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to

Anisotropic Residual Stresses in Arteries

by

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1 Microstructure

The scalar quantities $\kappa_{\mathrm{ip}}^{(\mathrm{L})}$ and $\kappa_{\mathrm{op}}^{(\mathrm{L})}$ describe the in-plane and out-of-plane dispersions of fibers, respectively [1]. Their closed form expressions are

$$\kappa_{\mathrm{ip}}^{(\mathrm{L})} = \frac{1}{2} - \frac{I_1(\widehat{a}^{(\mathrm{L})})}{2I_0(\widehat{a}^{(\mathrm{L})})}, \quad \kappa_{\mathrm{op}}^{(\mathrm{L})} = \frac{1}{2} - \frac{1}{8\widehat{b}^{(\mathrm{L})}} + \frac{1}{4}\sqrt{\frac{2}{\pi\widehat{b}^{(\mathrm{L})}}}\frac{\exp(-2\widehat{b}^{(\mathrm{L})})}{\mathrm{erf}(\sqrt{2\widehat{b}^{(\mathrm{L})}})}, \quad (1)$$

where L = I stands for intima, L = M for media, L = A for adventitia, and $\widehat{a}^{(\mathrm{L})}$ and $\widehat{b}^{(\mathrm{L})}$ are the concentration parameters defining the shape of the von Mises distributions that describe the corresponding in-plane and out-of-plane fiber dispersions in each layer. The modified Bessel functions of the first kind of order 0 and 1 are denoted by I_1 and I_0, respectively. The intervals of the quantities $\kappa_{\mathrm{ip}}^{(\mathrm{L})}$ and $\kappa_{\mathrm{op}}^{(\mathrm{L})}$ are $\kappa_{\mathrm{ip}}^{(\mathrm{L})} \in [0,1]$ and
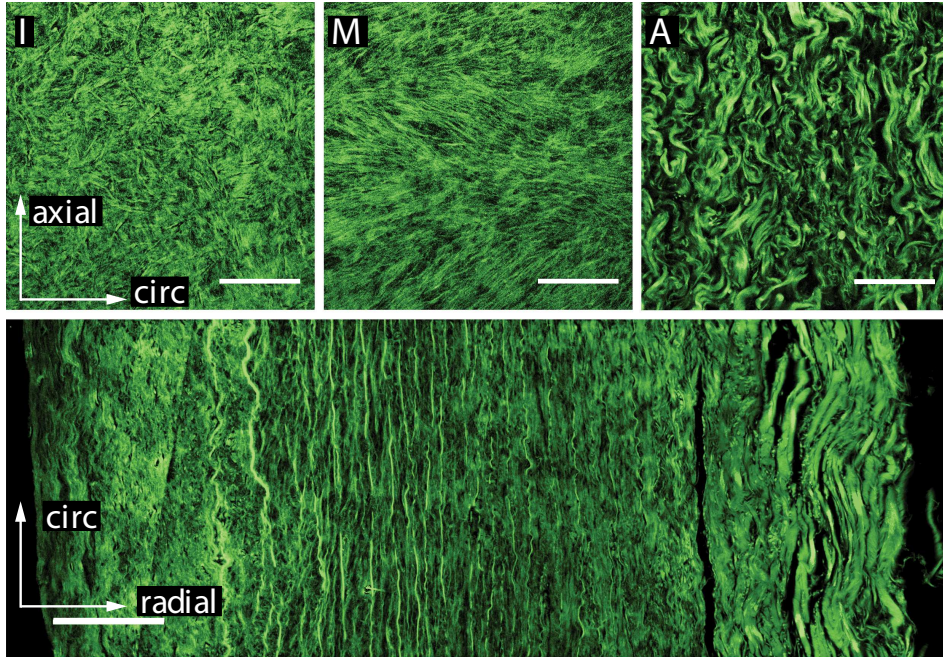

Figure 1: Microstructure of a representative healthy abdominal aorta. Top three figures illustrate in-plane fiber distributions in intima (I), media (M) and adventitia (A), while fibers through the thickness (out-of-plane) are shown in the bottom figure (reprinted from [2]).

$\kappa_{\text{op}}^{(\text{L})} \in [0, 1/2]$, respectively. When $\kappa_{\text{op}}^{(\text{L})} \to 1/2$, it indicates that there is an insignificant number of out-of-plane fibers as they tend to be in-plane. When $\kappa_{\text{ip}}^{(\text{L})} = 1/2$, the in-plane fibers are isotropic. When $\kappa_{\text{op}}^{(\text{L})} = 1/3$, the out-of-plane fibers are isotropic. Figure 1, taken from [2], indicates a typical distribution of the in-plane fibers in the intimal, medial and adventitial layers of a healthy abdominal aorta as well as the out-of-plane fiber distribution through the thickness. As can clearly be seen, the fibers mostly lie in the θz-plane and have some specific directions, which, although not clearly visible, are far from being isotropic. This allows us to expect that $\kappa_{\text{op}}^{(\text{L})} \to 1/2$ and $\kappa_{\text{ip}}^{(\text{L})} \neq 1/2$ for all layers (L = I, M, A) in the given healthy abdominal aorta.

2 Residual stress state: boundary/interface conditions and equilibrium equations

The Cauchy stresses for each layer (L = I, M, A) are

$$\sigma_{rr}^{(\text{L})} = [c^{(\text{L})} + 4(1 - 2A^{(\text{L})} - B^{(\text{L})})\psi_4'^{(\text{L})}]\lambda_r^{(\text{L})^2} - p^{(\text{L})},$$
$$\sigma_{\theta\theta}^{(\text{L})} = [c^{(\text{L})} + 4(A^{(\text{L})} + B^{(\text{L})}\cos^2\alpha^{(\text{L})})\psi_4'^{(\text{L})}]\lambda_\theta^{(\text{L})^2} - p^{(\text{L})}, \quad (2)$$
$$\sigma_{zz}^{(\text{L})} = [c^{(\text{L})} + 4(A^{(\text{L})} + B^{(\text{L})}\sin^2\alpha^{(\text{L})})\psi_4'^{(\text{L})}]\lambda_z^{(\text{L})^2} - p^{(\text{L})},$$

where $p^{(\text{I})}$, $p^{(\text{M})}$ and $p^{(\text{A})}$ are functions of the radial coordinate r to be determined from the boundary and interface conditions. The equilibrium equation valid for all the layers (L = I, M, A) can be written as

$$\frac{\mathrm{d}\sigma_{rr}^{(\text{L})}}{\mathrm{d}r} + \frac{\sigma_{rr}^{(\text{L})} - \sigma_{\theta\theta}^{(\text{L})}}{r} = 0. \quad (3)$$

It is convenient to write this expression in the forms of definite and semi-definite integrals (L = I, M, A), i.e.

$$\sigma_{rr}^{(\text{L})}(b^{(\text{L})}) - \sigma_{rr}^{(\text{L})}(a^{(\text{L})}) = \int_{a^{(\text{L})}}^{b^{(\text{L})}} \frac{\sigma_{\theta\theta}^{(\text{L})} - \sigma_{rr}^{(\text{L})}}{r}\mathrm{d}r,$$
$$\sigma_{rr}^{(\text{L})}(r) - \sigma_{rr}^{(\text{L})}(a^{(\text{L})}) = \int_{a^{(\text{L})}}^{r} \frac{\sigma_{\theta\theta}^{(\text{L})} - \sigma_{rr}^{(\text{L})}}{r}\mathrm{d}r, \quad (4)$$
$$\sigma_{rr}^{(\text{L})}(b^{(\text{L})}) - \sigma_{rr}^{(\text{L})}(r) = \int_{r}^{b^{(\text{L})}} \frac{\sigma_{\theta\theta}^{(\text{L})} - \sigma_{rr}^{(\text{L})}}{r}\mathrm{d}r.$$

The conditions of no stresses on the inner and outer surface of the composite artery and the equilibrium equations in the semi-indefinite form allow the computation of $p^{(\text{I})}$ and $p^{(\text{A})}$:

$$\sigma_{rr}^{(\text{I})}(a^{(\text{I})}) = 0 \implies p^{(\text{I})} = [c^{(\text{I})} + 4(1 - 2A^{(\text{I})} - B^{(\text{I})})\psi_4'^{(\text{I})}]\lambda_r^{(\text{I})^2} + \int_{a^{(\text{I})}}^{r} \frac{\sigma_{rr}^{(\text{I})} - \sigma_{\theta\theta}^{(\text{I})}}{r}\mathrm{d}r, \quad (5)$$

$$\sigma_{rr}^{(\text{A})}(b^{(\text{A})}) = 0 \implies p^{(\text{A})} = [c^{(\text{A})} + 4(1 - 2A^{(\text{A})} - B^{(\text{A})})\psi_4'^{(\text{A})}]\lambda_r^{(\text{A})^2} + \int_{r}^{b^{(\text{A})}} \frac{\sigma_{\theta\theta}^{(\text{A})} - \sigma_{rr}^{(\text{A})}}{r}\mathrm{d}r.$$

Equilibrium equations for separate layers in the form of definite integrals are expressed as follows

$$\sigma_{rr}^{(\mathrm{I})}(b^{(\mathrm{I})}) - \underbrace{\sigma_{rr}^{(\mathrm{I})}(a^{(\mathrm{I})})}_{0} = \int_{a^{(\mathrm{I})}}^{b^{(\mathrm{I})}} \frac{\sigma_{\theta\theta}^{(\mathrm{I})} - \sigma_{rr}^{(\mathrm{I})}}{r} \mathrm{d}r,$$

$$\sigma_{rr}^{(\mathrm{M})}(b^{(\mathrm{M})}) - \sigma_{rr}^{(\mathrm{M})}(a^{(\mathrm{M})}) = \int_{a^{(\mathrm{M})}}^{b^{(\mathrm{M})}} \frac{\sigma_{\theta\theta}^{(\mathrm{M})} - \sigma_{rr}^{(\mathrm{M})}}{r} \mathrm{d}r, \qquad (6)$$

$$\underbrace{\sigma_{rr}^{(\mathrm{A})}(b^{(\mathrm{A})})}_{0} - \sigma_{rr}^{(\mathrm{A})}(a^{(\mathrm{A})}) = \int_{a^{(\mathrm{A})}}^{b^{(\mathrm{A})}} \frac{\sigma_{\theta\theta}^{(\mathrm{A})} - \sigma_{rr}^{(\mathrm{A})}}{r} \mathrm{d}r.$$

The interface conditions $\sigma_{rr}^{(\mathrm{I})}(b^{(\mathrm{I})}) = \sigma_{rr}^{(\mathrm{M})}(a^{(\mathrm{M})})$ and $\sigma_{rr}^{(\mathrm{M})}(b^{(\mathrm{M})}) = \sigma_{rr}^{(\mathrm{A})}(a^{(\mathrm{A})})$, in turn, allow to reduce the above system to the following single integral equation

$$\int_{a^{(\mathrm{I})}}^{b^{(\mathrm{I})}} \frac{\sigma_{\theta\theta}^{(\mathrm{I})} - \sigma_{rr}^{(\mathrm{I})}}{r} \mathrm{d}r + \int_{a^{(\mathrm{M})}}^{b^{(\mathrm{M})}} \frac{\sigma_{\theta\theta}^{(\mathrm{M})} - \sigma_{rr}^{(\mathrm{M})}}{r} \mathrm{d}r + \int_{a^{(\mathrm{A})}}^{b^{(\mathrm{A})}} \frac{\sigma_{\theta\theta}^{(\mathrm{A})} - \sigma_{rr}^{(\mathrm{A})}}{r} \mathrm{d}r = 0, \qquad (7)$$

which needs to be satisfied and, thus, it imposes some restrictions on the geometry of the problem. The last unknown of the system $p^{(\mathrm{M})}$ can be determined from either of the interface conditions as a function of the radial coordinate r. Here we use $\sigma_{rr}^{(\mathrm{I})}(b^{(\mathrm{I})}) = \sigma_{rr}^{(\mathrm{M})}(a^{(\mathrm{M})})$, which yields

$$p^{(\mathrm{M})} = [c^{(\mathrm{M})} + 4(1 - 2A^{(\mathrm{M})} - B^{(\mathrm{M})})\psi_4'^{(\mathrm{M})}]\lambda_r^{(\mathrm{M})2} + \int_{a^{(\mathrm{I})}}^{b^{(\mathrm{I})}} \frac{\sigma_{rr}^{(\mathrm{I})} - \sigma_{\theta\theta}^{(\mathrm{I})}}{r} \mathrm{d}r + \int_{a^{(\mathrm{M})}}^{r} \frac{\sigma_{rr}^{(\mathrm{M})} - \sigma_{\theta\theta}^{(\mathrm{M})}}{r} \mathrm{d}r. \qquad (8)$$

3 Loaded stress state: solution procedure

To describe the loaded state of the composite aorta we employ the following deformation field

$$\begin{cases} \rho = \sqrt{\dfrac{r^2 - a_I^2}{\lambda_z} + \rho_a^2}, \\ \vartheta = \theta, \\ \zeta = \lambda_z z, \end{cases} \qquad (9)$$

which captures both inflation and extension. Particularly, parameter λ_z will be used to descibe the deformations due to axial loadings. The unloaded geometry can be described as $r \in [a_I, b_A]$, $\theta \in [0, 2\pi]$, $z \in [-l, l]$, and the deformed geometry as $\rho \in [\rho_a, \rho_b] = [\rho(a_I), \rho(b_A)]$, $\vartheta \in [0, 2\pi]$, $\zeta \in [-\eta, \eta]$. Cylindrical coordinate system $\{\rho, \vartheta, \zeta\}$ describes the deformed body configuration. The deformation gradient $\mathbf{F}_{\mathrm{load}}$ is expressed as

$$\mathbf{F}_{\mathrm{load}} = \frac{r}{\rho\lambda_z}\mathbf{e}_\rho \otimes \mathbf{e}_r + \frac{\rho}{r}\mathbf{e}_\vartheta \otimes \mathbf{e}_\theta + \lambda_z\mathbf{e}_\zeta \otimes \mathbf{e}_z. \qquad (10)$$

In order to account for residual stresses, it is necessary to consider residual deformations of individual layers described via the following deformation gradients

$$\mathbf{F}_{\text{res}}^{(\text{I})} = \frac{R}{rk^{(\text{I})}\lambda_z^{(\text{I})}}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{k^{(\text{I})}r}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z^{(\text{I})}\mathbf{e}_z \otimes \mathbf{E}_Z,$$

$$\mathbf{F}_{\text{res}}^{(\text{M})} = \frac{\pi L^{(\text{M})}R}{rk^{(\text{M})}\beta l^{(\text{M})}}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r\beta}{L^{(\text{M})}}\mathbf{e}_\theta \otimes \mathbf{E}_Z + \frac{l^{(\text{M})}k^{(\text{M})}}{\pi R}\mathbf{e}_z \otimes \mathbf{E}_\Theta, \qquad (11)$$

$$\mathbf{F}_{\text{res}}^{(\text{A})} = \frac{L_2^{(\text{A})}}{\pi r\lambda_z^{(\text{A})}}\mathbf{e}_r \otimes \mathbf{E}_X + \frac{\pi r}{L_2^{(\text{A})}}\mathbf{e}_\theta \otimes \mathbf{E}_Y + \lambda_z^{(\text{A})}\mathbf{e}_z \otimes \mathbf{E}_Z.$$

Then, the multiplicative decomposition rule is used to describe the stress state of the loaded and residually stressed aortic layers, i.e.

$$\mathbf{F}_{\text{res\&load}}^{(\text{L})} = \mathbf{F}_{\text{load}}\mathbf{F}_{\text{res}}^{(\text{L})} \quad (\text{L} = \text{I, M, A}), \qquad (12)$$

so that the left Cauchy-Green tensor is

$$\mathbf{B}_{\text{res\&load}}^{(\text{L})} = \mathbf{F}_{\text{res\&load}}^{(\text{L})}\left(\mathbf{F}_{\text{res\&load}}\right)^{(\text{L})\text{T}} = \lambda_\rho^{(\text{L})^2}\mathbf{e}_\rho \otimes \mathbf{e}_\rho + \lambda_\vartheta^{(\text{L})^2}\mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta + \lambda_\zeta^{(\text{L})^2}\mathbf{e}_\zeta \otimes \mathbf{e}_\zeta. \quad (13)$$

Following the same approach as for the residually stressed configuration, the stresses for the loaded and residually stressed configuration can then be expressed as

$$\sigma_{\rho\rho}^{(\text{L})} = [c^{(\text{L})} + 4(1 - 2A^{(\text{L})} - B^{(\text{L})})\psi_4'^{(\text{L})}]\lambda_\rho^{(\text{L})^2} - \tilde{p}^{(\text{L})},$$

$$\sigma_{\vartheta\vartheta}^{(\text{L})} = [c^{(\text{L})} + 4(A^{(\text{L})} + B^{(\text{L})}\cos^2\alpha^{(\text{L})})\psi_4'^{(\text{L})}]\lambda_\vartheta^{(\text{L})^2} - \tilde{p}^{(\text{L})}, \qquad (14)$$

$$\sigma_{\zeta\zeta}^{(\text{L})} = [c^{(\text{L})} + 4(A^{(\text{L})} + B^{(\text{L})}\sin^2\alpha^{(\text{L})})\psi_4'^{(\text{L})}]\lambda_\zeta^{(\text{L})^2} - \tilde{p}^{(\text{L})}.$$

The corresponding equilibrium equations are

$$\frac{\text{d}\sigma_{\rho\rho}^{(\text{L})}}{\text{d}\rho} + \frac{\sigma_{\rho\rho}^{(\text{L})} - \sigma_{\vartheta\vartheta}^{(\text{L})}}{\rho} = 0, \qquad (15)$$

with the interface conditions between the layers of the aorta

$$\sigma_{\rho\rho}^{(\text{I})}(\rho(b^{(\text{I})})) = \sigma_{\rho\rho}^{(\text{M})}(\rho(a^{(\text{M})})), \quad \sigma_{\rho\rho}^{(\text{M})}(\rho(b^{(\text{M})})) = \sigma_{\rho\rho}^{(\text{A})}(\rho(a^{(\text{A})})), \qquad (16)$$

and the boundary conditions, which imply blood flow pressure P on the inner surface of the cylinder and no stress on its outer surface

$$\sigma_{\rho\rho}^{(\text{I})}(\rho_a) = P, \quad \sigma_{\rho\rho}^{(\text{A})}(\rho_b) = 0. \qquad (17)$$

For given values of pressure P and axial stretch λ_z, this boundary-value problem can be resolved using the same approach as described above (Section 2 of the supplementary material).

4 Residual stress state: classical opening angle method

In the classical opening angle method, aortic rings are cut along the thickness to release stress. To model that we assume that the aortic wall is homogeneous (labeled as (W) and standing for "wall") and its zero-stress state is given by the sector

$$R \in [A^{(W)}, B^{(W)}], \quad \Theta \in [\alpha^{(W)}, 2\pi - \alpha^{(W)}], \quad Z \in [-L^{(W)}, L^{(W)}]. \tag{18}$$

The sector is then closed into the ring

$$r \in [a^{(W)}, b^{(W)}], \quad \theta \in [0, 2\pi], \quad z \in [-l, l]. \tag{19}$$

The deformation field necessary to deform a sector into the ring is given by

$$\begin{cases} r = \sqrt{\dfrac{R^2 - A^{(W)^2}}{k^{(I)}\lambda_z^{(W)}} + a^{(W)^2}}, \\ \theta = k^{(W)}(\Theta - \alpha_0^{(W)}), \\ z = \lambda_z^{(W)} Z. \end{cases} \tag{20}$$

Here $\lambda_z^{(W)} = l/L^{(W)}$ is a constant stretch in the axial direction, while $k^{(W)} = \pi/(\pi - \alpha_0^{(W)})$ is the opening angle measure. The deformation gradient $\mathbf{F}^{(W)}$ can be expressed as

$$\mathbf{F}^{(W)} = \frac{R}{r k^{(W)} \lambda_z^{(W)}} \mathbf{e}_r \otimes \mathbf{E}_R + \frac{k^{(W)} r}{R} \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z^{(W)} \mathbf{e}_z \otimes \mathbf{E}_Z. \tag{21}$$

Since $\mathbf{F}^{(W)}$ is diagonal, the left Cauchy-Green deformation tensor $\mathbf{B}^{(W)} = \mathbf{F}^{(W)}\mathbf{F}^{(W)T}$ has the same diagonal elements as $\mathbf{C}^{(W)}$, and the principal stretches in the radial and the circumferential directions are

$$\lambda_r^{(W)} = \frac{R}{r k^{(W)} \lambda_z^{(W)}}, \quad \lambda_\theta^{(W)} = \frac{k^{(W)} r}{R}. \tag{22}$$

The corresponding stresses can be determined as

$$\begin{aligned} \sigma_{rr}^{(W)} &= [c^{(W)} + 4(1 - 2A^{(W)} - B^{(W)})\psi_4'^{(W)}]\lambda_r^{(W)^2} - p^{(W)}, \\ \sigma_{\theta\theta}^{(W)} &= [c^{(W)} + 4(A^{(W)} + B^{(W)} \cos^2 \alpha^{(W)})\psi_4'^{(W)}]\lambda_\theta^{(W)^2} - p^{(W)}, \\ \sigma_{zz}^{(W)} &= [c^{(W)} + 4(A^{(W)} + B^{(W)} \sin^2 \alpha^{(W)})\psi_4'^{(W)}]\lambda_z^{(W)^2} - p^{(W)}. \end{aligned} \tag{23}$$

The equilibrium equation is

$$\frac{d\sigma_{rr}^{(W)}}{dr} + \frac{\sigma_{rr}^{(W)} - \sigma_{\theta\theta}^{(W)}}{r} = 0. \tag{24}$$

Zero-stress boundary condition on the inner surface of the aorta $\sigma_{rr}^{(W)}(a^{(W)}) = 0$ allows to determine the Lagrange multiplier $p^{(W)}$ as a function of r

$$p^{(W)} = [c^{(W)} + 4(1 - 2A^{(W)} - B^{(W)})\psi_4'^{(W)}]\lambda_r^{(W)^2} + \int_{a^{(W)}}^r \frac{\sigma_{rr}^{(W)} - \sigma_{\theta\theta}^{(W)}}{r} dr, \tag{25}$$

while the remaining zero-stress boundary condition on the outer surface of the aorta $\sigma_{rr}^{(W)}(b^{(W)}) = 0$ has to be satisfied.

References

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