

Supplementary Material:

Quantum Stopwatch: How To Store Time in a Quantum Memory.

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Supplementary Note 1: error bound for frequency projection [Eq. (4.2) of the main text].

Here we prove Eq. (4.2) of the main text, regarding the error of frequency projection of the compressor. We prove the result in a general setting, where the states to be compressed are of the form

$$\rho_{\phi,p} := p|\phi\rangle\langle\phi| + (1-p)|\phi_{\perp}\rangle\langle\phi_{\perp}|, \quad (1)$$

with $p \in (1/2, 1]$, $|\phi\rangle = \sqrt{s}|0\rangle + \sqrt{1-s}e^{-i\phi}|1\rangle$, and $|\phi_{\perp}\rangle := \sqrt{1-s}|0\rangle - \sqrt{s}e^{-i\phi}|1\rangle$ for some fixed $s \in (0, 1)$. Note that the clock states $\rho_{t,\gamma}$ considered in the main text are a special case of states of the form (1), with $p = (1 + e^{-\gamma t})/2$ and $\phi = t$.

To begin with, we recall a few basic facts from the main text. First, the n -fold clock state $\rho_{\phi,p}^{\otimes n}$ can be decomposed as

$$\rho_{\phi,p}^{\otimes n} \simeq \sum_{J=0}^{n/2} q_J \left(|J\rangle\langle J| \otimes \rho_{\phi,p,J} \otimes \frac{I_{m_J}}{m_J} \right), \quad (2)$$

where \simeq denotes the unitary equivalence implemented by the Schur transform, J is the quantum number of the total spin, $q_{J,t,\gamma}$ is a probability distribution, $|J\rangle$ is the state of the index register, $\rho_{t,\gamma,J}$ is the state of the representation register, and I_{m_J}/m_J is the maximally mixed state in a suitable subspace of the multiplicity register [1, 2]. The state $\rho_{\phi,p,J}$ can be expressed in the form

$$\rho_{\phi,p,J} = U_{\phi}^{\otimes n} (\rho_{p,J}) U_{\phi}^{\dagger \otimes n} \quad U_{\phi} = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|, \quad (3)$$

where the fixed state $\rho_{p,J}$ has the form

$$\rho_{p,J} := (N_J)^{-1} \sum_{m=-J}^J p^{J+m}(1-p)^{J-m} |J, m\rangle_s \langle J, m|_s \quad (4)$$

$$N_J := \sum_{k=-J}^J p^{J+k}(1-p)^{J-k} \quad (5)$$

where $|J, m\rangle_s$ is the orthonormal basis defined as

$$|J, m\rangle_s := \frac{\sum_{\pi \in \mathcal{S}_{2J}} V_{\pi} |\phi_0\rangle^{\otimes (J+m)} |\phi_{0,\perp}\rangle^{\otimes (J-m)}}{\sqrt{(2J)!(J+m)!(J-m)!}} \quad (6)$$

with $|\phi_0\rangle = \sqrt{s}|0\rangle + \sqrt{1-s}|1\rangle$, $|\phi_{0,\perp}\rangle = \sqrt{1-s}|0\rangle - \sqrt{s}|1\rangle$, \mathcal{S}_{2J} being the $(2J)$ -symmetric group and V_{π} being the unitary implementing the permutation π .

The frequency projection channel $\mathcal{P}_{\text{proj},J}$ is defined as

$$\begin{aligned} \mathcal{P}_{\text{proj},J}(\rho) &:= P_{\text{proj},J} \rho P_{\text{proj},J} + (1 - \text{Tr}[\rho P_{\text{proj},J}]) \rho_0 \\ P_{\text{proj},J} &:= \sum_{|m-(2s-1)J| \leq \frac{\sqrt{J} \log J}{2}} |J, m\rangle \langle J, m|, \end{aligned} \quad (7)$$

where ρ_0 is a fixed state of the representation register. What we need to prove is exactly the following theorem.

Theorem 1. *For large J , the frequency projection error $\epsilon_{\text{proj},J} := \frac{1}{2} \|\mathcal{P}_{\text{proj},J}(\rho_{\phi,p,J}) - \rho_{\phi,p,J}\|_1$ is upper bounded as*

$$\epsilon_{\text{proj},J} \leq (3/2) J^{-\frac{1}{8} \ln(\frac{p}{1-p})} + O\left(J^{-\frac{1}{8} \ln J}\right) \quad (8)$$

for every t .

The property below is useful in our proof of the error bound.

Lemma 1. *The basis $\{|J, m\rangle_s\}$ defined by Eq. (6) satisfies the property*

$$|\langle J, m|_s |J, k\rangle| \leq \begin{cases} \sqrt{\binom{2J}{J+k} \binom{2J}{J-m} s^{2J+k-m} (1-s)^{m-k}}, & s \geq 1/2 \\ \sqrt{\binom{2J}{J+k} \binom{2J}{J-m} s^{m+k} (1-s)^{2J-m-k}}, & s < 1/2. \end{cases} \quad (9)$$

where

$$|J, k\rangle := \frac{\sum_{\pi \in \mathcal{S}_{2J}} V_\pi |0\rangle^{\otimes (J+k)} |1\rangle^{\otimes (J-k)}}{\sqrt{(2J)!(J+k)!(J-k)!}}$$

is the symmetric basis.

Proof. Exploiting the symmetry of both bases, we have the following chain of inequalities.

$$|\langle J, m|_s |J, k\rangle| = \frac{1}{(2J)!} \left| \sum_{\pi, \pi' \in \mathcal{S}_{2J}} \frac{\langle \phi_0 |^{\otimes (J+m)} \langle \phi_{0,\perp} |^{\otimes (J-m)} V_\pi V_{\pi'} |0\rangle^{\otimes (J+k)} |1\rangle^{\otimes (J-k)}}{\sqrt{(J+m)!(J-m)!(J+k)!(J-k)!}} \right| \quad (10)$$

$$= \left| \sum_{\pi \in \mathcal{S}_{2J}} \frac{\langle \phi_0 |^{\otimes (J+m)} \langle \phi_{0,\perp} |^{\otimes (J-m)} V_\pi |0\rangle^{\otimes (J+k)} |1\rangle^{\otimes (J-k)}}{\sqrt{(J+m)!(J-m)!(J+k)!(J-k)!}} \right| \quad (11)$$

$$= \sqrt{\frac{(J+m)!(J-m)!}{(J+k)!(J-k)!}} \left| \sum_{l=\max\{0, -k-m\}}^{\min\{J-m, J-k\}} (-1)^l \binom{J-k}{l} \binom{J+k}{J-m-l} s^{\frac{2l+k+m}{2}} (1-s)^{\frac{2J-m-k-2l}{2}} \right| \quad (12)$$

$$\leq \sqrt{\frac{(J+m)!(J-m)!}{(J+k)!(J-k)!}} \sum_l \binom{J-k}{l} \binom{J+k}{J-m-l} s^{\frac{2l+k+m}{2}} (1-s)^{\frac{2J-m-k-2l}{2}} \quad (13)$$

$$= \sqrt{\frac{(J+m)!(J-m)! s^{J+k} (1-s)^{J-k}}{(J+k)!(J-k)!}} \sum_l \binom{J-k}{l} \binom{J+k}{J-m-l} \left(\frac{s}{1-s}\right)^{l - \frac{J-m}{2}} \quad (14)$$

$$\leq \sqrt{\binom{2J}{J+k} s^{J+k} (1-s)^{J-k}} \cdot \sqrt{\binom{2J}{J-m} \left(\frac{s}{1-s}\right)^{J-m}}, \quad (15)$$

and thus we have reached Eq. (9). Notice that the last inequality is for the case $s \geq 1/2$. For $s < 1/2$, the last term in the second square-root should be replaced by $[(1-s)/s]^{J-m}$. \square

Proof of Theorem 1. First observe that since $[U_\phi^{\otimes n}, P_{\text{proj},J}] = 0$, the error is independent of ϕ . Then we can rewrite

the error as

$$\begin{aligned}\epsilon_{\text{proj},J} &\leq \frac{1}{2} \left\{ \|P_{\text{proj},J} \rho_{p,J} P_{\text{proj},J} - \rho_{p,J}\|_1 + (1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}]) \right\} \\ &\leq \frac{1}{2} \left\{ 2\sqrt{(1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}])} + (1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}]) \right\} \\ &\leq \frac{3}{2} \sqrt{1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}]},\end{aligned}$$

having used the gentle measurement lemma [3, 4] to derive the second last inequality. We now simply have to estimate $1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}]$. First, we expand the quantity as

$$\begin{aligned}1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}] &= \sum_{m=-J}^J \frac{p^{J+m}(1-p)^{J-m}}{N_J} \langle J, m |_s (I - P_{\text{proj},J}) | J, m \rangle_s \\ &= \sum_{m=-J}^J \frac{p^{J+m}(1-p)^{J-m}}{N_J} \sum_{|k-(2s-1)J| > \frac{\sqrt{J} \log J}{2}} |\langle J, m |_s | J, k \rangle|^2.\end{aligned}$$

Applying Lemma 1 (for $s \geq 1/2$, and the case $s < 1/2$ can be treated in the same way), we have

$$\begin{aligned}1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}] &\leq \sum_{m=J-a}^J \frac{p^{J+m}(1-p)^{J-m}}{N_J} \left(\frac{s}{1-s}\right)^{J-m} \binom{2J}{J-m} \sum_{|k-(2s-1)J| > \frac{\sqrt{J} \log J}{2}} s^k (1-s)^{2J-k} \binom{2J}{k} \\ &\quad + \sum_{m=-J}^{J-a-1} \frac{p^{J+m}(1-p)^{J-m}}{N_J}\end{aligned}$$

where $a \geq 2$ is a parameter to be specified later. We continue bounding the term as

$$1 - \text{Tr}[P_{\text{proj},J} \rho_{p,J}] \leq \left(\frac{s}{1-s}\right)^a \binom{2J}{a} \sum_{|k-(2s-1)J| > \frac{\sqrt{J} \log J}{2}} s^k (1-s)^{2J-k} \binom{2J}{k} + \left(\frac{1-p}{p}\right)^{a+1} \quad (16)$$

$$\leq \frac{2(2J)^a}{a!} \exp\left[-\frac{\log^2 J}{4} + a \ln\left(\frac{s}{1-s}\right)\right] + \left(\frac{1-p}{p}\right)^{a+1} \quad (17)$$

$$\leq \exp\left[-\frac{\ln^2 J}{4 \ln^2 2} + a \ln(2J) + a \ln\left(\frac{s}{1-s}\right)\right] + \left(\frac{1-p}{p}\right)^{a+1}, \quad (18)$$

having used Hoeffding's bound and $a \geq 2$. Choosing, for instance, $a = \lfloor (\ln J)/4 \rfloor$ guarantees that

$$\begin{aligned}\epsilon_{\text{proj},J} &\leq \frac{3}{2} \sqrt{\exp\left[-\frac{\ln^2 J}{4 \ln^2 2} + \frac{\ln^2 J}{4} + \frac{\ln J}{4} \ln\left(\frac{s}{1-s}\right)\right] + \left(\frac{1-p}{p}\right)^{(\ln J)/4+1}} \\ &\leq \frac{3}{2} \left(\frac{1-p}{p}\right)^{(\ln J)/8} + O\left(J^{-\frac{1}{8} \ln J}\right) \\ &= (3/2) J^{-\frac{1}{8} \ln\left(\frac{p}{1-p}\right)} + O\left(J^{-\frac{1}{8} \ln J}\right).\end{aligned}$$

□

Supplementary Note 2: bound on the storage error.

Here we prove the compressor mentioned in the main text has a vanishing error. First, notice that the trace distance between the original state $\rho_{\phi,p}^{\otimes n}$ and the output of the compression protocol, denoted as $\rho'_{\phi,p,n}$, is

$$\begin{aligned}\epsilon_{\phi,p} &= \frac{1}{2} \left\| \rho'_{\phi,p,n} - \rho_{\phi,p}^{\otimes n} \right\|_1 \\ &= \frac{1}{2} \left\| \sum_J q_J |J\rangle\langle J| \otimes [\mathcal{P}_{\text{proj},J}(\rho_{\phi,p,J}) - \rho_{\phi,p,J}] \otimes \frac{I_{m_J}}{m_J} \right\|_1 \\ &= \sum_J q_J \cdot \epsilon_{\text{proj},J}, \quad \epsilon_{\text{proj},J} = \frac{1}{2} \left\| \mathcal{P}_{\text{proj},J}(\rho_{\phi,p,J}) - \rho_{\phi,p,J} \right\|_1.\end{aligned}\tag{19}$$

Since we already have a bound for $\epsilon_{\text{proj},J}$, the only ingredient needed for the error bound is the concentration property of q_J . Notice that the probability distribution q_J in Eq. (2) has the explicit form [5]

$$q_J = \frac{2J+1}{2J_0} \left[B\left(\frac{n}{2} + J + 1\right) - B\left(\frac{n}{2} - J\right) \right]\tag{20}$$

where $B(k) = p^k(1-p)^{n-k} \binom{n}{k}$ and $J_0 = (p-1/2)(n+1)$, which is a Gaussian distribution concentrated in an interval of width $O(\sqrt{n})$ around J_0 when n is large. Picking an interval much larger than \sqrt{n} , we get from Hoeffding's inequality that

$$\sum_{|J-J_0| \leq n^{2/3}} q_J \geq 1 - 2 \exp\left[-\frac{2n^{1/3}}{p^2}\right].\tag{21}$$

Substituting the above inequality into Eq. (19), we have

$$\begin{aligned}\epsilon_{\phi,p} &\leq \max_{|J-J_0| \leq n^{2/3}} \epsilon_{\text{proj},J} + \sum_{|J-J_0| > n^{2/3}} q_J \\ &\leq \frac{3}{2} \left(\frac{2}{(2p-1)n} \right)^{\frac{1}{8} \ln\left(\frac{p}{1-p}\right)} + O\left(n^{-\frac{1}{8} \ln n}\right).\end{aligned}$$

Finally, substituting $p = (1 + e^{-\gamma t})/2$ into the above inequality, the error for the compressor in the depolarizing model is bounded by

$$\epsilon_{\gamma,t} \leq \frac{3}{2} \left(\frac{2e^{\gamma t}}{n} \right)^{\frac{1}{8} \ln \cosh \frac{\gamma t}{2}} + O\left(n^{-\frac{1}{8} \ln n}\right).$$

From the above bound, we can directly get the bound for the overall error of compression in a quantum stopwatch as

$$\epsilon_{T,\gamma}^{(k)} \leq \frac{3k}{2} \left(\frac{2e^{\gamma T}}{n} \right)^{\frac{1}{8} \ln \coth \frac{\gamma T}{2}} + O\left(kn^{-\frac{1}{8} \ln n}\right),\tag{22}$$

where k is the number of events and T is the total duration.

Supplementary Note 3: coherent versus incoherent protocols.

Here we compare the accuracy of the coherent protocols with the accuracy of incoherent protocols using repeated measurements.

Suppose that we want to measure the total duration of k intervals, each of which has length T/k . In the coherent protocol, each clock qubit starts with a pure state and ends up in a mixed state with maximum eigenvalue depending

on the total duration T as $p(T, \gamma) = (1 + e^{-\gamma T})/2$. The inaccuracy for the quantum stopwatch thus has leading order $1/\sqrt{nF_{\text{loc}}}$, where

$$F_{\text{loc}} = 1 - \gamma^2 - \sqrt{1 - e^{-2\gamma T}} + \frac{\gamma^2}{\sqrt{1 - e^{-2\gamma T}}}$$

is the classical Fisher information. On the other hand, an incoherent protocol involves k measurements of time and k initializations of the clock qubits, resulting in an of leading order $\sqrt{k/(nF'_{\text{loc}})}$, where F'_{loc} is the classical Fisher information of the individual step, given by

$$F'_{\text{loc}} = 1 - \gamma^2 - \sqrt{1 - e^{-2\gamma T/k}} + \frac{\gamma^2}{\sqrt{1 - e^{-2\gamma T/k}}}.$$

Comparing the two error terms, we conclude that the coherent protocol outperforms the incoherent one when $F_{\text{loc}} \geq (F'_{\text{loc}})/k$, namely that

$$k \left[1 - \gamma^2 - \sqrt{1 - e^{-2\gamma T}} + \frac{\gamma^2}{\sqrt{1 - e^{-2\gamma T}}} \right] \geq 1 - \gamma^2 - \sqrt{1 - e^{-2\gamma T/k}} + \frac{\gamma^2}{\sqrt{1 - e^{-2\gamma T/k}}}.$$

Finally we comment on the case of large k , fixing γ and T . In this case, the left hand side of the above inequality scales linearly in k , while the right hand side scales as \sqrt{k} (which can be seen by taking the Taylor expansion of the last term). Therefore, the left hand side dominates the right hand side in the large k limit, and thus the coherent protocol is always better for large k .

Supplementary Note 4: inaccuracy of the local strategy for unknown γ .

Here we analyze the precision of local time measurement strategy when γ is unknown and is treated as a nuisance parameter, which is not directly of interest but affects the analysis of our estimation. For this analysis, we parameterize the distribution of our interest as $P_{T,\gamma}(\tau)$ because the dependence on γ is crucial in the following discussion. Then, we introduce Fisher information matrix F_{loc} defined by

$$(F_{\text{loc}})_{x,y} := \int_0^{2\pi} d\tau \frac{\partial \log P_{T,\gamma}(\tau)}{\partial x} \cdot \frac{\partial \log P_{T,\gamma}(\tau)}{\partial y} P_{T,\gamma}(\tau),$$

where $x, y \in \{T, \gamma\}$. And we need to evaluate $(F_{\text{loc}}^{-1})_{TT}$, namely the (T, T) -entry of the inverse Fisher information matrix F_{loc}^{-1} .

To simplify the calculation, we first adapt a re-parameterization of the nuisance parameter from γ to $p = (e^{-\gamma T} + 1)/2$. The T component $(F_{\text{loc}}^{-1})_{TT}$ of the inverse Fisher information is invariant under such a re-parameterization of the nuisance parameter, which can be proved as follows.

The inverse Fisher information F_{loc}^{-1} with respect to the parametrization (T, γ) is given by

$$F_{\text{loc}}^{-1} = \begin{bmatrix} (F_{\text{loc}})_{TT} & (F_{\text{loc}})_{T,\gamma} \\ (F_{\text{loc}})_{\gamma,T} & (F_{\text{loc}})_{\gamma\gamma} \end{bmatrix}^{-1}.$$

Now, consider a re-parameterization of the nuisance parameter $\gamma \rightarrow p$, and the new Fisher information matrix is

$$F_{\text{loc},p}^{-1} = A^T F_{\text{loc}}^{-1} A$$

where

$$A = \begin{pmatrix} 1 & 0 \\ \frac{\partial \gamma}{\partial T} & \frac{\partial \gamma}{\partial p} \end{pmatrix}$$

is the Jacobi matrix. Therefore, we have

$$F_{\text{loc}}^{-1} = A \left(F_{\text{loc},p}^{-1} \right) A^T$$

and it can be checked by straightforward calculation that

$$(F_{\text{loc}}^{-1})_{TT} = (F_{\text{loc},p}^{-1})_{TT}.$$

It is therefore enough to calculate the Fisher information matrix for the parameterization (T, p) , when the input state is $\rho_{T,p} = p|\varphi_T\rangle\langle\varphi_T| + (1-p)|\varphi_T^\perp\rangle\langle\varphi_T^\perp|$ with $|\varphi_T\rangle = (|0\rangle + e^{-iT}|1\rangle)/\sqrt{2}$ and $|\varphi_T^\perp\rangle = (|0\rangle - e^{-iT}|1\rangle)/\sqrt{2}$. That is, in the following, we employ the parametrization $P_{T,p}(\tau)$ by the pair (T, p) for the probability of the measurement to yield outcome τ . When the input state is $\rho_{T,p}$, it can be expressed as

$$\begin{aligned} P_{T,p}(\tau) &:= \text{Tr}[\rho_{T,p}M_\tau] \\ &= \frac{1}{2\pi}(1 + (2p-1)\cos(\tau-T)). \end{aligned} \quad (23)$$

The Fisher information of $P_{T,p}(\tau)$ can be derived from a lengthy but straightforward calculation, and we have

$$\begin{aligned} (F_{\text{loc},p})_{TT} &= \int_0^{2\pi} \left(\frac{\partial \log P_{T,p}(\tau)}{\partial T} \right)^2 P_{T,p}(\tau) d\tau \\ &= 1 - \sqrt{1 - (2p-1)^2} \\ &= 1 - \sqrt{1 - e^{-2\gamma T}}. \end{aligned} \quad (24)$$

To evaluate the inverse of the Fisher information matrix, we also need to calculate the off-diagonal elements. We define the logarithmic likelihood derivative $l_p(\tau)$ with respect to p as

$$\begin{aligned} l_p(\tau) &:= \frac{\partial \log P_{T,p}(\tau)}{\partial p} \\ &= \frac{2\cos(\tau-T)}{1 + (2p-1)\cos(\tau-T)}. \end{aligned}$$

Then the off-diagonal components $F_{T,p}$ and $F_{p,T}$ are given as

$$(F_{\text{loc},p})_{T,p} = (F_{\text{loc},p})_{p,T} := \int_0^{2\pi} l_p(\tau) \frac{\partial \log P_{T,p}(\tau)}{\partial T} P_{T,p}(\tau) d\tau = 0. \quad (25)$$

Since the off-diagonal components are zero, we have

$$(F_{\text{loc}}^{-1})_{TT} = (F_{\text{loc},p}^{-1})_{TT} = 1/(F_{\text{loc},p})_{TT}.$$

The value of $(F_{\text{loc}}^{-1})_{TT}$ is thus given by the reciprocal of the r.h.s. of Eq. (24).

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