Electronic Supplementary Materials

1 Montonicity of \( \hat{g}_k \) for \( k \geq k_m \) and calculation of \( \epsilon_g \)

Since Theorem 2.11 in the main text relies on calculation of \( \epsilon_g := \sup_{k \geq K+1} |\hat{g}(k)| \) and we cannot calculate \( |\hat{g}(k)| \) indefinitely, using the conclusion of Proposition 2.2 main paper, whose proof is completed in the following sub-section, we seek to show that there exists some integer \( k_m \) so that \( |\hat{g}(k)| \) monotonically decreases for integer \( k \geq k_m \). By showing monotonicity, we will have deduced

\[
\epsilon_g = \sup_{k_m \geq k \geq K+1} |\hat{g}(k)|, 
\]

which is readily calculated.

We notice first that

\[
|\hat{g}(k)|^2 = \frac{1}{\nu^2 k^6 \left((1 + a(k))^2 + b^2(k)\right)},
\]

\[
a(k) = \frac{1}{\nu k^4} \Re N[k], \quad b(k) = \frac{C_0}{\nu k^3} - \frac{1}{\nu k^4} \Im N[k],
\]

and therefore monotonic decrease of \( \hat{g}(k) \) with increasing integer \( k \) is equivalent to showing

\[
(k + 1)\left((1 + a(k + 1))^2 + b^2(k + 1)\right) - k^6 \left((1 + a(k))^2 + b^2(k)\right) > 1,
\]

which is equivalent to showing that

\[
\frac{(k + 1)^6 - k^6}{k^5} \left(1 + 2a(k + 1) + a^2(k + 1) + b^2(k + 1)\right) > (2k|a(k + 1)| - 2ka(k) + ka^2(k + 1) - ka^2(k) + kb^2(k + 1) - kb^2(k)).
\]

It is clearly enough to ensure that

\[
\frac{(k + 1)^6 - k^6}{k^5} \left(1 - 2|a(k + 1)|\right) > (2k|a(k + 1)| + 2ka(k) + ka^2(k + 1) + kb^2(k + 1)).
\]

We choose

\[
k_m = \max \left\{ \sqrt{\frac{3}{\nu}} R, 6\nu^{-1/2} \right\}
\]

Using Proposition 2.2, the inequalities in equation (2.7) in the main text hold, and therefore,

\[
|N[k]| \leq \Lambda k^2 \gamma_0, \quad \text{where} \quad \gamma_0 = \frac{(1 + 2.24 \times 10^{-4})}{(1 - 1.61 \times 10^{-3})(1 - 0.051)}, \quad \Lambda = \frac{R}{4\nu}\gamma_0 + (\gamma_0 - 1)\Lambda k^2. \quad (8)
\]

It follows at once that for \( k \geq k_m \),

\[
|ka(k)| \leq \frac{RA\gamma_0}{4\nu^2 k^3} + (\gamma_0 - 1)\frac{\Lambda}{\nu k}, \quad \leq \frac{\Lambda\gamma_0}{4\nu^{3/2} R^2} + \frac{(\gamma_0 - 1)\Lambda}{\sqrt{3}\nu R}, \quad |b(k)| \leq \frac{|C_0|}{\nu k^3} + \frac{\gamma_0\Lambda}{\nu k^2},
\]

\[
\text{(10)}
\]

We choose

\[
k_m = \max \left\{ \sqrt{\frac{3}{\nu}} R, 6\nu^{-1/2} \right\}
\]

Using Proposition 2.2, the inequalities in equation (2.7) in the main text hold, and therefore,

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|ka(k)| \leq \frac{RA\gamma_0}{4\nu^2 k^3} + (\gamma_0 - 1)\frac{\Lambda}{\nu k}, \quad \leq \frac{\Lambda\gamma_0}{4\nu^{3/2} R^2} + \frac{(\gamma_0 - 1)\Lambda}{\sqrt{3}\nu R}, \quad |b(k)| \leq \frac{|C_0|}{\nu k^3} + \frac{\gamma_0\Lambda}{\nu k^2},
\]

\[
\text{(10)}
\]
and
\[ ka^2(k) + kb^2(k) \leq \frac{C^2}{v^2k^5} + \frac{2|C_0|\gamma_0\Lambda}{v^2k^4} + \frac{\Lambda^2\gamma_0^2}{v^2k^3} \]  
(12)

Therefore, using (10)-(12), (6) is confirmed for \( k \geq k_m \) if
\[ 6\left(1 - \frac{\Lambda\gamma_0}{18R^3} - 2(\gamma_0 - 1)\frac{\Lambda}{\sqrt{3vR}}\right) > \frac{\Lambda\gamma_0}{\sqrt{v^33^2/2R^2}} + \frac{4(\gamma_0 - 1)\Lambda}{\sqrt{3vR}} + \frac{C_0^2}{9k_mR^2} + \frac{2|C_0|\gamma_0\Lambda}{9R^4} + \frac{\Lambda^2\gamma_0^2}{3^3/2\sqrt{vR^3}}. \]
(13)

In the parameter space explored, (13) was always valid. We calculated \( \epsilon_g \) based on (1).

2 Determination of \( \mathcal{N}[k] \) and proof of Proposition 2.2, main text.

Recall that we need to calculate
\[ \mathcal{N}[k] = -\frac{iA}{v} \left( \frac{k\sqrt{v}}{2} \right) F''(0, k\sqrt{v}), \]
(14)

with \( \alpha = k\sqrt{v} \), and \( F \) satisfying the Orr-Sommerfield two point boundary value problem
\[ \left( \frac{d^2}{dy^2} - \alpha^2 \right)^2 F(y; \alpha) - i\alpha Ry \left( \frac{d^2}{dy^2} - \alpha^2 \right) F(y; \alpha) = 0, \]
(15)
\[ F(0; \alpha) = 0, \quad F'(0; \alpha) = 1, \quad F(1; \alpha) = 0, \quad F'(1; \alpha) = 0. \]
(16)

Using vorticity, \( w = \left( \frac{d^2}{dy^2} - \alpha^2 \right) F \), it is clear that we may write
\[ w(y) = C_1 \text{Ai}(z) + C_2 \text{Ai}(\omega z), \quad z = (\alpha R)^{1/3} \left(y - \frac{i\alpha}{R}\right), \quad \text{where} \quad \omega = e^{2i\pi/3}. \]
(17)

It follows that
\[ F(y; \alpha) = C_1 A_1(y; \alpha) + C_2 A_2(y; \alpha) + C_3 \sinh(\alpha y) + C_4 \cosh(\alpha y), \]
(18)

where, with \( \zeta' = (i\alpha R)^{1/3} (y' - i\alpha/R) \), we define
\[ A_1(y, \alpha) = \frac{1}{\alpha} \int_0^y \sinh \left( \alpha(y - y') \right) \text{Ai}(\zeta') dy', \quad A_2(y, \alpha) = \frac{1}{\alpha} \int_0^y \sinh \left( \alpha(y - y') \right) \text{Ai}(\omega z') dy'. \]
(19)

It is convenient to define images of \( y = 0 \) under the mapping \( z(y) \) to be \( z_0, z_1 \) respectively and similarly the images of those points under \( \omega z(y) \) to be \( z_2 \) and \( z_3 \) respectively. Calculation shows
\[ z_0 = e^{-i\pi/3} \alpha^4/3 R^{-2/3}, \quad z_1 = z_0 \left( 1 + \frac{iR}{\alpha} \right), \quad z_2 = e^{i\pi/3} \alpha^4/3 R^{-2/3}, \quad z_3 = z_2 \left( 1 + \frac{iR}{\alpha} \right). \]
(20)

It is to be noted that when \( \alpha^2 >> R \), each \( z_0 \) and \( z_1 \) are large, with arg \( z_0 = -\pi/3 \) while arg \( z_1 \in (-\pi, \pi) \), and indeed close to arg \( z_0 \) when \( \alpha >> R \). Note that \( A_1'(0, \alpha) = A_1(0, \alpha) = 0 = A_2(0, \alpha) = A_2'(0, \alpha) \). Satisfying boundary conditions (16) completely determines \( C_1, C_2, C_3 \) and \( C_4 \) in (18) and hence \( F(y; \alpha) \), which allows us to express
\[ F''(0; \alpha) = \frac{n(\alpha)}{D(\alpha)}, \]
(21)
where
\[ D(\alpha) = \alpha (A_2(1)A_1'(1) - A_1(1)A_2'(1)) \],
\[ n(\alpha) = \alpha \cosh(\alpha)N_1(\alpha) + \sinh(\alpha)N_2(\alpha) \],

and
\[ N_1(\alpha) = B_2A_1(1) - A_2(1)B_1 \], \quad N_2(\alpha) = B_1A_2'(1) - B_2A_1'(1),
\[ B_1(\alpha) = \text{Ai}(z_0), \quad B_2(\alpha) = \text{Ai}(z_2). \]

It is also convenient to define
\[ \lambda_1 = e^\alpha e^{-\frac{3}{2}z_0^{\frac{1}{2}}} \alpha^{-1}, \quad \lambda_2 = e^{-\alpha} e^{-\frac{3}{2}z_0^{\frac{1}{2}}} \alpha^{-1}, \]

and integrals
\[ I_1 = \int_{z_0}^{z_1} e^{-\frac{1}{2}z} \text{Ai}(z) \, dz, \quad I_2 = \int_{z_0}^{z_1} e^{\frac{1}{2}z} \text{Ai}(z) \, dz, \]
\[ I_3 = \omega^{-1} \int_{z_2}^{z_3} e^{\frac{1}{2}z} \text{Ai}(z) \, dz, \quad I_4 = \omega^{-1} \int_{z_2}^{z_3} e^{-\frac{1}{2}z} \text{Ai}(z) \, dz. \]

It follows from (19) that
\[ A_1(1; \alpha) = \frac{1}{2\alpha} (\lambda_1 I_1 - \lambda_2 I_2), \quad A_1'(1; \alpha) = \frac{1}{2} (\lambda_1 I_1 + \lambda_2 I_2), \]
\[ A_2(1; \alpha) = \frac{1}{2\alpha} (\lambda_1 I_3 - \lambda_2 I_4), \quad A_2'(1; \alpha) = \frac{1}{2} (\lambda_1 I_3 + \lambda_2 I_4). \]

Therefore,
\[ D(\alpha) = \frac{\lambda_1 \lambda_2}{2} (I_2 I_3 - I_1 I_4) \]
\[ n(\alpha) = \frac{1}{2} [\lambda_1 e^{-\alpha} (B_2 I_1 - B_1 I_3) - \lambda_2 e^\alpha (B_2 I_2 - B_1 I_4)] \]

and so, using \( \alpha = k\sqrt{\nu} \) and expression for \( F''(0, \alpha) \) in (21), it follows from (14) that
\[ N[k] = -\frac{iA}{2\nu} \left( \frac{\alpha n(\alpha)}{D(\alpha)} \right) = -\frac{iA}{2\nu} \left( \frac{\lambda_1 e^{-\alpha} (B_2 I_1 - B_1 I_3) - \lambda_2 e^\alpha (B_2 I_2 - B_1 I_4)}{\lambda_1 \lambda_2 (I_2 I_3 - I_1 I_4)} \right). \]

2.1 Details of the proof of Proposition 2.2, main text

Recall from the main part of the paper the functions
\[ H_0(z) = \exp \left[ \frac{2}{3} z^{3/2} \right] \text{Ai}(z), \quad H_j(z, z_0) = \frac{d}{dz} \left[ \frac{H_{k-1}(z, z_0)}{z^{1/2} + z_0^{1/2}} \right] \text{ for } j \geq 1, \]
\[ \mathcal{U}(z) = z^{-1/2} H_0(z), \quad \mathcal{V}(z) = z^{-1/2} \mathcal{U}'(z), \]
\[ H_1(z, z_0) = m \mathcal{U}'(z) + \frac{s}{2z} m^2 \mathcal{U}(z), \]
\[ s = z^{-1/2} \alpha_0^{1/2}, \quad m = (1 + s)^{-1}, \]
\[ H_2(z, z_0) = m^2 \mathcal{V}'(z) + \frac{3s}{2z^{3/2}} m^3 \mathcal{U}'(z) + \left( \frac{3s^2}{4z^{5/2}} m^4 - \frac{s}{2z^2} m^3 \right) \mathcal{U}(z). \]
We will find convenient sometimes to use $H_0(z, z_0) \equiv H_0(z)$. In addition, define

$$J(\tau) = \left(1 + \left[1 + \tau^2\right]^{1/4}\right) (1 + \tau^3)^{1/8}. \quad (38)$$

We will also need the functions $k_{0,t}$, $k_{1,t}$, $k_{1,v}$, $C_k$, $f_{v',v''}$ that are defined by equations (43)-(46) in Section 4 of the main article.

**Definition 1.** We define the straight line segments $L_0$ and $L_2$:

$L_0 := \{z : \Re z = z_0 + t(z_1 - z_0), t \in [0, 1]\}, \quad L_2 := \{z : \Re z = z_3 + t(z_2 - z_3), t \in [0, 1]\}. \quad (39)$

**Corollary 1.** $H_0(z)$ defined in (40) satisfies the following upper and lower bounds at any point on $L_0$ and $L_2$ in the regime $\alpha = k\sqrt{\nu} \geq \max \left\{\sqrt{3R}, \alpha_r\right\}$.

$$\frac{|z|^{-1/4}}{2\sqrt{\pi}} (1 - k_{0,t}(z_0)) \leq |H_0(z)| \leq \frac{|z|^{-1/4}}{2\sqrt{\pi}} (1 + k_{0,t}(z_0)) =: C_0|z|^{-1/4} \leq C_0|z_0|^{-1/4}. \quad (40)$$

**Proof.** First we note that either on $L_0$ or $L_2$, $|z| \geq |z_0|$, since $z_1 = z_0(1 + iR\alpha^{-1})$ and $z_3 = z_2(1 + iR\alpha^{-1})$ and $z_2 = z_0^\ast$. Regime $\alpha = k\sqrt{\nu} \geq \max \left\{\sqrt{3R}, \alpha_r\right\}$ ensures that $|z| \geq 2$ and arg $z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and using the definition (40) for $H_0(z)$, Lemma 4.7 of the main article is applicable and we obtain the given bounds, noting that $k_{0,t}(z) \leq k_{0,t}(z_0)$ (See Remark 4.6 in the paper).

**Lemma 1.** $H_j$ defined in (40) for $j = 1, 2$ satisfy the following bounds for any point on line segments $L_0$ and $L_2$ in the regime $\alpha = k\sqrt{\nu} \geq \max \left\{\sqrt{3R}, \alpha_r\right\}$, where we define $\hat{z}_0 = z_0$ on $L_0$ and $\hat{z}_2 = z_2$.

$$|H_1(z, \hat{z}_0)|, \leq C_1|z_0|^{-7/4}, \quad |H_2(z, \hat{z}_0)| \leq C_2|z_0|^{-13/4}, \quad (41)$$

where

$$C_1 = \frac{3}{8\sqrt{\pi}} \left(1 + \epsilon_{U'}(z_0)\right) + \frac{1}{4\sqrt{\pi}} \left(1 + \epsilon_U(z_0)\right), \quad (42)$$

$$C_2 = \frac{1}{\sqrt{\pi}} \left\{\frac{27}{32} \left(1 + \epsilon_V(z_0)\right) + \frac{9}{16} \left(1 + \epsilon_{U'}(z_0)\right) + \frac{7}{8} \left(1 + \epsilon_U(z_0)\right)\right\}. \quad (43)$$

For $|H_1(z_3, z_2)|$, we also have the sharper bound

$$|H_1(z_3, z_2)| \leq C_2|z_3|^{-7/4} \quad (44)$$

**Proof.** On $z \in L_0 \cup L_2$,

$$s = z^{-1/2}z_0^{1/2} = (1 + iR\alpha^{-1})^{-1/2} \text{ for } t \in [0, 1] \quad (45)$$

and it is clear that $\Re s \geq 0$ and $|s| \leq 1$ in both cases, and so $|m| = |1 + s|^{-1} \leq 1$. We also note that for any $\alpha \geq 0$, for $z$ on these straight line segments, $|z|^{-\alpha} = |z_0|^{-\alpha}|s|^{z_0} \leq |z_0|^{-\alpha}$. Combining (40) with bounds on $U$ and $U'$ in Lemma 4.7 of the main part, we obtain

$$|H_1(z, \hat{z})| \leq |U'(z)| + \frac{1}{2|z|} |U(z)| \leq \frac{3}{8\sqrt{\pi}|z_0|^{7/4}} \left(1 + \epsilon_{U'}(z_0)\right) + \frac{1}{4\sqrt{\pi}|z_0|^{7/4}} \left(1 + \epsilon_U(z_0)\right), \quad (46)$$

and using (47) we have

$$|H_2(z, \hat{z})| \leq |V'(z)| + \frac{3}{2|z|^{3/2}} |U'(z)| + \frac{7}{4|z|^{5/2}} |U(z)| \leq \frac{1}{\sqrt{\pi}|z_0|^{13/4}} \left\{\frac{27}{32} \left(1 + \epsilon_V(z_0)\right) + \frac{9}{16} \left(1 + \epsilon_{U'}(z_0)\right) + \frac{7}{8} \left(1 + \epsilon_U(z_0)\right)\right\} \quad (47)$$
For $H_1(z_3, z_2)$, we note from the definition of $H_1(z, z_0)$ that since $s = z_3^{-1/2}z_2^{1/2} = (1 + iR\alpha^{-1})$ and $m = 1/(1 + s)$ are each bounded by 1,

$$|H_1(z_3, z_2)| \leq |U'(z_3)| + \frac{1}{2|z_3|}|U'(z_3)|$$

(48)

The rest follows from bounds on $U$ and $U'$ in Lemma 4.7 in the main part, the observation $|z_3| > |z_2| = |z_0|$, and the fact that each of $\epsilon_{0,U}$, $\epsilon_{1,U}$ are decreasing with $|z|$. 

**Lemma 2.** $H_0(z_0)$ and $H_1(z_0, z_0)$ satisfy the following bound

$$\left| \frac{H_1(z_0, z_0)}{H_0(z_0)} + \frac{1}{4z_0^{3/2}} \right| \leq \frac{1}{4|z_0|^{3/2}} \epsilon_{1,0},$$

(49)

where

$$\epsilon_{1,0} = \frac{3}{2} \left\{ \left( 1 + \frac{5}{16|z_0|^{3/2}} + 8\frac{\sqrt{\pi}k_{1,U}(z_0)}{3} \right) \left( 1 - \frac{5}{48|z_0|^{3/2}} - 2\sqrt{\pi}k_{0,U}(z_0) \right)^{-1} - 1 \right\}$$

(50)

**Proof.** From (33) and $H_1(z, z_0) = mL'(z) + \frac{\pi}{2\sqrt{2}} m^2U(z)$,

$$\frac{H_1(z_0, z_0)}{H_0(z_0)} = \frac{1}{8z_0^{3/2}} + \frac{U'(z_0)}{2z_0^{1/2}U(z_0)}$$

(51)

Using Lemma 4.7 of the main part, we obtain

$$\frac{U'(z_0)}{U(z_0)} = -\frac{3}{4z_0} \left( 1 - \frac{5}{16z_0^{3/2}} + \frac{\sqrt{\pi}K_{1,U}(z_0)}{3} \right) \left( 1 - \frac{5}{48z_0^{3/2}} + 2\sqrt{\pi}K_{0,U}(z_0) \right)^{-1},$$

(52)

where $K_{1,U}, K_{0,U}$ are bounded, respectively, by $k_{1,U}$ and $k_{0,U}$, defined in equations (4.39)-(4.40). Therefore,

$$\left| \frac{H_1(z_0, z_0)}{H_0(z_0)} + \frac{1}{4z_0^{3/2}} \right| \leq \frac{1}{4|z_0|^{3/2}} \epsilon_{1,0}$$

(53)

with $\epsilon_{1,0}$ as defined above. 

**Lemma 3.** Define, for $t \in [0,1]$,

$$h_0(t) = \frac{-2i\alpha^2}{3R} \left( 1 + \frac{iR}{\alpha}t \right)^{3/2} + \frac{2i\alpha^2}{R},$$

(54)

$$h_2(t) = \frac{2i\alpha^2}{3R} \left( 1 + \frac{iR}{\alpha}(1-t) \right)^{3/2} - \frac{2i\alpha^2}{3R} \left( 1 + \frac{iR}{\alpha} \right)^{3/2}.$$  

(55)

Then, for any $t \in [0,1]$, and with $\tau = \frac{R\alpha}{\alpha}$,

$$\frac{d}{dt} \Re h_0(t) = \alpha \left( 1 + \frac{\tau^2}{\sqrt{2}\sqrt{1 + (1 + \tau^2)^{1/2} + 2}} (1 + (1 + \tau^2)^{1/2}) \right) \geq \alpha,$$

(56)

and

$$\frac{d}{dt} \Re h_2(t) \geq \alpha,$$

(57)

implying in each case that

$$\Re h_0(t), \Re h_2(t) \geq \alpha t.$$  

(58)
Proof. On differentiation and considering the first case we have

\[ \frac{d}{dt} \Re h_0(t) = \alpha \Re \left( 1 + \frac{iR}{\alpha} t \right)^{1/2}, \quad \frac{d}{dt} \Re h_2(t) = \alpha \Re \left( 1 + \frac{iR}{\alpha} (1 - t) \right)^{1/2}. \] (59)

The rest follows from trigonometric simplification of

\[ \Re (1 + i\tau)^{1/2} = (1 + \tau^2)^{1/4} \cos \left( \frac{1}{2} \arctan \tau \right), \] (60)

and using integration with initial condition \( h_0(0) = 0 = h_2(0). \) □

Corollary 2. On any point on the straight line segment \( L_0 \) parameterized by \( t \in [0, 1] \), define,

\[ g_0(t) = \frac{2}{3} \left( z^{3/2} - z_0^{3/2} \right) + z_0^{1/2} (z - z_0). \] (61)

Similarly, on any point on the straight line \( L_2 \) parametrized by \( t \in [0, 1] \), define

\[ g_2(t) = \frac{2}{3} \left( z^{3/2} - z_0^{3/2} \right) + z_2^{1/2} (z - z_3). \] (62)

Then, in either case,

\[ \Re g_0(t), \Re g_2(t) \geq 2\alpha t \] (63)

Proof. On \( L_0 \) using \( z_0^{3/2} = -\frac{iR^2}{\alpha^2} \), \( z/z_0 = 1 + \frac{iR}{\alpha} t \), we get in terms of \( h_0 \) defined in the last Lemma,

\[ g_0(t) = h_0(t) + \alpha t \quad \Rightarrow \quad \Re g_0(t) \geq 2\alpha t. \] (64)

On \( L_2 \), using \( z_2^{3/2} = \frac{iR^2}{\alpha^2} \), \( z/z_2 = 1 + \frac{iR}{\alpha} (1 - t) \), we obtain in terms of \( h_2(t) \) defined in the last Lemma

\[ g_2(t) = h_2(t) + \alpha t \quad \Rightarrow \quad \Re g_2(t) \geq 2\alpha t. \] (65)

□

Lemma 4. \( I_1 \) defined in (27) may be expressed as

\[ I_1 = \frac{1}{2z_0^{1/2}} e^{-\frac{1}{2} z_0^{3/2}} \left[ H_0(z_0) + H_1(z_0, z_0) + \frac{R_1}{2\sqrt{\pi} z_0^{1/4}} \right] \] (66)

where \( R_1 \) satisfies the bound

\[ |R_1| \leq 4\sqrt{\pi} e^{-2\alpha} \left( C_0 + C_1 |z_0|^{-3/2} \right) + 2\sqrt{\pi} C_2 |z_0|^{-3} := k_1. \] (67)

In particular, we have the upper and lower bounds

\[ C_{m,1} |z_0|^{-3/4} \leq |I_1| \leq C_{I,1} |z_0|^{-3/4}, \] (68)

where

\[ C_{I,1} = \frac{1}{2} \left( C_0 + C_1 |z_0|^{-3/2} + \frac{k_1}{2\sqrt{\pi}} \right), \quad C_{m,1} = \frac{1}{2} \left( \frac{1 - \epsilon_U(z_0)}{2\sqrt{\pi}} - C_1 |z_0|^{-3/2} - \frac{k_1}{2\sqrt{\pi}} \right). \] (69)
Proof. From the definitions of $I_1$ and $H_0$ in [27] and (33), we note that

$$I_1 = \int_{z_0}^{z_1} e^{-\frac{1}{2} z^2 - \frac{2}{3} z^3} H_0(z) dz.$$  \hspace{1cm} (70)

On integration by parts twice, we obtain

$$I_1 = \left[ \frac{e^{-\frac{1}{2} z_0^2 - \frac{2}{3} z_0^3}}{z_0^{1/2} + z_0^{1/2}} (H_0(z) + H_1(z, z_0)) \right]_{z=z_0}^{z=z_1} + \int_{z_0}^{z_1} e^{-\frac{1}{2} z^2 - \frac{2}{3} z^3} H_2(z) dz.$$  \hspace{1cm} (71)

Therefore, we are able to write

$$I_1 = \frac{1}{2z_0^{1/2}} e^{\frac{2}{3} z_0^{3/2}} \left[ H_0(z_0) + H_1(z_0, z_0) + \frac{R_1}{2\pi z_0^{1/2}} \right],$$  \hspace{1cm} (72)

provided, we identify $R_1 = R_{1,1} + R_{1,2}$, where

$$R_{1,1} = -\frac{4\sqrt{\pi} z_0^{3/4}}{z_0^{1/2} + z_1^{1/2}} [H_0(z_1) + H_1(z_1, z_0)] e^{-g_0(1)},$$  \hspace{1cm} (73)

$$R_{1,2} = 4\sqrt{\pi} z_0^{3/4} \int_{z_0}^{z_1} e^{-g_0(t)} H_2(z) dz.$$  \hspace{1cm} (74)

We note that the exponents in $R_{1,1}, R_{1,2}$ are bounded by $e^{-2\alpha}$ and $e^{-2\alpha t}$, respectively. We also note the global bounds on $H_1$ and $H_2$ on any point on the straight line segment $L_0$ connecting $z_0$ to $z_1$ in Lemma [1]. Furthermore, since $z_1 = z_0 \left(1 + \frac{iR}{\alpha}\right)$, then $|z_1^{-1/2} z_0^{1/2}| \leq 1$ and $|1 + z_0^{-1/2} z_1^{1/2}|^{-1} \leq 1$. Further in the integral in $R_{1,2}$, with $t$ parametrization of $L_0$, we obtain $dz = (z_1 - z_0) dt = \frac{R_0 z_0}{\alpha} dt$, while $\int_{z_0}^{z_1} e^{-2\alpha t} dt \leq \frac{1}{2\alpha}$, $R_0 = |z_0|^{-3/2}$. With this information, we readily obtain

$$|R_{1,2}| \leq 2\sqrt{\pi} C_2 |z_0|^{-3}, \quad |R_{1,1}| \leq 4\sqrt{\pi} e^{-2\alpha} \left(C_0 + C_1 |z_0|^{-3/2}\right),$$  \hspace{1cm} (75)

from which the first statement of the Lemma follows. The second statement follows from the first after some algebraic manipulation. \[\square\]

Remark 1. Note that for large $\alpha$, $k_1$ becomes small and approaches zero. The point of the above Lemma is to show precise bounds when $\alpha$ is some finite number, and therefore makes it precise how large is large.

Lemma 5. $I_4$ defined in (27) may be expressed as

$$\omega I_4(z) = -\frac{e^{-\frac{1}{2} z_2^2 - \frac{2}{3} z_2^3}}{z_2^{1/2} + z_3^{1/2}} \left( H_0(z_3) + H_1(z_3, z_2) - \frac{1}{2\sqrt{\pi} |z_3|^{1/2}} R_4(z) \right),$$  \hspace{1cm} (76)

where $R_4$ satisfies the bound

$$|R_4| \leq \sqrt{\pi} J \left( \frac{R}{\alpha} \right) \left\{ e^{-2\alpha} \left(C_0 + C_1 |z_0|^{-3/2}\right) + C_2 |z_0|^{-3} \right\} =: k_4,$$  \hspace{1cm} (77)

and $J(\tau)$ is defined in (38). In particular $I_4$ satisfies the lower bound

$$|\exp \left[ z_2^{1/2} z_3 + \frac{2}{3} z_3^{3/2} \right] I_4| \geq \frac{C_{m,4}}{J \left( \frac{R}{\alpha} \right)} |z_0|^{-3/4},$$  \hspace{1cm} (78)

where

$$C_{m,4} = \frac{1 - \epsilon U(z_0)}{2\sqrt{\pi}} - C_1 |z_0|^{-3/2} - \frac{1}{2\sqrt{\pi}} k_4.$$  \hspace{1cm} (79)
Proof. Using (20), since \( z_2^{3/2} = \frac{\alpha t}{R} \) and \( z_3 = z_2 \left( 1 + \frac{iR}{\alpha} \right) \), it follows from (27) that
\[
I_4 = \omega^{-1} \int_{z_2}^{z_3} \exp \left[ -\frac{z_2^{3/2}}{3} - \frac{2}{3} z_3^{3/2} \right] H_0(z) dz. \tag{80}\]
On integration by parts twice, as for \( I_1 \), and using the straight line segment \( L_2 \) for integration, where \( dz = (z_2 - z_3) dt = -\frac{iR}{\alpha z_2} dt \), we obtain
\[
\omega \exp \left[ \frac{1}{2} z_2^{3/2} + \frac{2}{3} z_3^{3/2} \right] I_4 = -\frac{H_0(z_3) + H_1(z_3, z_2)}{z_3^{1/2} + z_2^{1/2}} \left( 1 - \frac{1}{2\sqrt{\pi} |z_3|^{3/2}} R_4 \right), \tag{81}\]
where
\[
R_4 = \sqrt{\pi} z_3^{1/4} \left( 1 + z_3^{1/2} z_2^{-1/2} \right) (H_0(z_2) + H_1(z_2, z_2)) e^{-g_2(t)}
+ \frac{2iR}{\alpha} \sqrt{\pi} z_2^{3/4} \left( 1 + z_2^{1/2} z_3^{-1/2} \right) \frac{1}{R} e^{-g_2(t)} H_2 \left( z_3 + t(z_2 - z_3), z_2 \right) dt, \tag{82}\]
with \( g_2 \) as defined in (82). Thus the exponential terms are bounded by \( e^{-2\alpha t} \) and \( e^{-2\alpha t} \), respectively. Using bounds on \( H_1 \) from Lemma 3 for any point \( z \in L_2 \), and noting \( \frac{2R}{\alpha} \int_0^1 e^{-2\alpha t} dt \leq \frac{R}{\alpha \gamma} = |z_0|^{-3/2} \), the first statement in the Lemma follows very much like the previous Lemma, except that we have an algebraic factor of
\[
\left| 1 + z_3^{1/2} z_2^{-1/2} \right| \left( \frac{z_2}{z_3} \right)^{1/4} \leq J \left( \frac{R}{\alpha} \right). \tag{83}\]
The second part of the Lemma clearly follows from the first on algebraic manipulation where we use \( \frac{z_2}{z_3} = 1 + \frac{iR}{\alpha} \) and \( |z_2| = |z_0| \).

Remark 2. Since
\[
\frac{2}{3} z_2^{3/2} - \frac{2}{3} z_3^{3/2} + z_2^{1/2} (z_2 - z_3) = g_2(1), \tag{84}\]
and \( \Re g_2(1) \geq 2\alpha \), while \( \Re z_2^{3/2} = 0 \), it follows that \( \left| \exp \left[ -\frac{2}{3} z_3^{3/2} - z_2^{1/2} z_3 \right] \right| \geq e^{2\alpha} \), and the lower bounds in the previous Lemma show that \( I_4 \) is exponentially large in \( \alpha \). This exponentially large lower bound for \( I_4 \) for \( \alpha \) large is significant, as it allows massive simplification of \( N(k) \) as we shall see shortly.

Lemma 6. \( I_2 \) and \( I_3 \) defined in (27) satisfy the following bounds
\[
|I_2| \leq C_0 R^{1/2}, \quad \left| \exp \left[ -\frac{1}{3} z_0^{3/2} \right] I_3 \right| \leq C_0 R^{1/2}. \tag{85}\]
Proof. We take the straight line path \( L_0 \) connecting \( z_0 \) to \( z_1 \) in \( I_2 \) in (27) and obtain
\[
\exp \left[ -\frac{1}{3} z_0^{3/2} \right] I_2 = (z_1 - z_0) \int_0^1 e^{-\hat{g}_0(t)} H_0 (z_0 + t(z_1 - z_0)) dt, \tag{86}\]
where
\[
\hat{g}_0(t) = -tz_0^{3/2} (z_1/z_0 - 1) + h_0(t) = -\alpha t + h_0(t), \tag{87}\]
and \( h_0 \) is defined in Lemma 3 from which we can conclude that since \( \Re h_0 \geq \alpha t \), we must have
\[
\Re \hat{g}_0 \geq 0, \quad \text{implying } \left| e^{-\hat{g}_0(t)} \right| \leq 1. \tag{88}\]
Proof. Using (93) in Lemma 4, we have
\[ |I_2| \leq C_0 |z_0|^{3/4} \frac{R}{\alpha} = C_0 \sqrt{R}. \quad (88) \]

For \( I_3 \) defined in (27), again using a straight line path of integration \( z = z_3 + t(z_3 - z_2) \) and \( z_3^{3/2} = \frac{i\alpha^2}{R}, \ z_3/z_2 = 1 + \frac{R}{\alpha}, \) we obtain
\[ e^{-z_2^{3/2}z_3 + \frac{3}{2}z_3^{3/2}} I_3 = \left( z_3 - z_2 \right) \int_0^1 e^{-\hat{g}_2(t)} H_0 (z_3 + t(z_2 - z_3)) \, dt, \quad (89) \]

where in this case
\[ \hat{g}_2(t) = -tz_2^{3/2} \left( 1 - \frac{z_3}{z_2} \right) + h_2(t) = -\alpha t + h_2(t), \quad (90) \]

with \( h_2 \) defined in Lemma 3. Using that Lemma, \( \Re \hat{g}_2(t) \geq 0, \) implying
\[ |e^{-\hat{g}_2(t)}| \leq 1. \quad (91) \]

Using bounds on \( H_0(z) \) on the line segment \( L_2 \) in Lemma 4 and \( dz = -\frac{iRz}{\alpha} \, dt, \) we obtain
\[ |e^{-z_2^{3/2}z_3 + \frac{3}{2}z_3^{3/2}} I_3| \leq C_0 |z_2|^{3/4} \frac{R}{\alpha} = C_0 R^{1/2}. \quad (92) \]

\[ \square \]

**Lemma 7.** Define
\[ \hat{I}_1 = \frac{2z_0^{1/2}I_1}{H_0(z_0)} e^{\frac{3}{2}z_0^{3/2}}. \quad (93) \]

Then,
\[ |\hat{I}_1 - 1 + \frac{1}{4z_0^{3/2}}| \leq \hat{C}_1 |z_0|^{-3}, \quad (94) \]

where
\[ \hat{C}_1 = \frac{|z_0|^3 k_1}{1 - c_U(z_0)} + \frac{1}{4} |z_0|^{3/2} \epsilon_{1,0}. \quad (95) \]

Proof. Using (93) in Lemma 4, we have
\[ \hat{I}_1 - 1 + \frac{1}{4z_0^{3/2}} = \left( \frac{H_1(z_0, z_0)}{H_0(z_0)} + \frac{1}{4z_0^{3/2}} \right) + \frac{R_1}{2\sqrt{\pi} z_0^{3/4} H_0(z_0)}. \quad (96) \]

Hence, from the upper bound on \( R_3 \) in Lemma 4, the lower bound on \( H_0(z_0) \) in Corollary 1, and the bound in Lemma 2, the Lemma follows. \[ \square \]

**Remark 3.** It is to be noted that for large \( |z_0|, \) \( \hat{C}_1 = O(1), \) since it is clear from (50) and (77) that \( \epsilon_{1,0} = O(|z_0|^{-3/2}) \) and \( k_1 = O(|z_0|^{-3}). \)

**Lemma 8.** \( \mathcal{N}(k) \) in (32) may be also expressed as
\[ \mathcal{N}(k) = \frac{i\Lambda \alpha^2}{\nu I_1} \left( \frac{1 + E_1}{1 + E_2} \right), \quad (97) \]
where
\[ E_1 = -\frac{B_2 I_2}{B_1 I_4} + \frac{B_2 B_1^{-1} I_1 - I_3}{I_4} e^{2z_{3/2}^3} , \quad E_2 = \frac{I_2 I_3}{I_4 I_1} , \] (98)

and have exponential bounds in \( \alpha \) as follows
\[ |E_1| \leq e^{-2\alpha} J \left( \frac{R}{\alpha} \right) \left( \frac{C_{1.1}}{C_{m,1}} + \frac{2\alpha C_0}{C_{m,1}} \right) , \] (99)
\[ |E_2| \leq \frac{C_0^2}{C_{m,1}} \alpha^2 e^{-2\alpha} J \left( \frac{R}{\alpha} \right) . \] (100)

**Proof.** Dividing the numerator of [32] by \( \lambda_2 e^\alpha B_1 I_4 \), and the denominator by \(-\lambda_1 \lambda_2 I_1 I_4\), and noting the definitions of \( \lambda_1 \) and \( \lambda_2 \),
\[ -\frac{\lambda_2 \lambda_3 I_1 I_4}{\lambda_2 e^\alpha B_1 I_4} = \frac{\hat{I}_1}{2\alpha} , \] (101)
we obtain [97], with \( E_1, E_2, \hat{I}_1 \) as defined above. To determine bounds, we observe that \( H_0(z) \) is real valued for \( z \in \mathbb{R} \) and thus has complex conjugate symmetries, and that \( z_2 = z_0^* \), with \( z_0^{3/2}, z_2^{3/2} \in i\mathbb{R} \), and
\[ |B_2| = |\text{Ai}(z_2)| = |\text{Ai}(z_0)| = |B_1|. \] (102)

Also, it is clear from upper bounds on \( I_3 \) and lower bounds on \( I_4 \) that
\[ \left| \frac{I_3}{I_4} \right| \leq \left| e^{2z_{3/2}^3} \right| \left| C_{m,1}^{-1} |z_0|^{3/4} C_0 R^{-1/2} J \left( \frac{R}{\alpha} \right) \right| e^{-2\alpha} C_0 C_{m,1}^{-1} J \left( \frac{R}{\alpha} \right) , \] (103)
and that the same bound applies to \( \frac{I_2}{I_4} \). We also note that
\[ \left| \frac{I_1}{I_4} \right| \leq \left| e^{2z_{3/2}^3} \right| \left| C_{m,1}^{-1} |z_0|^{3/4} C_0 |z_0|^{-3/4} e^{-Rg_2(1)} \right| e^{-2\alpha} C_0 C_{m,1}^{-1} J \left( \frac{R}{\alpha} \right) . \] (104)
Combining, we get the upper bound for \( E_1 \). For \( E_2 \) we use lower bounds on \( I_4 \) and \( I_1 \) from Lemmas [5] and [4], and combine with upper bounds on \( I_2, I_3 \) in Lemma [6] to find
\[ |E_2| = \left| \frac{I_2 I_3}{I_4 I_1} \right| \leq \frac{C_0^2 \alpha^2}{C_{m,1} C_{m,1}} J \left( \frac{R}{\alpha} \right) e^{-2\alpha} . \] (105)

**Proof of Proposition 2.2, main text.** The stated proposition follows from Lemma [8] if we define
\[ E_A = \left( 1 - \frac{1}{4z_0^{3/2}} \right)^{-1} \left( \hat{I}_1 - 1 + \frac{1}{4z_0^{3/2}} \right) , \] (106)
and the bounds on \( E_A \) as stated in Lemma [7]. The exponential bound for \( E_1, E_2 \) is obvious in Lemma [8]. For \( E_A \), from estimates in [7], we only have a bound that decays with \( |z_0|^{-3} \). Since all the constants are monotonically decreasing with \( |z_0| = R^{-2/3} \alpha^{4/3} \) and \( J(R\alpha^{-1}) \leq J(1/\sqrt{3}) \), it follows that we can calculate bounds in the regime \( \alpha = k\sqrt{\nu} \geq \max \{ \sqrt{3R}, \alpha_r \} \), precisely by evaluation at \( \alpha = \alpha_r \), \( \frac{R}{\alpha} = \frac{1}{\sqrt{3}} \) which results in the quoted values.
3 Behaviour of bifurcation point for $R >> 1$, $\nu << 1$, in the regime $R^3/2 >> 1$.

We denote $\mathcal{N}_1/2[k]$ as the evaluation of $\mathcal{N}[k]$ for $\Lambda = 1/2$, in which case $\mathcal{N}[k] = 2\Delta \mathcal{N}_1/2[k]$. We require the asymptotics of $\mathcal{N}_1/2[k]$ for fixed $k$ large $R$, small $\nu$ in the regime stated. We recall that

$$\mathcal{N}_1/2[k] = -i 4\nu \frac{\alpha(n(\alpha))}{D(\alpha)}$$

where $D(\alpha)$ and $n(\alpha)$ are defined in terms of integrals of Airy functions given in (12) in the ESM. Now, with the restriction given it is easy to note that $z_0, z_2$ defined in (20) are each small, since $\alpha = k/\sqrt{R}$ is small; however, $z_1, z_3$ are large since each is clearly $O \left( R^{1/3} \nu^{1/6} \right)$. We also note that $\arg z_1 \sim \frac{\pi}{3}$, $\arg z_3 \sim \frac{\pi}{2}$, and the Airy function $\text{Ai}(z)$ is exponentially small near $z_1$ and exponentially large near $z_3$. Furthermore, rewriting

$$I_{1,2}(z) = \int_0^\infty e^{\pm \frac{\alpha}{\sqrt{3}} \frac{z}{z_0}} \text{Ai}(z) dz + \int_{z_0}^0 e^{\mp \frac{\alpha}{\sqrt{3}} \frac{z}{z_0}} \text{Ai}(z) dz + \int_{0}^\infty e^{\pm \frac{\alpha}{\sqrt{3}} \frac{z}{z_0}} \text{Ai}(z) dz,$$

it is clear that the last integral gives an exponentially small contribution and the leading two-order contribution comes from the first integral so that we have

$$I_{1,2} = \frac{1}{3} + \frac{3\sqrt{2}}{2\pi} \Gamma \left( \frac{2}{3} \right) z_0^{1/2} + O(z_0).$$

It follows that

$$\frac{I_2}{I_1} = 1 + 6\hat{a}_1 z_0^{1/2} + O(z_0), \quad \text{where} \quad \hat{a}_1 = \frac{3\sqrt{2}}{2\pi} \Gamma \left( \frac{2}{3} \right).$$

On the other hand because of exponentially large behaviour at $z_3$ of the integrands for $I_3$ and $I_4$ in (27), on integrating the known leading order asymptotics of $\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}z_3^{3/4}} e^{-2/3z_3^{3/2}}$, we find

$$\omega I_{3,4} = -\frac{1}{2\sqrt{\pi}z_3^{3/4}} e^{-2/3z_3^{3/2}} \left( 1 + \alpha + O(\alpha^2) \right),$$

where we used $z_2^{1/2} z_3 = -\alpha << 1$. Therefore, it follows that

$$\frac{I_3}{I_4} = 1 - 2\alpha + O(\alpha^2).$$

We also have $\alpha e^{-\alpha} \lambda_1 = z_0^{1/2} e^{3/2}$ and $\alpha e^\alpha \lambda_2 = z_0^{1/2} e^{-3/2}$, hence we may write

$$\frac{\alpha(n(\alpha))}{D(\alpha)} \sim \frac{B_1 z_0^{1/2} \left( e^{-3/2} - e^{3/2} \frac{l_4}{l_3} \right)}{z_0 \alpha - 2 \left( \frac{l_3}{l_1 l_4} - 1 \right)}$$

$$\sim \frac{B_1 \alpha^2}{z_0^{1/2} I_1} \sim - \frac{3B_1 \alpha^2}{z_0^{1/2} I_1} = - \frac{3\sqrt{3}}{2\pi} \frac{\alpha^4/3 R^{1/3}}{1 + O \left( \nu^{1/3} R^{-1/3}, \nu^{1/2} \right)}.$$
Considering the bifurcation point

\[-2\Lambda_b R \{N_{1/2}[k]\} = \nu k^4,\]  

(115)

it follows that for fixed \(k\) we have the asymptotic balance

\[
\frac{\Lambda_b^{3/3} \Re \left\{ \frac{N_{1/2}[k]}{2} \right\}}{4\nu^{4/3} \Gamma \left( \frac{2}{3} \right)} = k^{4/3} R^{1/3} = \nu k^4
\]

(116)

implying that \(\Lambda_b\) scales as \(\nu^{4/3} R^{-1/3}\), whereas

\[C_b = 2k^{-1} \Lambda_b \Im \{N_{1/2}[k]\},\]  

(117)

which implies that \(C_b\) scales as \(\nu\), but is independent of \(R\) to the leading order.

4 Additional quasi-solutions and checking conditions of Theorem 1 from the main part.

4.1 Quasi-solution for \(k = 1\) branch for \(\Lambda = 1\), \(R = 20\) and \(\nu = \frac{1}{10}\) and details

We chose quasi-solution \((C_0, \{\hat{H}_0(k)\}_{k=1}^8)\), given by expressed as rationals so as to avoid any round off errors in the computation, given by

\[
\begin{bmatrix}
8554 & 12885 & -1043 & 435i & 1409 & 585i & 302 & 127i & 30 & 91i \\
1397 & 23828 & -4331 & 2339 & 55585 & 7199i & 18357 & 41559 & 16099 & 36906i \\
& & -36 & 77i & 9 & 3i & 4 & 13i & 152065 & 168821i \\
& & & & & & & & 111589 & 1407007i & 800731i & 1170328 \\
\end{bmatrix}
\]

(118)

with corresponding \(\{N_{1/2}[k]\}_{k=1}^8\) obtained from integrals of Airy function, obtained with the help of symbolic manipulation tool and expressed as rational numbers

\[
\begin{bmatrix}
21061 & 88807i & 35583 & 49758i & 126496 & 107673i & 24107 & 88338i \\
126378 & 27831i & 53450 & 7591i & 84899 & 10486i & 9219 & 6089 \\
& & & & & & & & & & & \\
56973 & 394296i & 109483 & 142793i & 76036 & 203838i & 67113 & 1427719i \\
14279 & 20285i & 19766 & 5668i & 10601 & 6397i & 7621 & 36144 \\
\end{bmatrix}
\]

(119)

and with choice \(K = 8\), with help of symbolic computational tools, it is easy to check that

\[
\epsilon_R \leq 2.416 \times 10^{-6}, \quad \|\hat{H}_0\|_1 \leq 0.9506, \quad M_g \leq 2.8703, \quad \epsilon_u \leq 0.014136, \\
\epsilon_q \leq 2.4785 \times 10^{-7}, \quad C_L \leq 2.7284, \quad \gamma_{1,1} \leq 8.450, \quad \beta_{1,2} \leq 12.623, \quad \beta_{1,2} \leq 34.933, \\
\beta_{2,1} \leq 0.18099 \quad \beta_{2,2} \leq 1.51523, \quad M_L \leq 36.45, \quad \epsilon \leq 0.8803 \times 10^{-4}, \quad \beta_c \leq 0.037
\]

(120)

implying that the condition for application of Theorem 2.11 in the main part is satisfied and hence there exists solution \((C, \hat{H})\) near quasi-solution \((C_0, \hat{H}_0)\) with

\[
|C - C_0| + \|\hat{H} - \hat{H}_0\|_1 \leq 2\epsilon \leq 1.7606 \times 10^{-4}
\]

(121)
4.2 Quasi-solution for $k = 1$ branch for $\Lambda = \frac{6}{5}$, $R = 50$, $\nu = \frac{1}{10}$ and details

For quasi-solution \( \left( C_0, \left\{ \hat{H}_0(k) \right\}_{k=1}^{12} \right) \), given by expressed as rationals so as to avoid any round off errors in the computation, given by

\[
\begin{bmatrix}
52299 & 34717 & 8178 & 26965i & 23247 & 21767i & 14284 & 39780i & 9473 & 8061i \\
10060 & 16727 & 5321 & 98492i & 26941 & 35052 & 105875 & 68137 & 68179 & 36734i \\
3386 & 6982i & 12791 & 4099i & 5668 & 2758i & 1580 & 2639i & 1470 & 1113i \\
35205 & 254943 & 374382 & 261327 & 1013251 & 218663 & 1035709 & 568600 & 1009301 & 1339817 \\
\end{bmatrix}
\]

with corresponding \( \left\{ \mathcal{N}_{1/2}[k] \right\}_{k=1}^{12} \) obtained from integrals of Airy function, obtained with the help of symbolic manipulation tool and expressed as rational numbers

\[
\begin{bmatrix}
34597 & 19739i & 79652 & 17607i & 94039 & 77003i & 4341 & 9232i & 96921 & 121883i \\
84279 & 6112 & 50645 & 2573 & 28408 & 6914 & 802 & 567 & 12583 & 5456, \\
-20407 & 291718i & 39064 & 165167i & -190290 & 315821i & -113933 & -8718i & -3143 & -4443 \\
2028 & 9953 & -3143 & -4443 & -12883 & 6878 & -6673 & -157 & -353192 & -324679i \\
& & & & & & & & & \\
18271 & 4919 & -5083 & -3320 & -6770 & -4243 & & & & \\
\end{bmatrix}
\]

and with choice $K = 12$, with help of symbolic computational tools, it is easy to check that

\[
\begin{align*}
\epsilon_R & \leq 9.316 \times 10^{-5}, \quad \| \hat{H}_0 \|_{l_1} \leq 5.7174, \quad M_\theta \leq 0.37019, \quad \epsilon_u \leq 0.004643, \\
\epsilon_q & \leq 1.1076 \times 10^{-6}, \quad C_L \leq 2.1165, \quad \gamma_{1,1} \leq 12.393, \quad \beta_{1,1} \leq 14.124, \quad \beta_{1,2} \leq 30.025, \\
\beta_{2,1} \leq 0.06587 & \quad \beta_{2,2} \leq 1.1448, \quad M_L \leq 31.17, \quad \epsilon \leq 2.91 \times 10^{-3}, \quad \beta_c \leq 0.13402
\end{align*}
\]

implying that condition for application of Theorem 2.11 in the main part is satisfied and hence there exists solution $(C, \hat{H})$ near quasi-solution $(C_0, \hat{H}_0)$ with

\[
|C - C_0| + \| \hat{H} - \hat{H}_0 \|_{l_1} \leq 2\epsilon \leq 5.82 \times 10^{-3}
\]

4.3 Quasi-solution for $k = 1$ branch $\Lambda = \frac{6}{5}$, $R = 100$ and $\nu = \frac{1}{10}$

We chose a quasi-solution was \( \left( C_0, \left\{ \hat{H}_0(k) \right\}_{k=1}^{20} \right) \), given by

\[
\begin{bmatrix}
48637 & 58399 & 45295 & 16699i & 102139 & 12263i & 57595 & 32714i & 10345 & 34465i & 33251 & 16258i \\
20794 & 15282 & 15473 & 36526 & 52823 & 11941 & 70024 & 25373 & 114736 & 36889 & 94515 & 47621 \\
12923 & 5977 & 18739 & 6232i & 4753 & 4116i & 2043 & 1415i & 2453 & 1090i & 2074 & 152i \\
55709 & 213589 & 202756 & 112919 & 252538 & 97519 & 370751 & 66602 & 323823 & 184447 & -503431 & 5441943 \\
1586 & 1313 & 223 & 775i & 267 & 473i & 288 & 841i & 993 & 71i & & \\
1143277 & 1251721 & 1150297 & 1108116 & 2384990 & 1699429 & 2723219 & 13412396 & 20003308 & 1203235 & & \\
217 & 121i & 25 & 71i & 47 & 42i & & & & & & \\
& & & & & & \end{bmatrix}
\]
The corresponding \( \{ N_{1/2}[k] \}_{k=1}^{20} \) obtained from integral of Airy function was

\[
\begin{bmatrix}
-24633 + 58771i & 40991 + 98653i & 29317 + 162055i & 33202i + 33202i & 53574 + 166797i \\
31499 - 15181 + 13005i & -5703 + 12583i & 30497 - 1735i & 5041 - 6346i \\
90446 + 73094i & 165113i & 179017i & 122002i & 136716i \\
6673 + 2928i & 4411 + 3824i & 9108 + 205i & 5992 + 3123i & 379703i \\
263396 - 159403i & 311356i & 174657i & 402611i + 407615i & 316605i \\
439061 - 1193144i & 182209i & 430539i & 64067 + 249188i \\
10562 - 12807i & -4085 + 1041i & -6418 + 2398i & -4085 + 1268i & -249188i \\
206295 + 274307i & -114909 + 321539i & -64067 + 1268i & -249188i \\
3863 + 1263i & 2044 + 1355i \\
\end{bmatrix}
\]

\( (127) \)

With choice \( K = 20 \), with help of symbolic computational tools, it is easy to check that

\[
\epsilon_R \leq 2.1521 \times 10^{-6}, \quad \| \hat{H}_0 \|_\nu \leq 12.361, \quad M_g \leq 0.1672, \quad \epsilon_u \leq 0.00109,
\]
\[
\epsilon_q \leq 4.072 \times 10^{-9}, \quad C_L \leq 2.0664, \quad \gamma_{1,1} \leq 13.746, \quad \beta_{1,1} \leq 15.345, \quad \beta_{1,2} \leq 29.342,
\]
\[
\beta_{2,1} \leq 0.54385, \quad \beta_{2,2} \leq 1.03303, \quad M_e \leq 30.3742, \quad \epsilon \leq 6.54 \times 10^{-5}, \quad \beta_c \leq 0.00133
\]

implying that condition for application of Theorem 2.11 in the main part is satisfied and hence there exists solution \((C, \hat{H})\) near quasi-solution \((C_0, \hat{H}_0)\) with

\[
|C - C_0| + \| \hat{H} - \hat{H}_0 \|_\nu \leq 2\epsilon \leq 1.308 \times 10^{-4}
\]

\( (129) \)

5  Computed travelling wave profiles

Here we give results of the computed wave profiles corresponding to the results of Figures 2 and 3. This is done for all marked points on each solution branch where existence of solutions was proved. In all the results shown we depict linearly stable solutions with a blue colour and unstable ones are coloured red. This way the reader can follow the bifurcations that take place along individual branches as \( \Lambda \) increases.

5.1 Wave profiles for \( \nu = 1/10 \) and different \( R \) and \( \Lambda \)

Results are shown in Figures [13] corresponding to \( R = 20, 50 \) and 100, respectively. The left panels show branch 1 \( k = 1 \) solutions, and the right panels the corresponding branch 2 solutions. This is clear from the figures because the former are \( 2\pi \)-periodic and the latter are \( \pi \)-periodic.
5.2 Wave profiles for $\nu = 1/20$ and different $R$ and $\Lambda$

Results are shown in Figures 4-6 corresponding to $R = 20$, 50 and 100, respectively. The left panels show branch 1 $k = 1$ solutions, and the right panels the corresponding branch 2 solutions. This is clear from the figures because the former are $2\pi$–periodic and the latter are $\pi$–periodic.
Figure 4: $H_0(x)$ vs. $x$ for $R = 20$, $\nu = 1/20$. Left: Branch 1, $\Lambda = 0.160, 0.2, 0.3, 0.6, \cdots 1.2$. Right: Branch 2, $\Lambda = 0.602, 0.7, 0.8, \cdots 1.3$. Blue - stable; Red - unstable.

Figure 5: $H_0(x)$ vs. $x$ for $R = 50$, $\nu = 1/20$. Left: Branch 1, $\Lambda = 0.07, 0.1, 0.2, 0.3, \cdots 1.2$. Right: Branch 2, $\Lambda = 0.25, 0.3, 0.4, \cdots 2.0$. Blue - stable; Red - unstable.

Figure 6: $H_0(x)$ vs. $x$ for $R = 100$, $\nu = 1/20$. Left: Branch 1, $\Lambda = 0.032, 0.1, 0.2, 0.3, \cdots 2.0$. Right: Branch 2, $\Lambda = 0.136, 0.2, 0.3, 0.4, \cdots 2.0$. Blue - stable; Red - unstable.