$ implies $T^*T = 0$ and hence $T = 0$. The observation $I \cap J = \{0\}$ if and only if $IJ = 0$ is often used without reference.

Proposition 3.1.6. *Let I be an ideal of a C^*-algebra A. The following are equivalent:*
(1) I is essential in A
(2) If $aI = 0$ for some $a \in A$, then $a = 0$.

Proof. Suppose that I is essential in A and that $aI = 0$ for some $a \in A$. Let J be the ideal generated by a, i.e. $J = \overline{\{bab' : b, b' \in A\}}$. For any $d \in I$, $db \in I$ and hence $d(bab') = (db)ab' = 0$. Since $IJ = 0$, the above observation shows that $I \cap J = \{0\}$. But since I is essential in J, $J = 0$ and hence $a = 0$. Conversely, if we assume that I is not essential in A, then there exists a nontrivial ideal J such that $I \cap J = \{0\}$. Let $a \in J$ be a nonzero element. Then $ab \in \ I \cap J$ for all $b \in J$, which shows that $aI = 0$ for a nonzero element a. ■

Definition 3.1.7. The *multiplier algebra* of a C^*-algebra A, denoted as $\mathcal{M}(A)$, is the universal, unital C^*-algebra that contains A as an essential ideal and satisfies the following universal property: Given any C^*-algebra D which contains A as an ideal, there exists a unique *-homomorphism $\varphi : D \to \mathcal{M}(A)$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r] \arrow[dr, hook] & D \arrow[d, dashed, "\varphi"] \\ & \mathcal{M}(A) \end{tikzcd}$$

The map φ is injective if and only if A is essential in D. Upon inspection of the definition it is not clear that multiplier algebras must exist. However, this does indeed hold to be true, see [20] for an in-depth construction of these objects. In general, a concise description of multiplier algebras is hard to provide, especially since they are typically nonseparable. However, there are a few examples that can be given:

Example 3.1.8. The multiplier algebra of any unital C^*-algebra is itself. Indeed since A is an essential ideal $\mathcal{M}(A)$,

$$(1_{\mathcal{M}(A)} - 1_A)A = 1_{\mathcal{M}(A)}A - A = 0$$

which by Proposition 3.1.6 yields $1_{\mathcal{M}(A)} = 1_A$ and hence $A = \mathcal{M}(A)$.

Example 3.1.9. Given a commutative C^*-algebra $C_0(X)$, its multiplier algebra is $C(\beta X)$, where βX is the Stone-Cech compactification of X. For a proof, see [21, Example 3.1.3].

Fix an ideal I and quotient B. In order to classify the possible extensions E, Busby observed that extensions could be placed in a 1-1 correspondence with *-homomorphisms from the quotient B to the *corona algebra* $\mathcal{Q}(I)$, which is the quotient $\mathcal{M}(I)/I$. Consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & I \arrow[r] & \mathcal{M}(I) \arrow[r, "\pi"] & \mathcal{Q}(I) \arrow[r] & 0
\end{tikzcd}$$

The map $\varphi : E \to \mathcal{M}(I)$ exists by the universal property of multiplier algebras, and moreover is injective if and only if I is essential in E. Define the map η to be the composition of φ with the quotient map $\pi : \mathcal{M}(I) \to \mathcal{Q}(I)$, which will make the above diagram commute.

Definition 3.1.10. The map $\eta : B \to \mathcal{Q}(I)$ given above is called the *Busby invariant* of the extension $0 \to I \to E \to B \to 0$.

It should be noted that, similar to the universal property for multiplier algebras, the map η is injective if and only if I is essential in E. Since every extension can be associated to a Busby invariant, it is natural to question whether a given map $\eta : B \to \mathcal{Q}(I)$ induces an extension of B by I.

Definition 3.1.11. Let A_1, A_2, B, be C^*-algebras and let $\varphi_i : A_i \to B$ be *-homo−morphisms. The *pullback* $(C, \{\psi_i\})$ of (A_1, A_2) along (φ_1, φ_2) is a C^*-algebra C and set of *-homomorphisms $\psi_i : C \to A_i$ that satisfy the following universal property: For any C*-algebra D with *-homomorphisms $\sigma_i : D \to A_i$ such that $\varphi_1 \circ \sigma_1 = \varphi_2 \circ \sigma_2$, there exists a *-homomorphism $\theta : C \to D$ such that $\sigma_i = \psi_i \circ \theta$.

The pullback of $(\mathcal{M}(I), B)$ along (π, η) is isomorphic to the subalgebra $E(\eta) := \{(x, b) \in \mathcal{M}(I)) \oplus B : \pi(x) = \eta(b)\}$. In this way a Busby invariant η induces an extension and $\varphi : E \to E(\eta)$ in the following diagram is a *-isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & E(\eta) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

This establishes a bijection between the extensions of B by I and the *-homomorphisms from B to $\mathcal{Q}(I)$. Often extensions and Busby invariants are referred to interchangeably.

Let us now assume that I is stable, i.e. $I \otimes \mathcal{K} \simeq I$. Given an isomorphism between $\mathbb{M}_2 \otimes \mathcal{K} \simeq \mathcal{K}$, there is an induced isomorphism between $\mathbb{M}_2 \otimes \mathcal{M}(I \otimes \mathcal{K}) \simeq \mathcal{M}(I \otimes \mathcal{K})$ and hence one for the corona algebras $\mathbb{M}_2 \otimes \mathcal{Q}(I \otimes \mathcal{K}) \simeq \mathcal{Q}(I \otimes \mathcal{K})$ using the identification $\mathcal{M}(\mathbb{M}_2(I)) \simeq \mathbb{M}_2 \otimes \mathcal{M}(I)$. Given two Busby invariants $\eta_1, \eta_2 : B \to \mathcal{Q}(I \otimes \mathcal{K})$, we can define their sum in the following way:

$$(\eta_1 \oplus \eta_2)(b) := \begin{pmatrix} \eta_1(b) & 0 \\ 0 & \eta_2(b) \end{pmatrix} \in \mathbb{M}_2 \otimes \mathcal{Q}(I \otimes \mathcal{K}) \simeq \mathcal{Q}(I \otimes \mathcal{K})$$

In order to ensure that this sum does not depend on the particular isomorphism $\mathbb{M}_2 \otimes \mathcal{K} \simeq \mathcal{K}$, we will define an equivalence relation on the set of Busby invariants. Two invariants η_1 and η_2 are said to be *strongly equivalent* if there exists a unitary $u \in \mathcal{M}(I)$ such that $\pi(u)^* \eta_1(b) \pi(u) = \eta_2(b)$ for all $b \in B$. In this case we write $\eta_1 \sim_{SE} \eta_2$. Strongly equivalent Busby invariants induce isomorphic extensions.

Definition 3.1.12. A Busby invariant $\eta : B \to \mathcal{Q}(I)$ is called *trivial* if it lifts to a *-homomorphism $\varphi : B \to \mathcal{M}(I)$. If $\eta \oplus \tau$ is strongly equivalent to η for every trivial extension τ, then η is said to be *absorbing.*

If two extensions τ_1 and τ_2 are both trivial and absorbing, then τ_1, $\tau_1 \oplus \tau_2$, and τ_2 are all strongly equivalent and therefore $E(\tau_1) \simeq E(\tau_2)$. Trivial extensions have another nice property. Let τ be a trivial extension and $\psi : B \to \mathcal{M}(I)$ be a lift. Given an arbitrary extension η, recall that $E(\eta) \simeq \{(x, b) \in \mathcal{M}(I)) \oplus B : \pi(x) = \eta(b)\}$. For $x \in \mathcal{M}(I)$ and $b \in B$, we can define an embedding of $E(\eta) \hookrightarrow E(\eta \oplus \tau)$ via the mapping

$$x \oplus b \mapsto \begin{pmatrix} x & 0 \\ 0 & \psi(b) \end{pmatrix} \oplus b.$$

Strong equivalence between Busby invariants does not give them enough structure to be interesting. To remedy this, we will define a second equivalence relation on the set of Busby invariants. We say that $\eta_1 \sim \eta_2$ if there exists trivial extensions τ_1, τ_2 such that $\eta_1 \oplus \tau_1$ is strongly equivalent to $\eta_2 \oplus \tau_2$. The quotient of this relation is an abelian semigroup denoted as $Ext(B, I)$ and we write that $[\eta_1] = [\eta_2]$. This technique was first introduced by Brown-Douglas-Fillmore when they studied extensions of commutative C^*-algebras by the compact operators [22].

The last theorem we will need is a generalization Kasparov gave of Voiculescu's theorem, which asserts that under certain conditions, trivial absorbing extensions exist.

Theorem 3.1.13. *([23, Thm. 15.12.3]) Assume that B is separable, I is σ-unital, and either B or I is nuclear. Let $\rho : B \to B(H)$ be a faithful representation such that H is separable, $\rho(B) \cap \mathcal{K} = \{0\}$ and the orthogonal complement of the nondegeneracy subspace of $\rho(B)$ (i.e. $H \ominus \overline{\rho(B)H}$) is infinite dimensional. Regarding $B(H) \simeq B(H) \otimes 1 \subset \mathcal{M}(\mathcal{K} \otimes I)$ as scalar operators we get a short exact sequence*

$$0 \longrightarrow \mathcal{K} \otimes I \longrightarrow (\rho(B) \otimes 1) + (\mathcal{K} \otimes I) \longrightarrow B \longrightarrow 0.$$

If η is the induced Busby invariant, then η is both trivial and absorbing.

The existence of a trivial absorbing extension η ensures that $Ext(B, I)$ has an identity element. Indeed, by the properties of trivial and absorbing we have that the trivial invariant $\eta \oplus \eta$ is strongly equivalent to η. Using this fact shows that $\gamma \sim \gamma \oplus \eta$ for any arbitrary extension γ since $(\gamma) \oplus \eta \oplus \eta$ is strongly equivalent to $(\gamma \oplus \eta) \oplus \eta$. Under the assumptions of Theorem 3.1.13, the semigroup $Ext(B, I)$ contains an identity and is therefore an abelian monoid. In general, $Ext(B, I)$ is not a group. Let $Ext^{-1}(B, I)$ denote the subgroup of invertible elements within $Ext(B, I)$. When B is nuclear and both I and B are separable, the Choi-Effros theorem implies that $Ext^{-1}(B, I) = Ext(B, I)$ by using a generalized version of Stinespring's theorem to dilate the ccp lift to a *-homomorphism [23, 15.7]. Every Busby invariant $\eta : B \to \mathcal{Q}(I)$ corresponds to a unique extension of the form

$$0 \longrightarrow I \otimes \mathcal{K} \longrightarrow E \longrightarrow B \longrightarrow 0 \tag{3.1}$$

Since quasidiagonality and AF embedability pass to subalgebras, it suffices to work with extensions of the form displayed in Equation 3.1, i.e. elements in $Ext(B, I \otimes \mathcal{K})$.

3.2 SPIELBERG'S CHARACTERIZATION OF STABLY FINITE EXTENSIONS

Spielberg's result characterizes stably finite extensions in terms of K-theory. In order to understand this result, we give a brief overview of K-theory, which is a pair of functors K_0 and K_1 that associates to every C^*-algebra A abelian groups $K_0(A)$ and $K_1(A)$. $K_0(A)$ is created using homotopy classes of projections and $K_1(A)$ is similarly constructed using homotopy classes of unitaries. These groups serve as topological invariants for C^*-algebras and not only can be used to tell C^*-algebras apart, but even serves as a complete invariant for certain classes of C^*-algebras. Readers looking for an in-depth treatment of this subject should consult [23] or [24].

The word projection will always refer to a self-adjoint idempotent, i.e. an element that satisfies $p = p^2 = p^*$. Define $P_n(A)$ to be the set of all projections which belong to $M_n(A)$, $M_\infty(A) = \cup_{n=1}^\infty M_n(A)$, and $P_\infty(A) = \cup_{n=1}^\infty P_n(A)$. Given projections $p, q \in P_\infty(A)$, there is a binary operation on $P_\infty(A)$ defined by

$$p \oplus q = \mathrm{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in P_\infty(A).$$

The projections p and q are said to be *Murray-von Neumann equivalent*, written as $p \sim_0 q$, if there exists a partial isometry $v \in M_{n,m}(A)$ such that $v^*v = p$ and $vv^* = q$. Assume that A is unital. Let $D(A)$ denote the abelian semigroup obtained by quotienting $P_\infty(A)$ under the relationship $\sim_0$. The addition on $D(A)$ is given by

$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}}.$$

Applying the Grothendieck construction to $(\mathcal{D}(A), +)$ yields the abelian group $K_0(A)$, whose elements are denoted as the difference of classes of projections $[p]_0 - [q]_0$ for $p, q \in P_\infty(A)$. Two classes $[p]_0, [q]_0 \in K_0(A)$ are equal if and only if there exists an $r \in P_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. Given a *-homomorphism $\varphi : A \to B$ between unital C^*-algebras, there is an induced map $K_0(\varphi) : K_0(A) \to K_0(B)$ given by $[p]_0 \mapsto [\varphi(p)]_0 \in K_0(B)$. For brevity, we usually write φ_* in place of $K_0(\varphi)$. This construction is also valid in the nonunital case.

If A is nonunital, we will define $K_0(A)$ in the following way. Consider the following split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & \tilde{A} \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, dashed, "\lambda"] \arrow[r] & 0 \end{tikzcd}$$

where $\tilde{A}$ is the unitization of A. $K_0(A)$ is defined to be the kernel of the group homomorphism $K_0(\pi) : K_0(\tilde{A}) \to K_0(\mathbb{C}) = \mathbb{Z}$.

Similar to how $K_0(A)$ is constructed via equivalence classes of projections, $K_1(A)$ is constructed via equivalence classes of unitaries. Let A be a unital C^*-algebra and let $\mathcal{U}(A)$ denote the unitaries in A, i.e. all elements $u \in A$ that satisfy $u^*u = uu^* = 1$. Let $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ and $\mathcal{U}_\infty(A) = \cup_{n=1}^\infty \mathcal{U}_n(A)$. $\mathcal{U}_n(A)$ comes equipped with a natural abelian semigroup structure:

$$u \oplus v = \mathrm{diag}(u, v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_n(A),$$

where $u, v \in \mathcal{U}_n(A)$. Two unitaries $u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A)$ are said to be equivalent (written as $u \sim_1 v$) if there exists a positive integer k such that $u \oplus 1_{k-n}$ is homotopic in $\mathcal{U}_k(A)$ to $v \oplus 1_{k-m}$. Recall that u is homotopic to v in a topological space X if there exists a continuous function $F : [0,1] \to X$ such that

$$F(0) = u$$
$$F(1) = v.$$

For any C^*-algebra A, define

$$K_1(A) = \mathcal{U}_\infty(\tilde{A})/ \sim_1.$$

The functors K_0 and K_1 are split exact but not exact. However, associated to any short exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

is the *6-term exact sequence*

$$\begin{tikzcd} K_1(I) \arrow[r, "K_1(\iota)"] & K_1(A) \arrow[r, "K_1(\pi)"] & K_1(B) \arrow[d, "\partial"] \\ K_0(B) \arrow[u, "\partial"] & K_0(B) \arrow[l, "K_0(\pi)"] & K_0(I) \arrow[l, "K_0(\iota)"] \end{tikzcd}$$

where the two maps $\partial : K_1(B) \to K_0(I)$ and $\partial : K_0(B) \to K_1(I)$ are referred to as the *boundary maps.*

In addition to the group structure of $K_0(A)$, we can also show that in specific cases, $K_0(A)$ is an *ordered* group.

Definition 3.2.1. An *ordered* group is a pair (G, G^+), where G is an abelian group and G^+ is a subset of G that satisfies the following three properties:

(i) $G^+ + G^+ = G^+$

(ii) $G^+ - G^+ = G$

(iii) $G^+ \cap -G^+ = \{0\}$

The subset G^+ is called the *positive cone* of G and induces a partial ordering "$\leq$" on G: $x \leq y$ if $y - x \in G^+$. In this case, the elemental $y - x$ is said to be *positive*. If $y - x \in G^+ \backslash \{0\}$, then $x - y$ is said to be *strictly positive* and we write $x < y$.

Definition 3.2.2. An *order unit* u for an ordered group (G, G^+) is a positive element such that for every $x \in G$, there exists a positive integer n such that $x \leq nu$. In this case we will emphasize the existence of u by writing (G, G^+, u) and say that G is a *scaled* ordered group. If every nonzero positive element in an ordered group is an order unit, then (G, G^+) is said to be *simple*.

Example 3.2.3. By far the most well-known examples are the different possible positive cones of $\mathbb{R}^n$ (or $\mathbb{Z}^n$). The two that occur most frequently are the standard cone $\{v \in \mathbb{R}^n | v_i \geq 0 \text{ for } 1 \leq i \leq n\}$ and the strict cone $\{v \in \mathbb{R}^n | v_i > 0 \text{ for } 1 \leq i \leq n\}$. For both choices of positive cones, the canonical order unit is $(1, 1, \ldots, 1)$, but notice that any strictly positive vector is also an order unit. Therefore, $\mathbb{R}^n$ equipped with the strict cone is a simple ordered group.

Proposition 3.2.4. *Let A be a unital stably finite C^*-algebra. Then the group $K_0(A)$ is an ordered group with positive cone*

$$K_0(A)^+ = \{[p]_0 \in K_0(A) : p \in M_n(A) \text{ is a projection}\}$$

and order unit $[1_A]_0$.

Proof. Part (i) of Definition 3.2.1 holds trivially by the construction of $K_0(A)$. Given two projections $p, q \in P_\infty(A)$, $[p]_0 + [q]_0 = [p \oplus q]_0$.

Part (ii) follows from A being unital. For unital C^*-algebras, $K_0(A)$ is given by the Grothendieck construction applied to the abelian monoid $D(A)$, which are equivalence classes of path-connected projections in $P_\infty(A)$. Since every element of an element in the Grothendieck group is represented by the difference of two classes of elements in $D(A)$, we have that $K_0(A)^+ - K_0(A)^+ = K_0(A)$.

To show (iii), let $x \in K_0(A)^+ \cap -K_0(A)^+$. Then there exist projections $p, q \in P_\infty(A)$ such that $[p]_0 = -[q_0]$ and hence $[\mathrm{diag}(p,q)]_0 = 0$. Being equivalent to 0 in a unital C^*-algebra means that there exists a projection $r \in P_\infty(A)$ such that $p \oplus q \oplus r$ is Murray-von Neumann equivalent to r. Find an $n \in \mathbb{N}$ large enough that contains both $p \oplus q \oplus r$ and r. $M_n(A)$ is finite, which implies that no projection is equivalent to a proper subprojection of itself. We immediately have that $p \oplus q = 0$. This shows that $p = q = 0$, which is to say that $x = 0$.

Fix $x \in K_0(A)$ and find projections $p, q \in P_n(A)$ such that $x = [p]_0 - [q]_0$. Using the identity that $n[1_A]_0 = [1_n]_0$, where 1_n is the unit in $M_n(A)$, we have the following inequalities.

$$\begin{aligned}
-n[1_A]_0 = [1_n]_0 &= -[1_n - q]_0 - [q]_0 \\
&\leq [q]_0 \\
&\leq [p]_0 - [q]_0 = x \\
&\leq [p]_0 \\
&\leq [1_n - p]_0 + [p]_0 \\
&\leq [1_n]_0 = n[1_A]_0
\end{aligned}$$

Therefore $[1_A]_0$ is an order unit of $(K_0(A), K_0(A)^+)$. ■

It should be noted that Part (iii) holds without the assumption of A unital. One must simply pass to the unitization and the proof is identical.

AF algebras always have ordered K_0 groups. Indeed, since unital AF algebras are stably finite, the above proposition ensures that unital AF algebras have ordered K_0 groups. However, using the continuity of the functor K_0 one can reach the same conclusion for AF algebras. Indeed, for any AF algebra A, $K_0 = \lim\limits_{\rightarrow} \mathbb{Z}^{n_i}$. Any such group which is given by the direct limit of finitely many copies of $\mathbb{Z}$ is called a *dimension group*. These groups are obviously countable. In addition, dimension groups are unperforated and have the Riesz interpolation property. An ordered group (G, G^+) is said to be *unperforated* if $nx \geq 0$ implies that $x \geq 0$ for some positive integer n. G is *weakly unperforated* if $nx > 0$ implies that $x > 0$ for some positive integer n. A consequence an ordered group being unperforated is that it must be torsion free. The *Riesz interpolation property* says that given any elements $x_i, y_j \in G$ such that $x_i \leq y_j$ for $i, j, = 1, 2$, there exists a $z \in G$ such that $x_i \leq z \leq y_j$. Effros-Handelman-Shen proved in [25] that these three properties fully characterize dimension groups.

Theorem 3.2.5 (Effros-Handelman-Shen)**.** *Let (G, G^+) be an ordered group. Then G is a dimension group if and only if G is countable, unperforated, and has the Riesz interpolation property.*

One large class of C^*-algebras whose ordered K_0 group is simple is due to Cuntz [23, Prop. 6.3.5]. In particular, as noted in [23, 6.3.6], a consequence of the following proposition is that if A is a unital stably finite simple C^*-algebra or $A \simeq C(X)$ for X compact and connected, then $(K_0(A), K_0(A)^+)$ is simple.

Proposition 3.2.6. *Let A be a unital stably finite C^*-algebra. If every nonzero idempotent in $M_\infty(A)$ is not contained in any proper two-sided ideal, then $K_0(A)$ is a simple ordered group.*

We now give a result about the structure of $K_0(C(X))$ that is due to Husemöller. Let X be a compact, connected Hausdorff space. We can identify $K_0(C(X))$ with the topological K-theory $K^0(X)$ by realizing that any projection $p \in P_n(C(X))$ carves out a complex vector bundle η over X. Under this identification, we can decompose $K_0(C(X))$ as $\mathbb{Z} \oplus \widetilde{K^0(X)}$, where $\widetilde{K^0(X)}$ is the reduced K-theory of X. The class $[p]_0$ of a projection is thus identified with the pair (k, α), where k is the rank of p. When X is a finite dimensional CW-complex, we have the following result regarding the ordering on $K_0(C(X))$.

Theorem 3.2.7 ([26], Thm. 10.1.2)**.** *Let X be a compact, connected finite dimensional CW-complex. There exists a natural number r that depends on the dimension of X such that for every element $(k, \alpha) \in K_0(C(X))$, $k > r$ implies that $(k, \alpha) > 0$.*

There is one last structure regarding ordered groups that we will need. The ordering on $K_0(A)$ is closely relates to traces on a C^*-algebra A.

Definition 3.2.8. A *state* on a simple ordered group (G, G^+, u) is a group homomorphism $\varphi : G \to \mathbb{R}$ such that $\varphi : (G^+) \subset [0, \infty)$ and $\varphi(u) = 1$.

The collection of all states on G is the *state space* $S(G)$ and is convex and compact in the topology of pointwise convergence. The state space is intimately related to the ordering on G, and sometimes even completely determines it.

Theorem 3.2.9 ([27], Thm. 4.12)**.** *Let (G, G^+, u) be an ordered group with order unit u. For every $x \in G$, $s(x) > 0$ for all states $s \in S(G)$ if and only if there exists a positive integer n such that nx is an order unit in G.*

Theorem 3.2.10 ([23], 6.8.5)**.** *Let (G, G^+, u) be a simple weakly unperforated ordered group with order unit u. Then $G^+ = \{0\} \cup \{x \in G : s(x) > 0$ for all $s \in S(G)\}$.*

All of the real-valued affine functions over $S(G)$ form a Banach space $\mathrm{Aff}(S(G))$. For every group element x, there is an induced element in $\mathrm{Aff}(S(G))$ given by $\hat{x}(f) = f(x)$. The infinitesimals of G form the subgroup

$$\mathrm{Inf}(G) = \{x \in G : \hat{x} = 0\}.$$

A more convenient description of this subgroup was given by Dadarlat in [28].

Lemma 3.2.11. *Let (G, G^+, u) be a simple ordered group. Then*

$$\mathit{Inf}(G) = \{x \in G : \forall n \in \mathbb{Z}\ \exists m > 0, m(u + nx) > 0\}.$$

Traces are intimately related to the structure of the state space. A *trace* τ is a positive linear functional on a C^*-algebra such that $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$. If $\|\tau\| = 1$, then τ is called a *tracial state*. The collection of all tracial states is denoted as $T(A)$. Every trace τ on A induces a map $\hat{\tau} : K_0(A) \to \mathbb{R}$ given by

$$\hat{\tau}([p]_0) = \sum_{i=1}^{n} \tau(p_{ii})$$

for $p \in P_n(A)$. In this way we have a map $\chi : T(A) \to S(K_0(A))$. In fact, the domain of this map can be extended to include quasitraces, which are functions from $M_\infty(A) \to \mathbb{C}$ that are linear on commutative subalgebras and satisfy $0 \leq \tau(a^*a) = \tau(aa^*)$ for all a. Denote the collection of all quasitraces on A as $QT(A)$. Clearly then $T(A) \subset QT(A)$. For exact C^*-algebras, $QT(A) = T(A)$ [29] and the map χ is surjective [23]. In the case that A is a stably finite unital C^*-algebra with real rank zero, χ is a bijection [30].

With this information about ordered groups, we are ready to present Spielberg's result that is central to studying the Blackadar-Kirchberg conjecture in the context of extensions.

Theorem 3.2.12 (Spielberg, [5])**.** *Let E be a C^*-algebra, I an ideal in E, and suppose that I and $B = E/I$ are stably finite. Then E is stably finite if and only if*

$$\partial(K_1(B)) \cap K_0(I)^+ = \{0\}. \tag{3.2}$$

Proof. Suppose there exists an element $x \in K_1(B)$ such that $\partial(x) > 0$. Find a unitary $u \in M_n(\widetilde{B})$ such that $[u]_1 = x$, and lift that unitary to a unitary $w \in M_{2n}(\widetilde{E})$ such that $\pi(w) = u \oplus u^*$. Hence $\partial(x) = [w^*(1_n \oplus 0_n)w]_0 - [1_n \oplus 0_n]_0 = [p]_0$ for some projection $p \in M_k(\widetilde{I})$ and therefore $[w(1_n \oplus 0_n)w^*]_0 = [1_n \oplus 0_n \oplus p]_0$. There exist positive integers r, s and unitary $v \in M_{2n+k+r+s}(\widetilde{I})$ such that

$$v(w(1_n \oplus 0_n)w^* \oplus 0_k \oplus 1_r \oplus 0_s)v^* = 1_n \oplus 0_n \oplus p \oplus 1_r \oplus 0_s.$$

Let $R = v(w \oplus 1_{k+r+s})(1_n \oplus 0_n \oplus 0_k \oplus 1_r \oplus 0_s)$. Then $R^*R = (1_n \oplus 0_n \oplus 0_k \oplus 1_r \oplus 0_s) < RR^* = (1_n \oplus 0_n \oplus p \oplus 1_r \oplus 0_s)$ and hence E is not finite by Prop. 2.3.5.

Conversely, if E is not stably finite, then let s be a non-unitary coisometry in $M_n(\widetilde{E})$. Since B is stably finite, $u = \pi(s)$ is a unitary in $M_n(\widetilde{B})$ and therefore $p = 1_n - s^*s$ is a nonzero projection in $M_n(I)$. This implies that $[p] \neq 0$ in $K_0(I)$. Indeed, if $[p]_0 = 0$, then $p \oplus 1_r \oplus 0_s$ is unitarily equivalent to $0_n \oplus 1_r \oplus 0_s$ in $M_{n+r+s}(\widetilde{I})$ and we can use the method in the preceding paragraph to create a non-unitary isometry in $M_{n+r+s}(\widetilde{I})$, contradicting the assumption that I is stably finite. Hence $\partial([u]_1) = [1 - s^*s]_0 - [1 - ss^*]_0 = [p]_0$ is a nonzero element of $K_0(I)^+$. ■

3.3 UNIVERSAL COEFFICIENT THEOREM AND KASPAROV'S KK-THEORY

Rosenberg and Schochet introduced the universal coefficient theorem (UCT), an immensely important tool especially used in classification theory that relates Kasparov's KK-theory to the K-theory of C^*-algebras. Kasparov's KK-theory is an additive bivariate functor that associates an abelian group $KK(A, B)$ to a pair of C^*-algebras (A, B) that is defined by homotopy classes of certain (A, B)-Hilbert bimodules. An in-depth treatment of this subject can be found in [23]. For the purpose of this thesis, we use the following facts. Given separable C^*-algebras A and B, $KK_n(A, B)$ is an abelian group for every $n \in \mathbb{N}$ and they satisfy $KK_n(A, B) \simeq KK_{n+2}(A, B)$. We often write $KK(A, B)$ for $KK_0(A, B)$. One powerful feature of KK-theory is the *Kasparov product*, which is a pairing

$$KK_i(A, D) \;\times\; KK_j(D, B) \longrightarrow KK_{i+j}(A, B)$$

that is an associative composition of KK-elements $(x, y) \mapsto x \cdot y$. Critical to what follows are the following natural isomorphims involving K-theory and $Ext^{-1}(A, B)$.

$$K_*(A) \simeq KK_*(\mathbb{C}, A) \tag{3.3}$$

$$KK_1(A, B) \simeq Ext^{-1}(A, B) \tag{3.4}$$

Note that when A is nuclear, Equation 3.4 implies that $KK_1(A, B) \simeq Ext(A, B)$ using the Choi-Effros lifting theorem. In addition to these identifications, these groups also satisfy Bott periodicity.

Theorem 3.3.1. *For separable C^*-algebras A and B, we have*

$$KK^1(A, B) \simeq KK(A, SB) \simeq KK(SA, B)$$

and $KK(A, B) \simeq KK^1(A, SB) \simeq KK^1(SA, B) \simeq KK(S^2A, B) \simeq KK(A, S^2B) \simeq KK(SA, SB)$, where $S^2A = S(SA)$.

Definition 3.3.2. An element $x \in KK(A, B)$ is said to be a *KK-equivalence* if there exists a $y \in KK(B, A)$ such that $x \cdot y = 1_B$ and $y \cdot x = 1_A$. If there exists a KK-equivalence between separable C^*-algebras A and B, then they are said to be *KK-equivalent.*

It follows that a KK-equivalence between A and B implies that $K_*(A) = K_*(B)$. Indeed a KK-equivalence implies that for all D, $KK_*(A, D) = KK_*(B, D)$, which in turn can be used to show that $K_*(A) = K_*(B)$. The converse holds under the UCT. The proof of this will be provided after the UCT. Let $\mathcal{N}$ denote the family of all separable C^*-algebras that are KK-equivalent to a commutative C^*-algebra.

Theorem 3.3.3 (Universal Coefficient Theorem)**.** *Let $B \in \mathcal{N}$. Then for every separable C^*-algebra A, the following short exact sequence holds.*

$$0 \to Ext^1_{\mathbb{Z}}(K_*(B), K_*(A)) \to KK_*(B, A) \to \mathrm{Hom}(K_*(B), K_*(A)) \to 0$$

In this case, we say that B satisfies the UCT. The above sequences is compactly written and contains information about two exact sequences, which are described below.

$$0 \to \left\{ \begin{array}{c} Ext^1_{\mathbb{Z}}(K_0(A), K_1(B)) \\ \oplus \\ Ext^1_{\mathbb{Z}}(K_1(A), K_0(B)) \end{array} \right\} \to KK(B, A) \to \left\{ \begin{array}{c} \mathrm{Hom}(K_0(B), K_0(A)) \\ \oplus \\ \mathrm{Hom}(K_1(B), K_1(A)) \end{array} \right\} \to 0$$

$$0 \to \left\{ \begin{array}{c} Ext^1_{\mathbb{Z}}(K_0(A), K_0(B)) \\ \oplus \\ Ext^1_{\mathbb{Z}}(K_1(A), K_1(B)) \end{array} \right\} \to KK_1(B, A) \to \left\{ \begin{array}{c} \mathrm{Hom}(K_0(B), K_1(A)) \\ \oplus \\ \mathrm{Hom}(K_1(B), K_0(A)) \end{array} \right\} \to 0$$

Proposition 3.3.4. *Let A, B satisfy the UCT. Then A and B are KK-equivalent if $K_*(A) = K_*(B)$.*

Proof. Since A and B satisfy the UCT, there exists an $\alpha \in KK_0(B, A)$ that induces $K_*(A) = K_*(B)$, i.e.

$$KK_*(\mathbb{C}, A) \simeq K_*(A) \xrightarrow{\alpha_*} K_*(B). \simeq KK_*(\mathbb{C}, B)$$

By functoriality of the UCT, we have the following diagram for all C^*-algebras D

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ext_1^{\mathbb{Z}}(K_*(A), K_*(D)) & \longrightarrow & KK(A, D) & \longrightarrow & \mathrm{Hom}(K_*(A), K_*(D)) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha_*} & & \Big\downarrow{\scriptstyle \circ\alpha} & & \Big\downarrow{\scriptstyle \alpha_*} & & \\ 0 & \longrightarrow & Ext_1^{\mathbb{Z}}(K_*(B), K_*(D)) & \longrightarrow & KK(B, D) & \longrightarrow & \mathrm{Hom}(K_*(B), K_*(D)) & \longrightarrow & 0 \end{array}$$

In particular, since the left and the right side morphisms are isomorphisms induced by α_*, the middle must also be an isomorphism by the five lemma. When $D = B$, this yields an isomorphism between $KK(A, B)$ and $KK(B, B)$. Letting $\beta \in KK(A, B)$ be the preimage of $1_B \in KK(B, B)$ under this isomorphism, we have that $\beta \cdot \alpha = 1_A$. This implies that the induced maps $\beta_* : K_*(B) \to K_*(A)$ are isomorphisms. Repeating the argument using $\beta \in KK(A, B)$ instead of $\alpha \in KK(B, A)$ yields a $\gamma \in KK(B, A)$ that satisfies $\gamma \cdot \beta = 1_B$. But then clearly $\gamma = \gamma \cdot \beta \cdot \alpha = \alpha$, which yields the necessary KK-equivalence between A and B. ■

The last structure included in the statement of the UCT is the functor $Ext^1_{\mathbb{Z}}(-, G)$. Consider the following short exact sequence of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{3.5}$$

For any abelian group G, the contravariant functor $\mathrm{Hom}(-, G)$ is left exact and when applied to 3.5 yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G) \tag{3.6}$$

To continue this exact sequence, one must introduce the functor $Ext^1_{\mathbb{Z}}(-, G)$. We will omit the construction of this group and its connecting maps. For an in-depth exposition on the subject refer to [31]. However, it should be noted that the construction is independent of the choice of short exact sequence (3.5). The only property of $Ext^1_z(-, G)$ we will use in this thesis is the case when G is divisible. An abelian group G is called *divisible* if $nG = G$ for every $n \in \mathbb{N}$. Divisible groups, the most prominent example being the rational numbers $\mathbb{Q}$, are injective groups and therefore satisfy $Ext^1_{\mathbb{Z}}(-, G) = 0$ [31, Prop. 11.4.6].

3.4 RESULTS ON EXTENSIONS BY N. BROWN AND DADARLAT

We are now equipped with the preliminary tools necessary to examine Question 3.0.1: Given quasidiagonal C^*-algebras I and B and an extension E of B by I, does stably finiteness of E imply that it is quasidiagonal? In this section we will introduce results from N. Brown and Dadarlat in [7] that give affirmative answers under the context of extensions whose class in $Ext(B, I)$ is 0.

Proposition 3.4.1. *[7, Prop. 2.5] Let $0 \to I \to E \to B \to 0$ be an exact sequence with Busby invariant γ. Let γ^s denote the Busby invariant of the short exact sequence $0 \to I \otimes \mathcal{K} \to E \otimes \mathcal{K} \to B \otimes \mathcal{K} \to 0$ that is induced by γ. If both I and B are quasidiagonal, B is separable, I is σ-unital, either I or B is nuclear and $[\gamma^s] = 0 \in Ext(B \otimes \mathcal{K}, I \otimes \mathcal{K})$, then E is also quasidiagonal.*

Proof. By Theorem 3.1.13, there exists a trivial absorbing extension $\tau : B \otimes \mathcal{K} \to \mathcal{Q}(I \otimes \mathcal{K})$. Since $[\gamma^s] = 0$, $\gamma^s \oplus \tau$ is trivial and absorbing. Indeed, there exist trivial extensions $\tau_1, \tau_2 : B \otimes \mathcal{K} \to \mathcal{Q}(\mathcal{K} \otimes I)$ such that $\gamma^s \oplus \tau_1 \sim_{SE} \tau_2$. Therefore we have that

$$\gamma^s \oplus \tau \sim_{SE} \gamma^s \oplus \tau \oplus \tau_1 \sim_{SE} \tau_2 \oplus \tau \sim_{SE} \tau.$$

From this we see that $\gamma^s \oplus \tau$ must be trivial and absorbing since it's strongly equivalent to τ. This yields the following chain of embeddings.

$$E(\gamma) \hookrightarrow E(\gamma^s) \hookrightarrow E(\gamma^s \oplus \tau) \simeq E(\tau).$$

Since quasidiagonality passes to C^*-subalgebras, it suffices to show that $E(\tau)$ is quasidiagonal. Referencing the extension in Theorem 3.1.13 that induces τ, we have that $E(\tau) \hookrightarrow (\rho(B) + \mathcal{K}) \otimes \tilde{I}$, where $\tilde{I}$ is the unitization of I. The trivial intersection $\rho(B) \cap \mathcal{K} = \{0\}$ ensures that $\rho(B) + \mathcal{K}$ is quasidiagonal and hence $(\rho(B) + \mathcal{K}) \otimes \tilde{I}$ is quasidiagonal by Proposition 2.2.8. ■

Recall that for a short exact sequence $0 \to I \to E \to B \to 0$, there is an associated 6-term sequence that is exact at every term:

$$\begin{array}{ccccc} K_1(I) & \longrightarrow & K_1(E) & \longrightarrow & K_1(B) \\ \uparrow & & & & \downarrow \\ K_0(B) & \longleftarrow & K_0(E) & \longleftarrow & K_0(I) \end{array}$$

As discussed in Section 3.2, we know that E being stably finite places a restriction on the image of the boundary map $\partial : K_1(B) \to K_0(I)$. If we assume that B satisfies the UCT and that the boundary map is 0, then we are able to pass quasidiagonality to the extension.

Theorem 3.4.2. *Let $0 \to I \to E \to B \to 0$ be a short exact sequence where E is separable, I is quasidiagonal, and B is nuclear, quasidiagonal, and satisfies the UCT. If the induced map $\partial : K_1(B) \to K_0(I)$ is zero, then E is quasidiagonal.*

Proof. In order to use Proposition 3.4.1, we must construct a short exact sequence whose class in $Ext(B, I)$ is 0l. To do so, we first use Proposition 3.3 from [7] to find a σ-unital quasidiagonal C^*-algebra J such that $\iota : I \hookrightarrow J$ is an approximately unital embedding and $K_1(J) = 0$. After stabilizing and tensoring by the Universal UHF algebra $\mathcal{U}$ we have the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J \otimes \mathcal{K} \otimes \mathcal{U} & \longrightarrow & E' \otimes \mathcal{K} \otimes \mathcal{U} & \longrightarrow & B \otimes \mathcal{K} \otimes \mathcal{U} & \longrightarrow & 0 \end{array}$$

Let $\gamma : \mathcal{Q}(B) \to J$ denote the Busby invariant of the second short exact sequence. The boundary map $\partial : K_0(B) \to K_1(J)$ vanishes trivially and $\partial : K_1(B) \to K_0(J)$ vanishes by naturality of the boundary maps.

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\partial} & K_1(I) \\ \| & & \downarrow {\scriptstyle K_1(\iota)} \\ K_0(B) & \xrightarrow{\partial} & K_1(J) \end{array}$$

Denote the Busby invariant of the third extension by η. Again by naturality, the boundary maps associated to η vanish. Furthermore, we have that $K_0(B \otimes \mathcal{K} \otimes$

$\mathcal{U}) = K_0(B) \otimes \mathbb{Q}$, which is divisible. Hence $Ext^1_{\mathbb{Z}}(K_0(B) \otimes \mathbb{Q}, K_0(J) \otimes \mathbb{Q}) = 0$ and therefore the extension $[\eta] = 0$ since B satisfies the UCT. Proposition 3.4.1 finishes the proof. ■

Definition 3.4.3. A quasidiagonal C^*-algebra A has the *QD extension property* if for every separable nuclear quasidiagonal C^*-algebra B that satisfies the UCT and Busby invariant $\eta : B \to \mathcal{Q}(A \otimes \mathcal{K})$ we have that $E(\eta)$ is stably finite if and only if $E(\eta)$ is quasidiagonal.

Naturally, one would like to show that every separable C^*-algebra satisfies the QD extension property. As it turns out, this property is related to another definition which is more deeply connected with controlling the K-theory of embeddings. Recall that any short exact sequence $0 \to I \to E \to B \to 0$ has an associated 6-term exact sequence. If we assume that E is stably finite, then Spielberg's characterization says that $\partial(K_1(B)) \cap K_0(I)^+ = \{0\}$. These subgroups represent the subgroups of $K_0(I)$ that can be the image of the boundary map for stably finite extensions. If we expect E to be quasidiagonal, the embedding $\varphi : I \hookrightarrow E$ must satisfy $\varphi_*(\partial(K_1(B)) = 0$ by exactness of the 6-term exact sequence. As we will see in Prop. 3.4.6, this particular embedding existing for all such subgroups is sufficient to ensure than an extension is quasidiagonal.

Definition 3.4.4. Let (G, G^+) be an ordered group. The singular elements of G, denoted as $\mathrm{Sing}(G)$, is the set $\{x \in G : \mathbb{Z}x \cap G^+\} = \{0\}$. If a subgroup is contained inside $\mathrm{Sing}(G)$, then it is called a *singular subgroup*. If A is a C^*-algebra with ordered group $(K_0(A), K_0(A)^+)$, then for brevity we write $\mathrm{Sing}(A)$ in place of $\mathrm{Sing}(K_0(A))$.

Definition 3.4.5. A quasidiagonal C^*-algebra A has the *K_0-embedding property* if for any singular subgroup G of $K_0(A)$, there exists a quasidiagonal C^*-algebra C and an embedding $\rho : A \hookrightarrow C$ such that $\rho_*(G) = 0$.

Proposition 3.4.6 (Prop. 4.6 [7]). *A separable C^*-algebra A satisfies the QD extension property if and only if it has the K_0-embedding property.*

Prop. 3.4.6 demonstrates the critical role K-theory plays for quasidiagonal extensions. Given Voiculescu's characterization of quasidiagonality, one can ask how well the asymptotically multiplicative and asymptotically isometric ccp maps capture the information contained in the K-theory. This gives rise to property weaker than the K_0-embedding property known as the K_0-Hahn-Banach property

Definition 3.4.7. A quasidiagonal C^*-algebra A has the K_0-*Hahn-Banach property* if for every $x \in \mathrm{Sing}(A)$ there exists a sequence of asymptotically multiplicative and asymptotically isometric ccp maps $\varphi_n : A \to \mathbb{M}_{k(n)}$ such that $(\varphi_n)_\#(x) = 0$ for n large enough.

One should immediately note that for a ccp map $\varphi : A \to \mathbb{M}_n$ the notion of an induced morphism $\varphi_* : K_0(A) \to K_0(B)$ is not defined. However, given a sequence of asymptotically multiplicative and asymptotically isometric ccp maps $\varphi_n : A \to \mathbb{M}_{k(n)}$, there is a well-defined way to talk about the induced behavior on K_0. To do this, we need an elementary observation regarding perturbing elements to projections. For every $\epsilon > 0$ there exists a $\delta > 0$ such that for any C^*-algebra A and any $a \in A$, if $\|a^2 - a\| < \delta$ and $\|a^* - a\| < \delta$, there exists a projection $p \in A$ such that $\|a - p\| < \epsilon$.

Let $0 < \epsilon < 1/4$ and $p \in P_\infty(A)$. Recall that if $p \in M_k(A)$ then $\varphi_n(p)$ is understood to be the matrix $(\varphi_n(p_{i,j})) \in M_k(B)$. There exists a $\delta > 0$ such that whenever $\|\varphi_n(p) - \varphi_n(p)^2\| < \delta$, we have that $\|\varphi_n(p) - \widetilde{p_n}\| < \epsilon$, where $\widetilde{p_n}$ is the projection whose existence is guaranteed by the above fact. Since there exists an N such that $n \geq N$ implies that φ_n is δ-multiplicative, we may define $(\varphi_n)_\#(p) = \widetilde{p_n}$ for $n \geq N$ and 0 elsewhere. Note that regardless of which N is chosen to construct the above sequence, any two choices will produce sequences which are tail equivalent. This tail represents the information regarding the K-theortic properties of φ_n and its asymptotic behavior.

While for individual C^*-algebras it was shown in [7, Prop. 4.10] that the K_0-embedding property implies the K_0-Hahn-Banach property, it turns out they are equivalent in the following sense.

Theorem 3.4.8. *The following statements are equivalent.*

(i) *Every separable, nuclear, quasidiagonal C^*-algebra has the QD extension property.*

(ii) *Every separable, nuclear, quasidiagonal C^*-algebra has the K_0-embedding property.*

(iii) *Every separable, nuclear, quasidiagonal C^*-algebra has the K_0-Hahn-Banach property.*

(iv) *If A is any separable, nuclear, quasidiagonal C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is quasidiagonal (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*

(v) *If A is any separable, nuclear, quasidiagonal C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

4. EXTENSIONS BY CONNECTIVE C^*-ALGEBRAS AND BY $C(X) \otimes \mathcal{K}$

In this chapter we discuss connective C^*-algebras, a notion introduced by Dadarlat and Pennig in [9] [10], and stabilized commutative C^*-algebras. Section 4.2 answers Question 3.0.1 and 3.0.2 affirmatively when the ideals are exact and connective. Section 4.3 considers ideals of the form $C(X) \otimes \mathcal{K}$, which, unlike connective C^*-algebras, have lots of projections. In particular, for compact metrizable spaces X, we answer Question 3.0.1 and Question 3.0.2 affirmatively when the ideals are of the form $C(X) \otimes \mathcal{K}$ and with the additional assumption that quotient is nuclear and satisfies the UCT.

4.1 DEFINITIONS AND EXAMPLES

Theorem 4.2.7 is motivated by the following example due to N. Brown and Dadarlat [7]. Let $0 \to SI \to E \to B \to 0$ be a short exact sequence, where I is σ-unital and B separable, nuclear, and quasidiagonal. Then E is quasidiagonal. This result follows by the homotopy invariance of $\mathit{Ext}(-,-)$ and that the suspension SI is null-homotopic. In particular, this is done without the use of the UCT. The suspension of any separable C^*-algebra is connective, a notion introduced by Dadarlat and Pennig in [9]. Originally introduced for the purposes of unsuspending in E-theory, we will use an embedding result of Gabe and the homotopy invariance of $\mathit{Ext}(-,-)$ to generalize the result of N. Brown and Dadarlat to connective ideals.

Definition 4.1.1. A separable C^*-algebra A is *connective* if there exists a *-monomorphism

$$\Phi : A \hookrightarrow \prod_{n=1}^{\infty} CB(\mathcal{H}) \Big/ \bigoplus_{n=1}^{\infty} CB(\mathcal{H})$$

that lifts to a ccp map $\varphi : A \to \prod_n CB(\mathcal{H})$, where $\mathcal{H}$ is an separable, infinite dimensional Hilbert space.

Dadarlat and Pennig in [10] proved that the primitive spectrum of any connective C^*-algebra had no nonempty compact open sets. Gabe proved that this property fully characterizes connectivity [11].

Theorem 4.1.2. *A separable exact C^*-algebra is connective if and only if its primitive ideal space has no non-empty, compact, open subsets.*

Example 4.1.3. Given a compact, connected metrizable space X, the C^*-algebra $C_0(X \backslash x_0)$ will be connective, where $x_0 \in X$ is a base point.

Example 4.1.4. For any separable C^*-algebra A, the cone CA is connective. Since connectivity passes to subalgebras, the suspension SA is connective as well.

Here are some general properties concerning connective C^*-algebras.

Theorem 4.1.5. *Connective C^*-algebras have the following properties:*

(i) Connective C^*-algebras are quasidiagonal.

(ii) Connective C^*-algebras do not contain any nonzero projections.

(iii) The class of nuclear connective C^*-algebras is closed under (minimal) tensoring with any other C^*-algebra.

One additional remark about the structure of connective C^*-algebras is needed. If B is exact and A is exact and connective, then $A \otimes B$ is exact and connective by Remark 2.8 in [9].

4.2 EXTENSIONS BY CONNECTIVE C^*-ALGEBRAS

In this section we prove two theorems about the structure of extensions for connective ideals without assuming the UCT. When the ideal is connective and exact, Theorem 4.2.7 provides an affirmative answer for Question 3.0.1 and Theorem 4.2.9 provides an affirmative answer for 3.0.2. Before they are presented, we must introduce a fundamental technique that allows one to pass an from an extension in $Ext(B, I)$ to an extension in $Ext(B, J)$ using a suitable embedding $I \hookrightarrow J$.

An *approximate unit* of A is a net of positive elements (e_λ) in the closed unit ball of A such that for any $a \in A$, $\|e_\lambda a - a\|$ tends to 0 as $\lambda \to \infty$. Approximate units always exist and when A is separable they may be chosen to be a sequential.

Definition 4.2.1. Let A and B be C^*-algebras and $\varphi : A \to B$ an injective *-homomorphism. φ is said to be *approximately unital* if there exists an approximate unit e_λ of A whose image $\varphi(e_\lambda)$ is an approximate unit of B.

Approximately unital embeddings provide a method to embed an extension into a new one without changing the quotient. Any approximately unital inclusion $A \to B$ induces an inclusion of multiplier algebras $\mathcal{M}(A) \subset \mathcal{M}(B)$ and hence corona algebras [32, 3.12.12]. For a given short exact sequence $0 \to I \to E \to B \to 0$ and approximately unital embedding $I \hookrightarrow J$, this process yields the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & J & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

One way to ensure an embedding is approximately unital is by using hereditary C^*-subalgebras. A C^*-subalgebra B of A is called *hereditary* if for any $a \in A^+$ and $b \in B^+$ the inequality $a \leq b$ implies $a \in B$. Given a subset $H \subset A$, the hereditary C^*-subalgebra generated by H is the smallest hereditary subalgebra of A containing H. We give two theorems regarding the structure of such algebras.

Theorem 4.2.2 ([21], Thm. 3.2.2)**.** *Let B be a C^*-subalgebra of A. Then B is a hereditary subalgebra if and only if $bab' \in B$ for all $b, b' \in B$ and $a \in A$.*

Theorem 4.2.3 ([21], Thm. 3.2.5)**.** *For every element $a \in A^+$, the C^*-subalgebra $\overline{aAa}$ is a hereditary subalgebra of A. Conversely, if B is a separable hereditary subalgebra of A, there exists a positive element $s \in B^+$ such that $\overline{sAs} = B$.*

Proposition 4.2.4. *Let A be a separable C^*-algebra, H a C^*-subalgebra of A, and B the hereditary C^*-subalgebra of A generated by H. Then the embedding $H \hookrightarrow B$ is approximately unital.*

Proof. Let e_n be an approximate unit of H and let $s = \sum e_n/2^n$. We will show that the hereditary C^*-subalgebra $\overline{sAs} = B$. Since B contains H, $s \in B$. By Theorem 4.2.2, we know that $sas \in B$ for all $a \in A$, which implies that $\overline{sAs} \subset B$. For the other direction, we note that since B is hereditary, Theorem 4.2.3 ensures that there exists a $b \in B^+$ such that $B = \overline{bAb}$. Furthermore, $s \in \overline{sAs}$ implies that $e_n \in \overline{sAs}$ because $e_n/2^n \leq s$. For any $a \in A$, $e_n(bab)e_n \in \overline{sAs}$ by Theorem 4.2.2 and therefore $B = \overline{bAb} \subset \overline{sAs}$. The last thing to show is that e_n is an approximate unit for $\overline{sAs}$. Indeed, for any $a \in A$, $\|e_n sas - sas\| \leq \|(e_n s - s)\| \, \|as\| \to 0$. Since e_n is an approximate unit for sAs, it is for $\overline{sAs} = B$ as well. ∎

With Prop. 4.2.4 in mind, we examine a result due to Gabe.

Theorem 4.2.5 ([11], Thm. A)**.** *Let I be an exact connective C*-algebra. Then I embeds into the Rørdam algebra $\mathcal{A}[0,1]$.*

Rørdam's algebra $\mathcal{A}[0,1]$ is the direct limit of the C^*-algebras

$$C_0([0,1), M_2) \xrightarrow{\varphi_1} C_0([0,1), M_4) \xrightarrow{\varphi_2} C_0([0,1), M_8) \xrightarrow{\varphi_3} \cdots \to \mathcal{A}[0,1]$$

where

$$\varphi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(\max(t, t_n)) \end{pmatrix}$$

for a dense sequence $\{t_n\}$ in $[0,1)$. Since $\mathcal{A}[0,1]$ is the inductive limit of $C_0[0,1) \otimes M_{2^n}$ and the K-theory of $C_0[0,1)$ vanishes, the K-theory of both $\mathcal{A}[0,1]$ and $\mathcal{A}[0,1] \otimes \mathcal{K}$ vanish. Furthermore, $\mathcal{A}[0,1]$ is an inductive limit of C*-algebras that satisfy the UCT and therefore also satisfies the UCT. This implies that $\mathcal{A}[0,1]$ and $\mathcal{A}[0,1] \otimes \mathcal{K}$ are KK-equivalent to 0 by Prop. 3.3.4.

One crucial step for the results in this section is the fact that Gabe's embedding can be taken to be approximately unital.

Proposition 4.2.6. *Let I be an exact connective C^*-algebra. The embedding $I \hookrightarrow \mathcal{A}[0,1]$ can be taken to be approximately unital.*

Proof. Let B be a hereditary subalgebra of $\mathcal{A}[0,1]$, which is nuclear since $\mathcal{A}[0,1]$ is nuclear. We will first show that the ideal lattice $\mathcal{I}(B)$ is homeomorphic to the unit interval $[0,1]$. The ideals of $\mathcal{A}[0,1]$ are order isomorphic to $[0,1]$ and denoted as I_t for $t \in [0,1]$. The embedding $\iota : B \hookrightarrow \mathcal{A}[0,1]$ induces a continuous map on the ideal lattices $\hat{\iota} : \mathcal{I}(\mathcal{A}[0,1]) \to \mathcal{I}(B)$ given by $\hat{\iota}(I_t) := \iota^{-1}(I_t) = I_t \cap B$. Notice that $\hat{\iota}$ is surjective since the ideals of a hereditary subalgebra are of the form $I \cap B$ for an ideal I in $\mathcal{A}[0,1]$. This implies that $\mathcal{I}(B)$ is compact and has a single connected component. Additionally, one can see that $\mathcal{I}(B)$ is totally ordered under containment of ideals. By [33, Prop. 1.1.5], the topology induced by a total order coincides with the canonical topology on the ideal lattice. Since $\mathcal{I}(B)$ is a totally ordered space which is compact, connected, and metrizable in the order topology (B is separable), it must be isomorphic to a compact subset of $\mathbb{R}$ with a single connected component, i.e. $\mathcal{I}(B)$ is order isomorphic to $[0,1]$.

Note that B is purely infinite because it's a hereditary C^*-subalgebra of the purely infinite algebra $\mathcal{A}[0,1]$. Since $\mathcal{I}(B)$ and $\mathcal{I}(\mathcal{A}[0,1])$ are both order isomorphic to $[0,1]$ B is stable by [12, Prop. 5.1]. Additionally, $\mathcal{A}[0,1]$ is also stable, so [34, Cor. 6.14] implies that $B \otimes \mathcal{O}_2 \simeq \mathcal{A}[0,1] \otimes \mathcal{O}_2$. Hence the embedding $B \hookrightarrow B \otimes \mathcal{O}_2 \simeq \mathcal{A}[0,1] \otimes \mathcal{O}_2 \simeq \mathcal{A}[0,1]$ is approximately unital, where the last isomorphism holds by [35, Prop. 6.1]. Let B equal the hereditary subalgebra of $\mathcal{A}[0,1]$ generated by the image of I in the original embedding. Prop. 4.2.4 completes the proof. ■

Theorem 4.2.7 (D)**.** *Let $0 \to I \to E \to B \to 0$ be a short exact sequence of C^*-algebras such that I is exact and connective and B is separable, nuclear, and quasidiagonal. Then E is quasidiagonal.*

Proof. As remarked at the end of Section 3.1, we can without loss of generality assume our extension is of the form

$$0 \longrightarrow I \otimes \mathcal{K} \longrightarrow E \longrightarrow B \longrightarrow 0.$$

Let σ denote the Busy invariant of the above extension. Since I is connective and exact and $\mathcal{K}$ is nuclear, $I \otimes \mathcal{K}$ is connective and exact. By Proposition 4.2.6, there exists an embedding $\iota : I \otimes \mathcal{K} \to \mathcal{A}[0,1]$ that is approximately unital and therefore induces an extension $\eta \in \mathrm{Ext}(B, \mathcal{A}[0,1])$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & I \otimes \mathcal{K} \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}[0,1] \arrow[r] & E(\eta) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{A}[0,1]$ and B are quasidiagonal, showing that $[\eta] = 0$ is sufficient to conclude that $E \subset E(\eta)$ is quasidiagonal by Proposition 3.4.1. Because $\mathcal{A}[0,1]$ is KK-equivalent to 0 and B is nuclear, Theorem 3.3.1 implies that

$$Ext(B, \mathcal{A}[0,1]) \simeq KK^1(B, \mathcal{A}[0,1]) \simeq KK^1(B, 0) \simeq 0$$

and hence $[\eta] = 0$. ■

Similar to Theorem 3.4.1, we will require a result that ensures AF embeddability under suitable conditions.

Proposition 4.2.8. *[36, Prop. 2.1] Let $0 \to I \to E \to B \to 0$ be an essential semi-split extension of separable C^*-algebras whose class vanishes in $Ext^{-1}(B, I)$. Suppose that both I and B are AF embeddable. Then E is AF embeddable.*

Theorem 4.2.9 (D)**.** *Let $0 \to I \to E \to B$ be a short exact sequence of C^*-algebras such that I is exact and connective and B is nuclear and AF-embeddable. Then E is AF embeddable.*

Proof. Since AF embeddability passes to C^*-subalgebras, we are able to work with the extension

$$0 \longrightarrow I \otimes \mathcal{K} \longrightarrow E \longrightarrow B \longrightarrow 0.$$

$I \otimes \mathcal{K}$ is connective and exact, so by Proposition 4.2.6 we can ensure that the embedding $I \otimes \mathcal{K} \to \mathcal{A}[0,1]$ is approximately unital and therefore yields the second row of the diagram below. This extension is essential if and only if $I \otimes \mathcal{K}$ is essential in E, which in general is not true. To remedy this, we will apply Lemma 1.12 from [5] to obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes \mathcal{K} \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}[0,1] \arrow[r] \arrow[d, hook] & E(\eta) \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}[0,1] \otimes \mathcal{K} \arrow[r] & E(\gamma) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

By construction, $\mathcal{A}[0,1] \otimes \mathcal{K}$ is essential in $E(\gamma)$. Since $\mathcal{A}[0,1] \otimes \mathcal{K}$ is KK-equivalent to 0, we have that

$$[\gamma] \in Ext(B, \mathcal{A}[0,1] \otimes \mathcal{K}) \simeq KK_1(B, \mathcal{A}[0,1] \otimes \mathcal{K}) \simeq KK_1(B, 0) = 0$$

Now the third line of the above diagram is an essential extension of C^*-algebras whose class in $Ext(B, \mathcal{A}[0,1] \otimes \mathcal{K})$ vanishes. Additionally, since B and $\mathcal{A}[0,1] \otimes \mathcal{K}$ are nuclear, so must $E(\gamma)$ and therefore the extension is semisplit. $\mathcal{A}[0,1]$ and hence $\mathcal{A}[0,1] \otimes \mathcal{K}$ is AF embeddable [37, Cor. 6.4]. By Prop. 4.2.8, $E(\gamma)$ is AF embeddable. ■

4.3 EXTENSIONS BY $C(X) \otimes \mathcal{K}$

In this section we consider ideals of the form $C(X) \otimes \mathcal{K}$, which, unlike connective C^*-algebras, contain lots of projections. We show that ideals of this form also give an affirmative answer to Question 3.0.1 and Question 3.0.2 when the quotient is nuclear and satisfies the UCT. To do so, we will require Proposition 3.4.6 and an astounding lifting theorem due to C. Schafhauser [38].

Theorem 4.3.1 (Schafhauser)**.** *Suppose A is a separable, unital, exact C^*-algebra satisfying the UCT and B is a simple, unital, $\mathcal{U}$-stable C^*-algebra with unique trace τ_B such that every quasitrace on B is a trace and $K_1(B) = 0$. If τ_A is a faithful, amenable trace on A and $\sigma : K_0(A) \to K_0(B)$ is a group homomorphism such that $\hat{\tau}_B\sigma = \hat{\tau}_A$ and $\sigma([1_A]) = [1_B]$, then there is a unital, faithful, nuclear *-homomorphism $\varphi : A \to B$ such that $K_0(\varphi) = \sigma$ and $\tau_B\varphi = \tau_A$.*

In order to use Schafhauser's theorem, we will construct an AF algebra whose K_0 group is order isomorphic to $K_0(C(X) \otimes \mathcal{U})$. Spielberg demonstrated a way to construct a group homomorphism between dimension groups that contains a predetermined singular subgroup in its kernel. We will use this result to obtain a second AF algebra and our desired group homomorphism between their K_0 groups. Using a lifting theorem of Elliott's, we are able to construct a *-homomorphism between these AF algebras that induces the map on K-theory described above. These two results are presented below.

Lemma 4.3.2 ([5]). *Let G be a dimension group and $H \subset G$ be a subgroup such that $H \cap G^+ = \{0\}$. Then there is a dimension group G' and dimension group morphism $\varphi : G \to G'$ such that*

(i) $H \subset \ker \varphi$

(ii) $G^+ \cap \ker \varphi = \{0\}$.

Theorem 4.3.3 ([39]). *Let A and B be AF algebras and $\sigma : K_0(A) \to K_0(B)$ be a homomorphism of scaled ordered groups. Then there exists a *-homomorphism $\varphi : A \to B$ such that $\varphi_* = \sigma$.*

Lemma 4.3.4. *(D) Let X be a connected, finite dimensional CW-complex. There exists an AF algebra A and an embedding $C(X) \hookrightarrow C(X) \otimes \mathcal{U} \hookrightarrow A$ such that $K_0(C(X) \otimes \mathcal{U})$ is order isomorphic to $K_0(A)$.*

Proof. As discussed in the paragraph preceding Theorem 3.2.7, $K_0(C(X))$ can be identified with the topological K-theory $K^0(X)$. This yields the decomposition

$$K_0(C(X)) \simeq \mathbb{Z} \oplus \widetilde{K^0(X)}$$

where $\widetilde{K^0(X)}$ is the reduced K-theory of X obtained by removing an arbitrary base point in X. For brevity, we will denote this group as H. $C(X) \otimes \mathcal{U}$ is the inductive limit of $C(X) \otimes \mathbb{M}_{n!}$ and therefore

$$(K_0(C(X) \otimes \mathcal{U}), K_0(C(X) \otimes \mathcal{U})^+, [1]_0) = \varinjlim K_0(C(X) \otimes \mathbb{M}_{n!})$$

as scaled ordered groups. Let $G_n = K_0(C(X) \otimes \mathbb{M}_{n!})$ and observe that for all $n \in \mathbb{N}$, $G_n = \mathbb{Z} \oplus H$. This yields the following inductive limit

$$\mathbb{Z} \oplus H \xrightarrow{\alpha_1} \mathbb{Z} \oplus H \xrightarrow{\alpha_2} \mathbb{Z} \oplus H \xrightarrow{\alpha_3} \cdots \longrightarrow \mathbb{Q} \oplus (H \otimes \mathbb{Q})$$

where $\alpha_n((z,h)) = ((n+1)z, (n+1)h)$, $\alpha_{m,n} = \alpha_{m-1} \circ \alpha_{m-2} \circ \cdots \circ \alpha_n : G_n \to G_m$, and $\alpha_{\infty,n} : G_n \to \mathbb{Q} \oplus (H \otimes \mathbb{Q})$ are the maps induced by the universal property of inductive limits. The identification $K_0(C(X) \otimes \mathcal{U}) \simeq \mathbb{Q} \oplus (H \otimes \mathbb{Q})$ will be used frequently. We will first prove that $K_0(C(X) \otimes \mathcal{U})$ is a dimension group. To do this, we will show that an element $(q,h) \in K_0(C(X) \otimes \mathcal{U})$ is positive if and only if $q > 0$. Indeed if $(q,h) \in \mathbb{Q} \oplus (H \otimes \mathbb{Q})$ is such that $q > 0$, we can find an $n \in \mathbb{N}$ and element $(z,g) \in G_n$ such that $\alpha_{\infty,n}((z,g)) = (q,h)$. By the nature of the connecting maps α_n, $q > 0$ implies $z > 0$. Since X is a finite dimensional complex, Theorem 3.2.7 implies that there exists an $m \in \mathbb{N}$ such that the amplification $\alpha_{m,n}((z,g))$ belongs to G_m^+. Recall that

$$K_0(C(X) \otimes \mathcal{U})^+ = \bigcup_{k=1}^{\infty} \alpha_{\infty,k}(G_k^+)$$

and therefore $(q,h) = (\alpha_{\infty,m} \circ \alpha_{m,n})((z,g)) > 0$. Conversely, if $(q,h) > 0$ there exists an element $(z,h) \in G_n^+$ such that $\alpha_{\infty,n}((z,h)) = (q,h)$. Since X is connected, the ordering on $K_0(C(X) \otimes \mathbb{M}_{n!})$ ensures that $z > 0$ and therefore $q > 0$.

This observation on the ordering of $K_0(C(X) \otimes \mathcal{U})$ implies that its positive cone is the set $\{(q,h) \in \mathbb{Q} \oplus (H \otimes \mathbb{Q}) : q > 0\} \cup \{0\}$. Since $\mathbb{Q}$ has the Riesz interpolation property, so must $K_0(C(X) \otimes \mathcal{U})$. Furthermore, tensoring a group with $\mathbb{Q}$ removes perforation. Using the identification $K_0(C(X) \otimes \mathcal{U}) \simeq K_0(C(X) \otimes \mathbb{Q}$, we may conclude that $K_0(C(X) \otimes \mathcal{U})$ is unperforated and therefore a dimension group by Theorem 3.2.5. In fact, we can even conclude that the infinitesimals of $K_0(C(X) \otimes \mathcal{U})$ is the subgroup $H \otimes \mathbb{Q}$ and that $K_0(C(X) \otimes \mathcal{U})$ is a simple ordered group.

Let A be a unital AF algebra whose scaled ordered group $(K_0(A), K_0(A)^+, [1]_0)$ is order isomorphic to $(K_0(C(X) \otimes \mathcal{U}), K_0(C(X) \otimes \mathcal{U})^+, [1]_0)$. A must have a unique trace. Indeed, since all of the infinitesimals of $K_0(C(X) \otimes \mathcal{U})$ are of the form $(0,h)$ for $h \in H \otimes \mathbb{Q}$, any trace τ on A induces a state on $K_0(A)$ such that $\hat{\tau}((0,h)) = 0$.

This implies that for any element $(q,h) \in K_0(A)$, $\hat{\tau}((q,h))$ depends only on q. There is a unique group morphism from $\mathbb{Q}$ to $\mathbb{R}$ mapping 1 to 1 and hence the state space $S(K_0(A))$ consists of a unique state. Since A is a unital AF algebra, the canonical map $\chi : T(A) \to S(K_0(A))$ is a bijection and we conclude that there is a unique trace on A.

Observe that using a similar argument to the one listed above, we know that A is $\mathcal{U}$-stable, i.e. $A \simeq A \otimes \mathcal{U}$. Now A is simple, unital, and has a unique trace ρ. Being an AF algebra, every quasitrace on A is a trace and $K_1(A) = 0$. Consider the trace $\tau : C(X) \otimes \mathcal{U} \simeq C(X,\mathcal{U}) \to \mathbb{C}$ given by $f \mapsto \tau(f(x))$, where x is a fixed point in X and τ is the unique trace on $\mathcal{U}$. Letting σ denote the identity map between the order isomorphic groups $K_0(C(X) \otimes U)$ and $K_0(A)$, we have that $\hat{\rho} \circ \sigma = \hat{\tau}$. We may use Theorem 4.3.1 to lift σ to an embedding $\varphi : C(X) \otimes \mathcal{U} \hookrightarrow A$. Composing the trivial embedding $\iota : C(X) \hookrightarrow C(X) \otimes \mathcal{U}$ with φ finishes the proof. ■

We will need the following fact. Given a compact metrizable space consisting of finitely many connected components $X_1, \ldots, X_n$, each of which is a finite dimensional CW-complex, an element $x = (x_1, \ldots, x_n) \in K_0(C(\sqcup X_i)) = K_0(C(X_1)) \oplus \cdots \oplus K_0(C(X_n))$ is positive if and only if each x_i in positive in $K_0(C(X_i))$. Let A_i be the AF algebras obtained from Lemma 4.3.4 into which $C(X_i) \otimes \mathcal{U}$ embeds and set $A = A_1 \oplus \cdots \oplus A_n$. The Schafhauser embeddings $\varphi_i : C(X_i) \otimes \mathcal{U} \hookrightarrow A_i$ can be summed together to yield an embedding $\varphi : C(X) \otimes \mathcal{U} \hookrightarrow A$ that induces an order isomorphism on the K-theory. Indeed since $K_0(A) = K_0(A_1) \oplus \cdots \oplus K_0(A_n)$, the positive cone $K_0(A)^+$ equals $K_0(A_1)^+ \oplus \cdots \oplus K_0(A_n)^+$.

Lemma 4.3.5. *(D) Let X be a compact, metrizable space. Then $C(X) \otimes \mathcal{U}$ has the K_0-embedding property.*

Proof. Let X be a compact, metrizable space. Write $C(X)$ as the inductive limit of $C(X_n)$, where each X_n is a finite dimensional CW-complex with finitely many connected components [40, Thm. 10.1]. After tensoring everything with $\mathcal{U}$, we will denote the connecting maps from $C(X_n) \otimes \mathcal{U} \to C(X_{n+1}) \otimes \mathcal{U}$ as ψ_n. Let φ_i :

$C(X_i) \otimes \mathcal{U} \hookrightarrow A_i$ be the embeddings discussed in the preceding paragraph, where each A_i is an AF algebra. Let $\sigma_i : K_0(A_i) \to K_0(A_{i+1})$ be the group homomorphism that makes the following diagram commute.

$$\begin{tikzcd}
K_0(C(X_1)\otimes\mathcal{U}) \arrow[r, "(\psi_1)_*"] \arrow[d, equal] & K_0(C(X_2)\otimes\mathcal{U}) \arrow[r, "(\psi_2)_*"] \arrow[d, equal] & \cdots \arrow[r] & K_0(C(X)\otimes\mathcal{U}) \arrow[d, dashed, "\varphi_*"] \\
K_0(A) \arrow[r, "\sigma_1"] & K_0(A_2) \arrow[r, "\sigma_2"] & \cdots \arrow[r] & K_0(A)
\end{tikzcd}$$

Note that the map φ_* induced by the diagram is an order isomorphism since all the vertical maps are order isomorphisms. By Theorem 4.3.3, each σ_i lifts to a *-homomorphism $\rho_i : A_i \to A_{i+1}$ that induces σ_i. This yields the following diagram of C^*-algebras.

$$\begin{tikzcd}
C(X_1)\otimes\mathcal{U} \arrow[r, "\psi_1"] \arrow[d, hook, "\varphi_1"] & C(X_2)\otimes\mathcal{U} \arrow[r, "\psi_2"] \arrow[d, hook, "\varphi_2"] & \cdots \arrow[r] & C(X)\otimes\mathcal{U} \arrow[d, hook, dashed, "\varphi"] \\
A_1 \arrow[r, "\rho_1"] & A_2 \arrow[r, "\rho_2"] & \cdots \arrow[r] & A
\end{tikzcd}$$

where $\varphi : C(X) \otimes \mathcal{U} \hookrightarrow A$ is an embedding and induces φ_*.

Let G be any singular subgroup of $K_0(C(X) \otimes U)$. By Lemma 4.3.2, there exists a dimension group H and morphism $f : K_0(C(X)\otimes U) \to H$ such that $f(G) = 0$ and $\ker(f) \cap H^+ = \{0\}$. Let B be an AF algebra such that $K_0(B) = H$ and lift f to an injective *-homomorphism $\gamma : A \hookrightarrow B$. Recall that the embedding $\varphi : C(X)\otimes\mathcal{U} \hookrightarrow A$ induces an order isomorphism on K_0 and therefore the composition

$$C(X) \otimes \mathcal{U} \overset{\varphi}{\hookrightarrow} A \overset{\gamma}{\hookrightarrow} B$$

is injective and and satisfies $(\gamma_* \circ \varphi_*)(G) = 0$. ■

Theorem 4.3.6 (D). *Let* $0 \to C(X) \otimes \mathcal{K} \to E \to B \to 0$ *be a short exact sequence where* X *is a compact metrizable space and* B *is a separable nuclear quasidiagonal* C^**-algebra that satisfies the UCT. If* E *is stably finite, then* E *is quasidiagonal.*

Proof. Since $\mathcal{U}$ is an AF algebra, $E \otimes \mathcal{U}$ will be stably finite. This follows from writing $E \otimes \mathcal{U}$ as the inductive limit of $E \otimes \mathbb{M}_{n!}$ and using the stably finiteness of

E. Therefore, we can tensor the given extension with $\mathcal{U}$ and create the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & C(X)\otimes\mathcal{K} \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(X)\otimes\mathcal{K}\otimes\mathcal{U} \arrow[r] & E\otimes\mathcal{U} \arrow[r] & B\otimes\mathcal{U} \arrow[r] & 0
\end{tikzcd}$$

Since $E\otimes\mathcal{U}$ is stably finite, we have that Spielberg's condition on the boundary map is satisfied. Without loss of generality, it suffices to prove that $E\otimes\mathcal{U}$ must be quasidiagonal, for E embeds into it. Lemma 4.3.5 showed that $C(X)\otimes\mathcal{U}$ has the K_0 embedding property. This immediately implies that $C(X)\otimes\mathcal{U}\otimes\mathcal{K}$ has it as well. Indeed, let G be a singular subgroup of $K_0(C(X)\otimes\mathcal{U})$ and use Lemma 4.3.4 to obtain a *-homomorphism $\varphi : C(X)\otimes\mathcal{U}\hookrightarrow B$ such that $\varphi_*(G) = 0$. The induced map $(\varphi\otimes\mathrm{id})_* : K_0(C(X)\otimes\mathcal{U}\otimes\mathcal{K}) \to K_0(B\otimes\mathcal{K})$ arising from the embedding $\varphi\otimes\mathrm{id} : C_0(X)\otimes\mathcal{U}\otimes\mathcal{K} \to B\otimes\mathcal{K}$ will contain G in its kernel. Prop. 3.4.6 completes the proof. ∎

Theorem 4.3.7 (D)**.** *Let $0 \to C(X)\otimes\mathcal{K} \to E \to B \to 0$ be a short exact sequence where X is a compact metrizable space and B is a nuclear AF embeddable C^*-algebra that satisfies the UCT. If E is stably finite, then E is AF embeddable.*

Proof. We will show that E embeds into an AF embeddable C^*-algebra E'. Using the same arguments in the preceding theorem, we know there exists a unital AF algebra A and unital embedding $C(X)\otimes\mathcal{U}\hookrightarrow A$. Therefore the embedding $C(X)\otimes\mathcal{U}\otimes\mathcal{K}\hookrightarrow A\otimes\mathcal{K}$ is approximately unital. Tensor the given short exact sequence with the universal UHF algebra $\mathcal{U}$ and use the approximately unital embedding mentioned above to yield the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & C(X)\otimes\mathcal{K} \arrow[r] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(X)\otimes\mathcal{K}\otimes\mathcal{U} \arrow[r] \arrow[d, hook] & E\otimes\mathcal{U} \arrow[r] \arrow[d, hook] & B\otimes\mathcal{U} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A\otimes\mathcal{K} \arrow[r] & E' \arrow[r] & B\otimes\mathcal{U} \arrow[r] & 0
\end{tikzcd}$$

By assumption, E and hence $E \otimes \mathcal{U}$ is stably finite. In particular, Lemma 3.2.12 ensures that the image of boundary map $\partial : K_1(B \otimes \mathcal{U}) \to K_0(C(X) \otimes \mathcal{K} \otimes \mathcal{U})$ is a singular subgroup. By construction, $K_0(A)$ and $K_0(C(X) \otimes \mathcal{U})$ are order isomorphic and hence the naturality of the boundary map implies that E' is stably finite. Observe that since B is nuclear, AF embeddable, and satisfies the UCT, $B \otimes \mathcal{U}$ will also have these three properties. Additionally, $A \otimes \mathcal{K}$ is an AF algebra and is therefore nuclear and satisfies the UCT. These conditions on $B \otimes \mathcal{U}$ and $A \otimes \mathcal{K}$ imply that the extension E' is nuclear and satisfies the UCT. By [5, Thm. 1.15], E' is AF embeddable.

$\blacksquare$

5. OPEN QUESTIONS

Question 1 Is every nuclear stably finite C*-algebra necessarily quasidiagonal?

The Blackadar-Kirchberg conjecture is still open in its fullest generality. Even with remarkable progress by many researchers, there is still room for exploration in the nonsimple case. With the major progress made by Tikuisis-Winter-White in [4], extensions are a natural framework in which to examine this question.

Question 2 Do all separable nuclear quasidiagonal C^*-algebras have the K_0-embedding property?

As shown to be equivalent to the Blackadar-Kirchberg conjecture in the context of extensions in [7], this remains an interesting question to explore. Not only due to its inherent interest due to quasidiagonality, but also since it's related to controlling the K-theory of embeddings, which originated in the work of Pimsner and Voiculescu [6].

Question 3 Do all separable nuclear quasidiagonal C^*-algebras have the K_0-Hahn-Banach property?

On the class of separable nuclear quasidiagonal C^*-algebras, [7] showed this to be equivalent to Question 2. However, on an individual basis, the K_0-embedding property implies the K_0-Hahn-Banach property. This property is considerably easier to work with since it deals with only a single element in $\mathrm{Sing}(A)$ at a time, and as described earlier allows one to work with asymptotically multiplicative and asymptotically isometric ccp maps instead of *-homomorphisms.

Question 4 Is every separable, exact, quasidiagonal C^*-algebra AF-embeddable?

This is another open problem concerning the nature of quasidiagonality. Since AF algebras are one of the primary classes of examples in the field, having a complete characterization of their C^*-subalgebras would be immensely helpful. One of the most surprising results is due to Ozawa, stating that the cone (and hence suspension) of any separable exact C^*-algebra is AF embeddable [8]. Gabe's embedding of exact connective C^*-algebras into Rørdam's algebra, which is AF embeddable, generalizes this result.

REFERENCES

[1] D. Voiculescu. A non-commutative Weyl-von Neumann theorem. *Rev. Roumaine Math, Pures Appl*, 21:97–113, 1976.

[2] D. Voiculescu. A note on quasidiagonal C^*-algebras and homotopy. *Duke mathematical journal*, 62(2):267–271, 1991.

[3] B. Blackadar and E. Kirchberg. Generalized inductive limits of finite-dimensional C^*-algebras. *Mathematische Annalen*, 307(3):343–380, 1997.

[4] W. Winter A. Tikuisis and S. White. Quasidiagonality of nuclear C^*-algebras. *Annals of Mathematics*, 185:229–284, 2016.

[5] John S Spielberg. Embedding C^*-algebra extensions into AF algebras. *Journal of functional analysis*, 81(2):325–344, 1988.

[6] M. Pimsner and D. Voiculescu. Imbedding the irrational rotation algebras into an AF algebra. *Journal of Operator Theory*, 4:201–210, 1980.

[7] N. Brown and M. Dadarlat. Extensions of quasidiagonal C^*-algebras and K-theory. In *Operator algebras and applications*, pages 65–84. Mathematical Society of Japan, 2004.

[8] N. Ozawa. Homotopy invariance of AF-embeddability. *Geometric & Functional Analysis*, 13(1):216–222, 2003.

[9] M. Dadarlat and U. Pennig. Deformations of nilpotent groups and homotopy symmetric C^*-algebras. *Mathematische Annalen*, 367:121–134, 2017.

[10] M. Dadarlat and U. Pennig. Connective C^*-algebras. *J. Functional Analysis*, 272:4919–4939, 2017.

[11] J. Gabe. Traceless AF embeddings and unsuspended E-theory. *Geometric and Functional Analysis*, 30:323–333, 2020.

[12] Mikael Rørdam. A purely infinite AH-algebra and an application to AF-embeddability. *Israel Journal of Mathematics*, 141(1):61–82, 2004.

[13] L. Brown. Extensions of AF algebras: the projection lifting problem. In *Proc. Symp. Pure Math*, volume 38, pages 175–176, 1982.

[14] M-D. Choi and E. Effros. The Completely Positive lifting problem for C^*-algebras. *Annals of Mathematics*, 104(3):585–609, 1976.

[15] N. Brown and N. Ozawa. *C^*-algebras and Finite-Dimensional Approximations*, volume 88. American Mathematical Soc., 2008.

[16] P. R. Halmos. Ten problems in Hilbert spaces. *Bull. Amer. Math. Soc.*, 76(5):887–993, 1970.

[17] N. Brown. On quasidiagonal C^*-algebras. In *Operator algebras and applications*, pages 19–64. Mathematical Society of Japan, 2004.

[18] N. P. Clarke. A Finite but not Stably Finite C*-algebra. *Proceedings of the Amer. Math. Soc.*, 96(1):85–88, 1986.

[19] R. Busby. Double Centralizers and Extensions of C^*-algebras. *Transactions of the American Mathematical Society*, 132:79–99, 1968.

[20] E.C. Lance. *Hilbert C^*-modules: a toolkit for operator algebraists*. Cambridge ; New York : Cambridge University Press, 1995.

[21] G. J. Murphy. *C^*-algebras and Operator Theory*. Academic Press, 2014.

[22] R.G. Douglas L. Brown and P. Fillmore. Extensions of C^*-algebras and K-homology. *Annals of Mathematics*, pages 265–324, 1977.

[23] B. Blackadar. *K-Theory for Operator Algebras*. Cambridge University Press, 1998.

[24] M. Rørdam, F. Larsen, and N.J. Laustsen. *An Introduction to K-theory for C*-algebras*, volume 49. Cambridge University Press, 2000.

[25] D. Handelman E. Effros and C. Shen. Dimension groups and their representations. *American Journal of Mathematics*, 102(2):385–407, 1980.

[26] D. Husemöller. *Fibre Bundles*. Springer, 1966.

[27] K.R. Goodearl. *Partially Ordered Abelian Groups With Interpolation*. American Mathematical Society, 1986.

[28] M. Dadarlat. Morphisms of Simple Tracially AF Algebras. *International Journal of Mathematics*, 15:919–957, 2004.

[29] U. Haagerup. Quasitraces on exact C^*-algebras are traces. Preprint version arXiv:1403.7653 [math-OA] , 1992.

[30] B. Blackadar and D. Handelman. Dimension functions and traces on C^*-algebras. *Journal of Functional Analysis*, 45:297–340, 1982.

[31] P. Bland. *Rings and Their Modules*. Walter de Gruyter, 2011.

[32] G. Pedersen. *C^*-algebras and their Automorphism Groups*. Academic Press, 1979.

[33] J. Mortensen. Classification of certain non-simple C^*-algebras. *J. Operator Theory*, 41:223–259, 1999.

[34] J. Gabe. A new proof of Kirchberg's $\mathcal{O}_2$-stable classification theorem. *Journal fuer die Reine und angewandte Mathematik*, 2018.

[35] E. Kirchberg and M. Rordam. Purely infinite C^*-algebras: Ideal-preserving zero homotopies. *Geometric and Functional Analysis*, 15:377–415, 2005.

[36] M. Dadarlat. On AF embeddability of continuous fields. *C. R. Math. Acad. Sci. Soc. R. Can.*, 31(1):16–19, 2009.

[37] M. Rørdam. A Purely Infinite AH-algebra and an application to AF-embeddability. *Israeli Journal of Mathematics*, pages 61–82, 2004.

[38] C. Schafhauser. Subalgebras of simple AF-algebras. *Ann. Of Math.* (to appear). Preprint version arXiv:1807.07381 [math-OA], 2018.

[39] G. Elliott. On the classification of inductive limits of sequences of semisimple finite-diemnsional algebras. *J. Algebra*, pages 29–44, 1976.

[40] E. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology.* Princeton University Press, 1952.

VITA

Jacob R. Desmond was born in Baton Rouge, LA, on January 15th, 1992. After receiving his B.S. in Mathematics with a minor in German from Louisiana State University, he moved to Purdue University to pursue a Ph.D. in Mathematics. After completing his degree in August 2020, he will work at the Cold Regions Research and Engineering Laboratory in Hanover, New Hampshire.