

SUPPLEMENTARY MATERIAL

APPENDIX B: Hermite from of a polynomial

Consider a polynomial defined as $p(x) = a_k x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \dots + a_1 x + a_0$. Let $\alpha_1, \alpha_2, \dots, \alpha_k$ are the roots of the polynomial. The relations between these roots and the coefficients of the polynomial are

$$\begin{aligned} e_1 &= \sum \alpha_j = -\frac{a_{k-1}}{a_k} \\ e_2 &= \sum \alpha_i \alpha_j = \frac{a_{k-2}}{a_k} \\ e_3 &= \sum \alpha_i \alpha_j \alpha_k = -\frac{a_{k-3}}{a_k} \\ &\bullet \\ &\bullet \\ &\bullet \\ e_k &= \sum \prod \alpha_i = (-1)^k \frac{a_0}{a_k} \end{aligned}$$

The power sum of the roots is defined as $s_i = \sum_{j=1}^k \alpha_j^i$.

Newton's identities S[1] relate the power sum of the roots to the coefficients of the polynomial as for $i \leq k - 1$,

$$\begin{aligned} s_0 &= k \\ s_1 &= e_1 \times 1 \\ s_2 &= e_1 \times s_1 - e_2 \times 2 \\ s_3 &= e_1 \times s_2 - e_2 \times s_1 + e_3 \times 3 \\ &\bullet \\ &\bullet \\ &\bullet \end{aligned}$$

$$s_{k-1} = e_1 s_{k-2} - e_2 s_{k-3} + \dots + (-1)^{k-3} e_{k-2} s_1 + (-1)^{k-2} e_{k-1} (k - 1)$$

and for $i \geq k$,

$$s_i = \sum_{j=1}^k (-1)^{j+1} e_j s_{i-j} = e_1 s_{i-1} - e_2 s_{i-2} + \dots + (-1)^{k+1} e_k s_{i-k}$$

The Hermite form of the polynomial is written as $\mathbf{H}(p) = \begin{bmatrix} s_0 & s_1 & \cdot & \cdot & s_{n-1} \\ s_1 & s_2 & \cdot & \cdot & s_n \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ s_{n-1} & s_n & \cdot & \cdot & s_{2n-2} \end{bmatrix}$.

The signature of the Hermite matrix is equal to the difference between the number of positive and negative eigen-values; and this denotes the number of real roots of $p(x)$ S[1].

References

- S1. R. Basu, S. Pollack and M.F Roy. *Algorithms in Real Algebraic Geometry*. Springer, 2003.