ON THE DEFINING IDEALS OF REES RINGS FOR DETERMINANTAL AND PFAFFIAN IDEALS OF GENERIC HEIGHT

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ABSTRACT

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This dissertation is based on joint work with Monte Cooper and is broken into two main parts, both of which study the defining ideals of the Rees rings of determinantal and Pfaffian ideals of generic height. In both parts, we attempt to place degree bounds on the defining equations.

The first part of the dissertation consists of Chapters 3 to 5. Let $R = K[x_1, \ldots, x_d]$ be a standard graded polynomial ring over a field K, and let I be a homogeneous R-ideal generated by s elements. Then there exists a polynomial ring $\mathcal{S} = R[T_1, \ldots, T_s]$, which is also equal to $K[x_1, \ldots, x_d, T_1, \ldots, T_s]$, of which the defining ideal of $\mathcal{R}(I)$ is an ideal. The polynomial ring $\mathcal{S}$ comes equipped with a natural bigrading given by $\deg x_i = (1, 0)$ and $\deg T_j = (0, 1)$. Here, we attempt to use specialization techniques to place bounds on the x-degrees (first component of the bidegrees) of the defining equations, i.e., the minimal generators of the defining ideal of $\mathcal{R}(I)$. We obtain degree bounds by using known results in the generic case and specializing. The key tool are the methods developed by Kustin, Polini, and Ulrich in [45] to obtain degree bounds from approximate resolutions. We recover known degree bounds for ideals of maximal minors and submaximal Pfaffians of an alternating matrix. Additionally, we obtain x-degree bounds for sufficiently large T-degrees in other cases of determinantal ideals of a matrix and Pfaffian ideals of an alternating matrix. We are unable to obtain degree bounds for determinantal ideals of symmetric matrices due to a lack of results in the generic case; however, we develop the tools necessary to obtain degree bounds once similar results are proven for generic symmetric matrices.

The second part of this dissertation is Chapter 6, where we attempt to find a bound on the T-degrees of the defining equations of $\mathcal{R}(I)$ when I is a nonlinearly presented homogeneous perfect Gorenstein ideal of grade three having second analytic deviation one that is of linear type on the punctured spectrum. We restrict to the case where $\mathcal{R}(I)$ is not Cohen-Macaulay. This is a natural next step following the work of Morey, Johnson, and Kustin-Polini-Ulrich in [54], [39], and [48], respectively. Based on extensive computation in Macaulay2, we give a conjecture for the relation type of I and provide some evidence for the conjecture. In an attempt to prove the conjecture, we obtain results about the defining ideals of general fibers of rational maps, which may be of independent interest. We end with some examples where the bidegrees of the defining equations exhibit unusual behavior.

1. INTRODUCTION

Let R be a Noetherian ring and I be an R-ideal. The Rees ring of I, denoted $\mathcal{R}(I)$, is the subring $R[It] \subseteq R[t]$ and is isomorphic to $\bigoplus_{k=0}^{\infty} I^k$. Rees rings are of interest for a variety of reasons. For example, since $\mathcal{R}(I) \cong \bigoplus_{i=0}^{k} I^k$, $\mathcal{R}(I)$ encodes information about all powers of the ideal I. Many invariants of an ideal, such as multiplicities, depend on the asymptotic properties of the powers I^k. As such, studying the Rees ring of I can give insight into these invariants. Additionally, Rees rings are useful in the study of the integral closure of ideals. In particular, an R-ideal I is integrally closed in R if and only if $\mathcal{R}(I) = R[It]$ is integrally closed in $R[t]$ as a ring. Although there are algorithms to compute the integral closure of special classes of ideals, such as monomial ideals (see [30, 1.4.3] for monomial ideals and, more generally, [60, Chapter 15]), the only known general method to compute the integral closure of an ideal I is to compute the integral closure of the Rees ring $\mathcal{R}(I)$.

Rees rings also play an important role in algebraic geometry. One application of Rees rings is in resolution of singularities. Let V be a variety and W be a subvariety. One can perform the process of blowing up V along W. This process "pulls apart" V along W. The blowup projects onto the original variety V. After a finite number of blowups, one may achieve a nonsingular variety which projects onto the original variety V. For more details, see [27, pp. 28-30] or [21, Section 5.2]. To make this more precise, let K be a field and V be a variety in $\mathbb{A}^n_K$. Suppose W is a subvariety of V. Let R be the coordinate ring of V and $I = I(W)$ in R. Then $\mathcal{R}(I)$ is the homogeneous coordinate ring of the blowup of V along W.

On the other hand, suppose $R = K[x_1, \ldots, x_d]$ is a standard graded polynomial ring over the field K and

$$\Phi : \mathbb{P}^{d-1}_K \dashrightarrow \mathbb{P}^{s-1}_K$$

is a rational map between projective spaces defined by

$$[a_1 : \cdots : a_d] \mapsto [f_1(a_1, \ldots, a_d) : \cdots : f_s(a_1, \ldots, a_d)]$$

for some homogeneous polynomials $f_1, \ldots, f_s$ in R of the same degree D. Let $I = (f_1, \ldots, f_s)$. Then $\mathcal{R}(I)$ is the bihomogeneous coordinate ring of the graph of Φ. As such, the study of rational maps is aided by the study of Rees rings.

In most of the examples above, a great deal of insight can come from understanding the implicit equations defining the Rees ring $\mathcal{R}(I)$. Let R be a Noetherian ring and $I = (f_1, \ldots, f_s)$ be an R-ideal. The definition of $\mathcal{R}(I)$ as the subring $R[It] \subseteq R[t]$ gives a parametric definition of $\mathcal{R}(I)$. However, there exists a polynomial ring $\mathcal{S} = R[T_1, \ldots, T_s]$ and a natural R-algebra epimorphism $\pi : \mathcal{S} \to \mathcal{R}(I)$ with $T_i \mapsto f_i t$. Let $\mathcal{J}$ denote the kernel of π. A generating set of $\mathcal{J}$ is a set of implicit equations defining $\mathcal{R}(I)$. Often, implicit defining equations provide a great deal more insight than parametric defining equations. For instance, with implicit defining equations, it is much easier to detect if a point lives on the subvariety in question. Additionally, determining ring-theoretic properties and invariants is much easier with implicit defining equations than with parametric equations. Further, having a defining ideal, rather than a defining subring, allows the use of Gröbner basis techniques for computations. Indeed, there has even been use in studying Rees rings from the field of geometric modeling and image processing, specifically from the aspect of implicitization. See, for example, [18].

A considerable amount of work has been done to study the defining equations of certain types of determinantal and Pfaffian ideals.

In particular, many have contributed to the study of the defining equations in the setting of perfect ideals of grade two. Much can be said about such ideals thanks to the Hilbert-Burch structure theorem, which characterizes these ideals as the ideal generated by the maximal minors of an $n \times (n + 1)$ matrix [11]. Contributions to the study of defining equations in the case of perfect ideals of grade two include Herzog-Simis-Vasconcelos [28], Morey [54], Morey-Ulrich [55], Cox-Hoffman-Wang [17], Hong-Simis-Vasconcelos [32], Busé [13], Kustin-Polini-Ulrich [44], Cortadellas

Benítez-D'Andrea [14] and [15], Lan Nguyen [56], Madsen [51], Boswell-Mukundan [5], Kustin-Polini-Ulrich [47], and Kim-Mukundan [42].

Similarly, some work has been done to study the defining equations in the case of perfect Gorenstein ideals of grade three. Indeed, the Buchsbaum-Eisenbud structure theorem characterizes all such ideals as the ideal generated by the submaximal Pfaffians of an alternating matrix of odd size [10]. Contributions to the study of the defining equations in the case of perfect Gorenstein ideals of grade three include Morey [54], Johnson [39], and Kustin-Polini-Ulrich [48].

There has been some work to study the defining equations in the settings of other types of determinantal ideals as well. In particular, in [6], Bruns, Conca, and Varbaro study the defining equations of $\mathcal{R}(I)$ in the setting of non-maximal minors of a generic matrix and make a conjecture on the degrees of the defining equations. In [7], Bruns, Conca, and Varbaro study the defining equations of $\mathcal{R}(I)$ in the setting of maximal minors of a matrix with linear entries. Additionally, in [33], Huang, Perlman, Polini, Raicu, and Sammartano prove parts of the conjecture of [6] in the case of the 2×2 minors of a generic matrix over the field of complex numbers $\mathbb{C}$.

This dissertation is based on joint work with Monte Cooper. The aim of this work is to study the defining equations of $\mathcal{R}(I)$ when I is a determinantal ideal or Pfaffian ideal of generic height. The main techniques are the use of specialization and approximate resolutions. Eisenbud and Huneke developed the theory of specialization of Rees rings in [22]; however, we do not study the specialization of Rees rings in and of itself. Rather, given an ideal J and a specialization I of J, we approximate $\mathcal{R}(I)$ using the specialization of $\mathcal{R}(J)$, particularly in the case where J is a determinantal or Pfaffian ideal of a generic matrix. The tools we employ related to approximate resolutions come from the work of Kustin, Polini, and Ulrich in [45, 3.8, 4.8]. Specifically, we use approximate resolutions of the specialization of powers of determinantal and Pfaffian ideals of generic matrices to place degree bounds on the defining equations of the Rees ring $\mathcal{R}(I)$, where I is a determinantal or Pfaffian ideal of a matrix which is sufficiently "close"to being generic.

This dissertation is organized in the following manner.

Chapter 2 introduces terminology, notation, and preliminary results necessary for the work done in the subsequent chapters. We begin in Section 2.1 with gradings and bigradings, including terminology and notation for the generation and concentration degrees of graded modules. Next, in Section 2.2, we define the Rees ring, symmetric algebra, and special fiber ring of an ideal and discuss the relationships between them. We also discuss how the defining equations of each of these rings relate to each other. Additionally, the gradings and bigrading on these rings are made explicit. Section 2.3 deals primarily with determinantal and Pfaffian ideals. Some important properties of determinants and Pfaffians are discussed. We also discuss the height bounds of determinantal ideals for ordinary and symmetric matrices as well as for the Pfaffian ideals of alternating matrices, and define generic height in each case. We end the chapter with Section 2.4, a discussion on finite free complexes. In particular, we mention the Buchsbaum-Eisenbud criterion for a finite complex of finite rank free modules to be acyclic. We also make explicit the grading on resolutions of graded modules and define the Castelnuovo-Mumford regularity.

In Chapter 3, we discuss key properties of determinantal and Pfaffian ideals of generic matrices. The chapter is split into three main sections: Section 3.1 develops results for generic ordinary matrices, Section 3.2 does the same for generic symmetric matrices, and Section 3.3 develops these results for generic alternating matrices. All sections cover the same basic results but in the explicit context of the appropriate type of matrix. In each section, we compute the analytic spread for such ideals, depending on the size of the minors or Pfaffians. We also compute the maximum of the projective dimensions of powers of these ideals over a polynomial ring. We end with proving the containment lemmas, Lemmas 3.1.5, 3.2.5 and 3.3.6. Both the computations of the maximum of the projective dimensions and the containment lemmas are crucial tools in Chapter 4. The containment lemmas, in particular, allow us to control the specializations of resolutions of powers by knowing sufficient information about the matrices themselves.

The goal of Chapter 4 is to develop the specialization techniques and use approximate resolutions to show how degree bounds can be placed on the defining equations of the Rees ring in the case of determinantal and Pfaffian ideals of generic height. We prove Lemma 4.1.1 which shows under which circumstances specializations of resolutions of powers of a generic determinantal ideal are themselves resolutions or approximate resolutions of the specialization of the powers. Additionally, part (c) of Lemma 4.1.1 gives a way to "approximate" the defining equations of $\mathcal{R}(I)$ from the defining equations of $\mathcal{R}(J)$ (where I is a specialization of J) and from knowing how "close" $\mathcal{R}(I)$ is to being a specialization of $\mathcal{R}(J)$. In particular, it provides an exact sequence where the defining ideal of $\mathcal{R}(I)$ fits between the specialization of the defining ideal of $\mathcal{R}(J)$ and the kernel of the natural map from the specialization of $\mathcal{R}(J)$ to $\mathcal{R}(I)$. We then employ the techniques of approximate resolutions to place conditions on I in order to obtain degree bounds on the kernel of the map from the specialization of $\mathcal{R}(J)$ to $\mathcal{R}(I)$. Finally, we use the containment lemmas and projective dimenions computed in Chapter 3 to apply these specialization results to determinantal and Pfaffian ideals of generic height.

The results of Chapter 5 apply the tools of Chapter 4 to obtain specific degree bounds on the defining equations of $\mathcal{R}(I)$ where I is a determinantal or Pfaffian ideal of generic height. This chapter is broken into three main sections.

Section 5.1 proves degree bounds in the case of ideals of minors of an ordinary matrix. Our most complete and sharpest results in this section are given in Theorem 5.1.3, due to knowing explicit minimal free resolutions of the powers of the ideal of maximal minors of a generic matrix, thanks to the work of Akin, Buchsbaum, and Weyman in [1]. For the rest of the section, we are unable to use explicit free resolutions of the powers of determinantal ideals of a matrix, since they are generally unknown. Therefore, we rely on the computations of the Castelnuovo-Mumford regularity by Raicu in [59], in the case of characteristic zero. We obtain complete degree bounds for the case of 2×2 minors of a matrix in Theorem 5.1.6 since Raicu has the regularity for all powers of the ideal of 2×2 minors of a generic matrix. We obtain partial degree bounds in the case of submaximal minors of a square matrix in Theorem 5.1.9 and for arbitrary size minors in Theorem 5.1.11. We obtain partial results since Raicu has the regularity for sufficiently high powers of such ideals. Throughout the section, we analyze the degree bounds to state when I is of linear type or fiber type.

As far as we are aware, neither minimal free resolutions nor the Castelnuovo-Mumford regularity have been computed for the case of the ideal of minors of a symmetric matrix. As such, Section 5.2 consists of only a minor result which can be used to prove degree bounds in the future once resolutions or the regularity have been computed.

In Section 5.3, we obtain degree bounds in the case of ideals of Pfaffians of an alternating matrix. Our most complete and sharpest result is Theorem 5.3.4, which treats the case of the submaximal Pfaffians of an alternating matrix of odd size, i.e., the perfect Gorenstein ideals of grade three. The sharpness and completeness of this result is due to the construction of explicit resolutions of the powers of such ideals in the generic case by Kustin and Ulrich in [49]. For the rest of the section, we use computations of Perlman in [58] of the Castelnuovo-Mumford regularity of the powers of Pfaffian ideals of a generic alternating matrix since explicit resolutions are unknown in these cases. We obtain partial degree bounds for the cases of $(n-2) \times (n-2)$ Pfaffians in Theorem 5.3.7, 4×4 Pfaffians in Theorem 5.3.9, and of general size Pfaffians in Theorem 5.3.10. Throughout the section, we analyze the degree bounds to state when I is of linear type or fiber type.

The final chapter, Chapter 6, discusses some results concerning nonlinearly presented perfect homogeneous Gorenstein ideals of grade three having second analytic deviation one that are of linear type on the punctured spectrum. We provide a conjecture for the relation type of such ideals in Conjecture 6.2.5. Inspired by the conjecture, in Section 6.3, we introduce fiber row ideals and morphism fiber ideals to study the fibers of rational maps. Additionally, we prove Theorem 6.3.7 which gives new information about the defining ideals of generic fibers of rational maps where the base locus consists of finitely many points. Finally, we end with Examples 6.4.1 and 6.4.2 which demonstrate unusual behavior for the bidegrees of the defining equations. In particular, the x-degree can increase as the T-degree increases.

2. PRELIMINARIES

2.1 Graded Rings and Modules

2.1.1 Gradings

Definition 2.1.1 *Let R be a ring with a decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ into Abelian groups so that for each i and j in $\mathbb{Z}$, $R_iR_j \subseteq R_{i+j}$. We call R a graded ring. The Abelian groups R_i are called the graded components of R. An element $f \in R$ is said to be homogeneous of degree i if $f \in R_i$. If R is a graded ring, then R_0 is a subring of R. If $R_i = 0$ for $i < 0$, we say that R is nonnegatively graded. If R is nonnegatively graded and is generated in degree 1 as an R_0-algebra, we say that R is standard graded.*

Suppose M is an R-module with a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ into Abelian groups so that for each i and j in $\mathbb{Z}$, $R_iM_j \subseteq M_{i+j}$. Then M is called a graded R-module. The Abelian groups M_i are R_0-modules and are called the graded components of M. An element $f \in M$ is said to be homogeneous of degree i if $f \in M_i$. An R-submodule N of M is a graded submodule of M if N is a graded R-module with $N_i \subseteq M_i$ for all $i \in \mathbb{Z}$.

An R-ideal I is a homogeneous ideal if I is a graded R-submodule of R. An R-ideal $\mathfrak{m}$ is a maximal homogeneous ideal of R if $\mathfrak{m}$ maximal with respect to being a proper homogeneous ideal.

Let M and N be graded R-modules. An R-linear map $f : M \to N$ is said to be homogeneous if $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$.

We will sometimes use the notation $[M]_i$ to denote the ith graded component of M. This is mainly done in situations where the notation for the module is a bit more complicated, in order to make it clear that we are taking a graded component. There are some useful consequences of these definitions.

Observation 2.1.2 *Let R be a graded ring. An R-ideal I is homogeneous if and only if I has a homogeneous generating set.*

If $f : M \to N$ is a homogeneous R-linear map of graded R-modules, then $\ker f$ *and* $\operatorname{Im} f$ *are graded R-modules.*

Homogeneous R-linear maps are extremely useful. However, important R-linear maps may be slightly off from being homogeneous. For example, one might have an R-linear map $f : M \to N$ so that, for some $j \in \mathbb{Z}$ and for every $i \in \mathbb{Z}$, $f(M_i) \subseteq N_{i-j}$. In order to force f to be a homogeneous map, we may regrade M or N.

Definition 2.1.3 *Let R be a graded ring, M be a graded R-module, and $j \in \mathbb{Z}$. The jth twist of M, denoted $M(j)$, is the regrading of M so that $[M(j)]_i = M_{i+j}$. $M(j)$ is a graded R-module.*

If $f : M \to N$ is an R-linear map of graded R-modules as described above, i.e., such that $f(M_i) \subseteq N_{i-j}$, then $f : M(j) \to N$ is homogeneous.

Let R be a Noetherian graded ring. If R has a unique maximal homogeneous ideal $\mathfrak{m}$ which is an actual maximal ideal of R, then $(R, \mathfrak{m})$ behaves like a Noetherian local ring when restricting consideration to homogeneous R-ideals and graded R-modules. In such a setting, there are graded versions of Nakayama's Lemma, the Auslander-Buchsbaum Formula, etc. See [8] for more details.

One particular setting of interest is when R is a nonnegatively graded Noetherian ring with R_0 local. In such a setting, R has a unique maximal homogeneous ideal which is an actual maximal ideal of R. More specifically, let $\mathfrak{m}_0$ denote the unique maximal ideal of R_0, and define $R_+ = \bigoplus_{i=1}^{\infty} R_i$. Then the unique homogeneous maximal ideal of R is $\mathfrak{m}_0 + R_+$. A polynomial ring $R = K[x_1, \ldots, x_d]$ over a field K is an example of such a ring, provided that we give each variable a nonnegative degree, with unique homogeneous maximal ideal $(x_1, \ldots, x_d)$.

2.1.2 Generation and Concentration Degrees

We often care about the maximal degrees in which a module is generated or in which a module is concentrated. This leads to the following defintions.

Definition 2.1.4 *Let R be a Noetherian graded ring and M a graded R-module.*

$$b_0(M) = \inf\left\{ p \,\middle|\, R\left(\bigoplus_{j\le p} M_j\right) = M \right\}.$$

If the infemum is a minimum, the number $b_0(M)$ is the largest degree of a generator in a minimal homogeneous generating set of M. In general, M is generated in degrees less than or equal to $b_0(M)$. Note that if $M = 0$, then for all $p \in \mathbb{Z}$, we have $R\left(\bigoplus_{j\le p} M_j\right) = M$. Therefore, $b_0(0) = \inf \mathbb{Z} = -\infty$.

Definition 2.1.5 *Let R be a Noetherian graded ring and M a graded R-module.*

$$\operatorname{topdeg}(M) = \sup\{j \mid M_j \neq 0\}\,.$$

The number $\operatorname{topdeg}(M)$ *is called the top degree of M.*

If the supremum is a maximum, the number $\operatorname{topdeg}(M)$ is the largest degree in which M is nonzero. In general, M is concentrated in degrees less than or equal to $\operatorname{topdeg}(M)$. Note that if $M = 0$, then $\{j \mid M_j \neq 0\} = \varnothing$. Therefore, we have $\operatorname{topdeg}(0) = \sup \varnothing = -\infty$.

Observation 2.1.6 *For any graded R-module M, $b_0(M) \le \operatorname{topdeg}(M)$.*

2.1.3 Bigradrings

We will also often deal with bigraded rings and modules.

Definition 2.1.7 *Let R be a ring with a decomposition $R = \bigoplus_{(i,j)\in\mathbb{Z}^2} R_{(i,j)}$ into Abelian groups so that for each (i,j) and (i',j') in $\mathbb{Z}^2$, $R_{(i,j)}R_{(i',j')} \subseteq R_{(i+i',j+j')}$. We* *call R a bigraded ring. The Abelian groups $R_{(i,j)}$ are called the bigraded components of R. An element $f \in R$ is said to be bihomogeneous of degree (or bidegree) (i,j) if $f \in R_{(i,j)}$. If R is a bigraded ring, then $R_{(0,0)}$ is a subring of R. If $R_{(i,j)} = 0$ whenever $i < 0$ or $j < 0$, we say that R is nonnegatively bigraded. If R is nonnegatively bigraded and is generated in bidegrees $(1,0)$ and $(0,1)$ as an $R_{(0,0)}$-algebra, we say that R is standard bigraded.*

Suppose M is an R-module with a decomposition $M = \bigoplus_{(i,j)\in\mathbb{Z}^2} M_{(i,j)}$ into Abelian groups so that for each (i,j) and (i',j') in $\mathbb{Z}^2$, $R_{(i,j)}M_{(i',j')} \subseteq M_{(i+i',j+j')}$. Then M is called a bigraded R-module. The Abelian groups $M_{(i,j)}$ are $R_{(0,0)}$-modules and are called the bigraded components of M. An element $f \in M$ is said to be bihomogeneous of degree (or bidegree) (i,j) if $f \in M_{(i,j)}$. An R-submodule N of M is a bigraded submodule of M if N is a bigraded R-module with $N_{(i,j)} \subseteq M_{(i,j)}$ for all $(i,j) \in \mathbb{Z}^2$.

An R-ideal I is a bihomogeneous ideal if I is a bigraded R-submodule of R.

Let M and N be bigraded R-modules. An R-linear map $f : M \to N$ is said to be bihomogeneous if $f\big(M_{(i,j)}\big) \subseteq N_{(i,j)}$ for all $(i,j) \in \mathbb{Z}^2$.

Let M be a bigraded R-module. We often use the notation $M_{(*,j)} = \bigoplus_{i\in\mathbb{Z}} M_{(i,j)}$ and $M_{(i,*)} = \bigoplus_{j\in\mathbb{Z}} M_{(i,j)}$.

2.2 Blowup Algebras and Defining Equations

2.2.1 The Rees Ring and the Symmetric Algebra

Definition 2.2.1 *Let R be a Noetherian ring and I be an R-ideal. The Rees ring of I, denoted $\mathcal{R}(I)$, is the subring $R[It] \subseteq R[t]$.*

We note that $\mathcal{R}(I) \cong \bigoplus_{k=0}^{\infty} I^k$. From this, we see that $\mathcal{R}(I)$ has a natural grading, even if R is not graded. The case where R is graded will be discussed in Subsection 2.2.2.

Since R is Noetherian, I is finitely generated. For instance, $I = (f_1, \ldots, f_s)$ for some nonnegative integer s. Hence, $\mathcal{R}(I) = R[f_1t, \ldots, f_st]$. We may then define the polynomial ring $\mathcal{S} = R[T_1, \ldots, T_s]$ over R. There exists a natural homogeneous

R-algebra epimorphism $\pi : \mathcal{S} \to \mathcal{R}(I)$ given by $T_i \mapsto f_i t$. Let $\mathcal{J} = \ker \pi$. Then $\mathcal{R}(I) \cong \mathcal{S}/\mathcal{J}$. Because of this, we call $\mathcal{J}$ the defining ideal of $\mathcal{R}(I)$. An element of $\mathcal{S}$ which is part of a minimal generating set of $\mathcal{J}$ is called a defining equation of $\mathcal{R}(I)$. The defining equations of $\mathcal{R}(I)$ are the implicit equations defining $\mathcal{R}(I)$.

Definition 2.2.2 *Let R be a Noetherian ring and $M = Rf_1 + \ldots + Rf_s$ be a finitely generated R-module. Define $\mathcal{S} = R[T_1, \ldots, T_s]$. Suppose φ is a presentation matrix of M; that is, the sequence*

$$R^r \xrightarrow{\varphi} R^s \xrightarrow{[f_1,\ldots,f_s]} M \longrightarrow 0$$

is exact. Consider the vector $[\ell_1, \ldots, \ell_s] = [T_1, \ldots, T_s]\,\varphi$ with entries in $\mathcal{S}$. Let $\mathcal{L} = (\ell_1, \ldots, \ell_s)$. The symmetric algebra of M, denoted $\mathrm{Sym}(M)$*, is the quotient ring $\mathcal{S}/\mathcal{L}$.*

The elements $\ell_1, \ldots, \ell_s$ are linear forms in $\mathcal{S}$. Therefore, $\mathrm{Sym}(M)$ inherits a natural grading from $\mathcal{S}$. Typically, we consider the symmetric algebra of an ideal I. A useful property of the symmetric algebra is how it respects base change.

Remark 2.2.3 *If S is an R-algebra and M is an R-module, then* $\mathrm{Sym}_R(M) \otimes_R S \cong \mathrm{Sym}_S(M \otimes_R S)$.

For more details, see, for instance, [21, Proposition A.A2.b].

Observation 2.2.4 *Adopt the setting of Definition 2.2.2 and assume that $M = I$ is an R-ideal. Then the $\mathcal{S}$-ideal $\mathcal{L}$ is contained in $\mathcal{J}$. Therefore, there exists a natural R-algebra epimorphism $\alpha : \mathrm{Sym}(I) \to \mathcal{R}(I)$ induced by the identity on $\mathcal{S}$. Let $\mathcal{A} = \ker \alpha$. We will also use the notation $\mathcal{A}(I)$.*

Proof Apply the homomorphism $\pi : \mathcal{S} \to \mathcal{R}(I)$ to the vector $[\ell_1, \ldots, \ell_s]$.

$$\pi([\ell_1, \ldots, \ell_s]) = \pi([T_1, \ldots, T_s])\,\varphi = [f_1, \ldots, f_s]\,\varphi = 0.$$

$\blacksquare$

It is not common practice to denote $\mathcal{A}$ as $\mathcal{A}(I)$; however, throughout this dissertation, we consider Rees rings of multiple ideals simultaneously. The notation $\mathcal{A}(I)$ is designed to help keep the ideal in question clear.

We also note that the epimorphism α is homogeneous; thus, $\mathcal{A}$ is homogeneous as well.

Observation 2.2.5 *Adopt the setting of Observation 2.2.4. Then $\mathcal{A} = \mathcal{J}/\mathcal{L}$.*

Definition 2.2.6 *We use the notation $\mathrm{Sym}_k(I)$ to mean $[\mathrm{Sym}(I)]_k$. In other words, $\mathrm{Sym}_k(I)$ is the degree k component of $\mathrm{Sym}(I)$. Similarly, we use the notation $\mathcal{A}_k(I)$ to mean $[\mathcal{A}(I)]_k$. In other words, $\mathcal{A}_k(I)$ is the degree k component of $\mathcal{A}(I)$.*

Observation 2.2.7 *The homomorphism $\alpha : \mathrm{Sym}(I) \to \mathcal{R}(I)$ is an isomorphism when restricted to the degree 1 components. Therefore, $\mathcal{J}_1 = \mathcal{L}$ and $\mathcal{A}_1(I) = 0$.*

Since it is easy to compute the linear defining equations of $\mathcal{J}$ (the generators of $\mathcal{L}$) through the use of a presentation matrix of I, the problem of finding the defining equations of $\mathcal{R}(I)$ reduces to finding a minimal generating set of $\mathcal{A}$.

2.2.2 The Bigrading on the Rees Ring and Symmetric Algebra

Most of the work which has been done to study the defining equations of $\mathcal{R}(I)$ are in a setting where R is a polynomial ring over a field and I is a homogeneous ideal. We can then use the grading on R to place bigradings on the rings $\mathcal{R}(I)$ and $\mathrm{Sym}(I)$. We have to be careful how we define these bigradings, however.

Definition 2.2.8 *Let R be a nonnegatively graded Noetherian ring, and suppose $I = (f_1, \ldots, f_s)$ is a homogeneous R-ideal with $\deg f_i = D$ for all i satisfying $1 \leq i \leq s$.*

We give the ring $\mathcal{S} = R[T_1, \ldots, T_s]$ the bigrading so that $\mathcal{S}_{(i,0)} = R_i$ for all $i \in \mathbb{Z}$ and $\deg T_i = (0, 1)$ for each i satisfying $1 \leq i \leq s$.

In order to make the natural epimorphism $\pi : \mathcal{S} \to \mathcal{R}(I)$ bihomogeneous, we establish a bigrading on $R[t]$ given by $R[t]_{(i,0)} = R_i$ for all $i \in \mathbb{Z}$ and $\deg t = (-D, 1)$.

The bigrading on $\mathcal{R}(I)$ is induced by the bigrading on $R[t]$. If R is standard graded, then $\mathcal{R}(I)$ is standard bigraded. The isomorphism $\mathcal{R}(I) \cong \bigoplus_{k=0}^{\infty} I^k(kD)$ is bihomogeneous. Moreover, $\pi : \mathcal{S} \to \mathcal{R}(I)$ is bihomogeneous. Therefore, $\mathcal{J}$ is a bihomogeneous $\mathcal{S}$-ideal.

In order to make the natural epimorphism $\alpha : \mathrm{Sym}(I) \to \mathcal{R}(I)$ bihomogeneous, we take the ring $\mathrm{Sym}(I(D))$ instead. Since α is bihomogeneous, $\mathcal{A}$ is a bihomogeneous ideal. If R is standard graded, then $\mathrm{Sym}(I(D))$ is standard bigraded.

Whenever we discuss $\mathcal{R}(I)$, $\mathrm{Sym}(I(D))\,, \mathcal{J}, \mathcal{L}$, or $\mathcal{A}$ in the setting where R is nonnegatively graded, we use the bigradings in the definition above.

The remarks made above concerning the relationships between $\mathcal{J}$, $\mathcal{L}$, and $\mathcal{A}$ correspond to relationships in the bigraded setting as well. For instance, it is the case that $\mathcal{J}_{(*,1)} = \mathcal{L}$. Therefore, $\mathcal{A}_1(I) = \mathcal{A}_{(*,1)} = 0$.

Throughout this dissertation, we are mainly concerned with obtaining degree bounds on $\mathcal{A}$ with respect to the grading on R. As such, in the setting where R is nonnegatively graded, we use the following definitions.

Definition 2.2.9 *Let R be a nonnegatively graded Noetherian ring and I a homogeneous R-ideal generated by forms of the same degree D.*

$$b_0(\mathcal{A}) = \inf\left\{ p \;\middle|\; \mathrm{Sym}(I(D)) \left(\bigoplus_{j \le p} \mathcal{A}_{(j,*)} \right) = \mathcal{A} \right\}.$$

$$b_0(\mathcal{A}_k(I)) = \inf\left\{ p \;\middle|\; \mathrm{Sym}(I(D)) \left(\bigoplus_{j \le p} \mathcal{A}_{(j,k)} \right) = \mathcal{A}_k(I) \right\}.$$

Definition 2.2.10 *Let R be a nonnegatively graded Noetherian ring and I a homogeneous R-ideal generated by forms of the same degree D.*

$$\mathrm{topdeg}(\mathcal{A}) = \sup\left\{ j \;\middle|\; \mathcal{A}_{(j,*)} \neq 0 \right\}.$$

$$\mathrm{topdeg}(\mathcal{A}_k(I)) = \sup\left\{ j \;\middle|\; \mathcal{A}_{(j,k)} \neq 0 \right\}.$$

2.2.3 Ideals of Linear Type

An important class of ideals in the study of Rees rings are those of linear type.

Definition 2.2.11 *The ideal I is said to be of linear type if $\alpha : \mathrm{Sym}(I(D)) \to \mathcal{R}(I)$ is an isomorphism.*

The name comes from the fact that when α is an isomorphism, $\mathcal{J}$ is generated only by forms which are linear in the T's.

Observation 2.2.12 *Let R be a nonnegatively graded Noetherian ring, and consider the R-ideal $I = (f_1, \ldots, f_s)$ so that $\deg f_i = D$ for each i satisfying $1 \le i \le s$. The following are equvialent.*

a. *I is of linear type.*

b. $\mathcal{J} = \mathcal{L}$.

c. $\mathcal{A} = 0$.

d. $\mathrm{topdeg}(\mathcal{A}) = -\infty$.

e. $b_0(\mathcal{A}) = -\infty$.

Proof The equivalences of (a), (b), and (c) are clear from the definitions. Part (c) implies part (d) by the definition of top degree. Part (d) implies part (e) since $b_0(\mathcal{A}) \le \mathrm{topdeg}(\mathcal{A})$ is always true. All that remains to show is that part (e) implies part (c). Now, $b_0(\mathcal{A}) = -\infty$ implies that

$$\mathrm{Sym}(I(D)) \left(\bigoplus_{j \le p-1} \mathcal{A}_{(j,*)} \right) = \mathcal{A}$$

for all $p \in \mathbb{Z}$. Therefore,

$$\mathrm{Sym}(I(D))\,(0) = \mathcal{A},$$

giving $\mathcal{A} = 0$. ■

2.2.4 The Special Fiber Ring and Ideals of Fiber Type

Another ring of interest, particularly in algebraic geometry, is the special fiber ring of an ideal.

Definition 2.2.13 *Let K be a field and $R = K[x_1, \ldots, x_d]$ be a standard graded polynomial ring over K. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the unique maximal homogeneous ideal of R. Suppose $I = (f_1, \ldots, f_s)$ is a homogeneous R-ideal with $\deg f_i = D$ for each i satisfying $1 \leq i \leq s$. The special fiber ring of I, denoted $\mathcal{F}(I)$, is given by $\mathcal{R}(I) \otimes_R R/\mathfrak{m}$.*

Remark 2.2.14 *Let K be a field and $R = K[x_1, \ldots, x_d]$ be a standard graded polynomial ring over K. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the unique maximal homogeneous ideal of R. Suppose $I = (f_1, \ldots, f_s)$ is a homogeneous R-ideal with $\deg f_i = D$ for each i satisfying $1 \leq i \leq s$.*

Consider the rational map

$$\Phi : \mathbb{P}_K^{d-1} \dashrightarrow \mathbb{P}_K^{s-1}$$

with base locus $V(I)$ defined by

$$[a_1 : \cdots : a_d] \mapsto [f_1(a_1, \ldots, a_d) : \cdots : f_s(a_1, \ldots, a_d)]\,.$$

Let X denote the closed image of Φ, $\mathcal{T} = K[T_1, \ldots, T_s]$ be the homogeneous coordinate ring of $\mathbb{P}_K^{s-1}$, and $I(X)$ denote the defining ideal of X in $\mathcal{T}$.

Then $\mathcal{F}(I)$ is the homogeneous coordinate ring of X. In other words, $\mathcal{F}(I) \cong \mathcal{T}/I(X)$.

See, for instance, [21, 12.5].

Remark 2.2.15 *Adopt the assumptions of Remark 2.2.14. The Rees ring $\mathcal{R}(I)$ is the bihomogeneous coordinate ring of the graph of Φ. In particular, $I(X)\,\mathcal{S} \subseteq \mathcal{J}$.*

A typical explanation of $I(X)\,\mathcal{S} \subseteq \mathcal{J}$ is to consider the natural morphisms of projective varieties

$$\begin{tikzcd}
\mathbb{P}_K^{d-1} \times \mathbb{P}_K^{s-1} \arrow[d, two heads] & \operatorname{graph} \Phi \arrow[l, hook'] \arrow[d, two heads] \\
\mathbb{P}_K^{s-1} & \operatorname{Im} \Phi = X \arrow[l, hook']
\end{tikzcd}$$

and use the correspondence with bihomogeneous K-algebra homomorphisms to obtain the commutative diagram below.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, two heads, "\pi"] & \mathcal{R}(I) \\
\mathcal{T} \arrow[u, hook] \arrow[r, two heads, "\pi|_{\mathcal{T}}"] & \mathcal{F}(I) \cong \mathcal{T}/I(X) \arrow[u, hook]
\end{tikzcd}$$

Indeed, the above commutative diagram gives us something much stronger: $\mathcal{J}_{(0,*)} = I(X)\,\mathcal{S}$.

Definition 2.2.16 *Adopt the setting of Remark 2.2.14. The ideal I is said to be of fiber type if $\mathcal{J} = \mathcal{L} + I(X)\,\mathcal{S}$.*

Observation 2.2.17 *Adopt the setting of Remark 2.2.14. The following are equivalent.*

a. I is of fiber type.

b. $\mathcal{A} = I(X)\operatorname{Sym}(I(D))$

c. $\mathcal{J}$ is generated in bidegrees $(*,1)$ and $(0,*)$.

d. $b_0(\mathcal{A}) \leq 0$.

Proof The equivalence of (a), (b), and (c) follow from the facts previously discussed about the bigrading of $\mathcal{J}$ and the definition of fiber type.

Since $\mathcal{L} = \mathcal{J}_{(*,1)}$ and $\mathcal{A} = \mathcal{J}/\mathcal{L}$, condition (c) holds if and only if $\mathcal{A}$ is generated in bidegrees $(0,*)$ and possibly $\mathcal{A} = 0$ (if $\mathcal{J} = \mathcal{L}$). Recall our definition of $b_0(\mathcal{A})$ in the bigraded setting from Definition 2.2.9. Then $\mathcal{A}$ is generated in bidegrees $(0,*)$ if and only if

$$b_0(\mathcal{A}) = \inf\left\{p \;\middle|\; \mathrm{Sym}(I(D))\left(\bigoplus_{j\le p}\mathcal{A}_{(j,*)}\right) = \mathcal{A}\right\} \le 0.$$

■

2.2.5 Analytic Spread

Definition 2.2.18 *Adopt the setting of Remark 2.2.14. We denote* $\dim \mathcal{F}(I)$ *as* $\ell(I)$ *and refer to it as the analytic spread of* I*.*

The analytic spread of an ideal has been an important invariant in the study of reductions of ideals. See, for instance, [57].

Remark 2.2.19 *Adopt the setting of Remark 2.2.14.*

a. $\ell(I) \le \min\{\mu(I), \dim R\}$.

b. If I *is of linear type, then* $\ell(I) = \mu(I)$.

c. (Burch [12, Corollary pg 373])

$$\ell(I) + \inf\{\mathrm{depth}_R\, R/I^k\} \le \dim R.$$

d. (Eisenbud-Huneke [22, 3.3]) *If* $\mathcal{R}(I)$ *is Cohen-Macaulay and* $\mathrm{ht}\, I \ge 1$*, then*

$$\ell(I) + \inf\{\mathrm{depth}_R\, R/I^k\} = \dim R.$$

Proof To prove (a), notice that $\alpha : \mathrm{Sym}_R(I) \twoheadrightarrow \mathcal{R}(I)$ induces a surjective K-algebra homomorphism $\mathrm{Sym}_R(I) \otimes_R R/\mathfrak{m} \twoheadrightarrow \mathcal{F}(I)$. Therefore, $\ell(I) \le \dim \mathrm{Sym}_R(I) \otimes_R R/\mathfrak{m}$. Now, $\mathrm{Sym}_R(I) \otimes_R R/\mathfrak{m} \cong \mathrm{Sym}_{R/\mathfrak{m}}(I \otimes_R R/\mathfrak{m}) \cong (R/\mathfrak{m})[T_1, \ldots, T_s]$. Hence, $\ell(I) = s = \mu(I)$.

Furthermore, one notes that $\mathcal{R}(I)/I\mathcal{R}(I) \otimes_R R/\mathfrak{m} \cong \mathcal{R}(I) \otimes_R R/\mathfrak{m}$. Therefore, $\ell(I) \le \dim \mathcal{R}(I)/I\mathcal{R}(I)$. The ring $\mathcal{R}(I)/I\mathcal{R}(I)$ is typically known as the associated graded ring of R with respect to I. It is a well-known fact that $\dim \mathcal{R}(I)/I\mathcal{R}(I) = \dim R$ when I is a proper ideal (see, for instance, [53, 15.7]). Therefore, $\ell(I) \le \dim R$.

The proof of (b) follows quickly from the proof of (a). If I is of linear type, then the natural surjection $\alpha : \mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}(I)$ is an isomorphism. Hence, the induced map $\mathrm{Sym}(I) \otimes_R R/\mathfrak{m} \twoheadrightarrow \mathcal{F}(I)$ is an isomorphism. As we have already seen, $\dim \mathrm{Sym}(I) \otimes_R R/\mathfrak{m} = \mu(I)$. ∎

2.3 Determinantal and Pfaffian Ideals

2.3.1 Types of Matrices

Let R be a Noethering ring, and let m and n be integers satisfying $1 \leq m \leq n$. We consider three types of matrices with entries in R. Namely, ordinary matrices, symmetric matrices, and alternating matrices. The use of the term "ordinary matrix" is mainly to distinguish from the case of symmetric and alternating matrices.

Definition 2.3.1 *Let A be a matrix with entries in R. The matrix A is said to be symmetric if $A^T = A$. The matrix A is said to be skew-symmetric if $A^T = -A$. The matrix A is said to be alternating if A is skew-symmetric and if all diagonal entries of A are* 0.

We note that symmetric, skew-symmetric, and alternating matrices are all square matrices.

The distinction between skew-symmetric and alternating matrices depends on whether 2 is a non-zero-divisor in R. If 2 is a non-zero-divisor in R, then A is skew-symmetric if and only if A is alternating. Indeed, by the definition of skew-symmetric, the diagonal element $A_{ii} = -A_{ii}$; hence, $2A_{ii} = 0$. Since 2 is a non-zero-divisor, we obtain $A_{ii} = 0$, giving that A is alternating. On the other hand, if 2 is a zero-divisor in R, then skew-symmetric matrices need not be alternating. For example, over the ring $\mathbb{Z}/6\mathbb{Z}$, the matrix

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

is skew-symmetric but not alternating. The extreme version of this is when char $R = 2$. In this situation, for every $x \in R$, $x = -x$. Hence, skew-symmetric matrices are equivalent to symmetric matrices.

Many of the important and useful properties of alternating matrices fail for skew-symmetric matrices when 2 is a zero-divisor in R. This is why we discuss alternating matrices.

2.3.2 Determinants and Pfaffians

If A is an alternating matrix with entries in R, then $\det(A)$ is a perfect square in R. Indeed, there exists a polynomial in the entries of A, called the Pfaffian of A and denoted $\mathrm{Pf}(A)$, so that $\mathrm{Pf}(A)^2 = \det(A)$. If A is an $n \times n$ alternating matrix with n odd, then $\mathrm{Pf}(A) = \det(A) = 0$. Therefore, we typically restrict our attention to the Pfaffian of an $n \times n$ alternating matrix where n is even.

Pfaffians of an alternating matrix share many properties with determinants, as can be seen from the following two remarks.

Remark 2.3.2 *Let R be a ring and A be an $n \times n$ matrix with entries in R.*

a. *There exists a Laplace expansion to compute the determinant. In particular, for any fixed j with $1 \leq j \leq n$, one has*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det{}^{ij}(A),$$

where $\det^{ij}(A)$ denotes the determinant of the matrix obtained from A by deleting row i and column j.

b. *There exists an $n \times n$ matrix $\mathrm{Adj}(A)$, called the classical adjoint of A, such that $\mathrm{Adj}(A)\, A = \det(A)\, I_n$.*

c. $\det(\mathrm{Adj}(A)) = \det(A)^{n-1}$

d. If A is a block diagonal matrix

$$A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right)$$

where B and C are square matrices, then $\det(A) = \det(B)\det(C)$.

Remark 2.3.3 *Let R be a ring and A be an $n \times n$ alternating matrix with entries in R, where n is a positive even integer.*

a. There exists a Laplace expansion to compute the Pfaffian. In particular, for any fixed j with $1 \leq j \leq n$, one has

$$\operatorname{Pf}(A) = \sum_{i<j} (-1)^{i+j-1} A_{ij} \operatorname{Pf}^{ij}(A) + \sum_{i>j} (-1)^{i+j} A_{ij} \operatorname{Pf}^{ij}(A),$$

where $\operatorname{Pf}^{ij}(A)$ denotes the Pfaffian of the matrix obtained from A by deleting rows and columns i and j.

b. There exists an $n \times n$ alternating matrix $\operatorname{PfAdj}(A)$, called the Pfaffian adjoint of A, such that $\operatorname{PfAdj}(A)\, A = \operatorname{Pf}(A)\, I_n$.

c. $\det(\operatorname{PfAdj}(A)) = \operatorname{Pf}(A)^{n-2}$.

d. If A is a block diagonal matrix

$$A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right)$$

where B and C are alternating matrices, then $\operatorname{Pf}(A) = \operatorname{Pf}(B)\operatorname{Pf}(C)$.

For a more detailed look at Pfaffians, the following references may be consulted: [24, Appendix D], [10, Section 2], or [3, pp. 140-142].

Parts (a) and (d) of Remark 2.3.3 are standard results about Pfaffians which can be found in the above references. However, parts (b) and (c) are less common in the literature. As such, we provide a definition of $\operatorname{PfAdj}(A)$. In particular, for $1 \leq i < j \leq n$, one takes the (i,j)-entry of $\operatorname{PfAdj}(A)$ to be $(-1)^{i+j} \operatorname{Pf}^{ij}(A)$. Part

(b) then follows from part (a), and part (c) follows from part (b) and the fact that $\det(A) = \mathrm{Pf}(A)^2$.

An important consideration is studying the minors of a matrix.

Definition 2.3.4 *Let t, m, and n be integers satisfying $1 \leq t \leq m \leq n$. Let A be an $m \times n$ matrix with entries in R. Suppose $i_1, \ldots, i_t$ and $j_1, \ldots, j_t$ are integers satisfying $1 \leq i_1 < \cdots < i_t \leq m$ and $1 \leq j_1 < \cdots < j_t \leq n$. There exists a $t \times t$ submatrix of A given by selecting rows $i_1, \ldots, i_t$ and columns $j_1, \ldots, j_t$ of A. Moreover, all $t \times t$ submatrices of A can be obtained in this manner. A $t \times t$ minor of A is the determinant of a $t \times t$ submatrix.*

We now define determinantal ideals.

Definition 2.3.5 *Let R be a Noetherian ring and A be an $m \times n$ matrix with entries in R, where $1 \leq m \leq n$. We let $I_t(A)$ denote the ideal generated by the $t \times t$ minors of A, provided that $1 \leq t \leq m$. For $t \leq 0$, we define $I_t(A) = R$, and for $t > m$, we define $I_t(A) = 0$.*

2.3.3 Height Bounds on Determinantal Ideals

Proposition 2.3.6 (Eagon-Northcott [19, Theorem 3]) *Let R be a Noetherian ring, t, m, and n be integers satisfying $1 \leq t \leq m \leq n$, A be an $m \times n$ matrix with entries in R, and $I = I_t(A)$ a proper ideal. Then $\operatorname{ht} I \leq (m - t + 1)\,(n - t + 1)$.*

Proposition 2.3.7 (Eagon [20, Theorem 2]) *Let R be a Noetherian ring, t, m, and n be integers satisfying $1 \leq t \leq m \leq n$, let X be a matrix whose entries are distinct indeterminates over R, $S = R[X]$ the polynomial ring over R generated by the entries of X, and $J = I_t(X)$. Then $\operatorname{ht} J = (m - t + 1)\,(n - t + 1)$.*

The matrix X above is often referred to as a "generic (ordinary) matrix" since the entries are indeterminates over R. Generic matrices will be defined and studied in more detail in Chapter 3.

Definition 2.3.8 *Let R be a Noetherian ring, t, m, and n integers satisfying $1 \leq t \leq m \leq n$, A an $m \times n$ ordinary matrix with entries in R, $I = I_t(A)$. If $\operatorname{ht} I = (m-t+1)(n-t+1)$, then we say that I is of generic (ordinary) height.*

We now discuss the corresponding results for symmetric matrices.

Proposition 2.3.9 (Józefiak [40, 2.1]) *Let R be a Noetherian ring, t and n be integers satisfying $1 \leq t \leq n$, A be an $n \times n$ symmetric matrix with entries in R, and $I = I_t(A)$ a proper ideal. Then $\operatorname{ht} I \leq \binom{n-t+2}{2}$.*

Proposition 2.3.10 *Let R be a Noetherian ring, t and n be integers satisfying $1 \leq t \leq n$, X be a symmetric matrix whose entries in the upper triangle are distinct indeterminates over R, $S = R[X]$ the polynomial ring over R generated by the entries in the upper triangle of X, and $J = I_t(X)$. Then $\operatorname{ht} J = \binom{n-t+2}{2}$.*

The above result is an immediate consequence of Proposition 2.3.9 when combined with the earlier result of Kutz in [50, 6.2], which states that $\operatorname{grade} J = \binom{n-t+2}{2}$.

Like before, the matrix X above is often referred to as a "generic symmetric matrix" since the entries are indeterminates over R. Generic symmetric matrices will be defined and studied in more detail in Chapter 3.

Definition 2.3.11 *Let R be a Noetherian ring, t and n integers satisfying $1 \leq t \leq n$, A an $n \times n$ symmetric matrix with entries in R, $I = I_t(A)$. If $\operatorname{ht} I = \binom{n-t+2}{2}$, then we say that I is of generic symmetric height.*

2.3.4 Pfaffian Ideals

Definition 2.3.12 *Adopt the notation of Definition 2.3.4. We call a $t \times t$ submatrix principal if $i_1 = j_1, \ldots,$ and $i_t = j_t$.*

Example 2.3.13 *Let $R = \mathbb{Z}$, and*

$$A = \begin{pmatrix} 0 & 1 & -4 & 3 \\ -1 & 0 & 5 & -7 \\ 4 & -5 & 0 & 2 \\ -3 & 7 & -2 & 0 \end{pmatrix}$$

be a matrix with entries in $\mathbb{Z}$. Then

$$\begin{pmatrix} 0 & -4 & 3 \\ 4 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix}$$

is the principal 3×3 submatrix given by rows and columns $1, 3$, and 4 of A.

When A is symmetric or alternating, every principal submatrix of A is also symmetric or alternating, respectively.

Let A be an alternating matrix. In a bit of an abuse in terminology, we refer to the Pfaffian of a principal $t \times t$ submatrix of A as a $t \times t$ Pfaffian of A. We emphasize that $t \times t$ Pfaffians of an alternating matrix of A are restricted to the principal $t \times t$ submatrices of A.

Definition 2.3.14 *Let R be a Noetherian ring and A be an $n \times n$ alternating matrix with entries in R, where $1 \leq n$. We let $\mathrm{Pf}_t(A)$ denote the ideal generated by the $t \times t$ Pfaffians of A, provided that $1 \leq t \leq n$. For $t \leq 0$, we define $\mathrm{Pf}_t(A) = R$, and for $t > n$, we define $\mathrm{Pf}_t(A) = 0$.*

We recall that if t is odd, then $\mathrm{Pf}_t(A) = 0$. As such, from now on, we will restrict our attention to ideals of the form $\mathrm{Pf}_{2t}(A)$.

2.3.5 Height Bounds on Pfaffian Ideals

Proposition 2.3.15 (Józefiak-Pragacz [41, 2.1]) *Let R be a Noetherian ring, t and n be integers satisfying $2 \leq 2t \leq n$, A be an $n \times n$ alternating matrix with entries in R, and $I = \mathrm{Pf}_{2t}(A)$ a proper ideal. Then $\operatorname{ht} I \leq \binom{n-2t+2}{2}$.*

Proposition 2.3.16 (Józefiak-Pragacz [41, 2.3]) *Let R be a Noetherian ring, t and n be integers satisfying $2 \leq 2t \leq n$, X be an alternating matrix whose entries above the diagonal are distinct indeterminates over R, $S = R[X]$ the polynomial ring over R generated by the entries above the diagonal of X, and $J = \mathrm{Pf}_{2t}(X)$. Then* $\mathrm{ht}\, J = \binom{n-2t+2}{2}$.

Similarly to the ordinary and symmetric cases, the matrix X above is often referred to as a "generic alternating matrix" since the entries are indeterminates over R. Generic alternating matrices will be defined and studied in more detail in Chapter 3.

Definition 2.3.17 *Let R be a Noetherian ring, t and n integers satisfying $2 \leq 2t \leq n$, A an $n \times n$ alternating matrix with entries in R, $I = \mathrm{Pf}_{2t}(A)$. If $\mathrm{ht}\, I = \binom{n-2t+2}{2}$, then we say that I is of generic alternating height.*

2.4 Finite Free Complexes

2.4.1 Buchsbaum-Eisenbud Criterion

Let R be a Noetherian ring and

$$\mathbf{C}_\bullet = 0 \to R^{\beta_n} \xrightarrow{\partial_n} R^{\beta_{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \to 0$$

be a finite complex of R-modules where each β_i is finite. Then, after a choice of basis, each ∂_i can be viewed as an $\beta_{i-1} \times \beta_i$ matrix with entries in R. For any t, $I_t(\partial_i)$ is independent of the choice of basis.

Buchsbaum and Eisenbud studied when such complexes are acyclic in [9]. The proposition below is an improved version of their main result, which can be found in [8, 1.4.13].

Proposition 2.4.1 (Buchsbaum-Eisenbud) *Let R be a Noetherian ring and*

$$\mathbf{C}_\bullet = 0 \to R^{\beta_n} \xrightarrow{\partial_n} R^{\beta_{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \to 0$$

be a complex with each β_i finite. Set

$$r_i = \sum_{j=0}^{n-i} (-1)^j \, \beta_{i+j}.$$

Then the following are equivalent.

a. $\mathbf{C}_\bullet$ *is acyclic.*

b. $\operatorname{grade} I_{r_i}(\partial_i) \geq i$ *for all* i *satisfying* $1 \leq i \leq n$.

To make notation more concise, whenever we are in this setting, we will use $I(\partial_i)$ to mean $I_{r_i}(\partial_i)$.

We will frequently perform a base change on complexes such as these. Specifically, if S is an R-algebra, we will consider the complex $\mathbf{C}_\bullet \otimes_R S$. We note that the ranks of the free modules are preserved under base change. Therefore, for each i satisfying $1 \leq i \leq n$, $I(\partial_i \otimes_R S) = I_{r_i}(\partial_i \otimes_R S) = I_{r_i}(\partial_i)\, S = I(\partial_i)\, S$ as S-ideals.

2.4.2 Graded Resolutions and Regularity

Definition 2.4.2 *Let R be a Noetherian graded ring, M a graded R-module, and $(\mathbf{C}_\bullet, \partial_\bullet)$ a complex consisting of graded R-modules. The complex $(\mathbf{C}_\bullet, \partial_\bullet)$ is said to be homogeneous if ∂_i is homogeneous for each i.*

A free R-resolution $(\mathbf{F}_\bullet, \partial_\bullet)$ of M is said to be homogeneous if the exact sequence $\mathbf{F}_\bullet \to M \to 0$ is homogeneous.

We emphasize the definition of a homogeneous resolution of M. Namely, the natural map $\mathbf{F}_0 \to M$ must be homogeneous. This distinction is critical throughout this dissertation, as we often take homogeneous free resolutions of both a module M and of a twist $M(j)$.

If R a graded ring with unique homogeneous maximal ideal $\mathfrak{m}$ which is an actual maximal ideal of R and if M is a graded R-module, then M has a minimal homogeneous free R-resolution. The minimal homogeneous free R-resolution of M

is unique up to homogeneous isomorphism of complexes and is a direct summand of every homogeneous free R-resolution of M.

Definition 2.4.3 *Let R be a graded ring with unique maximal homogeneous ideal $\mathfrak{m}$ which is a homogeneous ideal of R, M a graded R-module, and $(\mathbf{F}_\bullet, \partial_\bullet)$ the minimal homogeneous resolution of M. The Castelnuovo-Mumford regularity of M (also called the regularity of M) is*

$$\operatorname{reg} M = \sup_i \{b_0(\mathbf{F}_i) - i\}\,.$$

Again, we stress the importance of the definition of a homogeneous resolution when it comes to the definition of regularity. In particular, $\operatorname{reg} M(j) = \operatorname{reg} M - j$.

3. GENERIC DETERMINANTAL AND PFAFFIAN IDEALS

The study of determinantal and Pfaffian ideals is considerably less challenging when the entries of the matrix are distinct variables. For instance, if X is an $m \times n$ matrix with $1 \leq m \leq n$ whose entries are distinct variables, then for any t with $1 \leq t \leq m$, the set of $t \times t$ minors of X are homogeneous polynomials in those variables whose terms are square free. As such, there has been a considerable amount of work done to study the structure of generic matrices and the Rees algebras associated with their determinantal ideals. The main technique of this dissertation is to use the known results about generic matrices to study determinantal and Pfaffian ideals of matrices which are, in some sense, “close” to being generic. This chapter is dedicated to the study of determinantal and Pfaffian ideals of different types of generic matrices.

This chapter is organized into three sections. Section 3.1 is dedicated to the study of the determinantal ideals of generic ordinary matrices, Section 3.2 is dedicated to the study of determinantal ideals of generic symmetric matrices, and Section 3.3 is dedicated to the study of Pfaffian ideals of generic alternating matrices.

The primary goal for this chapter is to prove the ultimate lemma in each section, Lemmas 3.1.5, 3.2.5 and 3.3.6, which we refer to as the containment lemmas. Loosely speaking, let X be a generic matrix, fix t, and let $J = I_t(X)$. These lemmas allow us to relate the determinantal ideals $I_j(X)$ for $j < t$ to the ideals $I\big(\partial^{\mathbf{k}}{}_i\big)$, where $\partial^{\mathbf{k}}{}_i$ is the ith boundary map in a free resolution of J^k. When we specialize the generic matrix X to a specific matrix A, the containments are preserved under base change of complexes to the specialized ring. Hence, if A is “close to” being generic, we will be able to determine information about the ideals of minors of A from the generic case. The specialization arguments are presented in Chapter 4.

Before we delve into the specifics of generic matrices, we need a result about resolutions.

Lemma 3.0.1 *Let R be a Noetherian ring and S be a free R-algebra. Suppose $f_1, \ldots, f_n$ are elements in R. Let $I = (f_1, \ldots, f_n)\, R$ and $J = (f_1, \ldots, f_n)\, S$. Then $\operatorname{pd}_S J = \operatorname{pd}_R I$.*

Proof Since S is a free R-algebra, $I \otimes_R S \cong IS = J$ as an S-module. Let $\mathbf{P}_\bullet$ be a projective R-resolution of I. Since S is a flat R-algebra, $\mathbf{P}_\bullet \otimes_R S$ is an S-resolution of J. Let P be a projective R-module. Then P is a direct summand of a free R-module. In other words, there exists an R-module M and an indexing set Γ such that $P \oplus M \cong \bigoplus_\Gamma R$. Hence, we have $(P \otimes_R S) \oplus (M \otimes_R S) \cong \bigoplus_\Gamma S$. Hence, $P \otimes_R S$ is a projective S-module. Therefore, $\mathbf{P}_\bullet \otimes_R S$ is a projective S-resolution of J. Thus, we see $\operatorname{pd}_S J \leq \operatorname{pd}_R I$.

On the other hand, let $\mathbf{Q}_\bullet$ be a projective S-resolution of J. By restriction of scalars, $\mathbf{Q}_\bullet$ is an R-resolution of I (since the generators of J as an S-module are equal to the generators of I as an R-module). Let Q be a projective S-module. Then Q is a direct summand of a free S-module. In other words, there exists an S-module N and an indexing set Λ such that $Q \oplus N \cong \bigoplus_\Lambda S$. Since S is a free R-algebra, there exists an indexing set Σ so that $S \cong \bigoplus_\Sigma R$ as an R-module. Therefore, $Q \oplus N \cong \bigoplus_\Lambda \bigoplus_\Sigma R$ as an R-module. Since Q is a direct summand of a free R-module, Q is projective as an R-module. Hence, $\mathbf{Q}_\bullet$ is a projective R-resolution of I. Therefore, $\operatorname{pd}_R I \leq \operatorname{pd}_S J$. ■

The primary use of this lemma is to “strip away” unnecessary variables from polynomial rings while bounding projective dimensions. In particular, given a polynomial ring $K[x_1, \ldots, x_d, X_1, \ldots, X_s]$ in two sets of variables over a field K, if we have an ideal whose generators are polynomials in $X_1, \ldots, X_s$, we will be able to remove the variables $x_1, \ldots, x_d$ while preserving projective dimension.

The above lemma also has the added benefit of allowing us to change the base field K to another field L of the same characteristic. For example, if K and L

are fields of characteristic 0, then $K[X_1, \ldots, X_s]$ and $L[X_1, \ldots, X_s]$ are free algebras over $\mathbb{Q}[X_1, \ldots, X_s]$. Therefore, if we have an ideal generated by polynomials in the variables $X_1, \ldots, X_s$ with rational coefficients in the ring $K[X_1, \ldots, X_s]$, then the projective dimension is preserved when considering the ideal generated by the same polynomials in the ring $L[X_1, \ldots, X_s]$.

In particular, these uses of the lemma apply when we take powers of determinantal ideals of generic matrices.

3.1 Determinantal Ideals of Ordinary Matrices

Let R be a Noetherian ring. As a reminder, by an ordinary matrix, we refer to a matrix without any special characteristics. The use of the phrase "ordinary matrix" is meant to distinguish this case from the other two cases we consider - namely, symmetric and alternating matrices.

Consider the following example of a 2×4 generic ordinary matrix over $\mathbb{Z}$.

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix}$$

Definition 3.1.1 *Let R be a Noetherian ring, and consider positive integers m and n with $1 \leq m \leq n$. Let $\mathcal{X} = \{X_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be a set of mn indeterminates over R, and let $S = R[\mathcal{X}]$ be the polynomial ring over R generated by the elements of $\mathcal{X}$. Suppose X is an $m \times n$ matrix with entries in S so that the (i, j)-entry of X is X_{ij}. We say that X is a generic (ordinary) matrix over R.*

Often, we will start with a ring R and say "let X be an $m \times n$ generic (ordinary) matrix over R." When doing this, we implicitly assume that one has fixed a set $\mathcal{X}$ of mn indeterminates as in the definition above.

A key feature of generic matrices is that if a submatrix is made invertible, then the ideal of $t \times t$ minors (for $1 \leq t \leq m$) is still an ideal of minors of a generic matrix in the localized ring. The following lemma makes this idea rigorous and allows us to study localizations of determinantal ideals of generic matrices.

Lemma 3.1.2 *Let R be a Noetherian ring. Suppose X is an $m \times n$ generic matrix over R which can be written as the following block matrix*

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where A is a $j \times j$ matrix. Let $\Delta = \det A$, and consider the ring $T = R[A, B, C, \Delta^{-1}]$. Let Y be a generic $(m-j) \times (n-j)$ matrix over T. Then there exists a T-algebra isomorphism $\varphi : T[D] \to T[Y]$ so that the extension of $I_t(X)^k T[D]$ along φ is equal to $I_{t-j}(Y)^k T[Y]$ for each k.

Proof It is well-known that multiplying a matrix by an invertible matrix (on either side) does not change the ideal of minors of the matrix. Now, consider the matrix product below where, by an abuse of notation, each instance of the symbol I denotes an appropriately sized identity matrix.

$$\left(\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} A^{-1} & -A^{-1}B \\ \hline 0 & I \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline CA^{-1} & D - CA^{-1}B \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & D - CA^{-1}B \end{array}\right)$$

Hence, $I_t(X) = I_t\left(\begin{array}{c|c} I & 0 \\ \hline 0 & D - CA^{-1}B \end{array}\right)$. Using the Laplace expansion of the determinant, one sees $I_t(X) = I_{t-j}(D - CA^{-1}B)$.

Next, we establish the T-algebra homomorphism $\varphi : T[D] \to T[Y]$ given by $D_{ij} \mapsto (Y + CA^{-1}B)_{ij}$. To see that this homomorphism is a bijection, we establish its inverse. Let $\psi : T[Y] \to T[D]$ be the T-algebra homomorphism defined by $Y_{ij} \mapsto (D - CA^{-1}B)_{ij}$.

Now, $\psi(\varphi(D_{ij})) = \psi\Big((Y + CA^{-1}B)_{ij}\Big) = (D - CA^{-1}B + CA^{-1}B)_{ij} = D_{ij}$. Similarly, $\varphi(\psi(Y_{ij})) = \varphi\Big((D - CA^{-1}B)_{ij}\Big) = (Y + CA^{-1}B - CA^{-1}B)_{ij} = Y_{ij}$.

Finally, we note that the entries of $D - CA^{-1}B$ are mapped via φ to the corresponding entries of Y. Hence, by the Laplace expansion of the determinant, the set of $(t-j) \times (t-j)$ minors of $D - CA^{-1}B$ are mapped to the set of $(t-j) \times (t-j)$ minors of Y. Therefore, we conclude that the extension of $I_t(X)^k\, T[D]$ through φ is $I_{t-j}(Y)^k\, T[Y]$ for each k. ∎

To establish bounds on the projective dimenion of $I_t(X)^k$, we will make use of the analytic spread of $I_t(X)$. The following lemma is due to Cowsik and Nori, but we provide a proof.

Lemma 3.1.3 (Cowsik-Nori [16]) *Let t, m, and n be integers satisfying $1 \leq t \leq m \leq n$, K a field, X an $m \times n$ generic matrix over K, $S = K[X]$, and $J = I_t(X)$.*

a. If $t = m$, then $\ell(J) = m\,(n-m) + 1$.

b. If $t < m$, then $\ell(J) = mn$.

Proof To prove part (a), one notes the analytic spread of ideal of the maximal minors of a generic matrix is the dimension of the Grassmannian in its Plücker embedding. Hence, $\ell(J) = m\,(n-m) + 1$.

For a proof of (b), we start with the special case that $m = n = t+1$. Let $M_1, \ldots, M_s$ be the set of $t \times t$ minors of X. Then

$$\ell(J) = \dim \mathcal{F}(J) = \operatorname{trdeg}_K K(M_1, \ldots, M_s)\,.$$

Now, consider the classical adjoint matrix $\operatorname{Adj}(X)$. The entries of this matrix are precisely $M_1, \ldots, M_s$. Hence, $\det(\operatorname{Adj}(X)) \in K(M_1, \ldots, M_s)$. However, $\det(X)^{n-1} = \det(\operatorname{Adj}(X))$. Therefore, $\det(X)$ satisfies the monic polynomial $Y^{n-1} - \det(\operatorname{Adj}(X))$ in $K(M_1, \ldots, M_s)[Y]$. Thus, $\det(X)$ is algebraic over $K(M_1, \ldots, M_s)$. Ergo,

$$\operatorname{trdeg}_K K(M_1, \ldots, M_s) = \operatorname{trdeg}_K K(M_1, \ldots, M_s, \det(X))\,.$$

Next, consider $\mathrm{Adj}(X)\,X = \det(X)\,I_n$. Let D_j denote the jth column vector of $\det(X)\,I_n$, Z_j denote the jth column vector of X, and A_j denote the matrix obtained by replacing column j of $\mathrm{Adj}(X)$ by D_j. Applying Cramer's rule to the equation $\mathrm{Adj}(X)\,Z_i = D_i$, we obtain $X_{ij} = \det(A_j)\,/\det(\mathrm{Adj}(X))$. Let $A^{\widehat{ij}}$ denote the matrix obtained from $\mathrm{Adj}(X)$ by deleting row i and column j. Then by the Laplace expansion for determinants, $\det(A_j) = (-1)^{i+j}\det(X)\det\left(A^{\widehat{ij}}\right)$. Therefore, we see $X_{ij} \in K(M_1, \ldots, M_s, \det(X))$ for each i and each j. Next, we consider $K(\mathcal{X})$, where $\mathcal{X}$ is the set of variables X_{ij}. Now, $M_i \in K(\mathcal{X})$ and $\det(X) \in K(\mathcal{X})$, so $K(M_1, \ldots, M_s, \det(X)) = K(\mathcal{X})$. Hence, we see

$$\operatorname{trdeg}_K K(M_1, \ldots, M_s) = \operatorname{trdeg}_K K(\mathcal{X})\,.$$

We finish the proof by returning to the general case: $1 \le t < m \le n$. By varying over the collection of $(t+1) \times (t+1)$ submatrices of X, and applying the above special case argument, we see

$$\ell(J) = \dim \mathcal{F}(J) = \operatorname{trdeg}_K K(\mathcal{X}) = mn.$$

■

We now use the analytic spread of J to compute the maximum of the projective dimensions of the powers J^k.

Lemma 3.1.4 *Let t, m, and n be integers satisfying $1 \le t \le m \le n$, K a field, $R = K[x_1, \ldots, x_d]$ a polynomial ring in d variables over K, X an $m \times n$ generic matrix over R, $S = R[X]$, and $J = I_t(X)$.*

a. If $t = m$, then $\max_k\left\{\operatorname{pd} J^k\right\} = m\,(n-m)$.

b. If $t < m$, then $\max_k\left\{\operatorname{pd} J^k\right\} = mn - 1$.

Proof We begin by applying Lemma 3.0.1. More specifically, J is an S-ideal whose generators are elements of the ring $K[X]$, and S is a polynomial ring over $K[X]$. Therefore, we may reduce to the case that $R = K$.

From the work of Eisenbud and Huneke in [22, 3.5], we know that $\mathcal{R}(J)$ is Cohen-Macaulay in the setting of (a), where $t = m$. Therefore, Burch's inequality is an equality. In other words, $\ell(J) + \inf_k\{\text{depth } S/J^k\} = \dim S$.

The Auslander-Buchsbaum Formula gives $\text{pd } S/J^k + \text{depth } S/J^k = \dim S$. In particular, $\dim S - \text{depth } S/J^k = \text{pd } S/J^k$. Therefore,

$$\ell(J) = \dim S - \inf_k\{\text{depth } S/J^k\} = \sup_k\{\dim S - \text{depth } S/J^k\} = \sup_k\{\text{pd } S/J^k\}.$$

Hence, by Lemma 3.1.3, we have $\max_k\{\text{pd}_S\, S/J^k\} = m\,(n-m)+1$. We note that a free S-resolution $\mathbf{D}_\bullet$ of J^k can be extended to a free S-resolution of S/J^k by appending $\mathbf{D}_\bullet \to S$ and that every free resolution of S/J^k may be obtained in this way. Therefore, we have $\text{pd } J^k = \text{pd } S/J^k - 1$. Consequently, we see $\max_k\{\text{pd } J^k\} = \max_k\{\text{pd } S/J^k\} - 1 = m\,(n-m)$.

We now prove (b). Since S is a polynomial ring in mn variables over a field, by Hilbert's Syzygy Theorem, we have $\max_k\{\text{pd } S/J^k\} \leq mn$. Hence,

$$\max_k\{\text{pd } J^k\} \leq mn - 1.$$

On the other hand, by Burch's inequality, we have $\ell(J) + \inf_k\{\text{depth } S/J^k\} \leq \dim S$. Then, using the Auslander-Buchsbaum Formula as above, $\ell(J) \leq \max_k\{\text{pd } S/J^k\}$. Therefore, $\ell(J) \leq \max_k\{\text{pd } J^k\} + 1$, so $\ell(J) - 1 \leq \max_k\{\text{pd } J^k\}$. We now apply Lemma 3.1.3 to obtain

$$mn - 1 \leq \max_k\{\text{pd } J^k\}.$$

■

We end the section with the following lemma, which is be critical to the specialization results in Chapter 4.

Lemma 3.1.5 (Containment Lemma for Ordinary Matrices) *Let t, m, and n be integers satisfying $1 \leq t \leq m \leq n$, K a field, $R = K[x_1, \dots, x_d]$ a polynomial ring in d variables over K, X an $m \times n$ generic matrix over R, $S = R[X]$, and $J = I_t(X)$. For each k, let $\left(\mathbf{F^k}_\bullet, \partial^{\mathbf{k}}{}_\bullet\right)$ be a finite free S-resolution of J^k where each module $\mathbf{F^k}_i$ is finitely generated.*

a. *If* $t = m$*, then for each* j *satisfying* $1 \leq j \leq t-1$*,* $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ *for all* k *whenever* $i \geq (m-j)(n-m)+1$*.*

b. *If* $t < m$*, then for each* j *satisfying* $1 \leq j \leq t-1$*,* $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ *for all* k *whenever* $i \geq (m-j)(n-j)$*.*

Proof Fix j with $1 \leq j \leq t-1$, and suppose $\mathfrak{p} \in \operatorname{Spec}(S) \setminus V(I_j(X))$. We wish to show that $\mathfrak{p} \notin V\big(I\big(\partial^{\mathbf{k}}{}_i\big)\big)$ for all i in the appropriate range ($i \geq (m-j)\,n - m + 1$ in (a) and $i \geq (m-j)(n-j)$ in (b)). To do this, it suffices to prove that $\operatorname{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} < i$ for the appropriate range of i values.

Since $\mathfrak{p} \notin V(I_j(X))$, it follows that $X_\mathfrak{p}$ must have a $j \times j$ submatrix with unit determinant in $S_\mathfrak{p}$. Since performing elementary row and column operations does not affect the ideal of minors of a matrix, we may assume that the upper left $j \times j$ block of $X_\mathfrak{p}$ has unit determinant.

To make notation consistent with Lemma 3.1.2, let

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where A is a $j \times j$ matrix. Let $\Delta = \det A$, and consider the ring $T = R[A, B, C, \Delta^{-1}]$. Let Y be a generic $(m-j) \times (n-j)$ matrix over T. Under this notation, $S = R[A, B, C, D]$.

Now, since A has unit determinant in $S_\mathfrak{p}$, it follows that $\Delta^{-1} \in S_\mathfrak{p}$. Therefore, $S_\mathfrak{p}$ is a localization of $T[D] = R[A, B, C, \Delta^{-1}, D]$. Hence, $\operatorname{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} \leq \operatorname{pd}_{T[D]} J^k$.

Recall from Lemma 3.1.2 that there exists a T-algebra isomorphism $\varphi : T[D] \to T[Y]$ so that the extension of $J^k T[D]$ along φ is equal to $I_{t-j}(Y)^k\, T[Y]$. Therefore, $\operatorname{pd}_{T[D]} J^k = \operatorname{pd}_{T[Y]} I_{t-j}(Y)^k$. Since $T[Y] = R[A, B, C, \Delta^{-1}, Y]$ is free over $R[A, \Delta^{-1}, Y]$ and since the generators of $I_{t-j}(Y)^k$ are elements of $R[A, \Delta^{-1}, Y]$, by Lemma 3.0.1, $\operatorname{pd}_{T[Y]} I_{t-j}(Y)^k = \operatorname{pd}_{R[A,\Delta^{-1},Y]} I_{t-j}(Y)^k$. Moreover, $R[A, \Delta^{-1}, Y]$ is a localization of $R[A, Y]$ obtained by inverting the powers of Δ. Hence, one has $\operatorname{pd}_{R[A,\Delta^{-1},Y]} I_{t-j}(Y)^k \leq \operatorname{pd}_{R[A,Y]} I_{t-j}(Y)^k$. Now, $R[A, Y]$ is free over $R[Y]$ and the generators of $I_{t-j}(Y)^k$ are elements of $R[Y]$. Therefore, by Lemma 3.0.1, we have $\operatorname{pd}_{R[A,Y]} I_{t-j}(Y)^k = \operatorname{pd}_{R[Y]} I_{t-j}(Y)^k$. Consequently, $\operatorname{pd}_{S_\mathfrak{p}} J_\mathfrak{p}^k \leq \operatorname{pd}_{K[Y]} I_{t-j}(Y)^k$.

Since Y is a $(m-j) \times (n-j)$ generic matrix over R, which is a polynomial ring over a field, we may apply Lemma 3.1.4.

For (a), we have

$$\operatorname{pd}_{S_\mathfrak{p}} J_\mathfrak{p}^k \leq \operatorname{pd}_{K[Y]} I_{t-j}(Y)^k \leq (m-j)\left((n-j)-(m-j)\right) = (m-j)\,(n-m)\,,$$

and for (b), we have

$$\operatorname{pd}_{S_\mathfrak{p}} J_\mathfrak{p}^k \leq \operatorname{pd}_{K[Y]} I_{t-j}(Y)^k \leq (m-j)\,(n-j) - 1.$$

Therefore, for (a), $\operatorname{pd}_{S_\mathfrak{p}} J_\mathfrak{p}^k < i$ whenever $i \geq (m-j)\,(n-m)+1$. Thus, $\mathfrak{p} \notin V\big(I\big(\varphi^{\mathbf{k}}{}_i\big)\big)$ for all $i \geq (m-j)\,(n-m)+1$. Hence, we conclude $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq (m-j)\,(n-m)+1$.

Similarly, for (b), $\operatorname{pd}_{S_\mathfrak{p}} J_\mathfrak{p}^k < i$ whenever $i \geq (m-j)\,(n-j)$. Thus, $\mathfrak{p} \notin V\big(I\big(\varphi^{\mathbf{k}}{}_i\big)\big)$ for all $i \geq (m-j)\,(n-j)$. Hence, we conclude $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq (m-j)\,(n-j)$. ■

3.2 Determinantal Ideals of Symmetric Matrices

Let R be a Noetherian ring. Recall that a symmetric matrix A is defined by the property $A^T = A$.

Definition 3.2.1 *Let R be a Noetherian ring, and n be a positive integer. Let $\mathcal{X} = \{X_{ij} \mid 1 \leq i \leq j \leq n\}$ be a set of $\binom{n+1}{2}$ indeterminates over R, and let $S = R[\mathcal{X}]$ be the polynomial ring over R generated by the elements of $\mathcal{X}$. Suppose X is an $n \times n$ symmetric matrix with entries in S so that, for $i \leq j$, the (i,j)-entry of X is X_{ij}. We say that X is a generic symmetric matrix over R.*

As an example, the following is a 4×4 generic symmetric matrix over $\mathbb{Z}$.

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12} & X_{22} & X_{23} & X_{24} \\ X_{13} & X_{23} & X_{33} & X_{34} \\ X_{14} & X_{24} & X_{34} & X_{44} \end{pmatrix}$$

Often, we will start with a ring R and say "let X be an $n \times n$ generic symmetric matrix over R." When doing this, we implicitly assume that one has fixed a set $\mathcal{X}$ of $\binom{n+1}{2}$ indeterminates as in the definition above. We, similarly, will often then define $S = R[X]$. This is shorthand for the polynomial ring over R in the entries of the upper triangle of X.

As with the ordinary case, a key feature of generic symmetric matrices is that if a submatrix is made invertible, then the ideal of $t \times t$ minors (for $1 \leq t \leq n$) is still the ideal of minors of a generic matrix in the localized ring.

Lemma 3.2.2 *Let R be a Noetherian ring. Suppose X is an $n \times n$ generic symmetric matrix over R which can be written as the following block matrix*

$$X = \left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array} \right),$$

where A is a $j \times j$ matrix. Let $\Delta = \det A$, and consider the ring $T = R[A, B, \Delta^{-1}]$. Let Y be an $(n-j) \times (n-j)$ generic symmetric matrix over T. Then there exists a T-algebra isomorphism $\varphi : T[C] \to T[Y]$ so that the extension of $I_t(X)^k\, T[C]$ along φ is equal to $I_{t-j}(Y)^k\, T[Y]$ for each k.

Proof It is well-known that multiplying by an invertible matrix (on either side) does not change the ideal of minors of a matrix. Now, consider the matrix product below where, by an abuse of notation, each instance of the symbol I denotes the appropriately sized identity matrix.

$$\left(\begin{array}{c|c} I & 0 \\ \hline -B^T A^{-1} & I \end{array} \right) \left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array} \right) \left(\begin{array}{c|c} A^{-1} & -A^{-1}B \\ \hline 0 & I \end{array} \right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline -B^TA^{-1} & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline B^TA^{-1} & C - B^TA^{-1}B \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & C - B^TA^{-1}B \end{array}\right)$$

Hence, $I_t(X) = I_t\left(\begin{array}{c|c} I & 0 \\ \hline 0 & C - B^TA^{-1}B \end{array}\right)$. Using the Laplace expansion of the determinant, one sees that $I_t(X) = I_{t-j}\big(C - B^TA^{-1}B\big)$.

Next, we establish the T-algebra homomorphism $\varphi : T[C] \to T[Y]$ given by $C_{ij} \mapsto \big(Y + B^TA^{-1}B\big)_{ij}$ for $i \leq j$. To see that this homomorphism is a bijection, we establish its inverse. Let $\psi : T[Y] \to T[C]$ be the T-algebra homomorphism defined by $Y_{ij} \mapsto \big(C - B^TA^{-1}B\big)_{ij}$ for $i \leq j$.

For $i \leq j$, $\psi(\varphi(C_{ij})) = \psi\Big(\big(Y + B^TA^{-1}B\big)_{ij}\Big) = \big(C - B^TA^{-1}B + B^TA^{-1}B\big)_{ij} = C_{ij}$. Similarly, $\varphi(\psi(Y_{ij})) = \varphi\Big(\big(C - B^TA^{-1}B\big)_{ij}\Big) = \big(Y + B^TA^{-1}B - B^TA^{-1}B\big)_{ij} = Y_{ij}$.

Finally, we note that the entries of $C - B^TA^{-1}B$ are mapped via φ to the corresponding entries of Y since $C - B^TA^{-1}B$ is a symmetric matrix and since the map φ sends entries of the upper triangle of $C - B^TA^{-1}B$ to the corresponding entry in the upper triangle of Y. Therefore, by the Laplace expansion of the determinant, the set of $(t-j) \times (t-j)$ minors of $C - B^TA^{-1}B$ are mapped to the set of $(t-j) \times (t-j)$ minors of Y. Thus, we conclude that the extension of $I_t(X)^k\, T[D]$ through φ is $I_{t-j}(Y)^k\, T[Y]$ for all k. ■

Like in the ordinary case, we compute the analytic spread of J in order to obtain a global bound on the projective dimension of the ideals J^k. We mimic the Cowsik-Nori argument, just like in the ordinary case.

Lemma 3.2.3 *Let t and n be integers satisfying $1 \leq t \leq n$, K a field, X an $n \times n$ generic symmetric matrix over K, $S = K[X]$, and $J = I_t(X)$.*

a. If $t = n$, then $\ell(J) = 1$.

b. If $t < n$, then $\ell(J) = \binom{n+1}{2}$.

Proof To prove part (a), one notes that $\det(X)$ is a regular element in S. Hence, $I_n(X) = (\det(X))$ is generated by a regular sequence. All such ideals are of linear type, according to the work of Huneke in [34, 3.1]. Hence, $\ell(J) = \mu(J) = 1$, as in Remark 2.2.19.

We now prove (b). Start with the special case that $n = t+1$. Let $M_1, \ldots, M_s$ be the set of $t \times t$ minors of X. Then

$$\ell(J) = \dim \mathcal{F}(J) = \operatorname{trdeg}_K K(M_1, \ldots, M_s)\,.$$

Now, consider the classical adjoint matrix $\operatorname{Adj}(X)$. The entries of this matrix are precisely $M_1, \ldots, M_s$. Hence, $\det(\operatorname{Adj}(X)) \in K(M_1, \ldots, M_s)$. However, $\det(X)^{n-1} = \det(\operatorname{Adj}(X))$. Therefore, $\det(X)$ satisfies the monic polynomial $Y^{n-1} - \det(\operatorname{Adj}(X))$ in $K(M_1, \ldots, M_s)[Y]$. Thus, $\det(X)$ is algebraic over $K(M_1, \ldots, M_s)$. Ergo,

$$\operatorname{trdeg}_K K(M_1, \ldots, M_s) = \operatorname{trdeg}_K K(M_1, \ldots, M_s, \det(X))\,.$$

Next, consider $\operatorname{Adj}(X)\, X = \det(X)\, I_n$. Let D_j denote the jth column vector of $\det(X)\, I_n$, Z_j denote the jth column vector of X, and A_j denote the matrix obtained by replacing column j of $\operatorname{Adj}(X)$ by D_j. Applying Cramer's rule to the equation $\operatorname{Adj}(X)\, Z_i = D_i$, we obtain $X_{ij} = \det(A_j) \,/\, \det(\operatorname{Adj}(X))$. Let $A^{\widehat{ij}}$ denote the matrix obtained from $\operatorname{Adj}(X)$ by deleting row i and column j. Then by the Laplace expansion for determinants, $\det(A_j) = (-1)^{i+j} \det(X) \det\left(A^{\widehat{ij}}\right)$. Therefore, we see $X_{ij} \in K(M_1, \ldots, M_s, \det(X))$ for each i and each j. Next, we consider $K(\mathcal{X})$, where $\mathcal{X}$ is the set of variables X_{ij}. Now, $M_i \in K(\mathcal{X})$ and $\det(X) \in K(\mathcal{X})$, so $K(M_1, \ldots, M_s, \det(X)) = K(\mathcal{X})$. Hence, we see

$$\operatorname{trdeg}_K K(M_1, \ldots, M_s) = \operatorname{trdeg}_K K(\mathcal{X})\,.$$

We finish the proof by returning to the general case: $1 \le t < n$. By varying over the collection of $(t+1) \times (t+1)$ submatrices of X, and applying the above argument in the special case, we see

$$\ell(J) = \dim \mathcal{F}(J) = \operatorname{trdeg}_K K(\mathcal{X}) = \binom{n+1}{2}.$$

■

We now use the analytic spread of J to find a global bound on $\operatorname{pd} J^k$.

Lemma 3.2.4 *Let t and n be integers satisfying $1 \leq t \leq n$, K a field, $R = K[x_1, \ldots, x_d]$ a polynomial ring in d variables over K, X an $n \times n$ generic symmetric matrix over R, $S = R[X]$, and $J = I_t(X)$.*

a. If $t = n$, then $\max_k\{\operatorname{pd} J^k\} = 0$.

b. If $t < n$, then $\max_k\{\operatorname{pd} J^k\} = \binom{n+1}{2} - 1$.

Proof We begin by applying Lemma 3.0.1. More specifically, J is an S-ideal whose generators are elements of the ring $K[X]$, and S is a polynomial ring over $K[X]$. Therefore, we may reduce to the case that $R = K$.

For (a), we know that J is a principal ideal generated by a regular element. Hence, for all k, J^k is a free R-module. Therefore, for all k, $\operatorname{pd} J^k = 0$.

To prove (b), we note that since S is a polynomial ring in $\binom{n+1}{2}$ variables over a field, by Hilbert's Syzygy Theorem, we have $\max_k\{\operatorname{pd} S/J^k\} \leq \binom{n+1}{2}$, so

$$\max_k\{\operatorname{pd} J^k\} \leq \binom{n+1}{2} - 1.$$

On the other hand, by Burch's inequality, $\ell(J) + \inf_k\{\operatorname{depth} S/J^k\} \leq \dim S$. The Auslander-Buchsbaum Formula gives us that $\operatorname{pd} S/J^k + \operatorname{depth} S/J^k = \dim S$. In particular, $\dim S - \operatorname{depth} S/J^k = \operatorname{pd} S/J^k$. Therefore,

$$\ell(J) \leq \dim S - \inf_k\{\operatorname{depth} S/J^k\} = \sup_k\{\dim S - \operatorname{depth} S/J^k\} = \sup_k\{\operatorname{pd} S/J^k\}.$$

Hence, $\ell(J) \leq \max_k\{\operatorname{pd} J^k\} + 1$, so $\ell(J) - 1 \leq \max_k\{\operatorname{pd} J^k\}$. Applying Lemma 3.2.3, we obtain

$$\binom{n+1}{2} - 1 \leq \max_k\{\operatorname{pd} J^k\}.$$

■

We conclude this section with the following lemma, which is a critical tool in the specialization arguments of Chapter 4.

Lemma 3.2.5 (Containment Lemma for Symmetric Matrices) *Let t and n be integers satisfying $1 \leq t \leq n$, K a field, $R = K[x_1, \ldots, x_d]$ a polynomial ring in d variables over K, X an $n \times n$ generic symmetric matrix over R, $S = R[X]$, and $J = I_t(X)$. For each k, let $(\mathbf{F^k}_\bullet, \partial^\mathbf{k}_\bullet)$ be a finite free S-resolution of J^k where each module $\mathbf{F^k}_i$ is finitely generated.*

a. If $t = m$, then for each j satisfying $1 \leq j \leq t-1$, $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^\mathbf{k}_i)}$ for all k whenever $i \geq 1$.

b. If $t < m$, then for each j satisfying $1 \leq j \leq t-1$, $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^\mathbf{k}_i)}$ for all k whenever $i \geq \binom{n-j+1}{2}$.

Proof Fix j with $1 \leq j \leq t-1$, and suppose $\mathfrak{p} \in \operatorname{Spec}(S) \setminus V(I_j(X))$. We wish to show that $\mathfrak{p} \notin V\big(I\big(\partial^\mathbf{k}_i\big)\big)$ for all i in the appropriate range ($i \geq 1$ in (a) and $i \geq \binom{n-j+1}{2}$ in (b)). To do this, it suffices to prove that $\operatorname{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} < i$ for the appropriate range of i values.

Since $\mathfrak{p} \notin V(I_j(X))$, it follows that $X_\mathfrak{p}$ must have a $j \times j$ submatrix with unit determinant in $S_\mathfrak{p}$. Since performing elementary row and corresponding column operations does not affect the ideal of minors of a matrix, we may assume that the upper left $j \times j$ block of $X_\mathfrak{p}$ has a unit determinant.

To make notation consistent with Lemma 3.2.2, let

$$X = \left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right),$$

where A is a $j \times j$ symmetric matrix. Let $\Delta = \det A$, and consider the ring $T = R[A, B, \Delta^{-1}]$. Let Y be an $(n-j)\times(n-j)$ generic symmetric matrix over T. Written in this notation, $S = R[A, B, C]$.

Since A has unit determinant in $S_\mathfrak{p}$, it follows that $\Delta^{-1} \in S_\mathfrak{p}$. Therefore, $S_\mathfrak{p}$ is a localization of $T[C]$. Hence, $\operatorname{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} \leq \operatorname{pd}_{T[C]} J^k$.

Recall from Lemma 3.2.2 that there exists a T-algebra isomorphism $\varphi : T[C] \to T[Y]$ so that the extension of $J^k T[C]$ along φ is equal to $I_{t-j}(Y)^k T[Y]$. Therefore,

$\operatorname{pd}_{T[C]} J^k = \operatorname{pd}_{T[Y]} I_{t-j}(Y)^k$. Since $T[Y] = R[A, B, \Delta^{-1}, Y]$ is free over $R[A, \Delta^{-1}, Y]$ and since the generators of $I_{t-j}(Y)^k$ are elements of $R[A, \Delta^{-1}, Y]$, by Lemma 3.0.1, $\operatorname{pd}_{T[Y]} I_{t-j}(Y)^k = \operatorname{pd}_{R[A,\Delta^{-1},Y]} I_{t-j}(Y)^k$. The ring $R[A, \Delta^{-1}, Y]$ is a localization of $R[A, Y]$ obtained by inverting powers of Δ. Consequently, $\operatorname{pd}_{R[A,\Delta^{-1},Y]} I_{t-j}(Y)^k \leq \operatorname{pd}_{R[A,Y]} I_{t-j}(Y)^k$. Now, $R[A, Y]$ is free over $R[Y]$ and the generators of $I_{t-j}(Y)^k$ are elements of $R[Y]$. Therefore, by Lemma 3.0.1, $\operatorname{pd}_{R[A,Y]} I_{t-j}(Y)^k = \operatorname{pd}_{R[Y]} I_{t-j}(Y)^k$. Hence, $\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k \leq \operatorname{pd}_{K[Y]} I_{t-j}(Y)^k$.

Since Y is an $(n-j) \times (n-j)$ generic symmetric matrix over R, which is a polynomial ring over a field, we may apply Lemma 3.2.4.

For (a),

$$\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k \leq \operatorname{pd}_{K[Y]} I_{t-j}(Y)^k \leq 0,$$

and for (b),

$$\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k \leq \operatorname{pd}_{K[Y]} I_{t-j}(Y)^k \leq \binom{n-j+1}{2} - 1.$$

Therefore, for (a), $\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k < i$ whenever $i \geq 1$. Thus, $\mathfrak{p} \notin V\big(I\big(\partial^{\mathbf{k}}{}_i\big)\big)$ for all $i \geq 1$. Hence, $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq 1$.

Similarly, for (b), $\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k < i$ whenever $i \geq \binom{n-j+1}{2}$. Thus, $\mathfrak{p} \notin V\big(I\big(\partial^{\mathbf{k}}{}_i\big)\big)$ for all $i \geq \binom{n-j+1}{2}$. Hence, $\sqrt{I_j(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq \binom{n-j+1}{2}$. ■

3.3 Pfaffian Ideals of Alternating Matrices

Let R be a Noetherian ring. Recall that a matrix A is alternating if it is skew-symmetric and the diagonal entries of A are 0. We reiterate that skew-symmetric implies alternating only in the case where 2 is a non-zero-divisor in R. Since the useful properties of alternating matrices do not, in general, hold for skew-symmetric matrices, we must take care when performing operations on alternating matrices.

Definition 3.3.1 *Let R be a Noetherian ring and n be a positive integer. Let $\mathcal{X} = \{X_{ij} \mid 1 \leq i < j \leq n\}$ be a set of $\binom{n}{2}$ indeterminates over R, and let $S = R[\mathcal{X}]$ be the polynomial ring over R generated by the elements of $\mathcal{X}$. Suppose X is an $n \times n$*

alternating matrix with entries in S so that, for $i < j$, the (i, j)-entry of X is X_{ij}. We say that X is a generic alternating matrix over R.

As an example, the following is a 4×4 generic alternating matrix over $\mathbb{Z}$.

$$\begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ -X_{12} & 0 & X_{23} & X_{24} \\ -X_{13} & -X_{23} & 0 & X_{34} \\ -X_{14} & -X_{24} & -X_{34} & 0 \end{pmatrix}$$

Often, we will start with a ring R and say "let X be an $n \times n$ generic alternating matrix over R." When doing this, we implicitly assume that one has fixed a set $\mathcal{X}$ of $\binom{n}{2}$ indeterminates as in the definition above. We, similarly, will often then define $S = R[X]$. This is shorthand for the polynomial ring over R in the entries above the diagonal of X.

As with the ordinary and symmetric cases, a key feature of generic alternating matrices is that if a submatrix is made invertible, then the ideal of $2t \times 2t$ Pfaffians (for $2 \le 2t \le n$) is still an ideal of Pfaffians of a generic alternating matrix in the localized ring. Before we can do this proof, however, we need a lemma.

Lemma 3.3.2 *Let R be a ring, n a positive integer, A an $n \times n$ alternating matrix with entries in R, and M an $n \times n$ matrix with entries in R. Then $M^T AM$ is an alternating matrix.*

Proof We have $\left(M^T AM\right)^T = M^T A^T \left(M^T\right)^T = -M^T AM$. Thus, $M^T AM$ is skew-symmetric. This is not sufficient to guarantee that $M^T AM$ is alternating, since 2 could be a zero-divisor in R.

Therefore, we consider the ring $\mathbb{Z}$. Let X be an $n \times n$ generic alternating matrix over $\mathbb{Z}$ and N be an $n \times n$ generic ordinary matrix over $\mathbb{Z}[X]$. As demonstrated above, the product $N^T XN$ is skew-symmetric. Moreover, 2 is a non-zero-divisor in $\mathbb{Z}[X, N]$. Therefore, $N^T XN$ is an alternating matrix.

We establish the $\mathbb{Z}$-algebra homomorphism $\varphi : \mathbb{Z}[X,N] \to R$ so that $X_{ij} \mapsto A_{ij}$ for $i < j$ and $N_{ij} \mapsto M_{ij}$. Then for all i and j, $X_{ij} \mapsto A_{ij}$. Since φ is a ring homomorphism, it follows that, for all i and j, $\left(N^T X N\right)_{ij} \mapsto \left(M^T A M\right)_{ij}$. In particular, $0 \mapsto \left(M^T A M\right)_{ii}$ for all i. Thus, $M^T A M$ is alternating. ∎

Lemma 3.3.3 *Let R be a Noetherian ring. Suppose X is an $n \times n$ generic alternating matrix over R. which can be written as the following block matrix*

$$X = \left(\begin{array}{c|c} A & B \\ \hline -B^T & C \end{array}\right),$$

where A is a $2j \times 2j$ block. Let $\Delta = \operatorname{Pf}(A)$, and consider the ring $T = R[A, B, \Delta^{-1}]$. Let Y be an $(n-2j) \times (n-2j)$ generic alternating matrix over T. Then there exists a T-algebra isomorphism $\varphi : T[C] \to T[Y]$ so that the extension of $\operatorname{Pf}_{2t}(X)^k\, T[C]$ along φ is equal to $\operatorname{Pf}_{2t-2j}(Y)^k\, T[Y]$ for all k.

Proof It is well-known that multiplying by an invertible matrix (on either side) does not change the ideal of Pfaffians of a matrix. Now, consider the matrix product below where, in an abuse of notation, each instance of the symbol I denotes the appropriately sized identity matrix.

$$\left(\begin{array}{c|c} I & 0 \\ \hline B^T A^{-1} & I \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline -B^T & C \end{array}\right) \left(\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline B^T A^{-1} & I \end{array}\right) \left(\begin{array}{c|c} A & 0 \\ \hline -B^T & C + B^T A^{-1} B \end{array}\right)$$

$$= \left(\begin{array}{c|c} A & 0 \\ \hline 0 & C + B^T A^{-1} B \end{array}\right) = L$$

The above matrix is alternating by Lemma 3.3.2.

Hence, $\operatorname{Pf}_{2t}(X) = \operatorname{Pf}_{2t}(L)$. We wish to show $\operatorname{Pf}_{2t}(L) = \operatorname{Pf}_{2t-2j}\left(C + B^T A^{-1} B\right)$.

From Remark 2.3.3, we know that if M and N are alternating matrices, then

$$\operatorname{Pf}\left(\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right) = \operatorname{Pf}(M)\operatorname{Pf}(N)\,. \qquad (3.3.3.A)$$

Therefore, setting $M = A$ and letting N be a $2t-2j$ principal submatrix of $C + B^TA^{-1}B$, we see that $\Delta \cdot \mathrm{Pf}(N) \in \mathrm{Pf}_{2t}(X)$. Since Δ is a unit in T, this implies that $\mathrm{Pf}(N) \in \mathrm{Pf}_{2t}(L)$. Thus, $\mathrm{Pf}_{2t-2j}\big(C + B^TA^{-1}B\big) \subseteq \mathrm{Pf}_{2t}(L)$. Let P be a $(2t)\times(2t)$ principal submatrix of L which isn't a principal submatrix of $C + B^TA^{-1}B$. Then P is of the form in Eq. (3.3.3.A), where M is a principal submatrix of A and N is a principal submatrix of $C + B^TA^{-1}B$. We may assume M and N have even dimensions otherwise $\mathrm{Pf}(P) = 0$. Also, we may assume the dimensions of N are larger than $2t-2j$, otherwise, $M = A$, which was handled earlier. Then $\mathrm{Pf}(P) = \mathrm{Pf}(M)\,\mathrm{Pf}(N) \in \mathrm{Pf}_{2t-2j}\big(C + B^TA^{-1}B\big)$, since $\mathrm{Pf}(N) \in \mathrm{Pf}_{2t-2j}\big(C + B^TA^{-1}B\big)$ by the Laplace expansion of Pfaffians. Therefore, $\mathrm{Pf}_{2t}(X) = I_{2t-2j}\big(C + B^TA^{-1}B\big)$.

Next, we establish the T-algebra homomorphism $\varphi : T[C] \to T[Y]$ given by $C_{ij} \mapsto \big(Y - B^TA^{-1}B\big)_{ij}$ for $i < j$. To see that this homomorphism is a bijection, we establish its inverse. Let $\psi : T[Y] \to T[C]$ be the T-algebra homomorphism defined by $Y_{ij} \mapsto \big(C + B^TA^{-1}B\big)_{ij}$ for $i < j$.

For $i < j$, $\psi(\varphi(C_{ij})) = \psi\Big(\big(Y - B^TA^{-1}B\big)_{ij}\Big) = \big(C + B^TA^{-1}B - B^TA^{-1}B\big)_{ij} = C_{ij}$. Similarly, $\varphi(\psi(Y_{ij})) = \varphi\Big(\big(C + B^TA^{-1}B\big)_{ij}\Big) = \big(Y - B^TA^{-1}B + B^TA^{-1}B\big)_{ij} = Y_{ij}$.

Finally, we note that the entries of $C + B^TA^{-1}B$ are mapped via φ to the corresponding entries of Y since both Y and $C + B^TA^{-1}B$ are alternating matrices and since the map φ sends entries of the upper triangle of $C + B^TA^{-1}B$ to the corresponding entry in the upper triangle of Y. Therefore, by the Laplace expansion of Pfaffians, the set of $(2t-2j)\times(2t-2j)$ Pfaffians of $C - B^TA^{-1}B$ are mapped to the set of $(2t-2j)\times(2t-2j)$ Pfaffians of Y. Thus, we conclude that the extension of $\mathrm{Pf}_{2t}(X)^k\,T[C]$ through φ is $\mathrm{Pf}_{2t-2j}(Y)^k\,T[Y]$ for all k. ■

As in the ordinary and symmetric cases, we apply the Cowsik-Nori argument to compute the analytic spread of J.

Lemma 3.3.4 *Let t and n be integers satisfying $2 \le 2t \le n$, K a field, X an $n \times n$ generic alternating matrix over K, $S = K[X]$, and $J = \mathrm{Pf}_{2t}(X)$.*

a. *If* $2t = n$*, then* $\ell(J) = 1$*.*

b. *If* $2t = n - 1$*, then* $\ell(J) = n$*.*

c. *If* $2t \leq n - 2$*, then* $\ell(J) = \binom{n}{2}$*.*

Proof To prove part (a), one notes that $\mathrm{Pf}(X)$ is a regular element in S. Hence, $\mathrm{Pf}_n(X) = (\mathrm{Pf}(X))$ is a complete intersection ideal. All complete intersection ideals are of linear type. Therefore, by Remark 2.2.19, $\ell(J) = \mu(J) = 1$.

To prove part (b), we note that J is of linear type, as proved by Huneke in [35, 2.2]. Therefore, as in the proof of part (a), $\ell(J) = \mu(J) = n$.

For a proof of (c), we start with the special case that $n = 2t + 2$. Let $M_1, \ldots, M_s$ be the set of $2t \times 2t$ Pfaffians of X. Then

$$\ell(J) = \dim \mathcal{F}(J) = \operatorname{trdeg}_K K(M_1, \ldots, M_s)\,.$$

Now, consider the Pfaffian adjoint matrix $\mathrm{PfAdj}(X)$. The entries of this matrix are either 0 or a unit multiple of one of $M_1, \ldots, M_s$. Hence, $\det(\mathrm{PfAdj}(X)) \in K(M_1, \ldots, M_s)$. From Remark 2.3.3, we know $\mathrm{Pf}(X)^{n-2} = \det(\mathrm{PfAdj}(X))$. Thus, $\mathrm{Pf}(X)$ satisfies the monic polynomial $Y^{n-2} - \det(\mathrm{PfAdj}(X))$ in $K(M_1, \ldots, M_s)[Y]$. Therefore, $\mathrm{Pf}(X)$ is algebraic over $K(M_1, \ldots, M_s)$. Ergo,

$$\operatorname{trdeg}_K K(M_1, \ldots, M_s) = \operatorname{trdeg}_K K(M_1, \ldots, M_s, \mathrm{Pf}(X))\,.$$

Next, consider $\mathrm{PfAdj}(X)\,X = \mathrm{Pf}(X)\,I_n$. Let P_j denote the jth column vector of $\mathrm{Pf}(X)\,I_n$, Z_j denote the jth column vector of X, and A_j denote the matrix obtained by replacing column j of $\mathrm{PfAdj}(X)$ by P_j. Applying Cramer's rule to the equation $\mathrm{PfAdj}(X)\,Z_j = P_j$, we obtain $X_{ij} = \det(A_j) \,/\, \det(\mathrm{PfAdj}(X))$. Let $A^{\widehat{ij}}$ denote the matrix obtained from $\mathrm{PfAdj}(X)$ by deleting row i and column j. Then by the Laplace expansion for determinants, $\det(A_j) = (-1)^{i+j}\,\mathrm{Pf}(X)\det\left(A^{\widehat{ij}}\right)$. Therefore, we see $X_{ij} \in K(M_1, \ldots, M_s, \mathrm{Pf}(X))$ for each i and each j. Next, we consider $K(\mathcal{X})$, where $\mathcal{X}$ is the set of variables X_{ij}. Now, $M_i \in K(\mathcal{X})$ and $\mathrm{Pf}(X) \in K(\mathcal{X})$, so $K(M_1, \ldots, M_s, \mathrm{Pf}(X)) = K(\mathcal{X})$. Hence,

$$\operatorname{trdeg}_K K(M_1, \ldots, M_s) = \operatorname{trdeg}_K K(\mathcal{X})\,.$$

We finish the proof by returning to the general case: $2 \leq 2t \leq n-2$. By varying over the collection of principal $(2t+2) \times (2t+2)$ submatrices of X, and applying the above special case argument, we see

$$\ell(J) = \dim \mathcal{F}(J) = \operatorname{trdeg}_K K(\mathcal{X}) = \binom{n}{2}.$$

■

We apply the above result to obtain projective dimension bounds on J^k.

Lemma 3.3.5 *Let t and n be integers satisfying $2 \leq 2t \leq n$, K a field, $R = K[x_1, \ldots, x_d]$ a polynomial ring in d variables over K, X an $n \times n$ generic alternating matrix over R, $S = R[X]$, and $J = \operatorname{Pf}_{2t}(X)$.*

a. If $2t = n$, then $\max_k\{\operatorname{pd} J^k\} = 0$.

b. If $2t = n-1$, then $\max_k\{\operatorname{pd} J^k\} = n-1$.

c. If $2t \leq n-2$, then $\max_k\{\operatorname{pd} J^k\} = \binom{n}{2} - 1$.

Proof We begin by applying Lemma 3.0.1. More specifically, J is an S-ideal whose generators are elements of the ring $K[X]$, and S is a polynomial ring over $K[X]$. Therefore, we may reduce to the case that $R = K$.

For (a), we know that J is a principal ideal generated by a regular element. Hence, for all k, J^k is a free R-module. Therefore, for all k, $\operatorname{pd} J^k = 0$.

For (b), from the work of Huneke in [35, 2.2], we know that $\mathcal{R}(J)$ is Cohen-Macaulay. Therefore, Burch's inequality is an equality. In other words, $\ell(J) + \inf_k\{\operatorname{depth} S/J^k\} = \dim S$.

The Auslander-Buchsbaum Formula gives $\operatorname{pd} S/J^k + \operatorname{depth} S/J^k = \dim S$. In particular, $\dim S - \operatorname{depth} S/J^k = \operatorname{pd} S/J^k$. Therefore,

$$\ell(J) = \dim S - \inf_k\{\operatorname{depth} S/J^k\} = \sup_k\{\dim S - \operatorname{depth} S/J^k\} = \sup_k\{\operatorname{pd} S/J^k\}.$$

Hence, by Lemma 3.3.4, we have $\max_k\{\operatorname{pd} S/J^k\} = n$. We note that a free S-resolution $\mathbf{F}_\bullet$ of J^k can be extended to a free S-resolution of S/J^k by appending

$\mathbf{F}_\bullet \to S$ and that every free resolution of S/J^k may be obtained in this way. Thus, $\operatorname{pd} J^k = \operatorname{pd} S/J^k - 1$. Therefore,

$$\max_k\{\operatorname{pd} J^k\} = \max_k\{\operatorname{pd} S/J^k\} - 1 = n - 1.$$

To prove (c), we note that since S is a polynomial ring in $\binom{n}{2}$ variables over a field, by Hilbert's Syzygy Theorem, $\max_k\{\operatorname{pd} S/J^k\} \leq \binom{n}{2}$, so

$$\max_k\{\operatorname{pd} J^k\} \leq \binom{n}{2} - 1.$$

On the other hand, by Burch's inequality, $\ell(J) + \inf_k\{\operatorname{depth} S/J^k\} \leq \dim S$. The Auslander-Buchsbaum Formula gives us that $\operatorname{pd} S/J^k + \operatorname{depth} S/J^k = \dim S$. In particular, $\dim S - \operatorname{depth} S/J^k = \operatorname{pd} S/J^k$. Therefore,

$$\ell(J) \leq \dim S - \inf_k\{\operatorname{depth} S/J^k\} = \sup_k\{\dim S - \operatorname{depth} S/J^k\} = \sup_k\{\operatorname{pd} S/J^k\}.$$

Thus, $\ell(J) \leq \max_k\{\operatorname{pd} J^k\} + 1$, so $\ell(J) - 1 \leq \max_k\{\operatorname{pd} J^k\}$. Applying Lemma 3.3.4, we obtain

$$\binom{n}{2} - 1 \leq \max_k\{\operatorname{pd} J^k\}.$$

■

We end the chapter with the containment lemma for alternating matrices. This lemma is an important tool in the specialization results of Chapter 4.

Lemma 3.3.6 (Containment Lemma for Alternating Matrices) *Let t and n be integers satisfying $2 \leq 2t \leq n$, K a field, $R = K[x_1, \ldots, x_d]$ a polynomial ring in d variables over K, X an $n \times n$ generic alternating matrix over R, $S = R[X]$, and $J = \operatorname{Pf}_{2t}(X)$. For each k, let $(\mathbf{F^k}_\bullet, \partial^{\mathbf{k}}_\bullet)$ be a finite free S-resolution of J^k where each module $\mathbf{F^k}_i$ is finitely generated.*

a. If $2t = n$, then for each j satisfying $2 \leq 2j \leq 2t-2$, $\sqrt{\operatorname{Pf}_{2j}(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}_i)}$ for all k whenever $i \geq 1$.

b. If $2t = n-1$, then for each j satisfying $2 \leq 2j \leq 2t-2$, $\sqrt{\operatorname{Pf}_{2j}(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}_i)}$ for all k whenever $i \geq n-2j$.

c. If $2t \le n-2$, then for each j satisfying $2 \le 2j \le 2t-2$, $\sqrt{\mathrm{Pf}_{2j}(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \ge \binom{n-2j}{2}$.

Proof Fix j with $2 \le 2j \le 2t-2$, and suppose $\mathfrak{p} \in \mathrm{Spec}(S) \setminus V(\mathrm{Pf}_{2j}(X))$. We wish to show that $\mathfrak{p} \notin V\big(I\big(\partial^{\mathbf{k}}{}_i\big)\big)$ for all i in the appropriate range ($i \ge 1$ in (a), $i \ge n-2j$ in (b), and $i \ge \binom{n-2j}{2}$ in (c)). To do this, it suffices to prove that $\mathrm{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} < i$ for the appropriate range of i values.

Since $\mathfrak{p} \notin V(\mathrm{Pf}_{2j}(X))$, it follows that $X_\mathfrak{p}$ must have a $2j \times 2j$ principal submatrix with unit Pfaffian in $S_\mathfrak{p}$. Since performing elementary row and corresponding column operations does not affect the ideal of Pfaffians of a matrix, we may assume that the upper left $2j \times 2j$ block of $X_\mathfrak{p}$ has a unit Pfaffian.

To make notation consistent with Lemma 3.3.3, let

$$X = \left(\begin{array}{c|c} A & B \\ \hline -B^T & C \end{array}\right),$$

where A is a $2j \times 2j$ alternating block. Let $\Delta = \mathrm{Pf}(A)$, and consider the ring $T = R[A, B, \Delta^{-1}]$. Let Y be an $(n-2j) \times (n-2j)$ generic alternating matrix over T. In this notation, $S = R[A, B, C]$.

Since A has a unit Pfaffin in $S_\mathfrak{p}$, it follows that $\Delta^{-1} \in S_\mathfrak{p}$. Therefore, $S_\mathfrak{p}$ is a localization of $T[C]$. Hence, $\mathrm{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} \le \mathrm{pd}_{T[C]} J^k$.

Recall from Lemma 3.3.3 that there exists a T-algebra isomorphism $\varphi : T[C] \to T[Y]$ so that the extension of $J^k T[C]$ along φ is equal to $\mathrm{Pf}_{2t-2j}(Y)^k\, T[Y]$. Therefore, $\mathrm{pd}_{T[C]} J^k = \mathrm{pd}_{T[Y]} \mathrm{Pf}_{2t-2j}(Y)^k$. Since $T[Y] = R[A, B, \Delta^{-1}, Y]$ is free over $R[A, \Delta^{-1}, Y]$ and since the generators of $\mathrm{Pf}_{2t-2j}(Y)^k$ are elements of $R[A, \Delta^{-1}, Y]$, by Lemma 3.0.1, we have $\mathrm{pd}_{T[Y]} \mathrm{Pf}_{2t-2j}(Y)^k = \mathrm{pd}_{R[A,\Delta^{-1},Y]} \mathrm{Pf}_{2t-2j}(Y)^k$. The ring $R[A, \Delta^{-1}, Y]$ is a localization of $R[A, Y]$ obtained by inverting powers of Δ. Hence, $\mathrm{pd}_{R[A,\Delta^{-1},Y]} \mathrm{Pf}_{2t-2j}(Y)^k \le \mathrm{pd}_{R[A,Y]} \mathrm{Pf}_{2t-2j}(Y)^k$. Now, $R[A, Y]$ is free over $R[Y]$ and the generators of $\mathrm{Pf}_{2t-2j}(Y)^k$ are elements of $R[Y]$. Therefore, by Lemma 3.0.1, $\mathrm{pd}_{R[A,Y]} \mathrm{Pf}_{2t-2j}(Y)^k = \mathrm{pd}_{R[Y]} \mathrm{Pf}_{2t-2j}(Y)^k$. Hence, $\mathrm{pd}_{S_\mathfrak{p}} J^k_\mathfrak{p} \le \mathrm{pd}_{K[Y]} \mathrm{Pf}_{2t-2j}(Y)^k$.

Since Y is an $(n-2j) \times (n-2j)$ generic alternating matrix over R, which is a polynomial ring over a field, we may apply Lemma 3.3.4.

For (a),

$$\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k \leq \operatorname{pd}_{K[Y]} \operatorname{Pf}_{2t-2j}(Y)^k \leq 0,$$

for (b),

$$\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k \leq \operatorname{pd}_{K[Y]} \operatorname{Pf}_{2t-2j}(Y)^k \leq n-2j-1,$$

and for (c),

$$\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k \leq \operatorname{pd}_{K[Y]} \operatorname{Pf}_{2t-2j}(Y)^k \leq \binom{n-2j}{2} - 1.$$

Therefore, for (a), $\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k < i$ whenever $i \geq 1$. Thus, $\mathfrak{p} \notin V\big(I(\partial^{\mathbf{k}}{}_i)\big)$ for all $i \geq 1$. Hence, $\sqrt{\operatorname{Pf}_{2j}(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq 1$.

Similarly, for (b), $\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k < i$ whenever $i \geq n-2j$. Thus, $\mathfrak{p} \notin V\big(I(\partial^{\mathbf{k}}{}_i)\big)$ for all $i \geq n-2j$. Hence, $\sqrt{\operatorname{Pf}_{2j}(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq n-2j$.

Similarly, for (c), $\operatorname{pd}_{S_{\mathfrak{p}}} J_{\mathfrak{p}}^k < i$ whenever $i \geq \binom{n-2j}{2}$. Thus, $\mathfrak{p} \notin V\big(I(\partial^{\mathbf{k}}{}_i)\big)$ for all $i \geq \binom{n-2j}{2}$. Hence, $\sqrt{\operatorname{Pf}_{2j}(X)} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k whenever $i \geq \binom{n-2j}{2}$. ■

4. APPROXIMATION OF REES RINGS THROUGH SPECIALIZATION

This chapter is concerned with using specialization to approximate Rees rings. Let S be a positively graded Cohen-Macaulay ring with S_0 local. Suppose J and N are S-ideals so that N is generated by a sequence which is weakly regular on S and S/J. Let $R = S/N$ and $I = JR$. Then I is a specialization of J. The critical property of I being a specialization of J is that $J \otimes_S R \cong I$. This will be demonstrated in the proof of Lemma 4.1.1, which is one of the main tools of the chapter. In particular, part (c) of Lemma 4.1.1 describes how $\mathcal{A}(I)$ fits between $\mathcal{A}(J) \otimes_S R$ and the kernel of the natural surjection $\psi : \mathcal{R}(J) \otimes_S R \to \mathcal{R}(I)$. Because of this, we are able to bound $b_0(\mathcal{A}(I))$ and $\operatorname{topdeg}(\mathcal{A}(I))$ in terms of $b_0(\mathcal{A}(J))$, $\operatorname{topdeg}(\mathcal{A}(J))$, $b_0(\ker \psi)$, and $\operatorname{topdeg}(\ker \psi)$.

In order to control $\ker \psi$, we use the concept of approximate resolutions.

Definition 4.0.1 *Let R be a nonnegatively graded Noetherian ring with R_0 local and $d = \dim R$. Suppose M is a graded R-module. A complex $\mathbf{C}_\bullet$ of graded R-modules is called an approximate resolution of M if the following three conditions hold:*

a. $\mathrm{H}_0(\mathbf{C}_\bullet) \cong M$,

b. $\dim \mathrm{H}_j(\mathbf{C}_\bullet) \leq j$ *for all* $1 \leq j \leq d-1$, *and*

c. $\operatorname{depth} \mathbf{C}_j \geq \min\{d, j+2\}$ *for all* $1 \leq j \leq d-1$.

An approximate resolution can be used to bound the generation and concentration degrees of the zeroth local cohomology of the module it is approximately resolving, as in the following lemma.

Lemma 4.0.2 (Kustin-Polini-Ulrich [45, 3.8, 4.8]) *Let K be a field, and suppose $R = K[x_1, \ldots, x_d]$ is a standard graded polynomial ring over K with $\mathfrak{m} = (x_1, \ldots, x_d)$ and $d > 0$. Let M be a graded R-module and $\mathbf{C}_\bullet$ be an approximate resolution of M where each $\mathbf{C}_i$ is finitely generated. Then*

$$b_0\big(\mathrm{H}^0_{\mathfrak{m}}(M)\big) \leq b_0(\mathbf{C}_{d-1}) - d + 1, \text{ and}$$

$$\operatorname{topdeg}\big(\mathrm{H}^0_{\mathfrak{m}}(M)\big) \leq b_0(\mathbf{C}_d) - d.$$

The majority of Section 4.1 is dedicated to showing the conditions under which we can use minimal homogeneous free resolutions of J^k to obtain degree bounds on $\ker \psi$.

Section 4.2 is dedicated to transferring the results of Section 4.1 to determinantal ideals $I_t(A)$ and Pfaffian ideals $\mathrm{Pf}_{2t}(A)$ of generic height by placing height bounds on $I_j(A)$ and $\mathrm{Pf}_{2j}(A)$ for $j < t$. The results of this section will be analyzed in Chapter 5.

4.1 General Tools Concerning Specialization

Lemma 4.1.1 *Let S be a nonnegatively graded Cohen-Macaulay ring with S_0 local, J be a homogeneous S-ideal generated by homogeneous elements of S of the same degree D. Suppose $Y_1, \ldots, Y_s$ is a sequence of homogeneous elements of S which is weakly regular on both S and S/J. Let $N = (Y_1, \ldots, Y_s)$, $R = S/N$, $I = JR$, and $d = \dim R$. For each k, let $\big(\mathbf{F^k}_\bullet, \partial^{\mathbf{k}}{}_\bullet\big)$ be a homogeneous finite free S-resolution of $J^k(kD)$ so that each $\mathbf{F}_i$ is finitely generated. Further, for each i, suppose there exists a family $\{K_i\}$ of S-ideals satisfying the condition $K_i \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$ for all k.*

a. If $\operatorname{ht} K_i R \geq i$ for all i, then for each k, $\mathbf{F^k}_\bullet \otimes_S R$ is a homogeneous free R-resolution of $J^k(kD) \otimes_S R$.

b. If $\operatorname{ht} K_i R \geq \min\{i, d-1\}$ for all i, then for each k, $\mathbf{F^k}_\bullet \otimes_S R$ is a homogeneous approximate R-resolution of $J^k(kD) \otimes_S R$.

c. *Let* $\psi_k : J^k \otimes_S R \to I^k$ *be the natural surjection. For each* k*, there exists a homogeneous exact sequence*

$$\mathcal{A}_k(J) \otimes_S R \to \mathcal{A}_k(I) \to (\ker \psi_k)(kD) \to 0.$$

Proof We begin with a proof of (a). By the right exactness of the tensor product, $\mathbf{F^k}_\bullet \otimes_S R$ is a homogeneous finite complex of finitely generated free S-modules with $\mathrm{H}_0\big(\mathbf{F^k}_\bullet\big) \cong J^k(kD) \otimes_S R$. By the Buchsbaum-Eisenbud criterion (see Proposition 2.4.1), it suffices to prove that grade $I\big(\partial^{\mathbf{k}}{}_i \otimes_S R\big) \geq i$ for each i. Since S is a Cohen-Macaulay ring and N is generated by a sequence which is weakly regular on S, it follows that $S/N \cong R$ is Cohen-Macaulay as well. Therefore, grade $I\big(\partial^{\mathbf{k}}{}_i \otimes_S R\big) = \operatorname{ht} I\big(\partial^{\mathbf{k}}{}_i \otimes_S R\big)$. Since $K_i \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i)}$, it follows that $K_i R \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i \otimes_S R)}$. Therefore, by assumption, $i \leq \operatorname{ht} K_i R \leq \operatorname{ht} I\big(\partial^{\mathbf{k}}{}_i \otimes_S R\big)$. Hence, $\mathbf{F^k}_\bullet \otimes_S R$ is acyclic, completing the proof of (a).

We now prove (b). We first discuss properties of S and R. Since S is nonnegatively graded with S_0 local, S has a unique maximal homogeneous ideal which is a maximal ideal of S. Call it $\mathfrak{M}$. Therefore, since $R = S/N$ where N is a homogeneous ideal, it follows that R has a unique maximal homogeneous ideal which is a maximal ideal of R, $\mathfrak{m} = \mathfrak{M}/N$. Therefore, we also have $\operatorname{ht} \mathfrak{m} = \dim R = d$.

Since R is a Cohen-Macaulay ring and each $\mathbf{F^k}_i \otimes_S R$ is a free R-module, the depth conditions in Definition 4.0.1 are automatically satisfied. Additionally, as in part (a), $\mathrm{H}_0\big(\mathbf{F^k}_\bullet \otimes_S R\big) \cong J^k(kD) \otimes_S R$. Therefore, all that remains to show is that $\dim \mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big) \leq j$ for all $1 \leq j \leq d-1$. By definition, $\dim \mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big) = \dim R/\operatorname{ann}\big(\mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big)\big)$. Since R is a Cohen-Macaulay ring with unique maximal homogeneous ideal of height equal to its dimension and since $\mathbf{F^k}_\bullet \otimes_S R$ is a homogeneous complex of R-modules, this is equal to $\dim R - \operatorname{ht}\operatorname{ann}\big(\mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big)\big)$. Therefore, it remains to show $d - \operatorname{ht}\operatorname{ann}\big(\mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big)\big) \leq j$, or equivalently it remains to show, $\operatorname{ht}\operatorname{ann}\big(\mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big)\big) \geq d - j$. Hence, we may show that $\mathrm{H}_j\big(\mathbf{F^k}_\bullet \otimes_S R\big)$ vanishes locally at primes of height at most $d - j - 1$.

Fix j with $1 \leq j \leq d-1$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht} \mathfrak{p} \leq d-j-1$. Consider the complex $\left(\mathbf{F^k}_\bullet \otimes_S R\right)_\mathfrak{p}$. As in part (a), we have $K_i R \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i \otimes_S R)}$. Therefore, $K_i R_\mathfrak{p} \subseteq \sqrt{I(\partial^{\mathbf{k}}{}_i \otimes_S R_\mathfrak{p})}$. Since heights cannot decrease after localization, $\operatorname{ht} K_i R_\mathfrak{p} \geq \min\{i, d-1\}$. Since $1 \leq j \leq d-1$, it follows that $d-j \leq d-1$. Moreover, because $\dim R_\mathfrak{p} \leq d-j-1$, for all i, we have $\operatorname{ht} K_i R_\mathfrak{p} \geq i$. Therefore, $\operatorname{ht} I\left(\partial^{\mathbf{k}}{}_i \otimes_S R_\mathfrak{p}\right) \geq i$. Consequently, $\left(\mathbf{F^k}_\bullet \otimes_S R\right)_\mathfrak{p}$ is a free resolution. Thus, $\mathrm{H}_i\left(\left(\mathbf{F^k}_\bullet \otimes_S R\right)_\mathfrak{p}\right) = 0$. Since exact functors commute with homology, $\mathrm{H}_i\left(\mathbf{F^k}_\bullet \otimes_S R\right)_\mathfrak{p} = 0$, completing the proof of (b).

We now prove (c). We begin by showing that $\psi_1 : J \otimes_S R \to I$ is an isomorphism. Note that $J \otimes_S R = J \otimes_S S/N \cong J/JN$. Moreover, by definition, $I = J/N$. Therefore, $\ker \psi_1 \cong \ker(J/JN \to J/N) = J \cap N/JN \cong \operatorname{Tor}_1^S(S/J, S/N)$. Now, since N is generated by a sequence which is weakly regular on S/J, $\operatorname{Tor}_1^S(S/J, S/N) = 0$. Hence, ψ_1 is an isomorphism.

For each k, the sequence

$$0 \to \mathcal{A}_k(J) \to \operatorname{Sym}_k(J(D)) \to J^k(kD) \to 0$$

is exact and homogeneous (recall the bigrading of the Rees ring and symmetric algebra from Definition 2.2.8). Therefore, by the right exactness of the tensor product, the sequence

$$\mathcal{A}_k(J) \otimes_S R \to \operatorname{Sym}_k(J(D)) \otimes_S R \to J^k(kD) \otimes_S R \to 0$$

is exact and homogeneous as well.

By the base change property of the symmetric algebra combined with the fact that ψ_1 is an isomorphism, $\operatorname{Sym}_k(J(D)) \otimes_S R \cong \operatorname{Sym}_k(J(D) \otimes_S R) \cong \operatorname{Sym}_k(I(D))$. Therefore, we obtain the following commutative diagram with exact rows and all maps homogeneous.

$$\begin{tikzcd}
 & \mathcal{A}_k(J) \otimes_S R \arrow[r] \arrow[d, "f"] & \operatorname{Sym}_k(I(D)) \arrow[r] \arrow[d, equal] & J^k(kD) \otimes_S R \arrow[r] \arrow[d, "\psi_k"] & 0 \\
0 \arrow[r] & \mathcal{A}_k(I) \arrow[r] & \operatorname{Sym}_k(I(D)) \arrow[r] & I^k(kD) \arrow[r] & 0
\end{tikzcd}$$

Hence, by the Snake Lemma, $\operatorname{coker} f \cong (\ker \psi_k)(kD)$ is a homogeneous isomorphism. Therefore, the sequence

$$\mathcal{A}_k(J) \otimes_S R \to \mathcal{A}_k(I) \to (\ker \psi_k)(kD) \to 0$$

is a homogeneous exact sequence for each k. ■

In order to use Lemma 4.0.2 to bound $b_0(\ker \psi_k(kD))$ and $\operatorname{topdeg}(\ker \psi_k(kD))$, we need to show that $\ker \psi_k = \mathrm{H}^0_{\mathfrak{m}}\big(J^k \otimes_S R\big)$. The following lemma gives a criterion for proving such statements.

Lemma 4.1.2 *Let R be a nonnegatively graded Cohen-Macaulay ring of dimension $d > 0$ with R_0 local. Let $\mathfrak{m}$ denote the unique maximal homogeneous ideal of R. Suppose M and L are graded R-modules so that M is finitely generated and $\operatorname{pd} M < \infty$. Suppose $f : M \to L$ is a homogeneous R-linear map. Let $K = \ker f$, and suppose $K = \tau(M)$, the R-torsion of M. Then $K = \mathrm{H}^0_{\mathfrak{m}}(M)$ if and only if $\operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \operatorname{ht} \mathfrak{p}$ for all homogeneous prime ideals $\mathfrak{p}$ of R with $\mathfrak{p} \neq \mathfrak{m}$ and $\operatorname{ht} \mathfrak{p} > 0$.*

Proof Since R is a Cohen-Macaulay ring of positive dimension, it follows that $\mathfrak{m}$ contains a non-zero-divisor in R. Indeed, if all elements of $\mathfrak{m}$ were zero-divisors in R, then $\mathfrak{m}$ would be an associated prime of R, giving that $\operatorname{depth} R_{\mathfrak{m}} = 0 < \dim R_{\mathfrak{m}}$, contradicting that R is Cohen-Macaulay. Hence, if $x \in M$ and $\mathfrak{m}^i x = 0$ for some i, then $x \in \tau(M)$. Since $\mathrm{H}^0_{\mathfrak{m}}(M)$ consists of all elements of M which are annihilated by a power of $\mathfrak{m}$, it follows that $\mathrm{H}^0_{\mathfrak{m}}(M) \subseteq \tau(M) = K$. Since R is Noetherian, $\mathrm{H}^0_{\mathfrak{m}}(M) = \{x \in M \mid \operatorname{Supp}(Rx) \subseteq \{\mathfrak{m}\}\}$. Therefore, $K \subseteq \mathrm{H}^0_{\mathfrak{m}}(M)$ if and only if $\operatorname{Supp}(K) \subseteq \{\mathfrak{m}\}$, or equivalently, $\mathrm{H}^0_{\mathfrak{m}}(M) = K$ if and only if $\operatorname{Supp}(K) \subseteq \{\mathfrak{m}\}$. Since $K = \ker f$ with f homogeneous, K is homogeneous. Hence, $\operatorname{Supp}(K) \subseteq \{\mathfrak{m}\}$ if and only if $K_{\mathfrak{p}} = 0$ for all homogeneous prime ideals $\mathfrak{p}$ of R with $\mathfrak{p} \neq \mathfrak{m}$.

We now prove that torsion localizes. In other words, we now prove that $\tau_R(M)_{\mathfrak{p}} = \tau_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Now, $\tau_R(M) = \ker(M \to \operatorname{Quot}(R) \otimes_R M)$, where $\operatorname{Quot}(R)$ is the localization of R with respect to the multiplicative set $R \setminus \bigcup_{\mathfrak{q} \in \operatorname{Ass}(R)} \mathfrak{q}$. Since R is Cohen-Macaulay, $\operatorname{Ass}(R) = \operatorname{Min}(R)$. Thus, $\operatorname{Quot}(R)$ is the localization of R with respect to the multiplicative set $R\backslash\bigcup_{\mathfrak{q}\in\mathrm{Min}(R)}\mathfrak{q}$. Similarly, for any prime ideal $\mathfrak{p}$, $\mathrm{Quot}(R_{\mathfrak{p}})$ is the localization of $R_{\mathfrak{p}}$ with respect to the multiplicative set $R_{\mathfrak{p}}\setminus\bigcup_{\mathfrak{q}\in\mathrm{Min}(R_{\mathfrak{p}})}\mathfrak{q}$. Hence, it follows that $\mathrm{Quot}(R)\otimes_R R_{\mathfrak{p}}\cong\mathrm{Quot}(R_{\mathfrak{p}})$. Therefore, for any prime ideal $\mathfrak{p}$, $\tau_R(M)_{\mathfrak{p}} = \ker(M_{\mathfrak{p}}\to R_{\mathfrak{p}}\otimes_R \mathrm{Quot}(R)\otimes_R M)\cong\ker(M_{\mathfrak{p}}\to\mathrm{Quot}(R_{\mathfrak{p}})\otimes_R M)=\tau_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$.

Recall that we have already shown that $\mathrm{H}^0_{\mathfrak{m}}(M)=K$ if and only if $K_{\mathfrak{p}}=0$ for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$. Since $K=\tau_R(M)$, from the above paragraph, we see that $K_{\mathfrak{p}}=0$ for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$ if and only if $\tau_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$ for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq 0$. Hence, $\mathrm{H}^0_{\mathfrak{m}}(M)=K$ if and only if $M_{\mathfrak{p}}$ is torsionfree for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$.

We next prove that $M_{\mathfrak{p}}$ is torsionfree for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$ if and only if $\operatorname{depth} M_{\mathfrak{p}}>0$ for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$ and with $\operatorname{ht}\mathfrak{p}>0$. Suppose $M_{\mathfrak{p}}$ is torsionfree for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$. Let $\mathfrak{p}$ be a homogeneous prime with $\mathfrak{p}\neq\mathfrak{m}$ and $\operatorname{ht}\mathfrak{p}>0$. Then $M_{\mathfrak{p}}$ is torsionfree. Therefore, if $\mathfrak{q}R_{\mathfrak{p}}\in\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, then $\mathfrak{q}R_{\mathfrak{p}}\subseteq\mathfrak{r}R_{\mathfrak{p}}$ for some $\mathfrak{r}R_{\mathfrak{p}}\in\mathrm{Ass}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Since $R_{\mathfrak{p}}$ is Cohen-Macaulay, $\mathfrak{r}R_{\mathfrak{p}}$ is a minimal prime of $R_{\mathfrak{p}}$. Thus, $\mathfrak{q}R_{\mathfrak{p}}=\mathfrak{r}R_{\mathfrak{p}}$ is a minimal prime of $R_{\mathfrak{p}}$. Hence, if it were the case that $\operatorname{depth} M_{\mathfrak{p}}=0$, then $\mathfrak{p}R_{\mathfrak{p}}\in\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, giving that $\mathfrak{p}R_{\mathfrak{p}}$ is a minimal prime of $R_{\mathfrak{p}}$, and hence, $\mathfrak{p}$ is a minimal prime of R. This would contradict $\operatorname{ht}\mathfrak{p}>0$. Thus, we conclude that $\operatorname{depth} M_{\mathfrak{p}}>0$. Conversely, suppose $\operatorname{depth} M_{\mathfrak{p}}>0$ for all homogeneous primes $\mathfrak{p}$ with $\mathfrak{p}\neq\mathfrak{m}$ and with $\operatorname{ht}\mathfrak{p}>0$. Let $\mathfrak{p}$ be a homogeneous prime with $\mathfrak{p}\neq\mathfrak{m}$. If $M_{\mathfrak{p}}$ had nonzero torsion, then there would exist $\mathfrak{q}R_{\mathfrak{p}}\in\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ with $\mathfrak{q}R_{\mathfrak{p}}$ not contained in any associated primes of $R_{\mathfrak{p}}$. Since the set of associated primes localizes, $\mathfrak{q}$ must be an associated prime of M. Since M is graded, the associated primes of M are all homogeneous; thus, $\mathfrak{q}$ is homogeneous. Since $R_{\mathfrak{p}}$ is Cohen-Macaulay, the associated primes of $R_{\mathfrak{p}}$ are minimal. Hence, $\operatorname{ht}\mathfrak{q}R_{\mathfrak{p}}>0$ (since $\mathfrak{q}R_{\mathfrak{p}}$ is not contained in any minimal primes of $R_{\mathfrak{p}}$). Thus, $\operatorname{ht}\mathfrak{q}>0$. Additionally, $\mathfrak{q}\neq\mathfrak{m}$ since $\mathfrak{p}\neq\mathfrak{m}$ and $\mathfrak{q}R_{\mathfrak{p}}$ is a prime ideal. Hence, by assumption, $\operatorname{depth} M_{\mathfrak{q}}>0$, contradicting that $\mathfrak{q}\in\mathrm{Ass}_R(M)$.

Finally, since R is a graded ring with unique maximal homogeneous ideal $\mathfrak{m}$ which is a maximal ideal of R and since M is a graded R-module with finite projective dimension, for any homogeneous prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$ and $\operatorname{ht}\mathfrak{p} > 0$, we have

$$\operatorname{depth} M_{\mathfrak{p}} + \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}}$$

by the Auslander-Buchsbaum Formula. Since $R_{\mathfrak{p}}$ is Cohen-Macaulay, $\operatorname{depth} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}} = \operatorname{ht}\mathfrak{p}$. Therefore, we have

$$\operatorname{depth} M_{\mathfrak{p}} + \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{ht}\mathfrak{p}.$$

Hence, we see that $\operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \operatorname{ht}\mathfrak{p}$ if and only if $\operatorname{depth} M_{\mathfrak{p}} > 0$. ■

We now use Lemma 4.0.2 and Lemma 4.1.2 to place bounds on $b_0(\mathcal{A}_k(I))$ and $\operatorname{topdeg}(\mathcal{A}_k(I))$ in terms of J^k and minimal homogeneous free resolutions of J^k.

Proposition 4.1.3 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let $A_1, \ldots, A_s$ be a sequence of elements of R which are each homogeneous of the same degree δ, $X_1, \ldots, X_s$ be a sequence of indeterminates over R, and define $S = R[X_1, \ldots, X_s]$. Give S the grading so that R is a graded subring and so that $\deg X_i = \delta$ for each i. Suppose J is an S-ideal which is generated by homogeneous elements of the same degree D. Define $Y_i = X_i - A_i$ for each i, let $N = (Y_1, \ldots, Y_s)$, and assume that N can be generated by a sequence which is weakly regular on S and S/J. We give R the S-algebra structure induced by the homogeneous isomorphism $R \cong S/N$. Let $I = JR$. For each k, let $\left(\mathbf{G^k}_{\bullet}, \mathbf{q^k}_{\bullet}\right)$ be a minimal homogeneous free S-resolution of $J^k(kD)$. Suppose that there exists a family $\{K_i\}$ of S-ideals having the property that, for each i, $K_i \subseteq \sqrt{I(\mathbf{q^k}_i)}$ for all k. If $\operatorname{ht} K_i R \geq \min\{i+1, d\}$ for all i, then*

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{b_0(\mathcal{A}_k(J))\,, b_0\left(\mathbf{G^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\operatorname{topdeg}(\mathcal{A}_k(J))\,, b_0\left(\mathbf{G^k}_{d}\right) - d\right\}.$$

Proof The hypotheses of Lemma 4.1.1 are satisfied. Let $\psi_k : J^k \otimes_S R \to I^k$ be the natural surjection. By part (c) of Lemma 4.1.1, the sequence

$$\mathcal{A}_k(J) \otimes_S R \to \mathcal{A}_k(I) \to (\ker \psi_k)(kD) \to 0.$$

is exact and homogeneous for each k.

We begin by proving

$$b_0(\mathcal{A}_k(I)) \leq \max\{b_0(\mathcal{A}_k(J))\,, b_0((\ker \psi_k)(kD))\}$$

for each k.

Suppose $g_1, \ldots, g_m$ and $h_1, \ldots, h_n$ are minimal homogeneous generating sets of $\mathcal{A}_k(J)$ and $(\ker \psi_k)(kD)$, respectively. Let $\overline{g_1}, \ldots, \overline{g_m}$ be the images of $g_1, \ldots, g_m$ in $\mathcal{A}_k(I)$, respectively. Similarly, let $\overline{h_1}, \ldots, \overline{h_n}$ be preimages of $h_1, \ldots, h_n$ in $\mathcal{A}_k(I)$ respectively. Then $\overline{g_1}, \ldots, \overline{g_m}, \overline{h_1}, \ldots, \overline{h_n}$ is a homogeneous generating set of $\mathcal{A}_k(I)$. The result follows from the homogeneity of the sequence.

Next, we prove

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\{\operatorname{topdeg}(\mathcal{A}_k(J))\,, \operatorname{topdeg}((\ker \psi_k)(kD))\}$$

for each k.

This, again, follows from the homogeneous exact sequence above. Indeed, if for some p one has $[\mathcal{A}_k(J)]_p = 0$ and $[(\ker \psi_k)(kD)]_p = 0$, then the sequence

$$0 \to [\mathcal{A}_k(I)]_p \to 0$$

is exact, giving $[\mathcal{A}_k(I)]_p = 0$.

Therefore, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\{b_0(\mathcal{A}_k(J))\,, b_0((\ker \psi_k)(kD))\}\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\{\operatorname{topdeg}(\mathcal{A}_k(J))\,, \operatorname{topdeg}((\ker \psi_k)(kD))\}\,.$$

We wish to characterize $b_0((\ker \psi_k)(kD))$ and $\operatorname{topdeg}((\ker \psi_k)(kD))$ in terms of the minimal homogeneous free resolution $\left(\mathbf{G^k}_\bullet, \mathbf{q^k}_\bullet\right)$ of $J^k(kD)$. Our strategy will be to use Lemma 4.1.2 to show that $\ker\psi_k = \mathrm{H}^0_{\mathfrak{m}}\big(J^k \otimes_S R\big)$, where $\mathfrak{m} = (x_1, \ldots, x_d)$. We will then use part (b) of Lemma 4.1.1 to show that $\mathbf{G^k}_{\bullet} \otimes_S R$ is an approximate resolution of $J^k \otimes_S R$. Lastly, we will use Lemma 4.0.2 to recast generation degree and concentration degree bounds of $(\ker\psi_k)(kD)$ in terms of the complexes $\mathbf{G^k}_{\bullet}$.

We begin by applying Lemma 4.1.2. Since R is a polynomial ring over a field, it is the case that R is a nonnegatively graded Cohen-Macaulay ring of dimension $d > 0$ with R_0 local. Choosing $M = J^k \otimes_S R$ and $L = I^k$, we have that $\psi_k : J^k \otimes_R S \to I^k$ is a homogeneous R-linear map between graded R-modules. Moreover, since R is a polynomial ring over a field, by Hilbert's Syzygy Theorem, $J^k \otimes_S R$ has finite projective dimension. We must prove that $\ker\psi_k$ is the torsion of $J^k \otimes_S R$. Consider the module $J^k \otimes_S R \otimes_R \mathrm{Quot}(R)$:

$$J^k \otimes_S \mathrm{Quot}(R) \cong J^k \otimes_S \frac{S_N}{NS_N} \cong J^k \otimes_S S_N \otimes_S \frac{S}{N} \cong J^k S_N \otimes_S \frac{S}{N}$$

Since N is a prime ideal (because R is a domain), if $J^k \subseteq N$, it would follow that $J \subseteq N$. Since N is generated by a sequence which is weakly regular on S/J, this is impossible. Thus, $J^k S_N = S_N$. Hence, we conclude $J^k \otimes_S R \otimes_R \mathrm{Quot}(R) \cong \mathrm{Quot}(R)$. Also, $I^k \otimes_R \mathrm{Quot}(R) \cong \mathrm{Quot}(R)$ since $I^k \neq 0$ and R is a domain. Therefore, since $\mathrm{Quot}(R)$ is flat over R, we have that the sequence

$$0 \to \ker\psi_k \otimes_R \mathrm{Quot}(R) \to \mathrm{Quot}(R) \to \mathrm{Quot}(R) \to 0$$

is exact. Hence, $\ker\psi_k \otimes_R \mathrm{Quot}(R) = 0$. Thus, $\ker\psi_k \subseteq \tau\big(J^k \otimes_S R\big)$. Finally, since I^k is torsionfree as an R-module, it follows that $\tau\big(J^k \otimes_S R\big) \subseteq \ker\psi_k$. Therefore, $\ker\psi_k$ is the torsion of $J^k \otimes_S R$.

Thus, by Lemma 4.1.2, $\ker\psi_k = \mathrm{H}^0_{\mathfrak{m}}\big(J^k \otimes_S R\big)$ if and only if $\mathrm{pd}_{R_{\mathfrak{p}}}\big(J^k \otimes_S R\big)_{\mathfrak{p}} < \mathrm{ht}\,\mathfrak{p}$ for all homogeneous primes $\mathfrak{p}$ of R with $\mathfrak{p} \neq \mathfrak{m}$ and $\mathrm{ht}\,\mathfrak{p} > 0$. Let $\mathfrak{p}$ be a homogeneous prime ideal of R with $\mathfrak{p} \neq \mathfrak{m}$ and $\mathrm{ht}\,\mathfrak{p} > 0$. Since $K_i \subseteq \sqrt{I(\mathbf{q^k}_i)}$, it follows that $K_i R_{\mathfrak{p}} \subseteq \sqrt{I(\mathbf{q^k}_i \otimes_S R_{\mathfrak{p}})}$. Since $\mathfrak{p} \neq \mathfrak{m}$, it follows that $\mathrm{ht}\,\mathfrak{p} < d$. Therefore, since $\mathrm{ht}\,K_i R \geq \min\{i+1, d\}$, it follows that $\mathrm{ht}\,K_i R_{\mathfrak{p}} \geq i+1$, giving $\mathrm{ht}\,I\big(\mathbf{q^k}_i \otimes_S R_{\mathfrak{p}}\big) \geq i+1$. Therefore, by the Buchsbaum-Eisenbud Criterion in Proposition 2.4.1, $\mathbf{G^k}_{\bullet} \otimes_S R_{\mathfrak{p}}$

is a homogeneous free $R_{\mathfrak{p}}$-resolution of $\left(J^k \otimes_S R\right)_{\mathfrak{p}}$. Moreover, for $i = \operatorname{ht}\mathfrak{p}$, we have $\operatorname{ht} I\left(\mathbf{q^k}_{\operatorname{ht}\mathfrak{p}} \otimes_S R_{\mathfrak{p}}\right) \geq \operatorname{ht}\mathfrak{p} + 1 > \dim R_{\mathfrak{p}}$. Therefore, $I\left(\mathbf{q^k}_{\operatorname{ht}\mathfrak{p}} \otimes_S R_{\mathfrak{p}}\right) = R_{\mathfrak{p}}$, giving that the resolution becomes split exact after position $\operatorname{ht}\mathfrak{p} - 1$. Hence, $\operatorname{pd}_{R_{\mathfrak{p}}} \left(J^k \otimes_S R\right)_{\mathfrak{p}} < \operatorname{ht}\mathfrak{p}$. Thus, we conclude $\ker\psi_k = \mathrm{H}^0_{\mathfrak{m}}\left(J^k \otimes_S R\right)$.

Next, we show that $\mathbf{G^k}_{\bullet} \otimes_S R$ is an approximate resolution of $J^k(kD) \otimes_S R$. By part (b) of Lemma 4.1.1, we just need $\operatorname{ht} K_i R \geq \min\{i, d-1\}$. However, we have a stronger condition, $\operatorname{ht} K_i R \geq \min\{i+1, d\}$.

Finally, we apply Lemma 4.0.2. Since R is a standard graded polynomial ring over a field and $\mathbf{G^k}_{\bullet} \otimes_S R$ is an approximate resolution of $J^k(kD) \otimes_S R$, we have

$$b_0\left(\mathrm{H}^0_{\mathfrak{m}}\left(J^k(kD) \otimes_S R\right)\right) \leq b_0\left(\mathbf{G^k}_{d-1} \otimes_S R\right) - d + 1, \text{ and}$$

$$\operatorname{topdeg}\left(\mathrm{H}^0_{\mathfrak{m}}\left(J^k(kD) \otimes_S R\right)\right) \leq b_0\left(\mathbf{G^k}_{d} \otimes_S R\right) - d.$$

Since $(\ker\psi_k)(kD) = \mathrm{H}^0_{\mathfrak{m}}\left(J^k(kD) \otimes_S R\right)$ and since base change cannot add generators, we have

$$b_0((\ker\psi_k)(kD)) \leq b_0\left(\mathbf{G^k}_{d-1}\right) - d + 1, \text{ and}$$

$$\operatorname{topdeg}((\ker\psi_k)(kD)) \leq b_0\left(\mathbf{G^k}_{d}\right) - d.$$

Since we have already shown

$$b_0(\mathcal{A}_k(I)) \leq \max\{b_0(\mathcal{A}_k(J)), b_0((\ker\psi_k)(kD))\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\{\operatorname{topdeg}(\mathcal{A}_k(J)), \operatorname{topdeg}((\ker\psi_k)(kD))\}$$

earlier in the proof, we conlcude

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{b_0(\mathcal{A}_k(J)), b_0\left(\mathbf{G^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\operatorname{topdeg}(\mathcal{A}_k(J)), b_0\left(\mathbf{G^k}_{d}\right) - d\right\}.$$

■

4.2 Determinantal and Pfaffian Ideals

We now apply the result to determinantal and Pfaffian ideals. We begin by showing that determinantal and Pfaffian ideals of generic height are specializations of generic determinantal and Pfaffian ideals.

Remark 4.2.1 *Let R be a nonnegatively graded Cohen-Macaulay ring of dimension $d > 0$ with R_0 local. Let t, m, and n be integers satisfying $1 \leq t \leq m \leq n$. Suppose A is an $m \times n$ matrix with entries in R and $I = I_t(A)$ is of generic height. Let X be an $m \times n$ generic matrix over R, $S = R[X]$, and $J = I_t(X)$. Then I is a specialization of J. In particular, let N be the S-ideal generated by the entries of the matrix $X - A$. Then N is generated by a sequence which is regular on S and on S/J, $R \cong S/N$, and $I = JR$.*

Proof The isomorphism $R \cong S/N$ is clear via the R-algebra epimorphism $\varphi : S \to R$ defined by $X_{ij} \mapsto A_{ij}$. Since the entries of X are mapped to the corresponding entries of A and because the $t \times t$ minors of a matrix are integer polynomials in the entries, we also have $I = JR$. Although it is fairly easy to see that N is generated by a sequence which is regular on S, we provide a more detailed proof, as the same technique will be used to show that N is generated by a sequence which is regular on S/J.

Since S is Cohen-Macaulay and N is generated by a sequence of length mn, to show that N is generated by a regular sequence on S, it suffices to show that $\dim S - \dim S/N = mn$. This follows since S is a polynomial ring in mn variables over R and $S/N \cong R$. More precisely, $\dim S = d + mn$ and $\dim S/N = d$. Thus, $\dim S - \dim S/N = d + mn - d = mn$. Therefore, N is generated by a sequence which is regular on S.

By the work of Hochster and Eagon in [31, Corollary 2], it is known that S/J is Cohen-Macaulay. Therefore, as above, to show that N is generated by a regular sequence on S/J, it suffices to show that $\dim S/J - \dim\,(S/J)\,/\,(N + J/J) = mn$. By the Third Isomorphism Theorem, $(S/J)\,/\,(N + J/J) \cong (S/N)\,/\,(J + N/N) \cong R/I$. Therefore, we just need to show $\dim S/J - \dim R/I = mn$. Since S and R are both Cohen-Macaulay, $\dim S/J = \dim S - \operatorname{ht} J$ and $\dim R/I = \dim R - \operatorname{ht} I$. Since $\operatorname{ht} I = \operatorname{ht} J$ by assumption, $\dim S/J - \dim R/I = \dim S - \dim R = d + mn - d = mn$. Therefore, N is generated by a sequence which is regular on S/J. ■

Observation 4.2.2 *The result in Remark 4.2.1 also applies to determinantal ideals of a symmetric matrix of generic symmetric height and for Pfaffian ideals of an alternating matrix of generic alternating height.*

There are three key aspects of the proof in Remark 4.2.1. The first is that determinants are polynomials in the entries of a matrix with integer coefficients. As such, $I = JR$. The second is that the ideal I has the same height as J. Both of these aspects are preserved in the symmetric and alternating cases. For the alternating case, the first aspect is preserved since Pfaffians are also polynomials in the entries of the matrix with integer coefficients. The third key aspect is that S/J is Cohen-Macaulay. For determinantal ideals of symmetric matrices, this was proven by Kutz in [50]. For Pfaffian ideals of alternating matrices, this was proven by Marinov in [52].

4.2.1 Ordinary Matrices

Corollary 4.2.3 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let t, m, and n be integers satisfying $1 \leq t \leq m \leq n$. Suppose A is an $m \times n$ ordinary matrix with entries in R so that every entry of A is homogeneous of the same degree δ. Let $I = I_t(A)$ be of generic height. Let X be an $m \times n$ generic ordinary matrix over R, $S = R[X]$, and $J = I_t(X)$. Give S the grading so that R is a graded subring and so that $\deg X_{ij} = \delta$ for each i and j. Define $Y_{ij} = X_{ij} - A_{ij}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$, let N be the S-ideal generated by the Y_{ij}. We give R the S-algebra structure induced by the homogeneous isomorphism $R \cong S/N$. For each k, let $\left(\mathbf{G^k}_\bullet, \mathbf{q^k}_\bullet\right)$ be a minimal homogeneous free S-resolution of $J^k(kt\delta)$. Assume one of the following two situations holds.*

a. Let $t = m$ and $\operatorname{ht} I_j(A) \geq \min\{(m - j + 1)(n - m) + 1, d\}$ for all $1 \leq j \leq t - 1$.

b. Let $t < m$ *and* $\operatorname{ht} I_j(A) \geq \min\{(m-j+1)\,(n-j+1)\,, d\}$ *for all* $1 \leq j \leq t-1$. *Then for each* k*,*

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{b_0(\mathcal{A}_k(J))\,, b_0\left(\mathbf{G^k}_{d-1}\right) - d + 1\right\}, \;\textit{and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\operatorname{topdeg}(\mathcal{A}_k(J))\,, b_0\left(\mathbf{G^k}_d\right) - d\right\}.$$

Proof We apply Proposition 4.1.3. We just have to check that all of the hypotheses have been satisfied. For the sequence of elements A_i in Proposition 4.1.3, we select the entries of A. For the sequence of indeterminates, we select the entries of X. Since $J = I_t(X)$, we note that J is generated by homogeneous elements of degree $t\delta$, since each entry of X has degree δ and since we are taking the $t \times t$ minors. By Remark 4.2.1, we know that N is generated by a sequence which is regular on S and on S/J and that $I = JR$. All that remains is to construct the family $\{K_i\}$ and prove that this family satisfies the desired properties.

Let $1 \leq i \leq \operatorname{ht} J - 1$. We take $K_i = \sqrt{J}$. For any k, $\sqrt{J} = \sqrt{J^k}$. We must show that, for each i with $1 \leq i \leq \operatorname{ht} J - 1$, $\sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_i)}$. Since $\mathbf{G^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(kt\delta)$, we can extend it to a free resolution of S/J^k by appending a new map $\mathbf{q^k}_0 : \mathbf{G^k}_0 \to S$ where $\mathbf{q^k}_0$ is given by multiplication with the row vector of a minimal homogeneous generating set of J^k. Since $\operatorname{ann}\left(S/J^k\right) \neq 0$, a structure theorem by David Buchsbaum and David Eisenbud on finite free resolutions in [10, Remarks pg 261] gives that $\sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_0)} \subseteq \cdots \subseteq \sqrt{I\left(\mathbf{q^k}_{\operatorname{grade} J^k - 1}\right)}$. Since S is a Cohen-Macaulay ring, $\operatorname{grade} J^k - 1 = \operatorname{ht} J^k - 1 = \operatorname{ht} J - 1$. Therefore, for each i with $1 \leq i \leq \operatorname{ht} J - 1$, we have $K_i = \sqrt{J} = \sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_i)}$. Moreover, $\operatorname{ht} K_i R = \operatorname{ht} JR = \operatorname{ht} I = \operatorname{ht} J$ since I has generic height. Therefore, $\operatorname{ht} K_i R \geq i+1$ for all i with $1 \leq i \leq \operatorname{ht} J - 1$.

Let $\operatorname{ht} J \leq i \leq \max_k\left\{\operatorname{pd} J^k\right\}$. The choice of K_i depends on whether we are in case (a) where $t = m$ or case (b) where $t < m$. For case (a) where $t = m$, let $j_i = \min\{j \mid i \geq (m-j)\,(n-m) + 1\}$. For case (b) where $t < m$, let $j_i = \min\{j \mid i \geq (m-j)\,(n-j)\}$. Then, in either case, define $K_i = \sqrt{I_{j_i}(X)}$.

We begin by proving that $1 \leq j_i \leq t-1$ for each i in the range $\operatorname{ht} J \leq i \leq \max_k\{\operatorname{pd} J^k\}$.

In case (a), when substituting $j = 0$ into $(m-j)(n-m)+1$, one obtains $m(n-m)+1$. By part (a) of Lemma 3.1.4, $m(n-m)+1 = \max_k\{\operatorname{pd} J^k\}+1$. Hence, this value is strictly greater than i. Since $(m-j)(n-m)+1$ is decreasing in j, by the definition of j_i as the minimum value of j with $i \geq (m-j)(n-m)+1$, it follows that $j_i \geq 1$. When substituting $j = t-1$ into $(m-j)(n-m)+1$, one obtains $(m-t+1)(n-m)+1$. In this case, $t = m$, so we have $n-t+1$. This is equal to $\operatorname{ht} J$ in this case, which is less than or equal to i. Hence, since j_i is defined as the minimum value j so that $i \geq (m-j)(n-m)+1$, it follows that $j_i \leq t-1$.

In case (b), when substituting $j = 0$ into $(m-j)(n-j)$, one obtains mn. By part (b) of Lemma 3.1.4, $mn = \max_k\{\operatorname{pd} J^k\}+1$. Thus, this value is strictly greater than i. Since $(m-j)(n-j)$ is decreasing in j, by the definition of j_i as the minimum value of j with $i \geq (m-j)(n-j)$, it follows that $j_i \geq 1$. When substituting $j = t-1$ into $(m-j)(n-j)$, one obtains $(m-t+1)(n-t+1)$. This is equal to $\operatorname{ht} J$ in this case, which is less than or equal to i. Hence, since j_i is defined as the minimum value j so that $i \geq (m-j)(n-j)$, it follows that $j_i \leq t-1$.

Finally, we end by proving that $\operatorname{ht} K_i R \geq \min\{i+1, d\}$. We have $\operatorname{ht} K_i R = \operatorname{ht} I_{j_i}(A)$. In part (a), by assumption, $\operatorname{ht} I_{j_i}(A) \geq \min\{(m-j_i+1)(n-m)+1, d\}$. But note that $(m-j_i+1)(n-m)+1 = (m-(j_i-1))(n-m)+1$. Since $j_i - 1 < j_i = \min\{j \mid i \geq (m-j)(n-m)+1\}$, it follows that $i < (m-j_i+1)(n-m)+1$. Therefore, $\operatorname{ht} K_i R \geq \min\{(m-j_i+1)(n-m)+1, d\} \geq \min\{i+1, d\}$. Similarly, in part (b), by assumption, $\operatorname{ht} I_{j_i}(A) \geq \min\{(m-j_i+1)(n-j_i+1), d\}$. But note that $(m-j_i+1)(n-j_i+1) = (m-(j_i-1))(n-(j_i-1))$. Since $j_i - 1 < j_i = \min\{j \mid i \geq (m-j)(n-j)\}$, it follows that $i < (m-j_i+1)(n-j_i+1)$. Therefore, $\operatorname{ht} K_i R \geq \min\{(m-j_i+1)(n-j_i+1), d\} \geq \min\{i+1, d\}$. ■

In the above corollary, we needed to put a fairly unnatural grading on the entries of X to preserve homogeneity. Typically, people take $\deg X_{ij} = 1$. We now recast the previous result in terms of this more natural grading. This difference of grading is also why we distinguished the notation of $\mathbf{G^k}_\bullet$ and $\mathbf{F^k}_\bullet$.

Corollary 4.2.4 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let t, m, and n be integers satisfying $1 \le t \le m \le n$. Suppose A is an $m \times n$ ordinary matrix with entries in R so that every entry of A is homogeneous of the same degree δ. Let $I = I_t(A)$ be of generic height. Let X be an $m \times n$ generic ordinary matrix over R, $S = R[X]$, and $J = I_t(X)$. Give S the grading so that R is a graded subring and so that $\deg X_{ij} = 1$ for each i and j. Define $Y_{ij} = X_{ij} - A_{ij}$ for each $1 \le i \le m$ and $1 \le j \le n$, let N be the S-ideal generated by the Y_{ij}. We give R the S-algebra structure induced by the isomorphism $R \cong S/N$. For each k, let $\left(\mathbf{F^k}_\bullet, \partial^\mathbf{k}_\bullet\right)$ be a minimal homogeneous free S-resolution of $J^k(kt)$. Assume one of the following two situations holds.*

a. Let $t = m$ and $\operatorname{ht} I_j(A) \ge \min\{(m-j+1)(n-m)+1, d\}$ for all $1 \le j \le t-1$.

b. Let $t < m$ and $\operatorname{ht} I_j(A) \ge \min\{(m-j+1)(n-j+1), d\}$ for all $1 \le j \le t-1$.

Then for each k,

$$b_0(\mathcal{A}_k(I)) \le \max\left\{\delta b_0(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J)), \delta b_0\left(\mathbf{F^k}_{d}\right) - d\right\}.$$

Proof We apply Corollary 4.2.3. The only difference is the grading on S due to the grading on the variables X_{ij}. To change from the grading in this setting to the grading in the setting of Corollary 4.2.3, we multiply the degrees of homogeneous polynomials in the variables X_{ij} by δ.

Recall that for each k, the exact sequence

$$0 \to \mathcal{A}_k(J) \to \operatorname{Sym}_k(J(t)) \to J^k(kt) \to 0$$

is homogeneous. Therefore, multiplying degrees by δ preserves the homogeneity. Hence, $b_0(\mathcal{A}_k(J))$ in the setting of Corollary 4.2.3 is the same number as $\delta b_0(\mathcal{A}_k(J))$

in the current setting. The same reasoning applies for $\operatorname{topdeg}(\mathcal{A}_k(J))$ in the setting of Corollary 4.2.3 and $\delta \operatorname{topdeg}(\mathcal{A}_k(J))$ in the current setting.

Finally, we wish to show that $b_0\big(\mathbf{G^k}_i\big)$ from Corollary 4.2.3 is the same number as $\delta b_0\big(\mathbf{F^k}_i\big)$ in the current setting for each i. To do this, it suffices to show that the the minimal free resolutions $\mathbf{F^k}_\bullet$ only depend on the variables X_{ij}. We employ the argument of Lemma 3.0.1. In particular, let $\mathbf{H^k}_\bullet$ be the minimal homogeneous free resolution of $J^k(kt)$ viewed as a graded $K[X]$-module. Then, since $S = K[X][x_1, \ldots, x_d]$, $\mathbf{H^k}_\bullet \otimes_{K[X]} S$ must be the minimal homogeneous free resolution of $J^k(kt)$ viewed as an S-module. Consequently, $b_0\big(\mathbf{G^k}_i\big) = \delta b_0\big(\mathbf{F^k}_i\big)$. ■

4.2.2 Symmetric Matrices

Corollary 4.2.5 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let t and n be integers satisfying $1 \leq t \leq n$. Suppose A is an $n \times n$ symmetric matrix with entries in R so that every entry of A is homogeneous of the same degree δ. Let $I = I_t(A)$ be of generic symmetric height. Let X be an $n \times n$ generic symmetric matrix over R, $S = R[X]$, and $J = I_t(X)$. Give S the grading so that R is a graded subring and so that $\deg X_{ij} = \delta$ for each i and j. Define $Y_{ij} = X_{ij} - A_{ij}$ for each $1 \leq i \leq j \leq n$, and let N be the S-ideal generated by the Y_{ij}. We give R the S-algebra structure induced by the homogeneous isomorphism $R \cong S/N$. For each k, let $\big(\mathbf{G^k}_\bullet, \mathbf{q^k}_\bullet\big)$ be a minimal homogeneous free S-resolution of $J^k(kt\delta)$. Assume one of the following two situations holds.*

a. Let $t = n$.

b. Let $t < n$ and $\operatorname{ht} I_j(A) \geq \min\big\{\binom{n-j+2}{2}, d\big\}$ for all $1 \leq j \leq t-1$.

Then for each k,

$$b_0(\mathcal{A}_k(I)) \leq \max\big\{b_0(\mathcal{A}_k(J))\,, b_0\big(\mathbf{G^k}_{d-1}\big) - d + 1\big\}\,,\ \textit{and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\big\{\operatorname{topdeg}(\mathcal{A}_k(J))\,, b_0\big(\mathbf{G^k}_d\big) - d\big\}\,.$$

Proof We apply Proposition 4.1.3. We just have to check that all of the hypotheses have been satisfied. For the sequence of elements A_i in Proposition 4.1.3, we select the entries in the upper triangle of A. For the sequence of indeterminates, we select the entries in the upper triangle of X. Since $J = I_t(X)$, we note that J is generated by homogeneous elements of degree $t\delta$, since each entry of X has degree δ and since we are taking the $t \times t$ minors. By Observation 4.2.2, we know that N is generated by a sequence which is regular on S and on S/J and that $I = JR$. All that remains is to construct the family $\{K_i\}$ and prove that this family satisfies the desired properties.

By part (a) of Lemma 3.2.4, we see $\max_k\{\operatorname{pd} J^k\} = 0$ in case (a). Therefore, the family $\{K_i\} = \varnothing$ and the height conditions are vacuously satisfied. Therefore, we only need to construct the family $\{K_i\}$ and prove the relevant height bounds in case (b).

Let $1 \leq i \leq \operatorname{ht} J - 1$. We take $K_i = \sqrt{J}$. For any k, $\sqrt{J} = \sqrt{J^k}$. We must show that, for each i with $1 \leq i \leq \operatorname{ht} J - 1$, $\sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_i)}$. Since $\mathbf{G^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(kt\delta)$, we can extend it to a free resolution of S/J^k by appending a new map $\mathbf{q^k}_0 : \mathbf{G^k}_0 \to S$ where $\mathbf{q^k}_0$ is given by multiplication with the row vector of a minimal homogeneous generating set of J^k. Since $\operatorname{ann}\big(S/J^k\big) \neq 0$, a structure theorem by Buchsbaum and Eisenbud on finite free resolutions in [10, Remarks pg 261] gives that $\sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_0)} \subseteq \cdots \subseteq \sqrt{I\big(\mathbf{q^k}_{\operatorname{grade} J^k - 1}\big)}$. Since S is a Cohen-Macaulay ring, $\operatorname{grade} J^k - 1 = \operatorname{ht} J^k - 1 = \operatorname{ht} J - 1$. Therefore, for each i with $1 \leq i \leq \operatorname{ht} J - 1$, we have $K_i = \sqrt{J} = \sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_i)}$. Moreover, $\operatorname{ht} K_i R = \operatorname{ht} JR = \operatorname{ht} I = \operatorname{ht} J$ since I has generic height. Thus, $\operatorname{ht} K_i R \geq i + 1$ for all i with $1 \leq i \leq \operatorname{ht} J - 1$.

Let $\operatorname{ht} J \leq i \leq \max_k\{\operatorname{pd} J^k\}$. Let $j_i = \min\big\{j \mid i \geq \binom{n-j+1}{2}\big\}$, and define $K_i = \sqrt{I_{j_i}(X)}$.

We begin by proving that $1 \leq j_i \leq t - 1$ for each i in the range $\operatorname{ht} J \leq i \leq \max_k\{\operatorname{pd} J^k\}$.

When substituting $j = 0$ into $\binom{n-j+1}{2}$, one obtains $\binom{n+1}{2}$. Applying part (b) of Lemma 3.2.4, we see $\binom{n+1}{2} = \max_k\{\operatorname{pd} J^k\} + 1$. Hence, this value is strictly greater than i. Since $\binom{n-j+1}{2}$ is decreasing in j, by the definition of j_i as the minimum value of j with $i \geq \binom{n-j+1}{2}$, it follows that $j_i \geq 1$. On the other hand, when substituting $j = t-1$ into $\binom{n-j+1}{2}$, one obtains $\binom{n-t+2}{2}$. This is equal to $\operatorname{ht} J$, which is less than or equal to i. Hence, since j_i is defined as the minimum value j so that $i \geq \binom{n-j+1}{2}$, it follows that $j_i \leq t-1$.

Finally, we end by proving that $\operatorname{ht} K_i R \geq \min\{i+1, d\}$. We have $\operatorname{ht} K_i R = \operatorname{ht} I_{j_i}(A)$. By assumption, $\operatorname{ht} I_{j_i}(A) \geq \min\left\{\binom{n-j_i+2}{2}, d\right\}$. But note that $\binom{n-j_i+2}{2} = \binom{n-(j_i-1)+1}{2}$. Since $j_i - 1 < j_i = \min\left\{j \mid i \geq \binom{n-j+1}{2}\right\}$, it follows that $i < \binom{n-j_i+2}{2}$. Therefore, $\operatorname{ht} K_i R \geq \min\left\{\binom{n-j_i+2}{2}, d\right\} \geq \min\{i+1, d\}$. ■

Like in the ordinary case, we placed a fairly unnatural grading on the entries of X in order to preserve homogeneity. The next result recasts the previous result in terms of this more natural grading, where $\deg X_{ij} = 1$.

Corollary 4.2.6 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let t and n be integers satisfying $1 \leq t \leq n$. Suppose A is an $n \times n$ symmetric matrix with entries in R so that every entry of A is homogeneous of the same degree δ. Let $I = I_t(A)$ be of generic symmetric height. Let X be an $n \times n$ generic symmetric matrix over R, $S = R[X]$, and $J = I_t(X)$. Give S the grading so that R is a graded subring and so that $\deg X_{ij} = 1$ for each i and j. Define $Y_{ij} = X_{ij} - A_{ij}$ for each $1 \leq i \leq j \leq m$, let N be the S-ideal generated by the Y_{ij}. We give R the S-algebra structure induced by the isomorphism $R \cong S/N$. For each k, let $\left(\mathbf{F^k}_\bullet, \partial^{\mathbf{k}}_\bullet\right)$ be a minimal homogeneous free S-resolution of $J^k(kt)$. Assume one of the following two situations holds.*

a. Let $t = n$.

b. Let $t < n$ and $\operatorname{ht} I_j(A) \geq \min\left\{\binom{n-j+2}{2}, d\right\}$ for all $1 \leq j \leq t-1$.

Then for each k,

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\, , \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\, , \delta b_0\left(\mathbf{F^k}_{d}\right) - d\right\}.$$

Proof We apply Corollary 4.2.5 and repeat the proof of Corollary 4.2.4. ■

4.2.3 Alternating Matrices

Corollary 4.2.7 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let t and n be integers satisfying $2 \le 2t \le n$. Suppose A is an $n \times n$ alternating matrix with entries in R so that every entry of A is homogeneous of the same degree δ. Let $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height. Let X be an $n \times n$ generic alternating matrix over R, $S = R[X]$, and $J = \operatorname{Pf}_{2t}(X)$. Give S the grading so that R is a graded subring and so that $\deg X_{ij} = \delta$ for each $1 \le i < j \le n$. Define $Y_{ij} = X_{ij} - A_{ij}$ for each $1 \le i < j \le n$, and let N be the S-ideal generated by the Y_{ij}. We give R the S-algebra structure induced by the homogeneous isomorphism $R \cong S/N$. For each k, let $\left(\mathbf{G^k}_\bullet, \mathbf{q^k}_\bullet\right)$ be a minimal homogeneous free S-resolution of $J^k(kt\delta)$. Assume one of the following three situations holds.*

a. Let $2t = n$.

b. Let $2t = n-1$ and $\operatorname{ht} \operatorname{Pf}_{2j}(A) \ge \min\{n-2j+2, d\}$ for all $1 \le j \le t-1$.

c. Let $2t \le n-2$ and $\operatorname{ht} I_j(A) \ge \min\left\{\binom{n-2j+2}{2}, d\right\}$ for all $1 \le j \le t-1$.

Then for each k,

$$b_0(\mathcal{A}_k(I)) \le \max\left\{b_0(\mathcal{A}_k(J))\,, b_0\left(\mathbf{G^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\left\{\operatorname{topdeg}(\mathcal{A}_k(J))\,, b_0\left(\mathbf{G^k}_{d}\right) - d\right\}.$$

Proof We apply Proposition 4.1.3. We just have to check that all of the hypotheses have been satisfied. For the sequence of elements A_i in Proposition 4.1.3, we select the entries above the diagonal of A. For the sequence of indeterminates, we select the entries above the diagonal of X. Since $J = \operatorname{Pf}_{2t}(X)$, we note that J is generated by homogeneous elements of degree $t\delta$, since each entry of X has degree δ and since the $t \times t$ Pfaffians are polynomials of degree t. By Observation 4.2.2, we know that

N is generated by a sequence which is regular on S and on S/J and that $I = JR$. All that remains is to construct the family $\{K_i\}$ and prove that this family satisfies the desired properties.

By part (a) of Lemma 3.3.5, we see $\max_k\{\operatorname{pd} J^k\} = 0$ in case (a). Therefore, the family $\{K_i\} = \varnothing$ and the height conditions are vacuously satisfied for case (a) here. Therefore, we only need to construct the family $\{K_i\}$ and prove the relevant height bounds in cases (b) and (c).

Let $1 \le i \le \operatorname{ht} J - 1$. We take $K_i = \sqrt{J}$. For any k, $\sqrt{J} = \sqrt{J^k}$. We must show that, for each i with $1 \le i \le \operatorname{ht} J - 1$, $\sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_i)}$. Since $\mathbf{G^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(kt\delta)$, we can extend it to a free resolution of S/J^k by appending a new map $\mathbf{q^k}_0 : \mathbf{G^k}_0 \to S$ where $\mathbf{q^k}_0$ is given by multiplication with the row vector of a minimal homogeneous generating set of J^k. Since $\operatorname{ann}\big(S/J^k\big) \neq 0$, a structure theorem by Buchsbaum and Eisenbud on finite free resolutions in [10, Remarks pg 261] gives that $\sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_0)} \subseteq \cdots \subseteq \sqrt{I\big(\mathbf{q^k}_{\operatorname{grade} J^k - 1}\big)}$. Since S is a Cohen-Macaulay ring, $\operatorname{grade} J^k - 1 = \operatorname{ht} J^k - 1 = \operatorname{ht} J - 1$. Therefore, for each i with $1 \le i \le \operatorname{ht} J - 1$, we have $K_i = \sqrt{J} = \sqrt{J^k} \subseteq \sqrt{I(\mathbf{q^k}_i)}$. Moreover, $\operatorname{ht} K_i R = \operatorname{ht} JR = \operatorname{ht} I = \operatorname{ht} J$ since I has generic height. Hence, $\operatorname{ht} K_i R \ge i + 1$ for all i with $1 \le i \le \operatorname{ht} J - 1$.

Let $\operatorname{ht} J \le i \le \max_k\{\operatorname{pd} J^k\}$. The definition of K_i depends on whether we are in case (b) where $2t = n - 1$ or in case (c) where $2t \le n - 2$. For case (b) where $2t = n - 1$, let $j_i = \min\{j \mid i \ge n - 2j\}$. For case (c) where $2t \le n - 2$, let $j_i = \min\big\{j \mid i \ge \binom{n-2j}{2}\big\}$. For either case, define $K_i = \sqrt{\operatorname{Pf}_{2j_i}(X)}$.

We begin by proving that $1 \le j_i \le t - 1$ for each i in the range $\operatorname{ht} J \le i \le \max_k\{\operatorname{pd} J^k\}$.

For case (b), when substituting $j = 0$ into $n - 2j$, one obtains n. By part (b) of Lemma 3.3.5, $n = \max_k\{\operatorname{pd} J^k\} + 1$. Thus, this value is strictly greater than i. Since $n - 2j$ is decreasing in j, by the definition of j_i as the minimum value of j with $i \ge n - 2j$, it follows that $j_i \ge 1$. When substituting $j = t - 1$ into $n - 2j$, one obtains $n - 2t + 2 = n - (n-1) + 2 = 3$. This is equal to $\operatorname{ht} J$ in this case, which is less than or equal to i. Hence, since j_i is defined as the minimum value j so that $i \geq n - 2j$, it follows that $j_i \leq t - 1$.

For case (c), when substituting $j = 0$ into $\binom{n-2j}{2}$, one obtains $\binom{n}{2}$. By part (c) of Lemma 3.3.5, $\binom{n}{2} = \max_k\{\operatorname{pd} J^k\} + 1$. Thus, this value is strictly greater than i. Since $\binom{n-2j}{2}$ is decreasing in j, by the definition of j_i as the minimum value of j with $i \geq \binom{n-2j}{2}$, it follows that $j_i \geq 1$. When substituting $j = t - 1$ into $\binom{n-2j}{2}$, one obtains $\binom{n-2t+2}{2}$. This is equal to $\operatorname{ht} J$ in this case, which is less than or equal to i. Hence, since j_i is defined as the minimum value j so that $i \geq \binom{n-2j}{2}$, it follows that $j_i \leq t - 1$.

Finally, we end by proving that $\operatorname{ht} K_iR \geq \min\{i + 1, d\}$. We have $\operatorname{ht} K_iR = \operatorname{ht} \operatorname{Pf}_{2j_i}(A)$.

For part (b), by assumption, $\operatorname{ht} \operatorname{Pf}_{2j_i}(A) \geq \min\{n - 2j_i + 2, d\}$. But note that $n - 2j_i + 2 = n - 2\,(j_i - 1)$. Since $j_i - 1 < j_i = \min\{j \mid i \geq n - 2j\}$, it follows that $i < n - 2j_i + 2$. Therefore, $\operatorname{ht} K_iR \geq \min\{n - 2j_i + 2, d\} \geq \min\{i + 1, d\}$.

For part (c), by assumption, $\operatorname{ht} \operatorname{Pf}_{2j_i}(A) \geq \min\left\{\binom{n-2j_i+2}{2}, d\right\}$. But note that $\binom{n-2j_i+2}{2} = \binom{n-2(j_i-1)}{2}$. Since $j_i - 1 < j_i = \min\left\{j \mid i \geq \binom{n-2j}{2}\right\}$, it follows that $i < \binom{n-2j_i+2}{2}$. Therefore, $\operatorname{ht} K_iR \geq \min\left\{\binom{n-2j_i+2}{2}, d\right\} \geq \min\{i + 1, d\}$. ■

Similarly to the ordinary and symmetric cases, the following result places a more natural grading on the entries of X, where $\deg X_{ij} = 1$, as opposed to the grading above which was necessary to preserve homogeneity.

Corollary 4.2.8 *Let K be a field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring in $d > 0$ variables over K. Let t and n be integers satisfying $2 \leq 2t \leq n$. Suppose A is an $n \times n$ alternating matrix with entries in R so that every entry of A is homogeneous of the same degree δ. Let $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height. Let X be an $n \times n$ generic alternating matrix over R, $S = R[X]$, and $J = \operatorname{Pf}_{2t}(X)$. Give S the grading so that R is a graded subring and so that $\deg X_{ij} = 1$ for each $1 \leq i < j \leq n$. Define $Y_{ij} = X_{ij} - A_{ij}$ for each $1 \leq i < j \leq n$, let N be the S-ideal generated by the Y_{ij}. We give R the S-algebra structure induced by the isomorphism*

$R \cong S/N$. For each k, let $\left(\mathbf{F^k}_\bullet, \partial^{\mathbf{k}}_\bullet\right)$ be a minimal homogeneous free S-resolution of $J^k(kt)$. Assume one of the following three situations holds.

a. Let $2t = n$.

b. Let $2t = n-1$ and $\operatorname{ht}\operatorname{Pf}_{2j}(A) \geq \min\{n-2j+2, d\}$ for all $1 \leq j \leq t-1$.

c. Let $2t \leq n-2$ and $\operatorname{ht} I_j(A) \geq \min\left\{\binom{n-2j+2}{2}, d\right\}$ for all $1 \leq j \leq t-1$.

Then for each k,

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_d\right) - d\right\}.$$

Proof We apply Corollary 4.2.7 and repeat the proof of Corollary 4.2.4. ■

5. DEGREE BOUNDS ON THE DEFINING EQUATIONS

Recall Corollaries 4.2.4, 4.2.6 and 4.2.8. In order to find explicit bounds on $b_0(\mathcal{A}_k(I))$ and $\operatorname{topdeg}(\mathcal{A}_k(I))$, we must evaluate or find upper bounds for $b_0(\mathcal{A}_k(J))$, $b_0\big(\mathbf{F^k}_{d-1}\big)$, $\operatorname{topdeg}(\mathcal{A}_k(J))$, and $b_0\big(\mathbf{F^k}_d\big)$ to the best of our ability. This chapter is dedicated to this task.

It is rarely known, specifically, what $b_0(\mathcal{A}_k(J))$ and $\operatorname{topdeg}(\mathcal{A}_k(J))$ are. However, there are a few cases where we have some information about these values. These will be discussed throughout the chapter.

We begin with a few results which describe whether we can find a meaningful bound on $\operatorname{topdeg}(\mathcal{A}(I))$ by determining when $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$ or $\operatorname{topdeg}(\mathcal{A}(J)) < \infty$.

Definition 5.0.1 *Let S be a nonnegatively graded Noetherian ring with S_0 local, $\mathfrak{M}$ be the unique maximal homogeneous ideal of S, and J be a homogeneous S-ideal. We say that J is of linear type on the punctured spectrum of S if for all homogeneous prime ideals $\mathfrak{p}$ of S with $\mathfrak{p} \neq \mathfrak{M}$ the ideal $J_{\mathfrak{p}}$ is of linear type.*

Remark 5.0.2 *Let S be a polynomial ring of dimension d with $0 < d < \infty$ over a field K, and let $\mathfrak{M}$ denote the unique maximal homogeneous ideal of S. Suppose J is a homogeneous S-ideal generated by homogeneous elements of S of the same degree D. Then* $\operatorname{topdeg}(\mathcal{A}(J)) < \infty$ *if and only if J is of linear type on the punctured spectrum of S.*

Proof Recall that the sequence

$$0 \to \mathcal{A}(J) \to \operatorname{Sym}(J(D)) \to \mathcal{R}(J) \to 0$$

is homogeneous exact sequence of S-modules. Since $\mathcal{R}(J) \subseteq S[t]$, $\mathcal{R}(J)$ is a torsionfree S-algebra. Therefore, $\tau_S(\operatorname{Sym}(J)) \subseteq \mathcal{A}(J)$. Since S is Cohen-Macaulay of positive dimension and $\mathfrak{M}$ is a maximal ideal of S, $\mathfrak{M}$ contains a non-zero-divisor of S. Therefore, $\mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J)) \subseteq \tau_S(\mathrm{Sym}(J)) \subseteq \mathcal{A}(J)$. Since S is a Noetherian ring, $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$ if and only if $\mathrm{Supp}_S(\mathcal{A}(J)) \subseteq \{\mathfrak{M}\}$. Because $\mathcal{A}(J)$ is a homogeneous S-module, $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$ if and only $\mathcal{A}(J)_{\mathfrak{p}} = 0$ for all homogeneous prime ideals $\mathfrak{p}$ of S with $\mathfrak{p} \neq \mathfrak{M}$.

Recall $\mathcal{R}(J) \cong \bigoplus_{i=0}^{\infty} J^i$. Thus, $\mathcal{R}(J) \otimes_S S_{\mathfrak{p}} \cong \bigoplus_{i=0}^{\infty} J_{\mathfrak{p}}^i \cong \mathcal{R}(J_{\mathfrak{p}})$. Furthermore, $\mathrm{Sym}(J) \otimes_S S_{\mathfrak{p}} \cong \mathrm{Sym}(J \otimes_S S_{\mathfrak{p}}) \cong \mathrm{Sym}(J_{\mathfrak{p}})$. Thus, for any prime ideal $\mathfrak{p}$ of S, the sequence

$$0 \to \mathcal{A}(J)_{\mathfrak{p}} \to \mathrm{Sym}(J_{\mathfrak{p}}) \to \mathcal{R}(J_{\mathfrak{p}}) \to 0$$

is exact. In particular, $\mathcal{A}(J)_{\mathfrak{p}} = 0$ if and only if $\mathrm{Sym}(J_{\mathfrak{p}}) \to \mathcal{R}(J_{\mathfrak{p}})$ is an isomorphism, or equivalently, $J_{\mathfrak{p}}$ is of linear type. Combining this with the previous paragraph, we see that $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$ if and only if J is of linear type on the punctured spectrum of S. Thus, we may show that $\mathrm{topdeg}(\mathcal{A}(J)) < \infty$ if and only if $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$.

Now, if $\mathrm{topdeg}(\mathcal{A}(J)) < \infty$, then there exists some $p \geq 0$ so that for all $q \geq p$ $[\mathcal{A}(J)]_q = 0$. Then $\mathfrak{M}^{p+1}\mathcal{A}(J) = 0$. Therefore, $\mathcal{A}(J) \subseteq \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$. Since we have already seen that $\mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J)) \subseteq \mathcal{A}(J)$, we have that $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$. Hence, if $\mathrm{topdeg}(\mathcal{A}(J)) < \infty$, then $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$.

Conversely, suppose $\mathcal{A}(J) = \mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))$. Since $\mathrm{Sym}(J)$ is a finitely generated S-algebra, there exists some p such that $\mathfrak{M}^p\mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J)) = 0$. Again, since $\mathrm{Sym}(J)$ is a finitely generated S-algebra, it follows that $\mathrm{topdeg}(\mathrm{H}^0_{\mathfrak{M}}(\mathrm{Sym}(J))) < \infty$. ∎

5.1 Ordinary Matrices

We begin with a list of known facts concerning $b_0(\mathcal{A}(J))$ and $\mathrm{topdeg}(\mathcal{A}(J))$.

Proposition 5.1.1 *Let, t, m, and n be integers satisfying $1 \leq t \leq m \leq n$, K be a field, X be an $m \times n$ generic ordinary matrix over K, $S = K[X]$, and $J = I_t(X)$.*

a. If $t = 1$, then J is of linear type. Hence,

$$b_0(\mathcal{A}(J)) = \mathrm{topdeg}(\mathcal{A}(J)) = -\infty.$$

b. [36, Proposition 1.1] *If* $n \le m+1$ *and* $t = m$*, then* J *is of linear type. Hence,*

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

c. If $n \ge m+2$ *and* $t = m$*, then* J *is not of linear type on the punctured spectrum of* S*. Hence,*

$$\operatorname{topdeg}(\mathcal{A}(J)) = \infty.$$

d. [22, 2.6] *If* $t = m$*, then* J *is of fiber type. Hence,*

$$b_0(\mathcal{A}(J)) \le 0.$$

e. [38, 2.4] *If* $n = m$ *and* $t = n-1$*, then* J *is of linear type. Hence,*

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

f. [33, 7.3] *If* $\operatorname{char} K = 0$*,* $m = 3$*,* $n \ge 3$*, and* $t = 2$*, then* J *is of fiber type. Hence,*

$$b_0(\mathcal{A}(J)) \le 0.$$

g. If $t = 2$*, then* J *is of linear type on the punctured spectrum of* S*. Hence,*

$$\operatorname{topdeg}(\mathcal{A}(J)) < \infty.$$

h. If $2 < t < m$ *and it is not the case that* $t+1 = m = n$*, then* J *is not of linear type on the punctured spectrum of* S*. Hence,*

$$\operatorname{topdeg}(\mathcal{A}(J)) = \infty.$$

Proof Part (a) follows from the fact that J is the ideal generated by the set of variables. Hence J is generated by a regular sequence. In [36], Huneke developed the theory of d-sequences. Every regular sequence is a d-sequence. Moreover, any ideal generated by a d-sequence is of linear type [34, Theorem 3.1]. This also proves the case of part (b) where $n = m$ and $t = m$. The case of part (b) where $n = m+1$ is again proven by d-sequences in [36, Proposition 1.1], which is the reference provided for part (b). Parts (d), (e), and (f) are proven in the provided references and by applying Observation 2.2.12 and Observation 2.2.17.

The proofs of (c), (g), and (h) are similar.

To prove (c), let X be an $m \times n$ generic matrix where $n \geq m+2$. Let $\mathfrak{p}$ be the homogeneous prime ideal of S generated by all of the variables of X with the exception of X_{11}. We apply Lemma 3.1.2. To make notation consistent, let A be the 1×1 matrix with $A = \begin{pmatrix} X_{11} \end{pmatrix}$. Since $S_{\mathfrak{p}}$ is a localization of the ring $T[D]$ (using the notation of Lemma 3.1.2), it suffices to show that the extension of J to $T[D]$ is not of linear type. By the isomorphism in Lemma 3.1.2, it suffices to show that $I_{m-1}(Y)$ is not of linear type where Y is a generic $(m-1) \times (n-1)$ matrix. But since $n \geq m+2$, it follows that $(n-1) \geq (m-1)+2$. Hence, we are taking the maximal minors of a matrix with at least two more columns than rows. By the classification of ideals of minors of linear type given by Huneke in [38, 2.6], $I_{m-1}(Y)$ is not of linear type. Hence, J is not of linear type on the punctured spectrum of S. Therefore, by Remark 5.0.2, $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$.

The same argument applies in case (h), localizing at the same prime ideal $\mathfrak{p}$ consisting of all variables of X with the exception of X_{11}. Then locally at $\mathfrak{p}$, this ideal is the same as the ideal generated by taking $(t-1) \times (t-1)$ minors of an $(m-1) \times (n-1)$ generic matrix. Again, by [38, 2.6], this ideal is not of linear type. Hence, by Remark 5.0.2, $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$.

Lastly, we prove (g). Let X be an $m \times n$ generic matrix and $t = 2$. Let $\mathfrak{p}$ be any homogeneous prime ideal of S with $\mathfrak{p} \neq \mathfrak{M}$. Then at least one entry of X must be invertible after localizing at $\mathfrak{p}$. If X contains a block of size 2×2 or higher that is invertible locally at $\mathfrak{p}$, then $J_{\mathfrak{p}} = S$, which is of linear type. Thus, we may assume that the largest invertible block locally at $\mathfrak{p}$ is a 1×1 matrix. Without loss of generality, we may assume X_{11} is invertible. We apply Lemma 3.1.2. To make notation consistent, let A be the 1×1 matrix with $A = \begin{pmatrix} X_{11} \end{pmatrix}$. By the isomorphism in Lemma 3.1.2, it suffices to show that $I_1(Y)$ is of linear type where Y is a generic $(m-1) \times (n-1)$

matrix. This is immediate. Thus, $J_{\mathfrak{p}}$ is of linear type. Hence, J is of linear type on the punctured spectrum of S. Therefore, by Remark 5.0.2, $\operatorname{topdeg}(\mathcal{A}(J)) < \infty$. ■

There are few cases in which explicit resolutions of J^k are known. However, we do know explicit resolutions for the case of maximal minors, i.e., when $t = m$.

Proposition 5.1.2 (Akin-Buchsbaum-Weyman [1, 5.4]) *Let R be a nonnegatively graded Noetherian ring with R_0 local, m and n integers satisfying $1 \le m \le n$, X an $m \times n$ generic matrix over R, $S = R[X]$, and $J = I_m(X)$. For any positive integer k, let $\mathbf{F^k}_\bullet$ be the minimal homogeneous free S-resolution of $J^k(kt)$. Then $\mathbf{F^k}_\bullet$ is a linear resolution of length $\min\{k, m\}(n - m)$. In other words, $b_0(\mathbf{F^k}_i) = i$ for all i with $1 \le i \le \min\{k, m\}(n - m)$ and $b_0(\mathbf{F^k}_i) = -\infty$ for all i with $i > \min\{k, m\}(n - m)$.*

This proposition is implicit from the proof of [1, 5.4] but is not directly stated. A direct statement and justification for the length of the resolution can be found in [7, 3.1], and a direct statement for the linearity of the resolution can be found in [7, 3.6].

Theorem 5.1.3 (Maximal Minors) *Let K be a field, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, m and n be integers satisfying $1 \le m \le n$, A be an $m \times n$ matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = I_m(A)$ be of generic height (that is, $\operatorname{ht} I = n - m + 1$). Further, suppose that $\operatorname{ht} I_j(A) \ge \min\{(m - j + 1)(n - m) + 1, d\}$ for all j in the range $1 \le j \le m - 1$.*

a. Let $n = m$. Then

$$b_0(\mathcal{A}(I)) = \operatorname{topdeg}(\mathcal{A}(I)) = -\infty.$$

b. Let $n = m + 1$.

If $d > \min\{k, m\}(n - m)$, then

$$b_0(\mathcal{A}_k(I)) = \operatorname{topdeg}(\mathcal{A}_k(I)) = -\infty.$$

If $d \le \min\{k, m\}(n - m)$, then

$$b_0(\mathcal{A}_k(I)) \le (d - 1)(\delta - 1) \;\; \textit{and} \;\; \operatorname{topdeg}(\mathcal{A}_k(I)) \le d(\delta - 1).$$

c. Let $n \geq m+2$.

If $d-1 > \min\{k,m\}\,(n-m)$, then

$$b_0(\mathcal{A}_k(I)) \leq 0.$$

If $d-1 \leq \min\{k,m\}\,(n-m)$, then

$$b_0(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1)\,.$$

Proof To prove (a), we note that R is a domain and that $\operatorname{ht} I = \operatorname{ht} I_m(A) = \operatorname{ht}(\det(A)) = 1$. Thus, $\det(A)$ is a non-zero-divisor in R. Therefore, I is generated by a regular sequence on R. By [34, Theorem 3.1], I is of linear type, giving the result.

For parts (b) and (c) we apply Corollary 4.2.4. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d}\right) - d\right\}.$$

We now prove (b).

By part (b) of Proposition 5.1.1, $b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty$. Therefore,

$$b_0(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{F^k}_{d}\right) - d.$$

By Proposition 5.1.2, if $d > \min\{k,m\}\,(n-m)$, then $b_0\left(\mathbf{F^k}_{d}\right) = -\infty$. Therefore, we have $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq -\infty$. Thus, $\operatorname{topdeg}(\mathcal{A}_k(I)) = -\infty$. It is always the case that $b_0(\mathcal{A}_k(I)) \leq \operatorname{topdeg}(\mathcal{A}_k(I))$; hence, we also conclude that $b_0(\mathcal{A}_k(I)) = -\infty$.

By Proposition 5.1.2, if $d \leq \min\{k,m\}\,(n-m)$, then $b_0\left(\mathbf{F^k}_{d}\right) = d$. Therefore, we have $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta d - d = d\,(\delta-1)$. Similarly, it must be the case that $d-1 \leq \min\{k,m\}\,(n-m)$. Hence, $b_0\left(\mathbf{F^k}_{d-1}\right) = d-1$. Therefore, $b_0(\mathcal{A}_k(I)) \leq \delta\,(d-1) - d + 1 = (d-1)\,(\delta-1)$.

We now prove (c). By part (c) of Proposition 5.1.1, $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$, so we do not draw any conclusions about $\operatorname{topdeg}(\mathcal{A}(I))$. On the other hand, by part (d) of Proposition 5.1.1, we know $b_0(\mathcal{A}(J)) \le 0$. Therefore, we have

$$b_0(\mathcal{A}_k(I)) \le \max\left\{0, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}.$$

By Proposition 5.1.2, if $d-1 > \min\{k, m\}\,(n-m)$, then $b_0\left(\mathbf{F^k}_{d-1}\right) = -\infty$. Thus, $b_0(\mathcal{A}_k(I)) \le 0$. On the other hand, if $d-1 \le \min\{k,m\}\,(n-m)$, then $b_0\left(\mathbf{F^k}_{d-1}\right) = d-1$. Therefore, $b_0(\mathcal{A}_k(I)) \le \delta\,(d-1) + d - 1 = (d-1)\,(\delta-1)$. ■

Combining the results of the above theorem with Observation 2.2.12 and Observation 2.2.17, we obtain the following corollary. Note that when $n = m+1$, $n - m = 1$.

Corollary 5.1.4 *Let K be a field, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, m and n be integers satisfying $1 \le m \le n$, A be an $m \times n$ matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = I_m(A)$ be of generic height (that is, $\operatorname{ht} I = n - m + 1$). Further, suppose that $\operatorname{ht} I_j(A) \ge \min\{(m-j+1)\,(n-m)+1, d\}$ for all j in the range $1 \le j \le m-1$.*

a. If $\delta = 1$, then I is of fiber type.

b. If $n = m+1$ and $d > m$, then I is of linear type.

c. If $n = m+1$, $d \le m$, and $\delta = 1$, then $(x_1, \ldots, x_d)\,\mathcal{A}(I) = 0$.

d. If $n \ge m+2$ and $d > m\,(n-m)+1$, then I is of fiber type.

Parts (a), (b), and (c) are known results that we recover. Part (a) was proven by Bruns, Conca, and Varbaro in [7, 3.7] using techniques from representation theory. Part (b) can be proven in the following way. The requirements in the corollary and in part (b) imply that I satisfies a condition known as G_∞ or $\mathcal{F}_1$. From the work of Apéry in [2] or of Gaeta in [25], it is known that I is in the linkage class of a complete intersection. Thus, I is strongly Cohen-Macaulay [35, 1.4]. Hence, I satisfies sliding depth. Any ideal which satisfies G_∞ and sliding depth is of linear type [29]. Part (c) follows from the work of Kustin, Polini, and Ulrich in [48, 6.1.a], where they applied the same technique we used here. We are unaware of whether part (d) is known from other methods.

Aside from maximal minors, resolutions of powers of J are not known. For these other situations, we use the Castelnuovo-Mumford regularity to approximate $b_0\left(\mathbf{F^k}_{d-1}\right)$ and $b_0\left(\mathbf{F^k}_{d}\right)$ (in the cases where $\operatorname{topdeg}(\mathcal{A}_k(J)) < \infty$).

Proposition 5.1.5 (Raicu [59, Theorem on Regularity]) *Let m and n be integers satisfying $2 \le m \le n$, X be an $m \times n$ generic matrix over $\mathbb{C}$, and $J = I_2(X)$.*

For $1 \le k \le m-2$,

$$\operatorname{reg} J^k = k + m - 1.$$

For $k \ge m-1$,

$$\operatorname{reg} J^k = 2k.$$

Although Raicu proved this result over $\mathbb{C}$, it holds over any field of characteristic zero. Indeed, let $\mathbf{F^k}_\bullet$ be the minimal homogeneous free resolution of J^k when viewed as an ideal of $\mathbb{Q}[X]$, and let K be any field of characteristic zero. Then $K[X]$ is a free algebra over $\mathbb{Q}[X]$ and the generators of J^k are elements of $\mathbb{Q}[X]$. Therefore, by Lemma 3.0.1, $\mathbf{F^k}_\bullet \otimes_{\mathbb{Q}[X]} K[X]$ is the minimal homogeneous free resolution of J^k when viewed as a $K[X]$-ideal.

Theorem 5.1.6 (2×2 Minors) *Let K be a field of characteristic zero. Let $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, m and n be integers satisfying $2 \le m \le n$, A be an $m \times n$ matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = I_2(A)$ be of generic height (that is, $\operatorname{ht} I = (m-1)(n-1)$). Let X be an $m \times n$ generic matrix over R and $J = I_2(X)$. Further, suppose that $\operatorname{ht} I_1(A) \ge \min\{mn, d\}$.*

a. Let $m = 3$. Then

$$b_0(\mathcal{A}(I)) \le (d-1)(\delta - 1), \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}(I)) \le \max\{\delta \operatorname{topdeg}(\mathcal{A}(J)), d(\delta-1)\}.$$

b. If $2 \le k \le m-2$*, then*

$$b_0(\mathcal{A}_k(I)) \le \max\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1) + \delta\,(m-k-1)\}\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, d\,(\delta-1) + \delta\,(m-k-1)\}\,.$$

If $k \ge m-1$*, then*

$$b_0(\mathcal{A}_k(I)) \le \max\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1)\}\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, d\,(\delta-1)\}\,.$$

Proof We apply Corollary 4.2.4. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \le \max\bigl\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\bigl(\mathbf{F^k}_{d-1}\bigr) - d + 1\bigr\}\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\bigl\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\bigl(\mathbf{F^k}_{d}\bigr) - d\bigr\}\,.$$

Furthermore, by part (g) of Proposition 5.1.1, $\operatorname{topdeg}(\mathcal{A}(J)) < \infty$. Thus, we can place meaningful bounds on $\operatorname{topdeg}(\mathcal{A}_k(I))$.

Since $\mathbf{F^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(2k)$, we have that $b_0\bigl(\mathbf{F^k}_{d-1}\bigr) \le \operatorname{reg} J^k(2k) + d - 1$ and $b_0\bigl(\mathbf{F^k}_{d}\bigr) \le \operatorname{reg} J^k(2k) + d$. Therefore,

$$\delta b_0\bigl(\mathbf{F^k}_{d-1}\bigr) \le \delta \operatorname{reg} J^k(2k) + (d-1)\,\delta, \text{ and}$$

$$\delta b_0\bigl(\mathbf{F^k}_{d}\bigr) \le \delta \operatorname{reg} J^k(2k) + d\delta.$$

Hence, we have

$$\delta b_0\bigl(\mathbf{F^k}_{d-1}\bigr) - d + 1 \le \delta \operatorname{reg} J^k(2k) + (d-1)\,(\delta-1)\,, \text{ and}$$

$$\delta b_0\bigl(\mathbf{F^k}_{d}\bigr) - d \le \delta \operatorname{reg} J^k(2k) + d\,(\delta-1)\,.$$

Note that $\operatorname{reg} J^k(2k) = \operatorname{reg} J^k - 2k$.

Now, $\mathcal{A}_1(I) = 0$ (for any ideal I), so we only concern ourselves with $\mathcal{A}_k(I)$ for $k \ge 2$.

By Proposition 5.1.5, if $2 \leq k \leq m-2$, then $\operatorname{reg} J^k = k+m-1$. Hence, $\operatorname{reg} J^k(2k) = k+m-1-2k = m-k-1$. Thus, for $2 \leq k \leq m-2$,

$$\delta b_0\big({\mathbf{F^k}}_{d-1}\big) - d + 1 \leq \delta\,(m-k-1) + (d-1)\,(\delta-1)\,, \text{ and}$$

$$\delta b_0\big({\mathbf{F^k}}_{d}\big) - d \leq \delta\,(m-k-1) + d\,(\delta-1)\,.$$

Thus, we conclude that for $2 \leq k \leq m-2$,

$$b_0(\mathcal{A}_k(I)) \leq \max\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1) + \delta\,(m-k-1)\}\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, d\,(\delta-1) + \delta\,(m-k-1)\}\,.$$

By Proposition 5.1.5, if $k \geq m-1$, then $\operatorname{reg} J^k = 2k$. Hence, $\operatorname{reg} J^k(2k) = 2k - 2k = 0$. Thus, for $2 \leq k \leq m-2$,

$$\delta b_0\big({\mathbf{F^k}}_{d-1}\big) - d + 1 \leq (d-1)\,(\delta-1)\,, \text{ and}$$

$$\delta b_0\big({\mathbf{F^k}}_{d}\big) - d \leq d\,(\delta-1)\,.$$

Therefore, we conclude that for $k \geq m-1$,

$$b_0(\mathcal{A}_k(I)) \leq \max\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1)\}\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, d\,(\delta-1)\}\,.$$

This concludes the proof of (b). The proof of (a) is a special case.

In the setting of (a), $m = 3$. Thus, $k \geq m-1$ if and only if $k \geq 2$. Therefore, we restrict our attention to the case where $k \geq m-1$ above. By part (f) of Proposition 5.1.1, we have $b_0(\mathcal{A}(J)) \leq 0$. Hence, we have the global conditions

$$b_0(\mathcal{A}(I)) \leq (d-1)\,(\delta-1)\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}(I)) \leq \max\{\delta \operatorname{topdeg}(\mathcal{A}(J))\,, d\,(\delta-1)\}\,.$$

$\blacksquare$

Corollary 5.1.7 *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n an integer satisfying $3 \leq n$, A a $3 \times n$ matrix whose entries are all homogeneous elements of R of degree 1, and $I = I_2(A)$ be of generic height (that is, $\operatorname{ht} I = 2(n-1)$). Further, suppose that $\operatorname{ht} I_1(A) \geq \min\{3n, d\}$. Then I is of fiber type.*

In other cases, Raicu is not able to compute the regularity for all powers of the ideal J, but is able to compute the regularity for sufficiently high powers.

Proposition 5.1.8 (Raicu [59, Theorem on Regularity]) *Let $t, m,$ and n be integers satisfying $1 < t < m \leq n$, X be an $m \times n$ generic matrix over $\mathbb{C}$, and $J = I_t(X)$.*

For $k \geq m-1$,

$$\operatorname{reg} J^k = tk + \begin{cases} \left(\frac{t-1}{2}\right)^2 & \text{if } t \text{ is odd} \\ \frac{(t-2)t}{4} & \text{if } t \text{ is even.} \end{cases}$$

As noted before, despite the fact that this computation was done over $\mathbb{C}$, the result holds over any field of characteristic zero.

Theorem 5.1.9 (Submaximal Minors of a Square Matrix) *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n be an integer satisfying $2 \leq n$, A be an $n \times n$ matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = I_{n-1}(A)$ be of generic height (that is, $\operatorname{ht} I = 4$). Further, suppose that $\operatorname{ht} I_j(A) \geq \min\left\{(n-j+1)^2, d\right\}$ for all j in the range $1 \leq j \leq n-2$.*

If $k \geq n-1$, then

$$b_0(\mathcal{A}_k(I)) \leq (d-1)(\delta-1) + \delta N(n)\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d(\delta-1) + \delta N(n)\,,$$

where $N(n) = \begin{cases} \left(\frac{n-2}{2}\right)^2 & \text{if } n \text{ is even} \\ \frac{(n-3)(n-1)}{4} & \text{if } n \text{ is odd.} \end{cases}$

Proof We apply Corollary 4.2.4. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \le \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d}\right) - d\right\}.$$

By part (e) of Proposition 5.1.1, $b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty$. Therefore, we have

$$b_0(\mathcal{A}_k(I)) \le \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \delta b_0\left(\mathbf{F^k}_{d}\right) - d.$$

Since $\mathbf{F^k}_{\bullet}$ is the minimal homogeneous free resolution of $J^k(k\,(n-1))$, we have that $b_0\left(\mathbf{F^k}_{d-1}\right) \le \operatorname{reg} J^k(k\,(n-1)) + d - 1$ and $b_0\left(\mathbf{F^k}_{d}\right) \le \operatorname{reg} J^k(k\,(n-1)) + d$. Therefore,

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) \le \delta \operatorname{reg} J^k(k\,(n-1)) + (d-1)\,\delta, \text{ and}$$

$$\delta b_0\left(\mathbf{F^k}_{d}\right) \le \delta \operatorname{reg} J^k(k\,(n-1)) + d\delta.$$

Hence, we have

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1 \le \delta \operatorname{reg} J^k(k\,(n-1)) + (d-1)\,(\delta-1)\,, \text{ and}$$

$$\delta b_0\left(\mathbf{F^k}_{d}\right) - d \le \delta \operatorname{reg} J^k(k\,(n-1)) + d\,(\delta-1)\,.$$

Note that $\operatorname{reg} J^k(k\,(n-1)) = \operatorname{reg} J^k - k\,(n-1)$.

By Proposition 5.1.8, if $k \ge n-1$, then

$$\operatorname{reg} J^k = k\,(n-1) + \begin{cases} \left(\frac{n-2}{2}\right)^2 & \text{if } n \text{ is even} \\ \frac{(n-3)(n-1)}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Let $N(n) = \begin{cases} \left(\frac{n-2}{2}\right)^2 & \text{if } n \text{ is even} \\ \frac{(n-3)(n-1)}{4} & \text{if } n \text{ is odd.} \end{cases}$

Thus, for $k \ge n-1$,

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1 \le \delta N(n) + (d-1)\,(\delta-1)\,, \text{ and}$$

$$\delta b_0\left(\mathbf{F^k}_d\right) - d \leq \delta N(n) + d\,(\delta - 1)\,.$$

Therefore, we conclude that for $k \geq n-1$,

$$b_0(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1) + \delta N(n)\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d\,(\delta - 1) + \delta N(n)\,.$$

■

Corollary 5.1.10 *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, A be a 3×3 matrix whose entries are all homogeneous elements of R of degree 1, and $I = I_2(A)$ be of generic height (that is, $\operatorname{ht} I = 4$). Further, suppose that $\operatorname{ht} I_1(A) \geq \min\{9, d\}$. Then I is of fiber type and $(x_1, \ldots, x_d)\,\mathcal{A}(I) = 0$.*

Theorem 5.1.11 (Minors of an Ordinary Matrix) *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, t, m, and n be integers satisfying $2 < t < m \leq n$, A be an $m \times n$ matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = I_t(A)$ be of generic height (that is, $\operatorname{ht} I = (m-t+1)\,(n-t+1)$). Let X be an $m \times n$ generic matrix over R and $J = I_t(X)$. Further, suppose that $\operatorname{ht} I_j(A) \geq \min\{(m-j+1)\,(n-j+1)\,, d\}$ for all j in the range $1 \leq j \leq t-1$.*

If $k \geq m-1$, then

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1) + \delta N(t)\right\},$$

where $N(t) = \begin{cases} \left(\frac{t-1}{2}\right)^2, & t \textit{ is odd} \\ \frac{(t-2)t}{4}, & t \textit{ is even.} \end{cases}$

Proof By part (h) of Proposition 5.1.1, $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$, so we do not draw conclusions about $\operatorname{topdeg}(\mathcal{A}(I))$.

We apply Corollary 4.2.4. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}$$

Since $\mathbf{F^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(k\,(n-1))$, we have that $b_0\big(\mathbf{F^k}_{d-1}\big) \leq \operatorname{reg} J^k(k\,(n-1)) + d - 1$. Therefore,

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) \leq \delta \operatorname{reg} J^k(kt) + (d-1)\,\delta, \text{ and}$$

Hence, we have

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1 \leq \delta \operatorname{reg} J^k(kt) + (d-1)\,(\delta - 1)\,, \text{ and}$$

Note that $\operatorname{reg} J^k(kt) = \operatorname{reg} J^k - kt$.

By Proposition 5.1.8, if $k \geq m-1$, then

$$\operatorname{reg} J^k = tk + \begin{cases} \left(\frac{t-1}{2}\right)^2 & \text{if } t \text{ is odd} \\ \frac{(t-2)t}{4} & \text{if } t \text{ is even.} \end{cases}$$

Let $N(t) = \begin{cases} \left(\frac{t-1}{2}\right)^2 & \text{if } t \text{ is odd} \\ \frac{(t-2)t}{4} & \text{if } t \text{ is even.} \end{cases}$

Thus, for $k \geq n-1$,

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1 \leq \delta N(t) + (d-1)\,(\delta - 1)\,, \text{ and}$$

Thus, we conclude that for $k \geq n-1$,

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1) + \delta N(t)\right\}.$$

■

5.2 Symmetric Matrices

We begin with a list of known facts concerning $b_0(\mathcal{A}(J))$ and $\operatorname{topdeg}(\mathcal{A}(J))$.

Proposition 5.2.1 *Let t and n be integers satisfying $1 \leq t \leq n$, K be a field, X be an $n \times n$ generic symmetric matrix over K, $S = K[X]$, and $J = I_t(X)$.*

a. If $t = 1$, then J is of linear type. Hence,

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

b. *If $t = n$, then J is of linear type. Hence,*

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

c. [43, 2.10] *If $t = n - 1$, then J is of linear type. Hence,*

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

d. *If $t = 2$, then J is of linear type on the punctured spectrum. Hence,*

$$\operatorname{topdeg}(\mathcal{A}(J)) < \infty.$$

e. *If $2 < t < n-1$, then J is not of linear type on the punctured spectrum. Hence,*

$$\operatorname{topdeg}(\mathcal{A}(J)) = \infty.$$

Proof Part (a) follows from the fact that J is the ideal generated by the set of variables. Hence J is generated by a regular sequence. Every regular sequence is a d-sequence. Moreover, any ideal generated by a d-sequence is of linear type [34, Theorem 3.1]. This also proves part (b).

The proofs of (d) and (e) are similar.

To prove (e), let X be an $n \times n$ generic matrix and $2 < t < n - 1$. Let $\mathfrak{p}$ be the homogeneous prime ideal of S generated by all of the variables of X with the exception of X_{11}. We apply Lemma 3.2.2. To make notation consistent, let A be the 1×1 matrix with $A = \begin{pmatrix} X_{11} \end{pmatrix}$. Since $S_\mathfrak{p}$ is a localization of the ring $T[D]$ (using the notation of Lemma 3.2.2), it suffices to show that the extension of J to $T[C]$ is not of linear type. By the isomorphism in Lemma 3.2.2, it suffices to show that $I_{t-1}(Y)$ is not of linear type where Y is an $(n-1) \times (n-1)$ generic symmetric matrix. Now, it follows that $1 < t-1 < (n-1)-1$. By the classification of ideals of minors of linear type given by Kotzev in [43, 3.1], $I_{t-1}(Y)$ is not of linear type. Hence, J is not of linear type on the punctured spectrum of S. By Remark 5.0.2, $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$.

Lastly, we prove (d). Let X be an $n \times n$ generic symmetric matrix and $t = 2$. Let $\mathfrak{p}$ be any homogeneous prime ideal of S with $\mathfrak{p} \neq \mathfrak{M}$. Then at least one entry of X

must be invertible after localizing at $\mathfrak{p}$. If X contains a block of size 2×2 or higher that is invertible locally at $\mathfrak{p}$, then $J_{\mathfrak{p}} = S$, which is of linear type. Thus, we may assume that the largest invertible block locally at $\mathfrak{p}$ is a 1×1 matrix. Without loss of generality, we may assume X_{11} is invertible (if a non-diagonal entry were invertible, there would be a 2×2 invertible block by symmetry). Then we are taking the 1×1 minors of a generic symmetric matrix, which is of linear type since it is generated by a sequence of variables. Thus, $J_{\mathfrak{p}}$ is of linear type. Hence, J is of linear type on the punctured spectrum of S. Therefore, by Remark 5.0.2, $\operatorname{topdeg}(\mathcal{A}(J)) < \infty$. ■

Unfortunately, we are unaware of any work which has been done to compute resolutions of J^k, nor are we aware of any work which has been done to calculate the regularity of J^k. Therefore, we are not able to obtain explicit degree bounds using our method. However, using the fact that J is of linear type when $t = n - 1$, we can state the following result for submaximal minors.

Proposition 5.2.2 (Submaximal Minors) *Let K be a field, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n be an integer satisfying $2 \leq n$, A an $n \times n$ symmetric matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = I_{n-1}(A)$ be of generic symmetric height (that is, $\operatorname{ht} I = 3$). Further, suppose that $\operatorname{ht} I_j(A) \geq \min\{\binom{n-j+2}{2}, d\}$ for all j in the range $1 \leq j \leq n-2$. Then*

$$b_0(\mathcal{A}_k(I)) \leq \delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1, \text{ and}$$
$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta b_0\big(\mathbf{F^k}_{d}\big) - d.$$

Proof We apply Corollary 4.2.6. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\big\{\delta b_0(\mathcal{A}_k(J)) \,, \delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1\big\}\,, \text{ and}$$
$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\big\{\delta \operatorname{topdeg}(\mathcal{A}_k(J)) \,, \delta b_0\big(\mathbf{F^k}_{d}\big) - d\big\}\,.$$

By part (c) of Proposition 5.2.1, $b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty$. Therefore, we have

$$b_0(\mathcal{A}_k(I)) \leq \delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta b_0\big(\mathbf{F^k}_d\big) - d.$$

■

Knowing this, if someone eventually computes resolutions of J^k or the regularity of J^k, for instance, we could use the above result to obtain explicit degree bounds.

5.3 Alternating Matrices

We begin with a list of known facts concerning $b_0(\mathcal{A}(J))$ and $\operatorname{topdeg}(\mathcal{A}(J))$.

Proposition 5.3.1 *Let t and n be integers satisfying $2 \leq 2t \leq n$, K be a field, X be an $n \times n$ generic alternating matrix over K, $S = K[X]$, and $J = \operatorname{Pf}_{2t}(X)$.*

a. If $2t = 2$, then J is of linear type. Hence,

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

b. If $2t = n$, then J is of linear type. Hence,

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

c. [35, 2.2] *If $2t = n - 1$, then J is of linear type. Hence,*

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

d. [4, 2.1] *If* $\operatorname{char} K \neq 2$ *and $2t = n - 2$, then J is of linear type. Hence,*

$$b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty.$$

e. If $2t = 4$, then J is of linear type on the punctured spectrum. Hence,

$$\operatorname{topdeg}(\mathcal{A}(J)) < \infty.$$

f. If $4 < 2t < n-2$, then J is not of linear type on the punctured spectrum. Hence,

$$\operatorname{topdeg}(\mathcal{A}(J)) = \infty.$$

Proof Part (a) follows from the fact that J is the ideal generated by the set of variables. Hence J is generated by a regular sequence. By the work of Huneke in [34, Theorem 3.1], J is of linear type. This also proves part (b).

The proofs of (e) and (f) are similar.

To prove (f), let X be an $n \times n$ generic alternating matrix and $4 < 2t < n-2$. Let $\mathfrak{p}$ be the homogeneous prime ideal of S generated by all of the variables of X with the exception of X_{12}. We apply Lemma 3.3.3. To make notation consistent, let A be the 2×2 matrix with $A = \begin{pmatrix} 0 & X_{12} \\ -X_{12} & 0 \end{pmatrix}$. Since $S_{\mathfrak{p}}$ is a localization of the ring $T[C]$ (using the notation of Lemma 3.3.3), it suffices to show that the extension of J to $T[C]$ is not of linear type. By the isomorphism in Lemma 3.3.3, it suffices to show that $\mathrm{Pf}_{2t-2}(Y)$ is not of linear type where Y is an $(n-2)\times(n-2)$ generic alternating matrix. Now, it follows that $2 < 2t-2 < (n-2)-2$. By the classification of ideals of Pfaffians of linear type given by Baetica in [4, 2.2], $\mathrm{Pf}_{2t-2}(Y)$ is not of linear type. Hence, J is not of linear type on the punctured spectrum of S. By Remark 5.0.2, $\mathrm{topdeg}(\mathcal{A}(J)) = \infty$.

Lastly, we prove (e). Let X be an $n \times n$ generic alternating matrix and $2t = 4$. Let $\mathfrak{p}$ be any homogeneous prime ideal of S with $\mathfrak{p} \neq \mathfrak{M}$. Then at least one entry of X must be invertible after localizing at $\mathfrak{p}$. If X contains a principal submatrix of size 4×4 or higher that is invertible locally at $\mathfrak{p}$, then $J_{\mathfrak{p}} = S_{\mathfrak{p}}$, which is of linear type. Thus, we may assume that the largest invertible principal submatrix locally at $\mathfrak{p}$ is a 2×2 matrix. Without loss of generality, we may assume X_{12} is invertible. Then we are taking the 2×2 Pfaffians of a generic alternating matrix, which is of linear type since it is generated by a sequence of variables. Thus, $J_{\mathfrak{p}}$ is of linear type. Hence, J is of linear type on the punctured spectrum of S. Therefore, by Remark 5.0.2, $\mathrm{topdeg}(\mathcal{A}(J)) < \infty$. ∎

In the case of submaximal Pfaffians, explicit resolutions of J^k are known thanks to the work of Kustin and Ulrich. Their computations are much more general than what is given below. However, in the case of the submaximal Pfaffians of a generic alternating matrix, we obtain explicit resolutions of $\mathrm{Sym}_k(J)$. Since J is of linear type in this case, $\mathrm{Sym}_k(J) \cong J^k$, giving us the desired result.

Proposition 5.3.2 (Kustin-Ulrich [49, 2.7, 4.7, 4.13.b]) *Let K be a field, n an odd integer with $3 \le n$, X an $n \times n$ generic alternating matrix over R, $S = K[X]$, and $J = \mathrm{Pf}_{n-1}(X)$. The minimal free resolution of $J^k(k\,(n-1)\,/2)$ is a complex $\mathcal{D}^{\mathbf{k}}{}_{\bullet}$ with*

$$\mathcal{D}^{\mathbf{k}}{}_i = \begin{cases} S(-i)^{\beta_i^k}, & \text{if } i \le \min\{k, n-1\} \\ S\big(-(i-1) - \frac{1}{2}(n-i+1)\big), & \text{if } i = k+1 \le n-1 \text{ and } k \text{ is odd} \\ 0, & \text{if } i = k+1 \text{ and } k \text{ is even} \\ 0, & \text{if } i \ge \min\{k+2, n\} \end{cases}$$

for some nonzero β_i.

The details are given in more detail in [48, Proof of 6.1.b], specifically in equations (6.1.4)-(6.1.7) of [48].

Corollary 5.3.3 *Let K be a field, n an odd integer with $3 \le n$, X an $n \times n$ generic alternating matrix over R, $S = K[X]$, and $J = \mathrm{Pf}_{n-1}(X)$.*

a. If $i \le \min\{k, n-1\}$, then $b_0\big(\mathcal{D}^{\mathbf{k}}{}_i\big) = i$.

b. If $i = k+1 \le n-1$ and k is odd, then $b_0\big(\mathcal{D}^{\mathbf{k}}{}_i\big) = (i-1) + \frac{1}{2}(n-i+1)$.

c. If $i = k+1$ and k is even or if $i \ge \min\{k+2, n\}$, then $b_0\big(\mathcal{D}^{\mathbf{k}}{}_i\big) = -\infty$.

Using the above results, we obtain the following degree bounds.

Theorem 5.3.4 (Submaximal Pfaffians) *Let K be a field, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n be an odd integer satisfying $3 \le n$, A be an $n \times n$ alternating matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = \mathrm{Pf}_{n-1}(A)$ be of generic alternating height (that is, $\mathrm{ht}\, I = 3$). Further, suppose that $\mathrm{ht}\,\mathrm{Pf}_{2j}(A) \ge \min\{n-2j+2, d\}$ for all j in the range $2 \le 2j \le n-3$.*

a. If $k \geq d$ and $d \leq n-1$, then

$$b_0(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1)\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d\,(\delta-1)\,.$$

b. If d is even, $d \leq n-1$, and $k = d-1$, then

$$b_0(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1)\,, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1) + \frac{\delta}{2}\,(n-d+1) - 1.$$

c. If d is odd and $k = d-1$, then

$$b_0(\mathcal{A}_k(I)) = -\infty, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) = -\infty.$$

d. If $d \geq n$ or $k \leq d-2$, then

$$b_0(\mathcal{A}_k(I)) = -\infty, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) = -\infty.$$

Proof We apply Corollary 4.2.8. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d}\right) - d\right\}.$$

By part (c) of Proposition 5.3.1, $b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty$. Therefore, we have

$$b_0(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{F^k}_{d}\right) - d.$$

If $k \geq d$ and $d \leq n-1$, then $d \leq \min\{k, n-1\}$. Hence, by part (a) of Corollary 5.3.3, $b_0\left(\mathbf{F^k}_{d}\right) = d$. Therefore, we have $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta d - d = d\,(\delta-1)$.

Similarly, it must be the case that $d-1 \leq \min\{k, n-1\}$. Hence, $b_0\left(\mathbf{F^k}_{d-1}\right) = d-1$. Therefore, we have $b_0(\mathcal{A}_k(I)) \leq \delta\,(d-1) - d + 1 = (d-1)\,(\delta-1)$. This proves (a).

If d is even, $d \leq n-1$, and $k = d-1$, then $d = k+1 \leq n-1$ and k is odd. Hence, by part (b) of Corollary 5.3.3, $b_0\left(\mathbf{F^k}_d\right) = (d-1) + \frac{1}{2}\,(n-d+1)$. Therefore, we have $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta\,\left((d-1) + \frac{1}{2}\,(n-d+1)\right) - d = \delta\,(d-1) + \frac{\delta}{2}\,(n-d+1) - d + 1 - 1 = (d-1)\,(\delta-1) + \frac{\delta}{2}\,(n-d+1) - 1$. Also, it must also be the case that $d-1 \leq \min\{k, n-1\}$. Hence, $b_0\left(\mathbf{F^k}_{d-1}\right) = d-1$. Therefore, we have $b_0(\mathcal{A}_k(I)) \leq \delta\,(d-1) - d + 1 = (d-1)\,(\delta-1)$. This proves (b).

If d is odd and $k = d-1$, then $d = k+1$ and k is even. Hence, by part (c) of Corollary 5.3.3, $b_0\left(\mathbf{F^k}_d\right) = -\infty$. Therefore, we have $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq -\infty$. Thus, $\operatorname{topdeg}(\mathcal{A}_k(I)) = -\infty$ It is always the case that $b_0(\mathcal{A}_k(I)) \leq \operatorname{topdeg}(\mathcal{A}_k(I))$. Hence, we also conclude that $b_0(\mathcal{A}_k(I)) = -\infty$. This proves (c).

If $d \geq n$ or $k \leq d-2$, then $d \geq n$ or $d \geq k+2$. Hence, $d \geq \min\{k+2, n\}$. Thus, by part (c) of Corollary 5.3.3, $b_0\left(\mathbf{F^k}_d\right) = -\infty$. Therefore, we have $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq -\infty$. Thus, $\operatorname{topdeg}(\mathcal{A}_k(I)) = -\infty$. It is always the case that $b_0(\mathcal{A}_k(I)) \leq \operatorname{topdeg}(\mathcal{A}_k(I))$. Hence, we also conclude that $b_0(\mathcal{A}_k(I)) = -\infty$. ■

Applying the above results, we draw the following conclusions.

Corollary 5.3.5 *Let K be a field, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n be an odd integer satisfying $3 \leq n$, A be an $n \times n$ alternating matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = \operatorname{Pf}_{n-1}(A)$ be of generic alternating height (that is, $\operatorname{ht} I = 3$). Further, suppose that $\operatorname{ht}\operatorname{Pf}_{2j}(A) \geq \min\{n-2j+2, d\}$ for all j in the range $2 \leq 2j \leq n-3$.*

a. *If $d \geq n$, then I is of linear type.*

b. *If $\delta = 1$, then I is of fiber type.*

c. *$\mathcal{A}_k(I) = 0$ for all $k \leq d-2$.*

d. *If d is odd, then $\mathcal{A}_k(I) = 0$ for all $k \leq d-1$.*

e. If d is odd and $\delta = 1$, then $(x_1, \ldots, x_d)\,\mathcal{A}(I) = 0$.

All of these results have already been proven. We restate them here for completeness. In particular, all of these are implications of [48, 6.1.b]. Kustin, Polini, and Ulrich used a similar technique to obtain the same results. The difference is a matter of perspective. Kustin, Polini, and Ulrich applied the approximate resolution arguments in settings where the ideal J is of linear type, a simpler context than the method employed in this dissertation. In the setting of submaximal Pfaffians, the broader perspective reduces to their technique and, therefore, recovers their results. Before [48], however, some of these results had already been recognized. Both (c) and (d) had been observed by Morey in [54, 4.1], also using the complexes in Proposition 5.3.2.

Moreover, (a) has been known for the longest amount of time. Indeed, due to the work of J. Watanabe in [61], we know that such ideals as in the above corollary are in the linkage class of a complete intersection. Hence, these ideals are strongly Cohen-Macaulay, and therefore, satisfy a condition known as sliding depth [35, 1.4]. In the presence of the sliding depth condition, a homogeneous ideal is of linear type if and only if the ideal satisfies a condition known as G_∞ or $\mathcal{F}_1$ [29]. However, condition (a) in the above corollary implies the condition G_∞. Thus, part (a) has been known from other methods.

For other values of $2t$, we do not know explicit resolutions of J^k. However, due to the work of Perlman, we know the regularity of J^k for sufficiently large k.

Proposition 5.3.6 (Perlman [58, Theorem A]) *Let t and n be integers satisfying $2 < 2t \leq n-2$, X be an $n \times n$ generic alternating matrix over $\mathbb{C}$, and $J = \mathrm{Pf}_{2t}(X)$.*

If n is even and $k \geq n-2$ or if n is odd and $k \geq n-3$, then

$$\operatorname{reg} J^k = tk + \begin{cases} t\left(\frac{t}{2} - 1\right) & \text{if } t \text{ is even} \\ \frac{1}{2}(t-1)^2 & \text{if } t \text{ is odd.} \end{cases}$$

As with Raicu's computation of regularity in the ordinary case (Proposition 5.1.5), despite the fact that this result was proven over $\mathbb{C}$, it holds over any field of characteristic zero.

For the rest of the section, we obtain degree bounds using the regularity above. As we are unaware of anyone else having proven degree bounds for Pfaffian ideals of alternating matrices like this before, we believe that the results proven in the rest of the section are novel.

Theorem 5.3.7 (Size $n-2$ Pfaffians) *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n be an even integer satisfying $4 \leq n$, A be an $n \times n$ alternating matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = \mathrm{Pf}_{n-2}(A)$ be of generic alternating height (that is, $\operatorname{ht} I = 6$). Further, suppose that $\operatorname{ht} \mathrm{Pf}_{2j}(A) \geq \min\left\{\binom{n-2j+2}{2}, d\right\}$ for all j in the range $2 \leq 2j \leq n-4$.*

a. If n is divisible by 4 and $k \geq n-2$, then

$$b_0(\mathcal{A}_k(I)) \leq (d-1)(\delta-1) + \frac{\delta\,(n-4)^2}{8}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d\,(\delta-1) + \frac{\delta\,(n-4)^2}{8}.$$

b. If n is not divisible by 4 and $k \geq n-2$, then

$$b_0(\mathcal{A}_k(I)) \leq (d-1)(\delta-1) + \frac{\delta\,(n-2)(n-6)}{8}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d\,(\delta-1) + \frac{\delta\,(n-2)(n-6)}{8}.$$

Proof We apply Corollary 4.2.8. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_d\right) - d\right\}.$$

By part (d) of Proposition 5.3.1, $b_0(\mathcal{A}(J)) = \operatorname{topdeg}(\mathcal{A}(J)) = -\infty$. Therefore, we have

$$b_0(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1, \text{ and}$$
$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \delta b_0\left(\mathbf{F^k}_{d}\right) - d.$$

Since $\mathbf{F^k}_{\bullet}$ is the minimal homogeneous free resolution of $J^k(k\,(n-2)\,/2)$, we have that $b_0\left(\mathbf{F^k}_{d-1}\right) \leq \operatorname{reg} J^k(k\,(n-2)\,/2) + d - 1$ and $b_0\left(\mathbf{F^k}_{d}\right) \leq \operatorname{reg} J^k(k\,(n-2)\,/2) + d$. Therefore,

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) \leq \delta \operatorname{reg} J^k(k\,(n-2)\,/2) + (d-1)\,\delta, \text{ and}$$
$$\delta b_0\left(\mathbf{F^k}_{d}\right) \leq \delta \operatorname{reg} J^k(k\,(n-2)\,/2) + d\delta.$$

Hence, we have

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1 \leq \delta \operatorname{reg} J^k(k\,(n-2)\,/2) + (d-1)\,(\delta-1)\,, \text{ and}$$
$$\delta b_0\left(\mathbf{F^k}_{d}\right) - d \leq \delta \operatorname{reg} J^k(k\,(n-2)\,/2) + d\,(\delta-1)\,.$$

Note that $\operatorname{reg} J^k(k\,(n-2)\,/2) = \operatorname{reg} J^k - k\,(n-2)\,/2$.

Since n is even, by Proposition 5.3.6, if $k \geq n-2$, then

$$\operatorname{reg} J^k = \frac{k\,(n-2)}{2} + \begin{cases} \frac{n-2}{2}\left(\frac{n-2}{4} - 1\right) & \text{if } \frac{n-2}{2} \text{ is even} \\ \frac{1}{2}\left(\frac{n-2}{2} - 1\right)^2 & \text{if } \frac{n-2}{2} \text{ is odd.} \end{cases}$$

One may check that $\frac{n-2}{2}$ is odd if and only if n is divisible by 4. Hence, in the case that n is divisible by 4, when $k \geq n-2$, we have $\operatorname{reg} J^k = \frac{k(n-2)}{2} + \frac{1}{2}\left(\frac{n-2}{2} - 1\right)^2 = \frac{k(n-2)}{2} + \frac{(n-4)^2}{8}$.

Thus, for $k \geq n-2$,

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1 \leq \delta\left(\frac{(n-4)^2}{8}\right) + (d-1)\,(\delta-1)\,, \text{ and}$$
$$\delta b_0\left(\mathbf{F^k}_{d}\right) - d \leq \delta\left(\frac{(n-4)^2}{8}\right) + d\,(\delta-1)\,.$$

Thus, we conclude that for $k \geq n-2$,

$$b_0(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1) + \frac{\delta\,(n-4)^2}{8}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d\,(\delta-1)+\frac{\delta\,(n-4)^2}{8}.$$

This proves (a).

Similarly, one may check that $\frac{n-2}{2}$ is even if and only if n is not divisible by 4 (since n is assumed to be even). Hence, in the case that n is not divisible by 4, when $k \geq n-2$, we have $\operatorname{reg} J^k = \frac{k(n-2)}{2} + \frac{n-2}{2}\left(\frac{n-2}{4}-1\right) = \frac{k(n-2)}{2} + \frac{(n-2)(n-6)}{8}$.

Thus, for $k \geq n-2$,

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1 \leq \delta\left(\frac{(n-2)\,(n-6)}{8}\right) + (d-1)\,(\delta-1)\,, \text{ and}$$

$$\delta b_0\big(\mathbf{F^k}_{d}\big) - d \leq \delta\left(\frac{(n-2)\,(n-6)}{8}\right) + d\,(\delta-1)\,.$$

Thus, we conclude that for $k \geq n-2$,

$$b_0(\mathcal{A}_k(I)) \leq (d-1)\,(\delta-1)+\frac{\delta\,(n-2)\,(n-6)}{8}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq d\,(\delta-1)+\frac{\delta\,(n-2)\,(n-6)}{8}.$$

This proves part (b). ■

Corollary 5.3.8 *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, A be a 6×6 alternating matrix whose entries are all homogeneous elements of R of degree 1, and $I = \operatorname{Pf}_4(A)$ be of generic alternating height (that is, $\operatorname{ht} I = 6$). Further, suppose that $\operatorname{ht}\operatorname{Pf}_2(A) \geq \min\{15, d\}$.*

Then I is of fiber type and $(x_1, \ldots, x_d)\,\mathcal{A}(I) = 0$.

Proof We apply Theorem 5.3.7. Since 6 is not divisible by 4, we see that $b_0(\mathcal{A}_k(I)) \leq 0$ and $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq 0$ for all $k \geq 4$.

We may then use the Macaulay2 computational software [26] to compute minimal homogeneous free resolutions of J^2 and J^3. These resolutions are linear. Hence, we also obtain $b_0(\mathcal{A}_k(I)) \leq 0$ and $\operatorname{topdeg}(\mathcal{A}_k(I)) \leq 0$ when $2 \leq k \leq 3$ and $\delta = 1$. Therefore, $b_0(\mathcal{A}(I)) \leq 0$ and $\operatorname{topdeg}(\mathcal{A}(I)) \leq 0$, giving that I is of fiber type and $\mathcal{A}(I)$ is annihilated by $(x_1, \ldots, x_d)$. ■

Theorem 5.3.9 (Size 4 Pfaffians) *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, n be an integer satisfying $4 \leq n$, A be an $n \times n$ alternating matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = \operatorname{Pf}_4(A)$ be of generic alternating height (that is, $\operatorname{ht} I = \binom{n-2}{2}$). Let X be an $n \times n$ generic alternating matrix over R and $J = \operatorname{Pf}_4(X)$. Further, suppose that $\operatorname{ht} \operatorname{Pf}_2(A) \geq \min\left\{\binom{n}{2}, d\right\}$.*

If n is even and $k \geq n-2$ or if n is odd and $k \geq n-3$, then

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J)), (d-1)(\delta-1)\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J)), d(\delta-1)\right\}.$$

Proof We apply Corollary 4.2.8. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \leq \max\Big\{\delta b_0(\mathcal{A}_k(J)), \delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1\Big\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\Big\{\delta \operatorname{topdeg}(\mathcal{A}_k(J)), \delta b_0\big(\mathbf{F^k}_d\big) - d\Big\}.$$

By part (e) of Proposition 5.3.1, $\operatorname{topdeg}(\mathcal{A}(J)) < \infty$. Therefore, we obtain meaningful bounds on $\operatorname{topdeg}(\mathcal{A}(I))$.

Since $\mathbf{F^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(2k)$, we have that $b_0\big(\mathbf{F^k}_{d-1}\big) \leq \operatorname{reg} J^k(2k) + d - 1$ and $b_0\big(\mathbf{F^k}_d\big) \leq \operatorname{reg} J^k(2k) + d$. Therefore,

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) \leq \delta \operatorname{reg} J^k(2k) + (d-1)\,\delta, \text{ and}$$

$$\delta b_0\big(\mathbf{F^k}_d\big) \leq \delta \operatorname{reg} J^k(2k) + d\delta.$$

Hence, we have

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1 \leq \delta \operatorname{reg} J^k(2k) + (d-1)(\delta-1), \text{ and}$$

$$\delta b_0\big(\mathbf{F^k}_d\big) - d \leq \delta \operatorname{reg} J^k(2k) + d(\delta-1).$$

Note that $\operatorname{reg} J^k(2k) = \operatorname{reg} J^k - 2k$.

Since 2 is even, by Proposition 5.3.6, if n is even and $k \geq n-2$ or if n is odd and $k \geq n-3$, then $\operatorname{reg} J^k = 2k$.

Thus, if n is even and $k \geq n-2$ or if n is odd and $k \geq n-3$,

$$\delta b_0\big(\mathbf{F^k}_{d-1}\big) - d + 1 \leq (d-1)\,(\delta - 1)\,, \text{ and}$$

$$\delta b_0\big(\mathbf{F^k}_{d}\big) - d \leq d\,(\delta - 1)\,.$$

Hence, if n is even and $k \geq n-2$ or if n is odd and $k \geq n-3$, then

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1)\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \leq \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, d\,(\delta-1)\right\}.$$

■

Although we are not aware $b_0(\mathcal{A}_k(J))$ in the above result, one may conjecture that $b_0(\mathcal{A}_k(J)) \leq 0$ for all k. This is due to the work of Huang, Perlman, Polini, Raicu, and Sammartano in [33]. They use techniques from representation theory to prove that the ideal of 2×2 minors of a generic ordinary matrix (of certain sizes) are of fiber type. Typically, representation theory techniques for ideals of minors of ordinary matrices transfer to the corresponding size of Pfaffian ideals of a generic alternating matrix. Therefore, one might suspect that in the near future, we may know whether or not $b_0(\mathcal{A}(J)) \leq 0$ for certain values of n in the above setting.

We end with the following general result.

Theorem 5.3.10 (Pfaffians of an Alternating Matrix) *Let K be a field of characteristic zero, $R = K[x_1, \ldots, x_d]$ be a polynomial ring in $d > 0$ variables over K, t and n be integers satisfying $4 < 2t < n-2$, A be an $n \times n$ alternating matrix whose entries are all homogeneous elements of R of the same degree δ, and $I = \operatorname{Pf}_{2t}(A)$ be of generic alternating height (that is, $\operatorname{ht} I = \binom{n-2t+2}{2}$). Let X be an $n \times n$ generic alternating matrix over R and $J = \operatorname{Pf}_{2t}(X)$. Further, suppose that $\operatorname{ht}\operatorname{Pf}_{2j}(A) \geq \min\left\{\binom{n-2j+2}{2}, d\right\}$ for all j in the range $2 \leq 2j \leq 2t-2$.*

If n is even and $k \geq n-2$ or if n is odd and $k \geq n-3$, then

$$b_0(\mathcal{A}_k(I)) \leq \max\left\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1) + \delta N(t)\right\},$$

where $N(t) = \begin{cases} t\left(\frac{t}{2}-1\right) & \textit{if } t \textit{ is even} \\ \frac{1}{2}(t-1)^2 & \textit{if } t \textit{ is odd.} \end{cases}$

Proof We apply Corollary 4.2.8. In particular, for each k, we have

$$b_0(\mathcal{A}_k(I)) \le \max\left\{\delta b_0(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1\right\}, \text{ and}$$

$$\operatorname{topdeg}(\mathcal{A}_k(I)) \le \max\left\{\delta \operatorname{topdeg}(\mathcal{A}_k(J))\,, \delta b_0\left(\mathbf{F^k}_{d}\right) - d\right\}.$$

By part (f) of Proposition 5.3.1, $\operatorname{topdeg}(\mathcal{A}(J)) = \infty$. Therefore, we do not obtain a meaningful bounds on $\operatorname{topdeg}(\mathcal{A}(I))$.

Since $\mathbf{F^k}_\bullet$ is the minimal homogeneous free resolution of $J^k(kt)$, we have that $b_0\left(\mathbf{F^k}_{d-1}\right) \le \operatorname{reg} J^k(kt) + d - 1$. Therefore,

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) \le \delta \operatorname{reg} J^k(kt) + (d-1)\,\delta.$$

Hence, we have

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1 \le \delta \operatorname{reg} J^k(kt) + (d-1)\,(\delta-1)\,.$$

Note that $\operatorname{reg} J^k(kt) = \operatorname{reg} J^k - kt$.

By Proposition 5.3.6, if n is even and $k \ge n-2$ or if n is odd and $k \ge n-3$, then

$$\operatorname{reg} J^k = tk + \begin{cases} t\left(\frac{t}{2}-1\right) & \text{if } t \text{ is even} \\ \frac{1}{2}(t-1)^2 & \text{if } t \text{ is odd.} \end{cases}$$

Let $N(t) = \begin{cases} t\left(\frac{t}{2}-1\right) & \text{if } t \text{ is even} \\ \frac{1}{2}(t-1)^2 & \text{if } t \text{ is odd.} \end{cases}$
Thus, if n is even and $k \ge n-2$ or if n is odd and $k \ge n-3$,

$$\delta b_0\left(\mathbf{F^k}_{d-1}\right) - d + 1 \le \delta N(t) + (d-1)\,(\delta-1)\,.$$

Hence, if n is even and $k \ge n-2$ or if n is odd and $k \ge n-3$, then

$$b_0(\mathcal{A}_k(I)) \le \max\left\{\delta b_0(\mathcal{A}_k(J))\,, (d-1)\,(\delta-1) + \delta N(t)\right\}.$$

∎

6. NONLINEARLY PRESENTED GRADE THREE GORENSTEIN IDEALS

Much like in the case of perfect ideals of grade two, there has been much work done to study the perfect Gorenstein ideals of grade three. The reason that both types of ideals have been studied so extensively rests largely upon the structure theorems for said ideals. This chapter concerns work done jointly with Monte Cooper to study the Rees rings $\mathcal{R}(I)$, and in particular, the defining equations of $\mathcal{R}(I)$ in the case that I is a perfect Gorenstein ideal of grade three.

6.1 Background

Definition 6.1.1 *Let R be a Noetherian local ring and I be a perfect R-ideal of grade g. Then I is said to be Gorenstein if* $\mathrm{Ext}_R^g(R/I,R) \cong R/I$.

An analogous definition can be given in the case that R is a nonnegatively graded Noetherian ring with R_0 local.

This definition is in analogue with properties of Gorenstein rings. A Cohen-Macaulay ring R is Gorenstein if and only if R has a canonical module ω_R with the property that $\omega_R \cong R$. Additionally, suppose $\varphi : R \to S$ is a local homomorphism of Cohen-Macaulay local rings and $g = \dim R - \dim S$. If R has a canonical module ω_R, then S has a canonical module $\omega_S \cong \mathrm{Ext}_R^g(S, \omega_R)$. Essentially, an ideal I is Gorenstein if R/I behaves like a canonical module for itself. Indeed, if we assume R is a local Gorenstein ring, then I is Gorenstein ideal if and only if R/I is a Gorenstein ring.

Proposition 6.1.2 (Serre) *Let R be a Noetherian local ring and I be a perfect Gorenstein ideal of grade at most two. Then I is a complete intersection.*

Proof Recall that an ideal is perfect if $\operatorname{pd} R/I = \operatorname{grade} I$.

If I is a perfect ideal of grade zero, then $\operatorname{pd} R/I = 0$, making R/I a free R-module. This is impossible unless $I = 0$. Hence, I is a complete intersection of grade zero.

If I is a perfect ideal of grade one, then $\operatorname{pd} R/I = 1$. Hence, there is an acyclic complex

$$0 \to R^n \to R^m \to 0$$

resolving R/I. Indeed, we may assume that the complex is a minimal free resolution of R/I. Since R/I is a cyclic module, $m = 1$. Further, because the above resolution is a finite free resolution with consisting of free modules of finite rank, it follows that $m - n = 0$. Therefore, $n = 1$ as well. Therefore,

$$0 \to R \to R \to 0$$

is a minimal free resolution of R/I. It follows then that

$$0 \to R \to 0$$

is a free resolution of I. Hence, I is a principal ideal. Since $\operatorname{grade} I = 1$, I can be generated by a regular sequence. Therefore, I is a complete intersection.

Finally, suppose I is a perfect Gorenstein ideal of grade two. Then $\operatorname{pd} R/I = 2$, and as before, we may assume we have a minimal free resolution of the cyclic module R/I. Therefore, there is an acyclic complex

$$0 \to R^n \to R^m \to R \to 0$$

resolving R/I. As before, $1 - m + n = 0$, so $m = n + 1$, giving that

$$0 \to R^n \to R^{n+1} \to R \to 0$$

is a free resolution of R/I. By the dualizing property of Gorenstein ideals, it follows that $n = 1$. Therefore,

$$0 \to R \to R^2 \to R \to 0$$

is a free resolution of R/I. Hence,

$$0 \to R \to R^2 \to 0$$

is a free resolution of I, giving that I can be generated by two elements. Since $\operatorname{grade} I = 2$, it follows that I is a complete intersection. ■

Serre also showed that perfect Gorenstein ideals of grade three are not complete intersections. Hence, the first interesting case occurs for perfect Gorenstein ideals of grade three. Buchsbaum and Eisenbud were able to develop the following structure theorem for these ideals.

Proposition 6.1.3 (Buchsbaum-Eisenbud [10, 2.1]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

a. *Let $n \geq 3$ be an odd integer, A be an $n \times n$ alternating matrix with entries in $\mathfrak{m}$, and $I = \operatorname{Pf}_{n-1}(A)$. Let A_i denote the alternating submatrix of A obtained by deleting row and column i. Define $f_i = (-1)^{i+1} \operatorname{Pf}(A_i)$, and let $\vec{v} = \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix}$. If $\operatorname{grade} I \geq 3$, then I is a perfect Gorenstein ideal of grade three, and*

$$0 \longrightarrow R \xrightarrow{\vec{v}^T} R^n \xrightarrow{A} R^n \xrightarrow{\vec{v}} R$$

is a minimal free resolution of R/I.

b. *Every perfect Gorenstein ideal of grade three is of the form described in part (a).*

The above structure theorem also yields a graded analogue for homogeneous perfect Gorenstein ideals of grade three.

Corollary 6.1.4 *Let R be a Noetherian nonnegatively graded ring with R_0 local.*

a. *Let $n \geq 3$ be an odd integer, A be an $n \times n$ alternating matrix with all entries homogeneous of the same degree δ, $D = \delta\,(n-1)/2$, and $I = \operatorname{Pf}_{n-1}(A)$. Let*

A_i denote the alternating submatrix of A obtained by deleting row and column i. Define $f_i = (-1)^{i+1}\operatorname{Pf}(A_i)$, and let $\vec{v} = \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix}$. If $\operatorname{grade} I \geq 3$, *then I is a homogeneous perfect Gorenstein ideal of grade three and*

$$0 \longrightarrow R(\delta+2D) \xrightarrow{\vec{v}^T} R(\delta+D)^n \xrightarrow{A} R(D)^n \xrightarrow{\vec{v}} R$$

is a minimal homogeneous free resolution of R/I.

b. *Every perfect homogeneous Gorenstein ideal of grade three is of the form described in part (a).*

A significant amount of work has been done to study the Rees algebras of perfect homogeneous Gorenstein ideals of grade three, particularly in the setting where $R = K[x_1, \ldots, x_d]$ is a polynomial ring over an infinite field K and I is of linear type on the punctured spectrum of R. The condition that I is of linear type on the punctured spectrum can be checked by a computational program, such as Macaulay2.

Proposition 6.1.5 *Let K be a field and $R = K[x_1, \ldots, x_d]$ be a polynomial ring. Let $n \geq 3$ be an odd integer, A an $n \times n$ alternating matrix such that every entry of A is a homogeneous element of R of the same degree $\delta \geq 1$, $I = \operatorname{Pf}_{n-1}(A)$, and* $\operatorname{ht} I = 3$. *Then I is of linear type on the punctured spectrum of R if and only if I satisfies the condition G_d. In other words, I is of linear type on the punctured spectrum if and only if* $\operatorname{ht} I_{n-j}(A) \geq j+1$ *for all j satisfying $1 \leq j \leq d-1$.*

Proof Due to the work of J. Watanabe in [61], we know that perfect Gorenstein ideals of grade three are in the linkage class of a complete intersection. Hence, such ideals are strongly Cohen-Macaulay, and therefore, satisfy a condition known as sliding depth [35, 1.4]. In the presence of the sliding depth condition, a homogeneous ideal is of linear type on the punctured spectrum if and only if the ideal satisfies the condition G_d, which means that $\mu(I_\mathfrak{p}) \leq \operatorname{ht}\mathfrak{p}$ for all prime ideals $\mathfrak{p} \in V(I)$ with $\operatorname{ht}\mathfrak{p} \leq d-1$ [29]. Since $\operatorname{ht} I \geq 1$, the condition G_d is equivalent to $\operatorname{ht}\operatorname{Fitt}_j(I) \geq j+1$ for all j satisfying $1 \leq j \leq d-1$. Recall that if φ is an $n \times n$ presentation matrix of I, then $\operatorname{Fitt}_j(I) = I_{n-j}(\varphi)$. By Proposition 6.1.3, A is a presentation matrix for I; therefore, G_d is equivalent to $\operatorname{ht} I_{n-j}(A) \geq j+1$ for all j satisfying $1 \leq j \leq d-1$. ∎

Because of the above proposition, instead of referring to I being of linear type on the punctured spectrum, we will instead say that I satisfies G_d.

As was mentioned earlier, some work has been done to study the defining equations of $\mathcal{R}(I)$ when I satisfies G_d. In [54], Morey studied the defining equations in the case that I has second analytic deviation one and the presentation matrix A of I has linear entries. In [39], Johnson studied the case where I has second analytic deviation one and $\mathcal{R}(I)$ is Cohen-Macaulay. In particular, Johnson determined the defining equations of $\mathcal{R}(I)$. More recently, in [48], Kustin, Polini, and Ulrich determined the defining equations of $\mathcal{R}(I)$ when the presentation matrix A of I consists of linear entries, or more generally, when $I_1(A)$ is a complete intersection.

A next natural step is to study the defining equations of $\mathcal{R}(I)$ when I has second analytic deviation one, the presentation matrix of I does not consist of linear entries, and $\mathcal{R}(I)$ is not Cohen-Macaulay. To make this more precise, we define second analytic deviation.

Definition 6.1.6 *Let R be a Noetherian nonnegatively graded ring with R_0 local and I be a homogeneous R-ideal. The second analytic deviation of I is $\mu(I) - \ell(I)$.*

The terminology comes from two other "deviations" of I: the deviation of I is $\mu(I) - \operatorname{grade} I$, and the analytic deviation of I is $\ell(I) - \operatorname{grade} I$. Thus, we see that the second analytic deviation of I is a form of "deviation" between the deviation and the analytic deviation.

The second analytic deviation has a very natural connection to studying the defining equations of the special fiber ring.

Remark 6.1.7 *Let $R = K[x_1, \ldots, x_d]$ be a polynomial ring over a field K. Let $I = (f_1, \ldots, f_n)$ be an R-ideal where $f_1, \ldots, f_n$ is a minimal homogeneous generating set of I and $\deg f_i = D$ for all i satisfying $1 \leq i \leq n$. Denote by $I(X)$ the defining ideal of the special fiber ring $\mathcal{F}(I)$.*

a. The second analytic deviation of I is equal to $\operatorname{ht} I(X)$.

b. If the second analytic deviation of I is zero, then $I(X) = 0$ and $\mathcal{F}(I)$ is a polynomial ring of dimension $\mu(I)$.

c. If the second analytic deviation of I is one, then $I(X)$ is principal. Moreover, the degree of the generator of $I(X)$ is $e(\mathcal{F}(I))$.

Proof Consider the natural R-algebra epimorphism $f\colon \mathcal{T} \to \mathcal{F}(I)$. The defining ideal $I(X) = \ker f$. By the definition of $\mathcal{T}$ as $K[T_1, \ldots, T_n]$, $\dim \mathcal{T} = n = \mu(I)$. Hence, by the dimension formula, $\ell(I) = \dim \mathcal{F}(I) = \dim \mathcal{T} - \operatorname{ht} I(X) = \mu(I) - \operatorname{ht} I(X)$. Therefore, $\mu(I) - \ell(I) = \operatorname{ht} I(X)$, proving part (a).

Next, since $\mathcal{F}(I)$ is a domain, it follows that $I(X)$ is a prime ideal of $\mathcal{T}$. Therefore, if $\operatorname{ht} I(X) = 0$, then $I(X) = 0$ and $\mathcal{F}(I) \cong \mathcal{T}$. Moreover, since $\mathcal{T}$ is a unique factorization domain, if $\operatorname{ht} I(X) = 1$, then $I(X)$ is principal.

Lastly, the multiplicity of the quotient of a standard graded polynomial ring by a principal ideal is the degree of the generator of the ideal. ■

Remark 6.1.8 *Let K be an infinite field, $R = K[x_1, \ldots, x_d]$ a polynomial ring over K, $n \geq 3$ an odd integer, A an $n \times n$ alternating matrix whose entries are homogeneous of the same degree δ. Suppose $I = \operatorname{Pf}_{n-1}(A)$ with $\operatorname{grade} I = 3$ and I satisfies G_d but is not of linear type. Then I has second analytic deviation one if and only if $n = \mu(I) = d + 1$. Consequently, d must be even.*

Proof Recall from Remark 2.2.19 that $\ell(I) \leq \min\{n, d\}$. If we can show that $\ell(I)$ is maximal, i.e., that $\ell(I) = \min\{n, d\}$, then the result follows quickly from there. Indeed, it must be the case that $n > d$. If $n \leq d$, then I is of linear type. Though this has been known for some time, this is also follows from part (a) of Corollary 5.3.5. Therefore, $\ell(I) = d$. Then, according to the definition of second analytic deviation, I has second analytic deviation one if and only if $n = d + 1$. Therefore, it suffices to show that $\ell(I) = d$.

Since K is infinite and since I is generated by homogeneous elements of the same degree, I has a homogeneous reduction J with $\mu(J) = \ell(I)$ (see [57, Theorem 1 on pg. 150]). To say that J is a reduction of I means that there exists a positive integer r so that $JI^r = I^{r+1}$. Since I is of linear type on the punctured spectrum by Proposition 6.1.5, it follows that $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for all homogeneous prime ideals $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$. Hence, $J_{\mathfrak{p}} : I_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all homogeneous prime ideals with $\mathfrak{p} \neq \mathfrak{m}$. Therefore, $\operatorname{ht} J : I \geq d$. However, since I satisfies G_d, $\operatorname{ht} J : I \leq \mu(J)$ (see [37, 3.1]). Hence, it follows that $d \leq \mu(J) = \ell(I)$, which implies that $\ell(I) = d$. ■

For our proofs later, we will use the following fact.

Remark 6.1.9 *Let d be an even integer with $d \geq 4$, K an infinite field, $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring, $\mathfrak{m} = (x_1, \ldots, x_d)$, A a $(d+1) \times (d+1)$ alternating matrix such that every entry of A is a homogeneous element of R of the same degree $\delta \geq 1$, $I = \operatorname{Pf}_d(A)$, and $\operatorname{ht} I = 3$. Suppose that I satisfies G_d. Then $I_1(A)$ is $\mathfrak{m}$-primary.*

Proof Since I satisfies G_d, by Proposition 6.1.5, $\operatorname{ht} I_{d+1-j}(A) \geq j+1$ for all j satisfying $1 \leq j \leq d-1$. Choosing $j = d-1$, we have $\operatorname{ht} I_2(A) \geq d$. Moreover, $I_2(A) \subseteq I_1(A)$. Therefore, $\operatorname{ht} I_1(A) \geq d$. However, the generators of $I_1(A)$ are homogeneous of positive degree. Thus, $I_1(A) \subseteq \mathfrak{m}$. Hence, it follows that $\operatorname{ht} I_1(A) = d$. Since $I_1(A)$ is a homogeneous ideal, its minimal primes are homogeneous. Thus, the only prime ideal containing $I_1(A)$ is $\mathfrak{m}$. Therefore, $\sqrt{I_1(A)} = \mathfrak{m}$, giving that $I_1(A)$ is $\mathfrak{m}$-primary. ■

6.2 A Conjecture on the Relation Type

Definition 6.2.1 *Let R be a Noetherian ring and I an R-ideal. The relation type of I is given by*

$$\operatorname{rt}(I) = \max\left\{ k \;\middle|\; \mathcal{J} = \mathcal{S}\left(\bigoplus_k \mathcal{J}_k\right)\right\}.$$

In other words, $\mathrm{rt}(I)$ is the maximal degree of a defining equation of $\mathcal{R}(I)$. In the case R is graded and $\mathcal{R}(I)$ is bigraded, the relation type only concerns the second component of the bidegree (the T-degree). This differs from the previous chapters in this dissertation, which were primarily concerned with bounding the first component (the x-degree) of the bidegrees of the defining equations.

There has been considerably less work done in the way of bounding the relation type of an ideal than there has been in bounding the x-degrees. Indeed, the techniques used in the previous chapters of this dissertation can only bound the relation type of I if one can prove that, for sufficiently large k, $\mathrm{topdeg}(\mathcal{A}_k(I)) = -\infty$ or $b_0(\mathcal{A}_k(I)) = -\infty$. Reviewing Chapter 5, one sees that obtaining such a result is a rare occurrence.

After extensive computations using the Macaulay2 computational software [26], we have developed a conjecture for the relation type of perfect homogeneous Gorenstein ideals of grade three having second analytic deviation one in which the Rees ring is not Cohen-Macaulay. Before we can state the conjecture, we need to give some definitions.

Definition 6.2.2 *Let K be a field and $R = K[x_1, \ldots, x_d]$ be a standard graded polynomial ring. Suppose I is an R-ideal. Then* $\mathrm{Mon}(I)$ *denotes the smallest monomial ideal containing I.*

Given such an ideal $I = (f_1, \ldots, f_n)$, computing $\mathrm{Mon}(I)$ is actually quite simple. Indeed, $\mathrm{Mon}(I) = (\mathrm{supp}(f_1), \ldots, \mathrm{supp}(f_n))$.

Example 6.2.3 *Let $R = \mathbb{Q}[x, y, z]$ be a standard graded polynomial ring. Suppose*

$$I = \left(x^3 - 2xy^2 + 3x, z^4 - 2yz, xyz + y^3\right).$$

Then

$$\mathrm{Mon}(I) = \left(x, z^4, yz, y^3\right).$$

To see why, we break each generator of I into its terms: x^3, xy^2, x, z^4, yz, xyz, and y^3. Then discard superfluous generators.

Definition 6.2.4 *Let K be a field and $R = K[x_1, \ldots, x_d]$ be a standard graded polynomial ring. Given a monomial $m = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$, define*

$$\text{deg-gcd}(m) = \gcd(\alpha_1, \ldots, \alpha_d)\,.$$

We are now ready to state the conjecture.

Conjecture 6.2.5 *Let d be an even integer with $d \geq 4$, K an infinite field, $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring, A a $(d+1) \times (d+1)$ alternating matrix such that every entry of A is a homogeneous element of R of the same degree $\delta \geq 1$, $I = \mathrm{Pf}_d(A)$, and $\operatorname{ht} I = 3$. Suppose I satisfies G_d and $\mathcal{R}(I)$ is not Cohen-Macaulay. Let $\mathcal{M}$ be the minimal monomial generating set of $\mathrm{Mon}(I_1(A))$. Then*

$$\mathrm{rt}(I) = (d-1) \prod_{m \in \mathcal{M}} \frac{\delta}{\text{deg-gcd}(m)}.$$

We will now provide some information to show that the conjecture is potentially plausible.

First, we note that the conjectured value of $\mathrm{rt}(I)$ is an integer.

Remark 6.2.6 *Adopt the setting of Conjecture 6.2.5. Then $\text{deg-gcd}(m)$ divides δ for all $m \in \mathcal{M}$.*

Proof Let $m = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$. Since all entries of A are homogeneous of degree δ, it follows that m has degree δ. Therefore, $\alpha_1 + \ldots + \alpha_d = \delta$. Since $\text{deg-gcd}(m) = \gcd(\alpha_1, \ldots, \alpha_d)$ divides α_i for each i satisfying $1 \leq i \leq d$, it follows that $\text{deg-gcd}(m)$ divides the sum $\alpha_1 + \ldots + \alpha_d = \delta$. ■

Next, we note that when $\delta = 1$, the conjectured value for $\mathrm{rt}(I)$ is $d-1$. Recall that Kustin, Polini, and Ulrich have determined the defining equations of $\mathcal{R}(I)$ in such a setting when $\delta = 1$. Using their classification in [48, 8.3], we know that the unique generator of $I(X)$ must have degree $d-1$ when $\delta = 1$. Therefore, in the $\delta = 1$ case, the conjectured value of $\mathrm{rt}(I)$ is correct.

Lastly, we will go through two examples, showing how the conjectured value of $\mathrm{rt}(I)$ is correct for those examples.

Example 6.2.7 *Let* $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$,

$$A = \begin{pmatrix} 0 & x_3^6 & x_1^3x_3^3 & x_2^2x_3^2x_4^2 & x_2^6 \\ -x_3^6 & 0 & x_4^6 & 0 & x_1^6 \\ -x_1^3x_3^3 & -x_4^6 & 0 & x_1^6 & 0 \\ -x_2^2x_3^2x_4^2 & 0 & -x_1^6 & 0 & x_3^6 \\ -x_2^6 & -x_1^6 & 0 & -x_3^6 & 0 \end{pmatrix},$$

and $I = \mathrm{Pf}_4(A)$. *The ideal* I *is a perfect homogeneous Gorenstein ideal of grade three having second analytic deviation one which satisfies* G_d, *and* $\mathcal{R}(I)$ *is not Cohen-Macaulay. Moreover,* $\mathrm{rt}(I) = 18$.

The above conditions were verified and the relation type was computed using the Macaulay2 computational software [26].

We note here that $d = 4$, $\delta = 6$, and $\mathcal{M} = \{x_1^6, x_2^6, x_3^6, x_4^6, x_1^3x_3^3, x_2^2x_3^2x_4^2\}$. Therefore, the conjectured value for $\mathrm{rt}(I)$ is

$$(4-1) \cdot \frac{6}{6} \cdot \frac{6}{6} \cdot \frac{6}{6} \cdot \frac{6}{6} \cdot \frac{6}{3} \cdot \frac{6}{2} = 18.$$

Example 6.2.8 *Let* $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$,

$$A = \begin{pmatrix} 0 & 0 & x_1^6 & x_4^6 & x_3^6 \\ 0 & 0 & 0 & x_2^6 & x_1^6 \\ -x_1^6 & 0 & 0 & 0 & x_4^6 \\ -x_4^6 & -x_2^6 & 0 & 0 & x_1^2x_2^4 \\ -x_3^6 & -x_1^6 & -x_4^6 & -x_1^2x_2^4 & 0 \end{pmatrix},$$

and $I = \mathrm{Pf}_4(A)$. *The ideal* I *is a perfect homogeneous Gorenstein ideal of grade three having second analytic deviation one which satisfies* G_d, *and* $\mathcal{R}(I)$ *is not Cohen-Macaulay. Moreover,* $\mathrm{rt}(I) = 9$.

The above conditions were verified and the relation type was computed using the Macaulay2 computational software [26].

We note here that $d = 4$, $\delta = 6$, and $\mathcal{M} = \{x_1^6, x_2^6, x_3^6, x_4^6, x_1^2x_2^4\}$. Therefore, the conjectured value for $\mathrm{rt}(I)$ is

$$(4-1) \cdot \frac{6}{6} \cdot \frac{6}{6} \cdot \frac{6}{6} \cdot \frac{6}{6} \cdot \frac{6}{2} = 9.$$

We have made numerous computations using Macaulay2 satisfying the relevant conditions, including scenarios with $d = 4$, $d = 6$, and $d = 8$, with δ ranging between 2 and 12, and even matrices where $I_1(A)$ is not a monomial ideal. In all such computations, the conjectured value for $\mathrm{rt}(I)$ was correct. For $d = 8$, computation times became prohibitive. Thus, we were only able to compute a few examples for $d = 8$ and no examples for $d = 10$.

We found some examples in which unusual behavior was observed. These examples will be mentioned in Section 6.4. As mentioned above, the conjectured value for $\mathrm{rt}(I)$ holds for these examples as well.

6.3 Row Ideals and Morphism Fiber Ideals

Unless otherwise stated, throughout this section, we will adopt the following setting.

Setting 6.3.1 *Let K be an infinite field and $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring. Suppose $I = (f_1, \ldots, f_n)$ is a homogeneous R-ideal with* $\deg f_i = D$ *for all i and that $f_1, \ldots, f_n$ is a minimal generating set of I. Let A be a minimal presentation matrix of I so that $[f_1, \ldots, f_n]\, A = 0$ and all entries of A are homogeneous of the same degree δ. Let Φ be the rational map*

$$\Phi : \mathbb{P}_K^{d-1} \dashrightarrow \mathbb{P}_K^{n-1}$$

with base locus $V(I)$ defined by

$$[a_1 : \cdots : a_d] \mapsto [f_1(a_1, \ldots, a_d) : \cdots : f_n(a_1, \ldots, a_d)]\,.$$

In order to investigate Conjecture 6.2.5, we introduce row ideals.

Definition 6.3.2 *Adopt Setting 6.3.1. Let* $P \in \mathbb{P}_K^{n-1}$ *with projective coordinates* $[a_1 : \cdots : a_n]$. *The row ideal of* A *at* P *is* $I_1([a_1, \ldots, a_n]\, A)$.

The vector $[a_1, \ldots, a_n]\, A$ is a generalized row of A. If P is a general point in $\mathbb{P}_K^{n-1}$, then the vector $[a_1, \ldots, a_n]\, A$, in some sense, captures the essence of "rows" in A since it is a general linear combination of the rows of A.

Definition 6.3.3 *Adopt Setting 6.3.1. Let* $P \in \mathbb{P}_K^{d-1} \setminus V(I)$. *The fiber row ideal of* A *at* P*, denoted* $\mathrm{FR}_P(A)$*, is* $I_1([f_1(P), \ldots, f_n(P)]\, A)$.

This name is motivated by the rational map Φ defined by the generating set of I. Given a point $P \in \mathbb{P}_K^{d-1} \setminus V(I)$, the fiber row ideal $\mathrm{FR}_P(A)$ is the ideal defining the fiber over $\Phi(P)$.

Definition 6.3.4 *Adopt Setting 6.3.1. Let* $P \in \mathbb{P}_K^{d-1} \setminus V(I)$. *The morphism fiber ideal of* A *at* P *is the saturation of* $\mathrm{FR}_P(A)$ *with respect to* I. *We denote the morphism fiber ideal of* A *at* P *as* $\mathrm{MF}_P(A)$. *Symbolically,* $\mathrm{MF}_P(A) = \mathrm{FR}_P(A) : I^\infty$.

Let X denote the closed image of the rational map Φ. Morphism fiber ideals have a very strong connection with X, as can be demonstrated by the following result of Kustin, Polini, and Ulrich.

Proposition 6.3.5 (Kustin-Polini-Ulrich [46, 3.7]) *Adopt Setting 6.3.1. If* P *is a general point in* $\mathbb{P}_K^{d-1}$*, then*

$$\deg X = e(\mathcal{F}(I)) = \frac{1}{e(R/\,\mathrm{MF}_P(A))} \cdot e\left(\frac{R}{(g_1, \ldots, g_{n-1} : I^\infty)}\right)$$

where $g_1, \ldots, g_{n-1}$ *are general linear combinations of* $f_1, \ldots, f_n$.

Recall that if I has second analytic deviation one, then the degree of the unique generator of $I(X)$ is $e(\mathcal{F}(I))$. Hence, to attempt to prove Conjecture 6.2.5, we would wish to connect $\mathrm{MF}_P(A)$ to $I_1(A)$ for a general point P. Since $\mathrm{FR}_P(A)$ is more likely to have a connection to $I_1(A)$ (as $\mathrm{FR}_P(A)$ is the ideal of entries of a linear combination of the rows of A), we may get closer to proving the conjecture (or understanding why it is false) if we can prove or disprove that $e(R/\operatorname{MF}_P(A)) = e(R/\operatorname{FR}_P(A))$ for a general point P. However, we note that one would need several more steps to prove Conjecture 6.2.5. If this approach could work, it would merely give us that the unique defining equation of $\mathcal{F}(I)$ has the conjectured degree, not that $\operatorname{rt}(I)$ has the conjectured value. Therefore, even if we could prove a connection between $e(\mathcal{F}(I))$ and the conjectured degree, one would still need to prove that the defining equation of $\mathcal{F}(I)$ has the highest T-degree of the defining equations of $\mathcal{R}(I)$, which is the case when the ideal I is of fiber type, for instance.

We are able to prove that $e(R/\operatorname{MF}_P(A)) = e(R/\operatorname{FR}_P(A))$ in the case that $d = \dim R = 4$. However, we need a lemma first.

Lemma 6.3.6 *Adopt Setting 6.3.1. Assume that $I_1(A)$ is an $\mathfrak{m} = (x_1, \ldots, x_d)$-primary ideal. Consider the ring $U = K(Y_1, \ldots, Y_d)[x_1, \ldots, x_d]$, and let $F_1, \ldots, F_m$ be $f_1(Y_1, \ldots, Y_d), \ldots, f_m(Y_1, \ldots, Y_d)$, respectively. Let $\mathfrak{p}$ be a prime ideal in R generated by all but one of the variables, and let $\mathfrak{q} = \mathfrak{p}U$. Then $I_1([F_1, \ldots, F_m]\, A)_{\mathfrak{q}} = U_{\mathfrak{q}}$.*

Proof Suppose that $\mathfrak{p}$ is generated by all of the variables except for x_i. Since $I_1(A)$ is $\mathfrak{m}$-primary, it follows that $A_{\mathfrak{q}}$ has at least one unit entry. Since A is a minimal presentation matrix of I, it follows that such an entry is of the form $cx_i^\delta + g$ for some $g \in \mathfrak{p}, c \in K$. Suppose that the column of $A_{\mathfrak{q}}$ containing this entry is

$$\begin{pmatrix} c_1 x_i^\delta + g_1 \\ \vdots \\ c_n x_i^\delta + g_m \end{pmatrix}$$

where each $c_i \in K$ and each $g_i \in \mathfrak{p}$. Define

$$u = \begin{bmatrix} F_1 & \ldots & F_n \end{bmatrix} \begin{pmatrix} c_1 x_i^\delta + g_1 \\ \vdots \\ c_n x_i^\delta + g_m \end{pmatrix} = x_i^\delta \sum_{j=1}^m c_j F_j + \sum_{j=1}^m F_j g_j.$$

Note that at least one of the c_j is nonzero, otherwise the column above could not contain a unit in $U_{\mathfrak{q}}$. We claim that $x_i^{\delta}\sum_{j=1}^{m} c_jF_j$ is not in $\mathfrak{q}$. Indeed, as $\mathfrak{q}$ is a prime ideal of U and $x_i \notin \mathfrak{q}$, the only way $x_i^{\delta}\sum_{j=1}^{m} c_jF_j$ could be in $\mathfrak{q}$ is if $\sum_{j=1}^{m} c_jF_j \in \mathfrak{q}$. Furthermore, since $\mathfrak{q} = \mathfrak{p}S$ and $\mathfrak{p}$ is generated by all variables of R except for x_i, the only way $\sum_{j=1}^{m} c_jF_j$ could be in $\mathfrak{q}$ is if $\sum_{j=1}^{m} c_jF_j = 0$. Since the f_j's are linearly independent over K (being a minimal generating set of I), the F_j's must also be linearly independent over K. Hence, it is impossible for $\sum_{j=1}^{m} c_jF_j = 0$, since at least one of the c_j is nonzero. On the other hand, the sum $\sum_{j=1}^{m} F_jg_j$ must be in $\mathfrak{q}$, as each $g_j \in \mathfrak{p}$.

Hence, u is a unit in $S_{\mathfrak{q}}$, so we conclude that $I_1([F_1, \ldots, F_m]\,A)_{\mathfrak{q}} = U_q$. ■

Theorem 6.3.7 *Let K be an infinite field, $R = K[x_1, \ldots, x_d]$ a standard graded polynomial ring, $\mathfrak{m} = (x_1, \ldots, x_d)$, I a homogeneous ideal with $\operatorname{ht} I = d-1$, A an $m \times n$ minimal presentation matrix of I in which all entries of A are homogeneous of the same degree $\delta > 0$. Let $f_1, \ldots, f_m$ be the homogeneous minimal generating set of I so that $[f_1, \ldots, f_m]\,A = 0$. Suppose that I is generically a complete intersection which is not a complete intersection. Let P be a general point in $\mathbb{P}_K^{d-1}$. Then $\operatorname{MF}_P(A) = \operatorname{FR}_P(A) : \mathfrak{m}^{\infty}$. In particular,*

$$e\left(\frac{R}{\operatorname{FR}_P(A)}\right) = e\left(\frac{R}{\operatorname{MF}_P(A)}\right) \text{ and } \operatorname{ht}\operatorname{FR}_P(A) = \operatorname{ht}\operatorname{MF}_P(A) = d-1.$$

Proof Without loss of generality, we may assume that K is algebraically closed. Indeed, passing to the algebraic closure of K will not change the generators or the heights of any of the relevant ideals.

To simulate all possible fiber row ideals, we pass to a ring extension and consider a generic fiber row ideal of A. Let $S = K[Y_1, \ldots, Y_d, x_1, \ldots, x_d]$ and $[F_1, \ldots, F_m] = [f_1(Y_1, \ldots, Y_d), \ldots, f_m(Y_1, \ldots, Y_d)]$. Let $[a_1 : \cdots : a_d]$ be the homogeneous coordinates of P. Specializing the variables $Y_1, \ldots, Y_d$ to $a_1, \ldots, a_d$, respectively, will map $I_1([F_1, \ldots, F_m]\,A)$ to $\operatorname{FR}_P(A)$. Define $T = K[Y_1, \ldots, Y_d]$. If we can show that $\operatorname{ht}(I_1([F_1, \ldots, F_m]\,A) + IS) \geq d$, then we will also have

$$\operatorname{ht}((I_1([F_1, \ldots, F_m]\,A) + IS) \otimes_T \operatorname{Quot}(T)) \geq d,$$

since height cannot decrease after localization. Hence, by the semicontinuity of fiber dimension (see [21, 14.8]), we will have $\operatorname{ht}(\mathrm{FR}_P(A)+I) \geq d$.

Indeed, let $\mathfrak{q}$ be a homogeneous prime ideal of S containing IS with $\operatorname{ht}\mathfrak{q} = d-1$. It suffices to show that $I_1([F_1,\ldots,F_m]\,A)_{\mathfrak{q}} = S_{\mathfrak{q}}$.

Contracting $\mathfrak{q}$ to R, we obtain a prime ideal $\mathfrak{p}$ containing I. Since S is a faithfully flat extension of R, it follows that $\operatorname{ht}\mathfrak{p} \leq \operatorname{ht}\mathfrak{q}$. As $\operatorname{ht}\mathfrak{q} = d-1$, $\operatorname{ht}\mathfrak{p} \leq d-1$. However, since $I \subseteq \mathfrak{p}$ and $\operatorname{ht} I = d-1$, it must be the case that $\operatorname{ht}\mathfrak{p} = d-1$. Again by faithful flatness, $\operatorname{ht}\mathfrak{p}S = d-1$. However, we have $\mathfrak{p}S \subseteq \mathfrak{q}$ (since $\mathfrak{p}S$ is the extension of the contraction of $\mathfrak{q}$) and $\operatorname{ht}\mathfrak{p}S = \operatorname{ht}\mathfrak{q}$. Consequently, $\mathfrak{p}S = \mathfrak{q}$. Therefore, $\mathfrak{q}$ is extended from a prime ideal of R of height $d-1$.

Since K is algebraically closed and $\dim R/\mathfrak{p} = 1$, $\mathfrak{p}$ is generated by linear forms; hence, by a linear change of coordinates, we may assume $\mathfrak{p}$ is generated by all but one of the indeterminates $x_1,\ldots,x_d$. Consider the ring $U = K(Y_1,\ldots,Y_d)[x_1,\ldots,x_d]$. Since $\mathfrak{q}$ is extended from R, $S_{\mathfrak{q}} = U_{\mathfrak{p}U}$. Hence, by Lemma 6.3.6, $I_1([F_1,\ldots,F_m]\,A)_{\mathfrak{q}} = S_{\mathfrak{q}}$. Therefore, $\operatorname{ht}(\mathrm{FR}_P(A)+I) \geq d$.

Recall that I and $\mathrm{FR}_P(A)$ are homogeneous ideals. Let $\mathfrak{p}$ be a homogeneous prime ideal of R containing both I and $\mathrm{FR}_P(A)$. Then $\mathfrak{p}$ contains $\mathrm{FR}_P(A)+I$. Since $\operatorname{ht}(\mathrm{FR}_P(A)+I) \geq d$, it follows that $\operatorname{ht}\mathfrak{p} \geq d$. Since $\dim R = d$, and $\mathfrak{p}$ is a homogeneous prime ideal of height d, it follows that $\mathfrak{p} = \mathfrak{m}$. Since $\mathrm{MF}_P(A) = \mathrm{FR}_P(A) : I^\infty$, but $\mathrm{FR}_P(A)$ and I are homogeneous ideals which share no homogeneous prime ideals except $\mathfrak{m}$, it follows that $\mathrm{MF}_P(A) = \mathrm{FR}_P(A) : \mathfrak{m}^\infty$. Therefore, $\mathrm{MF}_P(A)$ and $\mathrm{FR}_P(A)$ have the same minimal primes. Hence, by the Associativity Formula for multiplicities,

$$e\left(\frac{R}{\mathrm{FR}_P(A)}\right) = e\left(\frac{R}{\mathrm{MF}_P(A)}\right).$$

Also, since they have the same minimal primes, $\operatorname{ht}\mathrm{FR}_P(A) = \operatorname{ht}\mathrm{MF}_P(A)$. The fact that $\operatorname{ht}\mathrm{MF}_P(A) = d-1$ is a consequence of the following proposition, Proposition 6.3.8, since $\ell(I) = d$ as I is generically a complete intersection which is not a complete intersection. ∎

Proposition 6.3.8 (Eisenbud-Ulrich [23, 3.1]) *If P is a general point in $\mathbb{P}_K^{d-1}$, then* $\operatorname{ht}\operatorname{MF}_P(A) = \dim\overline{\operatorname{Im}\Phi} = \ell(I) - 1$.

The results in Theorem 6.3.7 may, generally, be of interest for the study of rational maps. If the base locus $V(I)$ only consists of a finite number of points, then the defining ideals of the generic fibers are simply the fiber row ideals $\operatorname{FR}_P(A)$ because the subschemes they define already do not meet the base locus.

We may apply the results in Theorem 6.3.7 to the setting of Conjecture 6.2.5 for $d = 4$. This then implies that for a general point $P \in \mathbb{P}_K^3$, $\operatorname{MF}_P(A) = \operatorname{FR}_P(A) : \mathfrak{m}^\infty$, $e\Big(\frac{R}{\operatorname{FR}_P(A)}\Big) = e\Big(\frac{R}{\operatorname{MF}_P(A)}\Big)$, and $\operatorname{ht}\operatorname{FR}_P(A) = \operatorname{ht}\operatorname{MF}_P(A) = 3$.

In order to make progress on Conjecture 6.2.5, one would hope to be able to extend the results of Theorem 6.3.7 to even dimensions $d \geq 6$, if possible. Additionally, one would like to make a connection between

$$e\left(\frac{R}{(g_1, \ldots, g_{n-1} : I^\infty)}\right)$$

in Proposition 6.3.5 and

$$e\left(\frac{R}{\operatorname{FR}_P(A)}\right)$$

with the conjectured value of $\operatorname{rt}(I)$ in Conjecture 6.2.5.

6.4 Interesting Examples

Typically, when studying the bidegrees of the defining equations of $\mathcal{R}(I)$, there is a pattern. If (a, h) and (b, k) are bidegrees of defining equations so that $1 < h < k$ and $a = \max\{j \mid \mathcal{R}(I) \text{ has a defining equation of bidegree } (j, h)\}$, then $b \leq a$. In other words, as the T-degrees of the defining equations increase, the maximum x-degree of the defining equations tends to weakly decrease.

In this section, we give two interesting examples satisfying the conditions of Conjecture 6.2.5 in which the property described above does not hold.

Example 6.4.1 *Let* $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$,

$$A = \begin{pmatrix} 0 & x_4^8 & x_2^8 & x_1^2x_2^2x_3^2x_4^2 & x_1^8 \\ -x_4^8 & 0 & 0 & x_3^8 & x_2^8 \\ -x_2^8 & 0 & 0 & x_1^8 & 0 \\ -x_1^2x_2^2x_3^2x_4^2 & -x_3^8 & -x_1^8 & 0 & x_3^8 \\ -x_1^8 & -x_2^8 & 0 & -x_3^8 & 0 \end{pmatrix},$$

and $I = \mathrm{Pf}_4(A)$. *The ideal* I *is a perfect homogeneous Gorenstein ideal of grade three having second analytic deviation one which is of linear type on the punctured spectrum of* R, *and* $\mathcal{R}(I)$ *is not Cohen-Macaulay. The bidegrees of the defining equations of* $\mathcal{R}(I)$ *are* $(8,1)$, $(6,3)$, $(8,6)$, $(6,9)$, *and* $(0,12)$.

The above conditions were verified and the bidegrees were computed using the Macaulay2 computational software [26].

Of particular note in this example is that the x-degree in T-degree 6 is larger than the maximum x-degree in T-degree 3. However, we do note that x-degrees are all bounded above by δ in this example.

Example 6.4.2 *Let* $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$,

$$A = \begin{pmatrix} 0 & x_3^6 & 0 & x_4^6 & 3x_1^6 + 4x_3^6 \\ -x_3^6 & 0 & x_4^6 & x_2^6 & 0 \\ 0 & -x_4^6 & 0 & x_1^6 & 0 \\ -x_4^6 & -x_2^6 & -x_1^6 & 0 & x_3^6 - x_1x_2^2x_3x_4^2 \\ -3x_1^6 - 4x_3^6 & 0 & 0 & -x_3^6 + x_1x_2^2x_3x_4^2 & 0 \end{pmatrix},$$

and $I = \mathrm{Pf}_4(A)$. *The ideal* I *is a perfect homogeneous Gorenstein ideal of grade three having second analytic deviation one which is of linear type on the punctured spectrum of* R, *and* $\mathcal{R}(I)$ *is not Cohen-Macaulay. The bidegrees of the defining equations of* $\mathcal{R}(I)$ *are* $(6,1)$, $(4,3)$, $(5,3)$, $(4,6)$, $(6,6)$, $(8,6)$, $(3,9)$, $(4,12)$, $(4,15)$, *and* $(0,18)$.

The above conditions were verified and the bidegrees were computed using the Macaulay2 computational software [26].

Of particular note in this example is that two of the x-degrees in T-degree 6 are larger than the maximum x-degree in T-degree 3. Moreover, the maximum x-degree is 8, which is larger than δ. Additionally, the x-degree in T-degree 12 is larger than the maximum x-degree in T-degree 9. Therefore, this example breaks the typical behavior twice.

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