S-I Scaling analysis

We begin our analysis of passive-pitching flapping flight with the classic approach for scaling flapping flight based on the quasi-steady aerodynamic model and blade element theory [1, 2, 3, 4]. The quasi-steady assumption states that the instantaneous forces experienced by a dynamically moving object are equivalent to those it would experience in steady motion at the same instantaneous velocity and angle of attack. In the blade element theory, a revolving wing or surface is divided into infinitesimally small slices along the radial coordinate. The component of force parallel to the flow is called drag and the component normal to the flow is called lift. Any span-wise flow (orthogonal to both lift and drag) is assumed to have no effect on the magnitudes of those forces [1]. Though the validity of the assumptions necessary for
the quasi-steady model and the blade-element theory is contested, they are sufficient as first
approximations of aerodynamic forces.

In aerodynamics, the lift coefficient, $C_L$, is defined as the ratio of the measured lift force,
$F_L$, to the force due to quasi-steady aerodynamic pressure, $F$. Since the aerodynamic pressure
is not constant across the entire wing in flapping flight due to varying translational velocity
along the length of the wing arising from the revolving motion, the pressure must be integrated
over the wing area:

$$C_L = \frac{F_L}{F} = \frac{F_L}{\int qdS}, \quad (S-1)$$

where $q$ is the dynamic pressure and $S$ is the area of the wing. Making use of the blade element
method, the force due to aerodynamic pressure, $F$, is computed by integrating along the length
of the wing as follows:

$$F = \int_0^R \frac{1}{2} \rho u(r)^2 c(r) dr, \quad (S-2)$$

where $R$ is the length of the wing, $\rho$ is the density of the fluid, $u(r)$ is the local translational
velocity, and $c$ is the chord length. Both $u(r)$ and $c(r)$ are a function of the span-wise coordi-
nate, $r$. At any given flapping frequency, the translational velocity of the wing is not constant
throughout the wingbeat cycle due the accelerations necessary to perform stroke reversals; for
the sake of simplicity, we will use the mean translational velocity,

$$u(r) = 2\Phi f(x_r + r), \quad (S-3)$$

where $\Phi$ is the stroke angle of the wing, $f$ is the flapping frequency, and $x_r$ is the location of
the wing root along the span-wise coordinate. Substituting (S-3) into (S-2), we have

$$F = \frac{1}{2} \rho (2\Phi f R)^2 R \hat{r}_2^2, \quad (S-4)$$

where

$$\hat{r}_2^2 = \int_0^1 (\hat{r} + \hat{x}_r)^2 \hat{c}(\hat{r}) d\hat{r} \quad (S-5)$$
is a dimensionless number referred to as the square of the second radius moment. The symbol $\bar{c}$ represents the mean chord length, which is defined

$$\bar{c} = \frac{A}{R}. \quad (S-6)$$

The symbols $\hat{c}$ and $\hat{r}$ represent the chord dimension, $c$, normalized by the mean chord length, $\bar{c}$; and the radial dimension, $r$, normalized by the wing length, $R$, respectively.

Finally, we combine (S-1) and (S-4) to obtain an expression for the lift coefficient:

$$C_L = \frac{F_L}{(1/2)\rho(2\Phi f R)^2 R\bar{c}^2} \hat{r}. \quad (S-7)$$

In addition, we will also define the Reynolds number, which measures the relative importance of inertial (fluid dynamic) forces to viscous fluid forces. The chord-based Reynolds number for a revolving wing is

$$Re = \frac{u(R)\bar{c}}{\nu} = \frac{2\Phi f (R + x_r)\bar{c}}{\nu}, \quad (S-8)$$

where $\nu$ is the kinematic viscosity, defined as the ratio of the dynamic viscosity, $\mu$ to the fluid density, $\rho$.

The quasi-steady analysis is extended to develop a definition for the Cauchy number, $Ch$, a non-dimensional number which relates the dynamic pressure force and the stiffness of the wing hinge. We begin with the equation relating static applied torque and resulting angular deflection for a linear torsion spring:

$$\tau = \kappa \Delta \psi, \quad (S-9)$$

where $\tau$ is the applied torque, $\kappa$ is the torsional stiffness of the spring, and $\Delta \psi$ is the angular deflection from a neutral state. In a pitching wing, the torque due to aerodynamic pressure which contributes to wing rotation can be approximated to first-order as

$$\tau_f = F\bar{c}(1 + \dot{y_r}), \quad (S-10)$$
where $\hat{y}_r$ is the wing root offset in the chord-wise direction, $y_c$, normalized by the mean chord length. Combining eq. S-9 and eq. S-10, we obtain

$$F\bar{c}(1 + \hat{y}_r) = \kappa \Delta \psi. \quad (S-11)$$

Since we are working with a passive-pitching system, we are interested in predicting the pitching kinematics from controllable or known parameters, such as geometry, fluid density, and flapping kinematics. We formulate the Cauchy number such that it is a function of the variables other than $\Delta \psi$: 

$$Ch = \Delta \psi = \frac{F\bar{c}(1 + \hat{y}_r)}{\kappa}. \quad (S-12)$$

Thus, each value of the Cauchy number can be considered to correspond with some characteristic pitch deflection angle. Combining eq. S-12 and eq. S-4, and we arrive at

$$Ch = \frac{2 \rho \Phi^2 f^2 R^3 \bar{c}^2 \hat{r}^2 (1 + \hat{y}_r)}{\kappa}, \quad (S-13)$$

As discussed in the main text, the Cauchy number can also be thought of as a scaled flapping frequency for a flapping wing system with given physical specifications. Note that the formulation presented here differs from those used in [5, 6] by a geometry-dependent factor.

The derivation of the second dimensionless parameter, the Inertial-Elastic number proceeds in a similar fashion. Now, we examine the ratio of the inertial and elastic effects in a passive-pitching flapping wing. Therefore, the Inertial-Elastic number, $IE$, is defined:

$$IE = \frac{\tau_I}{\kappa}. \quad (S-14)$$

where $\tau_I$ is the inertial torque which comes from both the surrounding fluid and the wing itself:

$$\tau_I = (I_f + I_w) \ddot{\psi}, \quad (S-15)$$

where $I_w$ and $I_f$ are the inertia due to the wing mass and the inertia due to the fluid mass, respectively. The acceleration of the pitch angle is represented by $\ddot{\psi}$. Here, we will define a
parameter called the Mass number, $M$, which describes the ratio of the contributions of the wing mass and the fluid mass to the total inertia of the flapping wing system:

$$M = \frac{I_w}{I_f} \quad \text{(S-16)}$$

such that we can rewrite eq. S-15 as

$$\tau_I = I_f(1 + M)\alpha. \quad \text{(S-17)}$$

Finally, we estimate the rotational acceleration of the wing by approximating the stroke profile as a sinusoid with amplitude $\Phi/2$, which yields a maximum angular acceleration at stroke reversal

$$\ddot{\psi} = 2\Phi \pi^2 f^2. \quad \text{(S-18)}$$

Combining eq. S-14, eq. S-17, and eq. S-18 we find that

$$IE = I_f(1 + M)(2\Phi \pi^2 f^2) \kappa. \quad \text{(S-19)}$$

The inertia due to fluid mass can be found by integrating the fluid density and the distance from the axis of rotation over the volume of added mass,

$$I_f = \int\int\int \rho(c + y_r)^2 d\hat{z} dr dc. \quad \text{(S-20)}$$

We assume that the added fluid mass in the transverse direction to the wing planform is proportional to the distance from the rotational axis, $z = (c + y_r)$[7] and after evaluating the integral (eq. S-20) and non-dimensionalizing, we arrive at

$$I_f = \rho c^4 R\hat{c}_3^3, \quad \text{(S-21)}$$

where $\hat{c}_3^3$ is the third chord moment cubed, defined in a manner reminiscent of $\hat{r}_2^2$ defined earlier:

$$\hat{c}_3^3 = \int_0^{\hat{c}_0} (\hat{c} + \hat{y}_r)^3 \hat{r}(\hat{c}) d\hat{c}. \quad \text{(S-22)}$$
Next, we will derive an expression for the inertia due to the wing mass. Since the axis of rotation of a pitching wing does not necessarily intersect the center of mass of the wing, the rotational inertia of the wing is

\[ I_w = m_w y_{ci}^2 + m_w (y_{cm} + y_r)^2. \] (S-23)

Here, \( m_w \) is the mass of the wing, \( y_{cm} \) is the location of the center of mass of the wing, and \( y_{ci} \) is the center of inertia of the wing with respect to rotation about the span-wise axis. The quantities \( y_{cm} \) and \( y_{ci} \) are defined

\[ y_{cm} = \frac{\int \int \int \rho \, c \, dV}{\int \int \int \rho \, dV}, \] (S-24)

and

\[ y_{ci}^2 = \frac{\int \int \int \rho \, c^2 \, dV}{\int \int \int \rho \, dV}, \] (S-25)

respectively.

By combining eq. S-21 and eq. S-23, we arrive at an expression for the Mass number,

\[ M = \frac{m_w (c_i^2 + (y_{cm} + y_r)^2)}{\rho \bar{c}^4 R_c^3}. \] (S-26)

Then assuming that \( M \) has the form in eq. S-26, we can combine eq. S-19 and eq. S-21 to write the following expression for \( IE \):

\[ IE = \frac{2 \rho \bar{c}^4 R \Phi \pi^2 f^2 \bar{c}_3}{\kappa} (1 + M). \] (S-27)
Experimental data regarding lift and drag coefficients for unidirectionally revolving wings was retrieved from [8] (fig. 6A,C,D; triangles). Six different data sets characterized by single wing aspect ratios (equivalently, Rossby numbers) and Reynolds numbers of relevance to the passive-pitching flapping wing experiments reported and reference in the current work were selected. The lift and drag coefficients ($C_L$ and $C_D$, respectively) as functions of the pitch deflection angle, $\psi$, were approximated by sinusoidal functions and later used in the models described in this paper. The data sets and sinusoidal approximations are shown below.
Re = 110, AR = \infty

Lift Coefficient, $C_L$

Experiment Data
$CL(\psi) = 0.8 \sin(2\psi/\pi)$

Drag Coefficient, $C_D$

Experiment Data
$CD(\psi) = 1.0 - 0.5 \cos(2\psi/\pi)$

Re = 14000, AR = 2.9

Lift Coefficient, $C_L$

Experiment Data
$CL(\psi) = 1.9 \sin(2\psi/\pi)$

Drag Coefficient, $C_D$

Experiment Data
$CD(\psi) = 1.9 - 1.9 \cos(2\psi/\pi)$
Re = 1400, AR = 3.6

Lift Coefficient, $C_L$

$CL(\psi) = 1.7 \sin(\pi/2 - \psi)$

Experiment Data

Drag Coefficient, $C_D$

$CD(\psi) = 1.5 \cdot 1.4 \cos(\pi/2 - \psi)$

Experiment Data
References


