

DAVID OLIVE

SCIENTIFIC CONTRIBUTIONS

BY EDWARD CORRIGAN¹ AND PETER GODDARD^{2,3}

¹*Department of Mathematics, University of York, YO10 5DD, UK*

²*Institute for Advanced Study, Princeton, NJ 08540, USA*

³*St John's College, Cambridge, CB2 1TP, UK*

A. *S*-matrix Theory

B. Fermions and the GSO projection

C. Monopoles and Duality

D. Toda Theories

E. Algebras, Lattices and Strings

where the lefthand side denotes the S -matrix element $\langle p'_a, p'_b | S | p_a, p_b \rangle$ and, on the righthand side, the first term denotes $\langle p'_a, p'_b | p_a, p_b \rangle$, essentially the product of delta functions between initial and final three-momenta for each of the two particles a and b , and, up to normalization factors, the second term equals $i\delta_4(p_a + p_b - p'_a - p'_b)A$. Here A is the ‘connected’ part of the scattering amplitude, a function of the invariants $s = (p_a + p_b)^2$ and $t = (p_a - p'_a)^2$, which is assumed to be as analytic as possible, consistent with the requirements of unitarity,

Unitarity for the S -matrix, S , takes the form $SS^\dagger = 1$, which in ‘bubble’ notation reads

$$\text{---} \circlearrowleft S \text{---} \circlearrowright S^\dagger \text{---} = \text{---} \text{---} \quad [2]$$

where the lines joining indicating integration over momenta and the particle labels have been omitted. So, substituting for S from [1],

$$\text{---} \circlearrowleft + \text{---} - \text{---} \circlearrowright - \text{---} = \text{---} \circlearrowleft + \text{---} \circlearrowright - \text{---} \quad [3]$$

where the second term on the lefthand side is the hermitian conjugate of the first. In (2), within the context of quantum field theory, David had argued that this hermitian conjugate, the ‘minus’ amplitude, is the analytic continuation of the first term in [3], the ‘plus’ amplitude, through the unphysical region below $s = (m_a + m_b)^2$, the two-particle normal threshold, more generally than had previously been established. [Here m_a, m_b denote the masses of a, b , and so on.] In (5), he produced a more general argument for the validity of this property, known as hermitian analyticity, within the context of S -matrix theory. [See also (12), p. 223.] For s real just below this threshold and for suitable real t , the ‘plus’ and ‘minus’ amplitudes are equal and real, and so [3] holds with the expression on the righthand side replaced by zero.

Above the two-particle threshold, [3] gives an expression for the discontinuity across a cut, with the ‘plus’ and ‘minus’ amplitudes being the boundary values of a single analytic function from different sides of the cut. If, at higher energy, it becomes possible to produce a particle, c , *i.e.* $a + b \rightarrow a + b + c$, an extra term appears in the unitarity equation, which takes the form

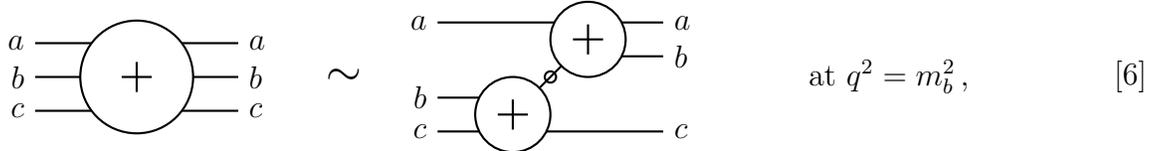
$$\text{---} \circlearrowleft + \text{---} - \text{---} \circlearrowright - \text{---} = \text{---} \circlearrowleft + \text{---} \circlearrowright - \text{---} + \text{---} \circlearrowleft + \text{---} \circlearrowright - \text{---} \quad [4]$$

in ‘bubble’ notation. The threshold at which this process becomes possible corresponds to another branch point at $s = (m_a + m_b + m_c)^2$, and [4] gives the total discontinuity across the two cuts along the real axis. And so, as and when new processes become possible at higher energies, the appearance of corresponding additional terms in the unitarity equations implies a sequence of branch point singularities, termed normal thresholds.

The unitarity equations can also imply the presence of poles, as well as cuts, in the physical region, *i.e.* for real, physical values of the momenta. At suitable values of momenta, the unitarity equation for three-to-three scattering, $a + b + c \rightarrow a + b + c$, the unitarity equation contains a term

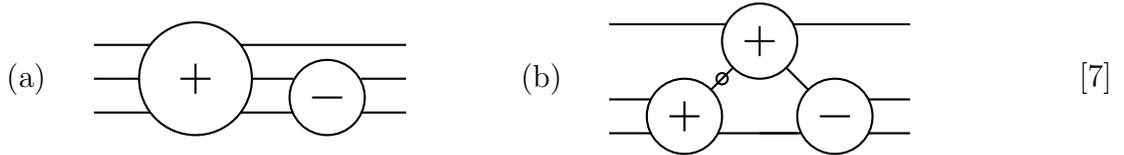


which corresponds to a $\delta(q^2 - m_b^2)$ discontinuity in the amplitude, where $q = p_a + p_b - p'_a$, where p_a, p_b denote incoming momenta, and p'_a an outgoing momentum, for the appropriate particles. David argued that this implies that the amplitude, viewed as an analytic function, has a pole at $q^2 = m_b^2$.



where the middle line on the right hand side denotes a factor of $(q^2 - m_b^2)^{-1}$, rather than a δ -function.

The unitarity equation for the $a + b + c \rightarrow a + b + c$ scattering amplitude also contains the term (a) shown in [7]. If the pole in [6] is inserted in this term, the structure corresponding to the term (b) in [7] is obtained [(11);(12), pp. 206, 266–278].



Related terms are generated by other terms in the unitarity equation and together they imply a cut in (s, t) -plane with discontinuity given by



just as in perturbative quantum field theory (QFT). In a series of three papers (15–17) with his student, Michael Bloxham, and John Polkinghorne, David was able to extend this approach to show that for physical values of the momenta, unitarity requires scattering amplitudes to be singular on the arcs of curves where Landau (1959) had determined that singularities occur for Feynman diagrams in QFT, and that the discontinuities associated with these singularities are given by the rules obtained for perturbative QFT by Cutkosky (1960).

Building on ideas of Gunson (later published in Gunson (1965); see p. 847), David also showed how similar arguments based on analyticity and unitarity imply the existence of antiparticles, and may allow the proof of crossing symmetry and the TCP theorem (5,12), as well as implying the

standard connection between spin and statistics (8) [*i.e.* that integral spin particles are bosons and half-odd-integral ones are fermions]. However, these results depend on the existence of suitable paths for analytic continuation and a knowledge of the singularity structure of the S -matrix outside the physical region and for complex values of the momenta.

Reviewing the state of S -matrix theory nearly a decade after he ceased working on it, David reiterated his belief that “Properties involving analyticity well away from [the physical region] should surely not be postulated but rather deduced from more fundamental principles involving the physical region.” The developments of the early 1960s, in which David had played a leading role, had shown how one might hope to re-establish the main achievements of Axiomatic QFT (see Streater and Wightman 1964) within the (arguably) more general framework of S -matrix theory, but, as David put it, “Although the general features of the physical region were appreciated, a detailed, precise and comprehensive mathematical treatment was lacking.”(48) Later approaches to providing such a mathematical treatment for discussing the S -matrix near the physical region were developed by Sato (1975) and by Iagolnitzer (1981), but no progress on establishing a rigorous analysis of non-physical region singularity structure has been made.

B. FERMIONS AND THE GSO PROJECTION

At early in 1971, Pierre Ramond proposed a description of free fermions in dual models (Ramond 1971) in a sort of generalization of the Dirac equation. This was quickly extended by André Neveu and John Schwarz who gave expressions for dual model amplitudes involving a single Ramond fermion interacting with a sequence of mesons (Neveu & Schwarz 1971). When he arrived at CERN some months later to take up his staff appointment, David began working with Edward Corrigan on a programme to construct a complete consistent theory of dual fermions and mesons working within the operator formalism.

The operator formalism of the original bosonic dual model of Veneziano, the space of states is created by bosonic oscillators, a_n^μ , where μ runs over the dimensions of space-time and n runs over the integers. Neveu and Schwarz enlarged this using anti-commuting fermionic oscillators, b_r^μ , where r runs over half odd integers, in addition to the a_n^μ , to create a space of space-time boson states, while the Ramond states, which are space-time fermions, are created by fermionic oscillators, d_n^μ , where n runs over the integers, in addition to the a_n^μ . These oscillators satisfy the commutation and anti-commutation relations,

$$[a_m^\mu, a_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu}, \quad \{b_r^\mu, b_s^\nu\} = \delta_{r,-s}\eta^{\mu\nu}, \quad \{d_m^\mu, d_n^\nu\} = \delta_{m,-n}\eta^{\mu\nu}, \quad [9]$$

where $\eta^{\mu\nu}$ is the space-time metric, and $a_m^{\mu\dagger} = a_{-m}^\mu$, $b_r^{\mu\dagger} = b_{-r}^\mu$, $d_n^{\mu\dagger} = d_{-n}^\mu$.

The Neveu-Schwarz states are created by the action of the oscillators a_m^μ, b_r^μ , on a vacuum state $|0\rangle$ that is annihilated by the operators $a_m^\mu, m > 0$, and $b_r^\mu, r > 0$. The Fock space created in this way has a natural scalar product which is not positive because the space-time metric $\eta^{\mu\nu}$ is not. The

consistency of a dual model, such as the original Veneziano model or the Ramond-Neveu-Schwarz model, requires that the physical states that couple in amplitudes belong to a positive definite subspace defined by an infinite set of gauge conditions, *i.e.* that there are no ‘ghost’ physical states.

The first step in constructing a complete dual theory of fermions and bosons was to rewrite the amplitudes of Neveu and Schwarz in a dual form corresponding to a boson annihilating into a fermion pair. In essence, this required the construction of a ‘fermion emission vertex’, an operator describing the process by which a fermion changes into a boson by the emission of a fermion. Such a construction had been found by Thorn (1971) and by Schwarz (1971). Corrigan and Olive (28) gave this a more elegant and manageable formulation, making the action of the vertex on the gauge conditions more transparent. A prime objective was the calculation of the amplitude for fermion-fermion scattering by combining a fermion emission vertex and its conjugate.

The defining properties of this vertex operator, $W_\chi(z)$, as formulated by Corrigan and Olive, are that it maps boson (or meson) states into fermion states in such a way that it intertwines between a Neveu-Schwarz field, $H^\mu(y) = \sum b_r^\mu y^{-r}$ and a Ramond field, $\Gamma^\mu(y) = \sum d_n^\mu y^{-n}$, according to an equation of the form,

$$W_\chi(z) \frac{H^\mu(y)}{\sqrt{y}} = \lambda \frac{\Gamma^\mu(y-z)}{\sqrt{y-z}} W_\chi(z), \quad [10]$$

where λ is a suitable constant, and that it creates the fermion state, $|\chi\rangle$, from the boson vacuum, $|0\rangle$: $W_\chi(z)|0\rangle = e^{zL_{-1}}|\chi\rangle$.

Whereas the vertices describing the emission of bosons essentially commute with the gauge conditions that eliminate ghost states, the behaviour of the fermion emission vertex is more complicated. It converts an infinite linear combination of gauge conditions in the boson sector into an infinite combination in the fermion sector. It did not immediately follow that ghost states would not couple in the four-point fermion amplitude.

Working with Lars Brink, David saw that it was necessary to introduce a projection operator onto the space of physical states to secure this (30). As a preparation, Brink and Olive looked at the algebraically related problem of calculating the one loop contribution in the original bosonic theory, giving a precise derivation (31) of the results earlier proposed heuristically by Lovelace (1971), that had first suggested that the space-time in bosonic string theory needed to be 26 for consistency. Next, with Claudio Rebbi and Joël Scherk, they worked out exactly how the gauge conditions related to the fermion emission vertex, under the requirement that the lowest fermion mass $m = 0$ and the dimension of space-time $D = 10$, and found that the calculation of the four-point fermion amplitude was surprising similar to the one loop calculation.

Using these results, Olive and Scherk (35) were able to calculate that the effect of projecting onto the physical states satisfying the gauge conditions was to introduce a factor $1/\Delta(x)$ into the integral expression for the four-point fermion amplitude. The function $\Delta(x)$ was defined as an infinite determinant which was not readily evaluated. After John Schwarz and Cheng-Chin Wu (1974) using a computer calculation to propose $\Delta(x) = (1-x)^{\frac{1}{4}}$, a result that was proved

analytically soon after by Corrigan, Goddard, Olive and Russell Smith (36). Meanwhile, Stanley Mandelstam (1973) had obtained the same result very efficiently using his mastery of the light-cone formulation of string theory.

The conclusion of the calculation of the four fermion amplitude not only in the s channel, where, by construction, the poles correspond to the propagation of physical states, but also in the t channel, where the poles correspond to the same spectrum. Before the calculation, it was not certain not that the amplitude would be meromorphic: conceivably it could have had cuts in the t channel. There was however a subtle discrepancy, which proved to be an unappreciated clue to a deeper structure: the parity of the boson states was changed between the s and t channels, so that if the lowest mass state in the s channel, a tachyon, is a pseudo-scalar, the lowest mass state in the t channel is a scalar tachyon.

The structure of the four fermion amplitude gave much encouragement that a consistent Ramond-Neveu-Schwarz model could be constructed, but the parity doubling was puzzling. By 1974, it had been realized by various people that it was possible to make a projection, consistent with interactions, that would remove something like half the states including the tachyon in the bosonic sector, and leave a massless chiral fermion as the lowest state in the fermionic sector. In 1974, the focus of David's interest shifted towards the study of monopoles in gauge theories, but in 1976 he was drawn back to dual fermion theories for what proved to be a very seminal interlude.

In the summer of 1975, David began thinking about the behaviour of spinors in different dimensions. A spinor in even dimension D of space-time in general has $2^{D/2}$ components. This can be halved by imposing either of two conditions; that the spinor be Weyl (chiral) or that it be Majorana (real). These conditions are incompatible unless $D = 2 \pmod{8}$. David was struck by the fact that $D = 10$, the dimension appropriate for the Ramond-Neveu-Schwarz model, is the smallest non-trivial dimension in which there are Majorana-Weyl spinors. Such spinors have $2^{10/2-2} = 8$ components.

In 1976, Joël Scherk, who had returned to the École Normale Supérieure in Paris, and Fernando Gliozzi, who was visiting, had begun studying the multiplicity of the states at various mass levels in the Ramond-Neveu-Schwarz model. They noticed, first by manual calculation, there was a remarkable coincidence between the number of states at a given mass level in the bosonic sector and the same level in the fermionic sector, provided that they could impose both Weyl and Majorana conditions. Such a projection would also eliminate the problem with parity doubling that had surfaced in the four fermion amplitude.

Soon they discovered that there was an identity in the literature, proved by Jacobi in 1829,

$$\frac{1}{2w^{\frac{1}{2}}} \left[\prod_{m=1}^{\infty} \left(\frac{1+w^{m-\frac{1}{2}}}{1-w^m} \right)^8 - \prod_{m=1}^{\infty} \left(\frac{1-w^{m-\frac{1}{2}}}{1-w^m} \right)^8 \right] = 8 \prod_{m=1}^{\infty} \left(\frac{1+w^m}{1-w^m} \right)^8, \quad [11]$$

that guarantees equal numbers of bosons and fermions at each mass level. The coefficient of w^n on the left hand side of this relation gives the number of bosonic states at the n -th mass level (above

$m = 0$) and the right hand side does the same for the fermionic sector, provided that the spinors are subject to two conditions. From the beginning, two-dimensional supersymmetry, on the world sheet of the string, had been at the heart of the structure of the Ramond-Neveu-Schwarz model, but the equality of the numbers of bosons at each mass level strongly suggested that the theory was also supersymmetric in space-time, and this was subsequently shown to be the case.

When Joël Scherk visited CERN, to give a talk on developments in supergravity, he told David of these results and David explained to Joël that the space-time dimension being 10 plays an essential role in being able to impose both Majorana and Weyl conditions, necessary for the factor 8 in [11]. Because of the importance of this insight, Joël invited David to join the collaboration, and the projection has become known as the Gliozzi-Scherk-Olive, or GSO, projection (44,45), and the theory it defined is what has become known as superstring theory. For the first time, there was a perturbatively unitary, tachyon free, space-time supersymmetric dual theory.

C. MONOPOLES AND DUALITY

In one of the most breathtakingly original contributions he made in the early years of Quantum Mechanics, Paul Dirac (1931) showed that the consistency of quantum mechanics for a system involving both electric charges, q , and magnetic charges, g , requires that each pair of such charges satisfy

$$\frac{qg}{4\pi\hbar} = \frac{n}{2}, \quad n \text{ an integer.} \quad [12]$$

This relation implies that the striking conclusion that the existence of a single magnetic charge requires the existence of a smallest electric charge, q_0 , with any other electric charge being an integral multiple of q_0 , and, similarly, that there is a smallest magnetic charge, g_0 , of which any other magnetic charge must be an integral multiple.

Dirac was discussing point charges, singularities in the electromagnetic field, but, the mid 1970s, in nearly simultaneous papers, 't Hooft (1974) and Polyakov (1974) showed that extended objects with magnetic charge can occur as smooth classical solutions in certain familiar gauge field theories. They considered an SO(3) gauge theory with an adjoint representation (*i.e.* triplet real) scalar field ϕ , whose self-interactions are described by the Higgs potential,

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - a^2)^2. \quad [13]$$

In the lowest energy state $\phi = \phi_0$, constant, a value on the surface \mathcal{M} of minima of V , which is the sphere $\phi^2 = 1$. All such choices of ϕ_0 are gauge-equivalent and each choice corresponds to a spontaneous breaking of the gauge group down to U(1), which can be identified with the gauge group of electromagnetism.

't Hooft and Polyakov showed that the classical field equations of this SU(2) Higgs model have a smooth spherically symmetric solution under simultaneous rotations in ordinary space and the

three-dimensional space in which the Higgs field ϕ lives. This implies making a particular identification between directions in these spaces; with this, ϕ is everywhere radial, vanishing at the origin. Asymptotically, $\phi(\mathbf{r}) \sim a\hat{\mathbf{r}}$, and its U(1) little group is identified with the electromagnetic gauge group. The corresponding component of the gauge field strength gives the electromagnetic field, which, for the 'tHooft-Polyakov solution is a purely magnetic field, $\mathbf{B} \sim (1/er^2)\hat{\mathbf{r}}$ asymptotically, where e is the gauge coupling constant; this is the field of a magnetic monopole of strength,

$$g = \frac{4\pi}{e}. \quad [14]$$

In the quantized SU(2) gauge theory, the electric charges q are multiples of $\frac{1}{2}\hbar e$ and so such charges satisfy the Dirac quantization condition [12] with the magnetic charge g given by [14].

To understand the 't Hooft-Polyakov monopole more deeply, David began to investigate whether similar solutions in gauge theories might provide theories of particles, such as hadrons, working with Edward Corrigan, David Fairlie and Jean Nuyts (40). They sought generalizations of it by looking for solutions to an SU(3) Higgs model that were spherically symmetric in an appropriate sense. In the case of SU(3), for a generic quartic (and so renormalizable) Higgs potential, V , the gauge group acts transitively on the vacuum manifold, \mathcal{M} , of minima of V , and the unbroken symmetry group, the little group of the Higgs field, ϕ , is SU(2) \times U(1)/ \mathbb{Z}_2 , and we can again identify the U(1) factor with electromagnetism.

The mapping between ordinary space and the space of the Higgs field, in the case of the 't Hooft-Polyakov monopole, determines an isomorphism between SU(2), the covering group of the group of spatial rotations SO(3), and the SU(2) gauge group. For other gauge groups, such as SU(3), given a homomorphism of SU(2) into the gauge group, spherical symmetry can be defined as invariance under the simultaneous applications of rotations and the corresponding gauge transformations under the homomorphism. If $t^i, i = 1, 2, 3$, are the images of a standard basis for the generators of SU(2) in the Lie algebra of the gauge group, spherical symmetry corresponds to invariance under transformations generated by $-i\mathbf{r} \wedge \nabla + \mathbf{t}$.

David and his collaborators noted that there are two essentially distinct homomorphisms of SU(2) into SU(3), that is ones not related by SU(3) gauge transformation: (i) one in which the image is SU(2); and (ii) another in which the image is SO(3). They found spherically symmetric solutions, possessing U(1) magnetic charge, for each of these cases, but, while in case (ii) the Dirac condition [12] was satisfied, for case (i) there were solutions for which the condition had to be relaxed to allow n to be half-integral. These solutions evaded the original Dirac condition by having a long range magnetic type SU(2) gauge field, not part of the context of Dirac's original argument.

In his next contribution, with Edward Corrigan (42), David derived a generalization of the Dirac condition which encompassed such cases. They considered a theory with gauge group G spontaneously broken, *e.g.* by an adjoint representation Higgs field, to a subgroup $H = K \times \text{U}(1)/Z$, where $Z = K \cap \text{U}(1)$ is discrete; K can be identified with a 'colour' gauge group and U(1) with electromagnetism. They showed that, for any smooth solution, the magnetic charge g associated

with the U(1) factor satisfies the generalized Dirac condition,

$$\exp(igQ) \in K, \quad [15]$$

where Q is the electric charge operator, which generates the U(1). If k , the element of K defined by [15], equals 1, then this condition is equivalent to the original Dirac condition [12] and this is the case for $G = \text{SU}(2)$, where K is trivial. For $G = \text{SU}(3)$, in case (ii) $k = 1$, but in case (i) $k \neq 1$, which is why the condition [12] can be violated in this case. Corrigan and Olive noted that, if $k \neq 1$, the generalized condition [15] implies that [12] holds for the electric charges of colour singlet particles, establishing a link between colour and fractional electric charges.

For spherically symmetric monopole solutions, David showed (43) that the image, in the Lie algebra of G , of the generator of rotations about the radial direction $\hat{\mathbf{r}} \cdot \mathbf{t} = -gQ + \kappa$, where κ is a generator of K . The condition [14] follows again from this relation because the eigenvalues of $\hat{\mathbf{r}} \cdot \mathbf{t}$ are half integral. Spherical symmetry implies that $\hat{\mathbf{r}} \cdot \mathbf{t}$ is a generator of the little group, H , of $\phi(\mathbf{r})$ and so $h(s) = \exp(4\pi is\hat{\mathbf{r}} \cdot \mathbf{t})$, $0 \leq s \leq 1$, defines a closed loop in H and an element of the homotopy group, $\Pi_1(H)$, which David show equalled the topological invariant or ‘topological quantum number’ (although it is a classical concept), which had earlier been associated with the solution (Tyupkin *et al.* 1975, Monastyrskii & Perelomov 1975). The asymptotic limit of the Higgs field, $\phi_\infty(\hat{\mathbf{r}}) = \lim_{r \rightarrow \infty} \phi(r\hat{\mathbf{r}})$, defines a map $S^2 \rightarrow \mathcal{M} \cong G/H$, and so an element of the homotopy group $\Pi_2(G/H)$, which is isomorphic to $\Pi_1(H)$ (assuming G to be simply connected). David showed that the element of $\Pi_1(H)$ associated with the solution in this way corresponds to the loop h .

Seeking to deepen our understanding of magnetic monopoles in gauge theories, working with Peter Goddard and Jean Nuyts (GNO), David next investigated (46) the general case of a theory with a compact gauge group G , spontaneously broken to an exact gauge group $H \subset G$ by a Higgs field, ϕ , in an arbitrary representation. They considered monopole solutions, that is solutions for which the spatial components of the gauge field strength $\mathcal{G}_{\alpha\beta}$, expressed as a generator of H , the little group of $\phi_\infty(\hat{\mathbf{r}})$, the asymptotic value of $\phi(\mathbf{r})$ in the radial direction,

$$\mathcal{G}_{ij}(\mathbf{r}) \sim \frac{\epsilon_{ijk} T_k}{r^3} \mathcal{G}(\hat{\mathbf{r}}), \quad \text{as } r \rightarrow \infty, \quad 1 \leq i, j, k \leq 3. \quad [16]$$

Generalizing [12] and [15], they established that

$$\exp(ie\mathcal{G}) = 1, \quad [17]$$

where $\mathcal{G} = \mathcal{G}(\hat{\mathbf{r}})$, and that the loop $h(s) = \exp(ise\mathcal{G})$, $0 \leq s \leq 1$, determines the topological invariant associated with the solution as an element of $\Pi_1(H)$. Taking a maximal set of commuting generators of H , T^a , $1 \leq a \leq m$, where m is the rank of H , *i.e.* a basis for a Cartan subalgebra of H , \mathcal{G} will be equivalent, under an H gauge transformation, to a linear combination, $\mathcal{G} = g_a T^a$, of the $\{T^a\}$. \mathcal{G} is the generalized magnetic charge of the solution, and GNO called $\mathbf{g} = (g_a)$ its magnetic weights.

The simultaneous eigenvalues $\mathbf{q} = (q^a)$ of $\hbar e T^a$, $1 \leq a \leq m$, in representations of H , are the generalized electric charges of quanta fields, transforming under those representations. The quantization condition [17] can be re-expressed in the form

$$\frac{\mathbf{q} \cdot \mathbf{g}}{4\pi\hbar} = \frac{n}{2}, \quad n \text{ an integer}, \quad [18]$$

more closely paralleling [12]. The possible values of \mathbf{q} are points of the lattice $\hbar e \Lambda_H$, where Λ_H is the weight lattice of the group H , implying that $\mathbf{g} \in (2\pi/e)\Lambda_H^*$, where Λ_H^* is the lattice dual to Λ_H . The weight lattice, $\Lambda_{\tilde{H}}$, of weights of the simply connected group, \tilde{H} , with the same Lie algebra as H , consists of those $\boldsymbol{\lambda}$ for which $2\boldsymbol{\alpha} \cdot \boldsymbol{\lambda}/\alpha^2$ is an integer for every root $\boldsymbol{\alpha}$ of H . This condition is satisfied by the roots themselves, and thus $\Lambda_R \subset \Lambda_H \subset \Lambda_{\tilde{H}}$, where Λ_R is the root lattice of H , the lattice generated by its roots, $\boldsymbol{\alpha}$. If H is a simple group all of whose roots have the same length (*i.e.* H is simply-laced), we can normalize so that $\alpha^2 = 2$ for each root, and then $\Lambda_R \subset \Lambda_H^* \subset \Lambda_{\tilde{H}}$. Thus Λ_H^* is the weight lattice of some group, H^\vee say, called the dual of H , with the same Lie algebra as H but a different global structure in general, *e.g.* $SU(2)^\vee = SO(3)$ and conversely. The relationship between a group and its dual is reflexive: $(H^\vee)^\vee = H$.

If H is not simply-laced, Λ_H^* is still the weight lattice of some group H^\vee , the dual of H , but now it is a group with roots $\boldsymbol{\alpha}^\vee = 2\boldsymbol{\alpha}/\alpha^2$, where $\boldsymbol{\alpha}$ is a root of H , and its algebra may not be isomorphic to that of H . The simple Lie algebras that are not simply-laced are the series B_n and C_n , corresponding to, *e.g.*, the groups $SO(2n+1)$ and $Sp(n)$, respectively, and the exceptional algebras G_2 and F_4 . For the groups with Lie algebras G_2 and F_4 , the dual groups have the same Lie algebra, but different global structure. However, the dual of a group with Lie algebra B_n is one with Lie algebra C_n , and *vice versa*.

Thus, GNO showed that, while the generalized electric charges are, up to a factor, the weights of the exact symmetry group, H , the generalized magnetic charges of the extended monopole solutions are, up to a factor, weights of the dual group H^\vee . They suggested that these solutions should form multiplets of H^\vee and that this group should be an exact symmetry group of the theory as well as H . Further, they speculated that the dual relationship between the generalized magnetic charges, \mathbf{g} , associated with the topological characteristics of the monopole solution and the group H^\vee , on the one hand, and the generalized electric charges, \mathbf{q} , associated with the quanta of the basic fields of the theory and the group H , on the other, was analogous to that between the solitons and the quanta of the basic boson field in the two-dimensional sine-Gordon theory.

GNO noted that the sine-Gordon theory had been shown to be equivalent to the massive Thirring model at the quantum level, with the solitons, which are extended solutions in the former boson theory, corresponding to the quanta of the basic fermion field in the Thirring model (Coleman 1975, Mandelstam 1975), ideas that go back to the work of T. H. R. Skyrme (Skyrme 1961, and references therein). In this correspondence, the topologically conserved soliton number in the sine-Gordon theory corresponds to a conserved Noether charge associated with a $U(1)$ symmetry of the Thirring model. But this exact equivalence of different theories is intrinsically quantum mechanical and so any such equivalence between dual theories, interchanging electric and magnetic

charges, is beyond the basically classical analysis of GNO (46). However, they speculated that the monopoles of the original theory might correspond to particles in a fundamental representation of H^\vee in a dual formulation; in his next investigations, David was led instead to a more elegant and profound conjecture.

David was determined to explore further what he acknowledged were highly speculative propositions, that H monopoles behave as irreducible representations of H^\vee , and that the theory has a H^\vee gauge symmetry. Working with Claus Montonen (47), he looked for evidence for the conjectures in the simplest monopole theory, that studied by 't Hooft and Polyakov, in which the gauge group $G = \text{SO}(3)$ is spontaneously broken to $H = \text{SO}(2) \cong \text{U}(1)$ by the choice of a particular vacuum value for ϕ minimizing the Higgs potential [13]. The mass, M_M , of the monopole solutions in this theory is of the form,

$$M_M = \frac{4\pi a}{e} f(\lambda/e^2), \quad [19]$$

where $f(\lambda/e^2) \rightarrow 1$ as $\lambda \rightarrow 0$ and following Prasad and Sommerfield (1975), Montonen and Olive considered the limiting value, $\lambda = 0$, in which the potential vanishes but the asymptotic condition $\phi^2 \rightarrow a^2$ is maintained at spatial infinity. Then, from [14], $M_M = ag$, which is of exactly the same form as the formula for the massive of the massive spin 1 particles, W^\pm , generated by the Higgs mechanism, namely $M_W = ae\hbar = aq$. Bogomolnyi (1976) showed that, for general λ , $M_M \geq ag$, so that the monopole mass reaches its lower bound at $\lambda = 0$, which is now known as the Bogomolnyi-Prasad-Sommerfield (BPS) limit.

Montonen and Olive conjectured that there should be dual equivalent versions of this theory, described by Lagrangians of the same form. Under this duality correspondence, electric and magnetic properties interchange, the massive ‘gauge’ particles, W^\pm , exchange rôles with monopole solutions, and topological and Noether quantum numbers are interchanged. In addition to the symmetry between the mass formulae $M_W = aq$ and $M_M = ag$, they noted that further circumstantial evidence for the duality conjecture was provided by Manton’s calculation of the classical magnetic force between monopoles as $g^2/4\pi r^2$, symmetric with the electric case (Manton 1977).

Because [14] implies that as e becomes small, g becomes large, this Montonen-Olive duality provides a (conjectured) equivalence between theories at strong and weak coupling. It was the first example of what became known as S-duality, which make a central concept in quantum field theory and string theory from the mid 1990s onwards.

Notwithstanding the suggestive evidence in favour of the extremely bold Montonen-Olive conjecture, there were significant potential obstacles to its validity: Why should the symmetry between the formulae for the masses of monopoles and massive gauge particles survive renormalization? Why should the monopole have spin 1 quantum mechanically? Answers to these two questions came quickly. With Edward Witten, David showed that in suitable $N = 2$ supersymmetric theories, the presence of monopole solutions results in the existence of central terms that modify the supersymmetry algebra, with the mass formulae following as a consequence of the algebra (50). Then Hugh Osborn (1979) observed that in $N = 4$ supersymmetric gauge theory, with sponta-

neous symmetry breaking, the monopole states would have spin 1 like the massive gauge particles, making it the most suitable theory for realizing the Montonen-Olive duality conjecture. Fifteen years later, Nathan Seiberg and Edward Witten (1994a,b) found a form of S-duality that worked in $N = 2$ supersymmetric theories.

David continued to study the characteristics of monopole solutions to gauge theories, motivated in large part by his duality conjecture. With Peter Goddard, he studied theories in a gauge group G is broken by an adjoint representation Higgs field to an exact symmetry gauge group, H , whose structure is locally of the form $U(1) \times K$ (60). They found that a necessary and sufficient condition for K to be semi-simple is that the vacuum value of the Higgs field be on the same orbit of G as a fundamental weight, and they gave prescriptions for determining K , and for calculating the basic units of electric and magnetic charge, from the Dynkin diagram of G . They also showed that the global structure of $H = (U(1) \times K)/Z$, where Z is a cyclic group whose order can also be calculated from the Dynkin diagram.

Next (61), they considered such theories in the BPS limit, analysing the charges of potentially stable magnetic monopoles. They showed that they have a structure consistent with their interpretation as heavy gauge particles of an overall symmetry group, G^\vee , dual to G , providing further circumstantial evidence for the some form of Montonen-Olive duality.

D. TODA THEORIES

David Olive continued his studies of solutions to gauge theories with symmetry broken by a Higgs field in the adjoint representation, working with Nikos Ganoulis and Peter Goddard (65), investigating static stable solutions to the field equations in the BPS limit that are spherically symmetric in a suitable sense. Andrej Leznov and Mikhail Saveliev (1980) had shown that they could be constructed from solutions to a spatial version of the so-called Toda molecule equations,

$$\frac{d^2\theta_i}{dr^2} = \exp\left(\sum_{j=1}^R K_{ij}\theta_j\right), \quad [20]$$

where $K_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2$ is the Cartan matrix of G , $\alpha_i, 1 \leq i \leq R$, being a basis of simple roots. The equations [20] are integrable and Ganoulis, Goddard and Olive showed how to determine the constants of integration so as to ensure regularity at the origin, thus obtaining solutions for any simple group G in terms of its root system. This work marked the beginning of David's fascination with Toda systems, which he kept coming back to, resulting in a series of contributions over the following fifteen years.

Toda's work (1967) was motivated by the numerical studies of Fermi, Pasta and Ulam (1955) of the behaviour of a system of particles joined by springs exerting forces nonlinear in their extensions. Toda made the important observation that if the polynomial forces considered by Fermi *et al.* were replaced by exponential ones, a one-dimensional lattice comprising an infinite

number of such particles could be solved analytically instead of numerically. It was found that this integrability extended to finite systems of particles, labelled by the simple roots of finite-dimensional Lie algebras, as in [20], replacing $d^2\theta_i/dr^2$ by $-d^2\theta_i/dt^2$ (Kostant, 1979), and also to corresponding systems of nonlinear relativistic wave equations, *i.e.* Toda field theories, replacing $d^2\theta_i/dr^2$ by $d^2\theta_i/dr^2 - d^2\theta_i/dt^2$ in [20]. There are also infinite lattice systems similarly related to the simple root systems of Kac-Moody algebras (Mikhailov, Olshanetsky and Perelomov 1981).

With Neil Turok (70), David showed how the symmetries of the Dynkin diagram of a simple Lie algebra led to new integrable systems of equations, which are reductions of the Toda molecule equations associated with the algebra, obtained by identifying variables related by the symmetry. Then, turning to the Toda lattice field theories associated with Kac-Moody algebras, termed affine Toda field theory, Olive and Turok systematically constructed an infinite number of commuting local conserved quantities, which could be used as Hamiltonians to define evolution in associated times (75, 86).

David returned to the study of Toda theories in the 1990s, following work by others on exact S -matrices for two-dimensional integrable quantum field theories, starting with the sine-Gordon quantum field theory, which is actually the simplest of the Toda field theories, being based on the group $SU(2)$. A key feature of the sine-Gordon theory is that classically it possesses soliton solutions, including soliton-anti-soliton bound states, known as breather solutions. These persist in the quantum field theory, becoming a finite discrete set of states whose number depends on the coupling constant. David's approach was to seek to use general Lie algebraic techniques, which he was familiar with from his work on magnetic monopoles and on conformal field theory, to provide unified and deeper understandings of mass spectra and couplings in affine Toda field theories (100, 101), in particular the rule describing three-point couplings discovered by Patrick Dorey (1991). Given David's background, it was natural for him to apply these insights to the conjectured S -matrix of the quantum field theory.

Typically the soliton solutions are complex rather than real, and so might seem uninteresting physically. However, with Turok and David's student, Jonathan Underwood (102, 103), he showed that the solutions have real energy and momentum, equal to the sum of contributions of the individual solitons. They further showed that the number of species of soliton equalled the rank of the Lie group with which it was associated and their masses were given in terms of its algebraic structure, and used a vertex operator formalism for constructing soliton solutions (105). With Saveliev and Underwood (104), and also with his students Andreas Fring, Peter Johnson, and Marco Kneipp (108, 110), David used vertex operators to study solitons and their scattering further, calculating time delays and showing that the process should be interpreted as transmission slowed down by attractive forces rather than reflection. No doubt one of the attractions of the study of affine Toda field theories for David was that aspects of them bore some resemblance to Yang-Mills-Higgs theories in four dimensions, with their monopole solutions analogous to the solitons of the Toda theories, and the mass spectrum of the solitons being equal to the mass spectrum of the dual theory based on the dual affine Lie algebra, but with the inverse coupling constant, recalling Montonen-Olive duality in gauge theories.

E. ALGEBRAS, LATTICES AND STRINGS

Having become aware of the profound connections that had been discovered between the vertex operators developed in dual models and string theory, on the one hand, and representations of Kac-Moody algebras, on the other (Frenkel & Kac 1980, Segal 1981), about 1980 David began, with Peter Goddard, to explore the possible role of these algebras in physics, and in string theory in particular. A further motivation was a proof by Graeme Segal (1980) of the Jacobi identity using a fermion-boson equivalence, the isomorphism between the space of states for a single boson field defined on a circle and that for a pair of fermion fields. The role of integral lattices (*i.e.* ones such that the scalar product between any two points is an integer) in these constructions provided links with concepts David had introduced in the theory of magnetic monopoles.

In Charlottesville in 1983, Goddard and Olive wrote an account synthesizing some of what they had learned and found out, under the title, *Algebras, Lattices and Strings* (72). They used dual model vertex operators to provide a unified construction of finite dimensional Lie algebras, affine Kac-Moody algebras, Lorentzian algebras and fermionic extensions of these algebras. Given an integral lattice, Λ , they associated to each point, $r \in \Lambda$, of squared length 2, the operator $e_r = c_r A_r$, where A_r is the contour integral about the origin of the dual model (or string) vertex operator, $V(r, z)$, for emitting a ‘tachyon’,

$$A_r = \frac{1}{2\pi i} \oint V(r, z) \frac{dz}{z}, \quad [21]$$

and c_r is a function of momentum such that $c_r c_s = (-1)^{r \cdot s} c_s c_r$. They considered the Lie algebra, \mathfrak{g}_Λ , generated by these e_r under commutation. If Λ contains points of squared length 1, \mathfrak{g}_Λ may be extended into a superalgebra by including operators associated with those points, which quite naturally satisfy anti-commutation relations.

If the scalar product on Λ is positive definite, \mathfrak{g}_Λ is a compact finite-dimensional Lie algebra, with a basis comprising the A_r , with $r \in \Lambda$ and $r^2 = 2$, together with the momenta. If the scalar product is positive semi-definite, \mathfrak{g}_Λ is an affine Kac-Moody algebra, which includes the contour integrals of vertex operators for emitting ‘photons’, associated with points $k \in \Lambda$ with $k^2 = 0$, as well as momenta. If the scalar product is indefinite, \mathfrak{g}_Λ involves the vertex operators associated with other dual model states.

In the context of string theory, if Λ were a lattice of momenta of physical states, discrete because the corresponding spatial dimensions were compactified, \mathfrak{g}_Λ would map the physical state space into itself, *i.e.* the physical states would fall into representations of \mathfrak{g}_Λ . Motivated in a general way by the ideas of electromagnetic duality that he had pioneered in gauge theories, where generalized electric and magnetic charges correspond to points of dual lattices, David formed the intuition that the cases where Λ is self-dual might be particularly interesting. Further, to concentrate on the bosonic case, Goddard and Olive considered the case where Λ is even, *i.e.* all the squared lengths of all its points are even, as well as self-dual.

The requirement of Λ being both even and self-dual is quite restrictive; for $\Lambda \subset \mathbb{R}^{m,n}$, it is necessary that $m - n$ be a multiple of 8. Thus, in the Euclidean case, the smallest possibilities for the dimension of Λ are 8, 18, 24, and these are the only dimensions relevant to bosonic string theory, for which the space-time dimension is bounded by 26. Goddard and Olive noted that the only such lattice with $\dim \Lambda = 8$ is the root lattice of E_8 , for $\dim \Lambda = 16$ there are two possibilities: the root lattice of $E_8 \times E_8$ and a sublattice of the weight lattice of the Lie algebra of $SO(32)$ (corresponding to the weight lattice of the group $Spin(32)/\mathbb{Z}_2$), while for $\dim \Lambda = 24$ there are 24 possibilities.

Applied to the quantum motion of a relativistic string, moving in a space some of whose dimensions are compactified to form a torus by identifying points of space related by displacements forming an even self-dual lattice (the length scale being set by the characteristic length of the string), \mathfrak{g}_Λ generates a gauge symmetry. For a closed string, the symmetry would be $\mathfrak{g}_\Lambda \oplus \mathfrak{g}_\Lambda$. To avoid this, Gross, Harvey, Martinec and Rohm (1985) found it was necessary to treat left and right moving waves on the closed string differently, so that, somewhat bizarrely, only those moving in one direction had motion in these compactified directions. The model they constructed, the heterotic string, enabled them to realize in string theory either of the gauge groups, $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$, for which Green and Schwarz (1984) had found anomaly cancellations for supersymmetric gauge theories coupled to gravity.

Rather than pursue these applications of Kac-Moody algebras, David returned to investigations of the equivalence of descriptions of two-dimensional quantum field theories in terms of either boson fields or fermion fields. Edward Witten (1984) had extended the known results on the equivalence in two dimensions between a single boson field and a pair of fermion fields to an equivalence between certain nonlinear boson theories and certain free fermion theories. Specifically, he demonstrated an equivalence between a nonlinear σ model, with a field g taking values in the group $SO(N)$, and a theory with N free fermion fields $\psi^i, 1 \leq i \leq N$.

To obtain suitable equations of motion for g , Witten needed to include a Wess-Zumino term in the σ -model Lagrangian, defining what is now known as the Wess-Zumino-Witten (WZW) theory, so that these then become $\partial_+(g^{-1}\partial_-g) = \partial_-(\partial_+gg^{-1}) = 0$, where ∂_\pm denote the partial derivatives with respect to $x^\pm = x \pm t$, implying that ∂_+gg^{-1} and $g^{-1}\partial_-g$ depend only on x^+ and x^- , respectively. Imposing periodic boundary conditions, identifying x with $x + L$, and expanding ∂_+gg^{-1} in powers of $z = e^{2\pi ix^+/L}$ and a basis, $\{t^a\}$, for the Lie algebra of $SO(N)$, the canonical commutation relations imply that, with suitable normalizations, its components, T_n^a , satisfy the Kac-Moody algebra,

$$[T_m^a, T_n^b] = if^{ab}_c T_{m+n}^c + km\delta^{ab}\delta_{m,-n}, \quad [22]$$

where k is an integer determined by the coefficient of the Wess-Zumino term, and f^{ab}_c are structure constants for $SO(N)$. Witten observed that a representation same algebra was provided by

$$T_n^a \rightarrow \frac{i}{2} \sum_r \psi_r^i M_{ij}^a \psi_{n-r}^j \quad [23]$$

where the M^a are the representation matrices for the defining representation of the Lie algebra of $\text{SO}(N)$ and ψ_r^i are the modes of the free fermion fields labelled by half odd integers r (for convenience taken to be odd under $x \rightarrow x + L$). The Kac-Moody algebra [22] with $k = 1$ follows from the canonical anti-commutation relations, $\{\psi_r^i, \psi_s^j\} = \delta_{r,-s} \delta^{ij}$. Witten observed that [22] has only a few positive energy unitary representations for $k = 1$, so that the isomorphism of the Kac-Moody algebras in the two theories effectively guarantees their equivalence in this case.

The requirements of positive energy, *i.e.* that the spectrum of L_0 be non-negative, and unitary require k in [22] to be a positive integer. (In the case of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated with a general compact simple Lie algebra \mathfrak{g} , the condition is that $x = 2k/\varphi^2$ should be a positive integer, called the level of the representation.) For $\text{SO}(N)$ WZW theories with $k > 1$, Witten proposed taking k copies of the N fermion theory, but, as David surmised might be the case, the equivalence is then no longer exact: there is more in the fermion theory than that WZW model.

David thought that exact equivalence would entail not only the isomorphism of the Kac-Moody algebras possessed by the two theories, but also this should extend to an equivalence of the energy-momentum tensors in the boson and fermion theories. Working in the general context of the WZW theory for a simple Lie group, G , with an N -dimensional real representation, M in [23], Goddard and Olive (74) sought to determine whether the energy-momentum tensors of the WZW theory and the N fermion theory were necessarily identical, and, if not, what conditions would ensure that they were equal.

The conformal symmetry of each of these theories means that the modes of the energy momentum tensor satisfy the Virasoro algebra,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}, \quad [24]$$

with $L_n = \mathcal{L}_n^{\mathfrak{g}}, c = c^{\mathfrak{g}}$, where

$$c^{\mathfrak{g}} = \frac{2k \dim \mathfrak{g}}{2k + Q^{\mathfrak{g}}} = \frac{x \dim \mathfrak{g}}{x + h^{\mathfrak{g}}}, \quad [25]$$

for the WZW theory, and $L_n = L_n^{\psi}, c = c^{\psi} = \frac{1}{2}N$ for the fermion theory, where

$$\mathcal{L}_n^{\mathfrak{g}} = \frac{1}{2k + Q_{\varphi}} \sum_m \circ T_{-m}^a T_{n+m}^a \circ, \quad L_n^{\psi} = \frac{1}{2} \sum_r : r b_{-r}^j b_{n+r}^j : . \quad [26]$$

[Here \circ and $:$ denote normal ordering with respect to the modes T_m^a and b_r^i , respectively, and $Q^{\mathfrak{g}}$ is the quadratic Casimir operator for the adjoint representation of \mathfrak{g} , and $h^{\mathfrak{g}} = Q^{\mathfrak{g}}/\varphi^2$, the dual Coxeter of \mathfrak{g} .] For the affine algebra $\hat{\mathfrak{g}}$ to be the same for the two theories, we need $k = \frac{1}{2}\kappa$, where $M_{ij}^a M_{ij}^b = \kappa \delta^{ab}$.

If the two energy-momentum tensors are equal, it is necessary that $c^{\mathfrak{g}} = c^{\psi}$. Goddard and Olive found that this is typically not the case, and, further, that the difference $K_n = L_n^{\psi} - \mathcal{L}_n^{\mathfrak{g}}$ itself satisfies a Virasoro algebra, commuting with $\mathcal{L}_m^{\mathfrak{g}}$, *i.e.* [24] holds for $L_n = K_n$ and $c = c^{\psi} - c^{\mathfrak{g}}$.

The requirement that the Virasoro algebra representations provided by L_n^ψ and $\mathcal{L}_n^{\mathfrak{g}}$ are unitarity and positive energy implies that K_n has the same property and so that $c^{\mathfrak{g}} \leq c^\psi$. If $c^{\mathfrak{g}} = c^\psi$, K_n vanishes because the Virasoro algebra does not have any non-vanishing positive energy unitary representations with $c = 0$. Thus $c^{\mathfrak{g}} = c^\psi$ provides a condition for the identity of the energy-momentum tensors and the equivalence of the theories that is both necessary and sufficient.

The cases where $c^K = c^\psi - c^{\mathfrak{g}} > 0$ also proved to be of great interest. The positive energy unitary representations of the Virasoro algebra [24] are labelled by two parameters, c and the lowest eigenvalue of L_0 , h , say. Some months before Goddard and Olive had begun their study of WZW models, Friedan, Qiu and Shenker (FQS) had shown that, for such representations, either $c \geq 1$ or c belongs to an infinite discrete sequence,

$$c = 1 - \frac{6}{(m+2)(m+3)}, \quad m = 0, 1, \dots, \quad [27]$$

for each value of which there are $\frac{1}{2}(m+1)(m+2)$ possible values of h (Friedan et al. 1984). At that time, it was known whether any of these discrete series representations exist, apart from $c = 0$, the trivial representation, and $c = \frac{1}{2}$, which corresponds to a single fermion field; the rest, with $c = \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, \dots$, were unknown. In most cases either $c^K = 0$ or $c^K \geq 1$, but Goddard and Olive found that taking M^a to be the 7-dimensional representation of $G = \text{SO}(3)$ gave the $c^K = \frac{7}{10}$ representation of the Virasoro algebra, with all the associated 6 values of h .

It turns out that the other values of c in the discrete series [27], those with $m \geq 4$, can not be obtained in this way and a more general approach is needed. This was found by Goddard, Kent and Olive (GKO) who changed the context from the free fermion representation of the affine algebra associated with $\mathfrak{so}(N)$ to that a positive energy unitary representation of the affine algebra, $\hat{\mathfrak{g}}$, associated with compact simple Lie algebra, \mathfrak{g}' , and a subalgebra $\mathfrak{g} \subset \mathfrak{g}'$. The representation of $\hat{\mathfrak{g}}'$ provides a representation of $\hat{\mathfrak{g}}$, whose level x is determined by the level y of the representation of $\hat{\mathfrak{g}}'$. (We obtain the previous construction by taking $\mathfrak{g}' = \mathfrak{so}(n)$.) We may define a Virasoro representation $\mathcal{L}_n^{\mathfrak{g}'}$ associated to the representation of $\hat{\mathfrak{g}}'$ in the same way the Virasoro representation $\mathcal{L}_n^{\mathfrak{g}}$ is associated to the representation of $\hat{\mathfrak{g}}$ by [26]. Now $K_n = \mathcal{L}_n^{\mathfrak{g}'} - \mathcal{L}_n^{\mathfrak{g}}$ commutes with the whole of $\hat{\mathfrak{g}}$ and so with $\mathcal{L}_n^{\mathfrak{g}}$. It follows that K_n provides a representation of the Virasoro algebra with central charge,

$$c^K = c^{\mathfrak{g}'} - c^{\mathfrak{g}} = \frac{y \dim \mathfrak{g}'}{y + h^{\mathfrak{g}'}} - \frac{x \dim \mathfrak{g}}{x + h^{\mathfrak{g}}}. \quad [28]$$

This construction, which has become known as the coset (or GKO) construction (76), extends straightforwardly to cases where \mathfrak{g}' or \mathfrak{g} are not simple (in which cases the various the level is defined for each summand). All of the values in the series [27], and all the associated values of h , can be obtained in this way, *e.g.* by taking $\mathfrak{g}' = \mathfrak{sp}(m+1)$, $x = 1$, $\mathfrak{g} = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$, for both of which the level is also 1 (81).

Given a conformal field theory (CFT) associated with the affine Kac-Moody algebra $\hat{\mathfrak{g}}'$, and $\mathfrak{g} \subset \mathfrak{g}'$, the coset construction defines a CFT associated with $\mathfrak{g}'/\mathfrak{g}$, on the subspace of states annihilated

by the positive modes of $\hat{\mathfrak{g}}$, with Virasoro algebra K_n . If $c^K = 0$, this is trivial, and the CFT associated with $\hat{\mathfrak{g}}$ is said to be conformally embedded in that associated with $\hat{\mathfrak{g}}'$. The coset construction has provided one of the principal ways of constructing interesting examples of CFTs in applications.

David returned to the investigation of the condition under which the energy-momentum tensors of a WZW theory for the group G , with an N -dimensional real representation, M , and an N fermion theory are equal, from which the coset construction arose as a by-product, and, with Peter Goddard and Werner Nahm, he found an elegant geometric interpretation, which enabled the classification of the cases where it is satisfied (78). The condition for the vanishing of $K_n = L_n^\psi - \mathcal{L}_n^{\mathfrak{g}}$ is

$$M_{ij}^a M_{kl}^a + M_{ik}^a M_{lj}^a + M_{il}^a M_{jk}^a = 0. \quad [29]$$

They observed that this is the condition that \mathfrak{g} can be extended by an N -dimensional space, \mathfrak{s} , to give a Lie algebra, $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{s}$, which is a symmetric space, *i.e.* $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, $[\mathfrak{g}, \mathfrak{s}] \subset \mathfrak{s}$, $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{g}$, with the action of \mathfrak{g} on \mathfrak{s} providing the representation M . Using $\{s_i\}$ to denote an orthonormal basis for \mathfrak{s} , for $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{s}$ to be a symmetric space, the commutation relations must have the form,

$$[t^a, t^b] = i f^{ab}{}_c t^c, \quad [t^a, s_i] = i M_{ij}^a s_j, \quad [s_i, s_j] = i M_{ij}^a t^a. \quad [30]$$

Given that \mathfrak{g} is a Lie algebra with representation M the condition for these to be consistent is the Jacobi identity for s_i, s_j, s_k , which is just [29]. Symmetric spaces have been classified and so provide a list of the instances in which $\mathcal{L}_n^{\mathfrak{g}} = L_n^\psi$.

REFERENCES TO OTHER AUTHORS

- Bogomol'nyi, E. B. 1976 The stability of classical solutions. *Sov. J. Nucl. Phys.* **24**, 389–394.
- Chew, G. F. 1966 *The Analytic S Matrix*. New York, 1966: Benjamin.
- Coleman, S. 1975 Quantum sine-Gordon equation as the massive Thirring model. *Phys. Rev.* **D11**, 2088–2097.
- Cutkosky, R. E. 1960 Singularities and discontinuities of Feynman amplitudes. *J. Math. Phys.* **1**, 429–433.
- Dirac, P. A. M. 1931 Quantised singularities in the electromagnetic field. *Proc. Roy. Soc.* **A33**, 60–72.
- Dorey, P. 1991 Root systems and purely elastic S -matrices. *Nucl. Phys.* **B358**, 654–676.
- Fermi, E., Pasta, J. & Ulam S. 1955 Studies of nonlinear problems. I. Los Alamos report LA-1940, published later in E. Segré (ed.) 1965 *Collected Papers of Enrico Fermi*, University of Chicago Press.
- Frenkel, I. B. & Kac, V. G. 1980 Basic representations of affine Lie algebras and dual resonance models. *Invent. Math.* **62**, 23–66.
- Green, M. B. & Schwarz, J. H. 1984 Anomaly cancellations in supersymmetric $D = 10$ gauge theory and superstring theory. *Phys. Lett.* **149B**, 117–122.
- Gross, D. J., Harvey, J. A., Martinec, E. & Rohm, R. 1985 Heterotic string theory (I) The free heterotic string. *Nucl. Phys.* **B256**, 253–284.
- Gunson, J. 1965 Unitarity and on-mass-shell analyticity as a basis for S -matrix theories. I, II, III. *J. Math. Phys.* **6**, 827–844; 845–851; 852–858.
- Iagolnitzer, D. 1981 Analyticity properties of the S -matrix: historical survey and recent results in S -matrix theory and axiomatic field theory. *Acta Phys. Austriaca, Suppl.*, **23**, 235–328.
- Kostant, B. 1979 The solution to a generalized Toda lattice and representation theory. *Advances in Mathematics* **34**, 195–338.
- Landau, L. D. 1959 On analytic properties of vertex parts in quantum field theory. *Nucl. Phys.* **13**, 181–192.
- Leznov, A. N. & Saveliev, M. V. 1980 Representation theory and integration of nonlinear spherically symmetric equations to gauge theories. *Commun. Math. Phys.* **74**, 111–118.
- Lovelace, C. 1971 Pomeron form factors and dual Regge cuts. *Phys. Lett.* **34B** 500–506.
- Manton, N. S. 1977 The force between 't Hooft-Polyakov monopoles. *Nucl. Phys.* **B126**, 525–541.

- Mandelstam, S. 1959 Analytic properties of transition amplitudes in perturbation theory. *Phys. Rev.* **115** 1741–1751; Construction of the Perturbation Series for Transition Amplitudes from their Analyticity and Unitarity Properties. *Phys. Rev.* **115** 1752–1762.
- Mandelstam, S. 1973 Manifestly dual formulation of the Ramond model. *Phys. Lett.* **46B** 447–451.
- Mandelstam, S. 1975 Soliton operators for the quantized sine-Gordon equation. *Phys. Rev.* **D11**, 3026–3030.
- Mikhailov, A. V., Olshanetsky, M. A. & Perelomov, A. M. 1981 Two dimensional generalized Toda lattice. *Commun. Math. Phys.* **79**, 473–488.
- Monastyrskii, M. I. & Perelomov, A. M. 1975 Concerning the existence of monopoles in gauge field theories. *ZhETF Pis. Red.* **21**, 94–96.
- Neveu, A. & Schwarz, J. H. 1971 Quark model of dual pions. *Phys. Rev.* **D4**, 1109–1111.
- Osborn, H. 1979 Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1. *Phys. Lett.* **83B**, 321–326.
- Prasad, M. K. & Sommerfield, C. M. 1975 An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon. *Phys. Rev. Lett.* **35**, 760–762.
- Ramond, P. 1971 Dual theory for free fermions. *Phys. Rev.* **D3**, 2415–2418.
- Sato, M. 1975 Recent development in hyperfunction theory and its applications to physics (microlocal analysis of S-matrices and related quantities) *Lect. Notes Math.* **449**, 13–29. Berlin: Springer-Verlag.
- Schwarz, J. H. 1971 Dual quark-gluon model of hadrons. *Phys. Lett.* **37B**, 315–319.
- Schwarz, J. H. & Wu, C. C. 1974 Functions occurring in dual fermion amplitudes. *Nucl. Phys.* **B73** 77–92.
- Segal, G. 1980 Jacobi's identity and an isomorphism between a symmetric algebra and an exterior algebra (unpublished).
- Segal, G. 1981 Unitary representations of some infinite dimensional groups. *Commun. Math. Phys.* **80**, 301–342.
- Seiberg, N. & Witten, E. 1994a Electromagnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nucl. Phys.* **B426**, 19–52; Erratum **B430**, 485–486.
- Seiberg, N. & Witten, E. 1994b Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nucl. Phys.* **B431**, 484–550.
- Sen, A 1994. Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $SL(2, \mathbb{Z})$ invariance in string theory. *Phys. Lett.* **B298**, 217–221.

- Skyrme, T. H. R. 1961 Particle states of a quantized meson field. *Proc. Roy. Soc.* **A262**, 237–245.
- Stapp, H. P. 1962 Derivation of the CPT theorem and the connection between spin and statistics from postulates of the S-Matrix theory. *Phys. Rev.* **125**, 2139–2162.
- Streater, R. & Wightman, A. 1964 PCT, Spin and Statistics, and All That. Princeton: Princeton University Press.
- 't Hooft, G. 1974 Magnetic monopoles in unified gauge theories. *Nucl. Phys.* **B79**, 276–281.
- Thorn, C. B. 1971 Embryonic dual model for pions and fermions *Phys. Rev.* **D4**, 1112–1116.
- Toda, M. 1967 Vibration of a chain with nonlinear interaction. *J. Phys. Soc. Japan* **22**, 431–436.
- Tyupkin, Yu. S., Fateev, V. A. & Shvarts, A. S. 1975 Existence of heavy particles in gauge field theories. *ZhETF Pis. Red.* **21**, 91–93.
- Witten, E. 1984 Non-abelian bosonization in two dimensions. *Commun. Math. Phys.* **95**, 455–472.

BIBLIOGRAPHY

- (1) 1962 (With J. C. Taylor) Singularities of scattering amplitudes at isolated real points. *Nuovo Cim.* **24**, 814–822.
- (2) 1962 Unitarity and evaluation of discontinuities. *Nuovo Cim.* **26**, 73–102.
- (3) 1963 On analytic continuation of scattering amplitude through a three-particle cut. *Nuovo Cim.* **28**, 1318–1336.
- (4) 1963 Unitarity and evaluation of discontinuities – II. *Nuovo Cim.* **29**, 326–335.
- (5) 1964 Exploration of S -matrix theory. *Phys. Rev.* **135**, B745–B760.
- (6) 1965 Unitarity and evaluation of discontinuities – III. *Nuovo Cim.* **37**, 1423–1445.
- (7) 1966 (With P. V. Landshoff & J. C. Polkinghorne) Hierarchical principle in perturbation theory. *Nuovo Cim.* **A43**, 444–453.
- (8) 1966 (With E. Y. C. Lu) Spin and statistics in S -matrix theory. *Nuovo Cim.* **45A**, 205–218.
- (9) 1966 (With P. V. Landshoff & J. C. Polkinghorne) Icecream-cone singularity in S -matrix theory. *J. Math. Phys.* **7**, 1600–1606.
- (10) 1966 (With P. V. Landshoff & J. C. Polkinghorne) Normal thresholds in subenergy variables *J. Math. Phys.* **7**, 1593–1599.
- (11) 1966 (With P. V. Landshoff) Extraction of singularities from S -matrix. *J. Math. Phys.* **7**, 1464–1477.
- (12) 1966 (With R. J. Eden, P. V. Landshoff & J. C. Polkinghorne) *The Analytic S-Matrix*. Cambridge, UK: Cambridge University Press.
- (13) 1968 (With J. C. Polkinghorne) Properties of Mandelstam cuts. *Phys. Rev.* **171**, 1475–1481.
- (14) 1969 (With R. E. Cutkosky, P. V. Landshoff & J. C. Polkinghorne) A non-analytic S -matrix. *Nucl. Phys.* **B12**, 281–300.
- (15) 1969 (With M. J. W. Bloxham, D. I. Olive & J. C. Polkinghorne) S -matrix singularity structure in the physical region. I. Properties of multiple integrals. *J. Math. Phys.* **10**, 494–502.
- (16) 1969 (With M. J. W. Bloxham, D. I. Olive & J. C. Polkinghorne) S -matrix singularity structure in the physical region. II. Unitarity integrals. *J. Math. Phys.* **10**, 545–552.
- (17) 1969 (With M. J. W. Bloxham, D. I. Olive & J. C. Polkinghorne) S -matrix singularity structure in the physical region. III. General discussion of simple Landau singularities. *J. Math. Phys.* **10**, 553–561.
- (18) 1969 (With D. K. Campbell & W. J. Zakrzewski) Veneziano amplitudes for Reggeons and spinning particles. *Nucl. Phys.* **B14**, 319–329.
- (19) 1969 (With W. J. Zakrzewski) A Veneziano amplitude for four spinning pions. *Phys. Lett.* **B30**, 650–652.
- (20) 1970 (With W. J. Zakrzewski) A possible Veneziano amplitude for many pions. *Nucl. Phys.* **B21**, 303–320.

- (21) 1970 The connection between Toller and Froissart-Gribov signed amplitudes. *Nucl. Phys.* **B15**, 617–627.
- (22) 1970 (With V. Alessandrini, D. Amati & M. Le Bellac) Duality and gauge properties of twisted propagators in multi-Veneziano theory. *Phys. Lett.* **B32**, 285–290.
- (23) 1970 (With D. Amati & M. Le Bellac) Twisting-invariant factorization of multiparticle dual amplitude. *Nuovo Cim.* **A66**, 815–830.
- (24) 1970 (With D. Amati & M. Le Bellac) The twisting operator in multi-Veneziano theory. *Nuovo Cim.* **A66**, 831–844.
- (25) 1971 Operator vertices and propagators in dual theories. *Nuovo Cim.* **A3**, 399–411.
- (26) 1971 (With V. Alessandrini, D. Amati & M. Le Bellac) The operator approach to dual multiparticle theory. *Phys. Rept.* **1C**, 269–345.
- (27) 1971 The dual approach to the strong interaction S -matrix. In *Proceedings of the Symposium on Basic Questions in Elementary Particle*. pp. 140–147. München, Germany: Max-Planck-Institut für Physik und Astrophysik.
- (28) 1971 (With E. Corrigan) Fermion meson vertices in dual theories. *Nuovo Cim.* **A11**, 749–773.
- (29) 1972 Clarification of the rubber string picture. In: *Proceedings of 16th International Conference on High-energy Physics, Batavia, IL* (ed. J. D. Jackson & A. Roberts), vol. 1, pp. 472–474. Batavia, IL, USA: National Accelerator Laboratory.
- (30) 1973 (With L. Brink & D. I. Olive) The physical state projection operator in dual resonance models for the critical dimension of space-time. *Nucl. Phys.* **B56**, 253–265.
- (31) 1973 (With L. Brink) Recalculation of the unitary single planar dual loop in the critical dimension of spacetime. *Nucl. Phys.* **B58**, 237–253.
- (32) 1973 (With L. Brink, C. Rebbi & J. Scherk) The missing gauge conditions for the dual fermion emission vertex and their consequences. *Phys. Lett.* **B45**, 379–383.
- (33) 1973 (With L. Brink & J. Scherk) The gauge properties of the dual model pomeron-reggeon vertex – their derivation and their consequences. *Nucl. Phys.* **B61**, 173–198.
- (34) 1973 (With J. Scherk) No ghost theorem for the pomeron sector of the dual model. *Phys. Lett.* **B44**, 296–300.
- (35) 1974 (With J. Scherk) Towards satisfactory scattering amplitudes for dual fermions. *Nucl. Phys.* **B64**, 334–348.
- (36) 1973 (With E. Corrigan, P. Goddard & R. A. Smith) Evaluation of the scattering amplitude for four dual fermions. *Nucl. Phys.* **B67**, 477–491.
- (37) 1975 Unitarity and discontinuity formulas. *Lect. Notes Math.* **449**, 133–142. Berlin: Springer-Verlag.
- (38) 1975 Dual Models. In: *Proceedings of 17th International Conference on High-energy Physics, London* (ed. J.R. Smith), pp. 269–280. Chilton, UK: Rutherford Laboratory.
- (39) 1975 (With D. Bruce & E. Corrigan) Group theoretical calculation of traces and determinants occurring in dual theories. *Nucl. Phys.* **B95**, 427–433.

- (40) 1976 (With E. Corrigan, D. B. Fairlie & J. Nuyts) Magnetic monopoles in SU(3) gauge theories. *Nucl. Phys.* **B106**, 475–492.
- (41) 1976 Some topics in dual theory. *Acta Universitatis Wratislaviensis* **368**, 293–310.
- (42) 1976 (With E. Corrigan) Color and magnetic monopoles. *Nucl. Phys.* **B110**, 237–247.
- (43) 1976 Angular momentum, magnetic monopoles and gauge theories. *Nucl. Phys.* **B113**, 413–420.
- (44) 1976 (With F. Gliozzi & J. Scherk) Supergravity and the spinor dual model. *Phys. Lett.* **B65**, 282–286.
- (45) 1977 (With F. Gliozzi & J. Scherk) Supersymmetry, supergravity theories and the dual spinor model. *Nucl. Phys.* **B122**, 253–290.
- (46) 1977 (With P. Goddard & J. Nuyts) Gauge theories and magnetic charge. *Nucl. Phys.* **B125**, 1–28.
- (47) 1977 (With C. Montonen) Magnetic monopoles as gauge particles? *Phys. Lett.* **B72**, 117–120.
- (48) 1977 Progress in analytic S -matrix theory. *Publ. Res. Inst. Math. Sci. Kyoto* **12**, 347–349.
- (49) 1978 (With P. Goddard) Magnetic monopoles in gauge field theories. *Rep. Prog. Phys.* **41**, 1357–1437.
- (50) 1978 (With E. Witten) Supersymmetry algebras that include topological charges. *Phys. Lett.* **B78**, 97–101.
- (51) 1979 Supersymmetric solitons. *Czech. J. Phys.* **B29**, 73–80.
- (52) 1979 Magnetic monopoles. *Phys. Rept.* **49**, 165–172.
- (53) 1979 The electric and magnetic charges as extra components of four momentum. *Nucl. Phys.* **B153**, 1–12.
- (54) 1979 (With S. Sciuto & R. J. Crewther) Instantons in field theory. *Rev. Nuovo Cim.* **2**, 1–117.
- (55) 1979 Magnetic monopoles and nonabelian gauge theories. In: *Proceedings of the International Conference on Mathematical Physics, Lausanne, Switzerland, 1979* (ed. K. Osterwalder), pp. 249–262. Berlin: Springer-Verlag.
- (56) 1980 Magnetic monopoles. In: *Proceedings of the International Conference On High Energy Physics, 1979* (ed. A. Zichichi), pp. 953–957. Geneva: CERN.
- (57) 1980 Magnetic monopoles and grand unified theories. In: *Unification of the Fundamental Particle Interactions* (ed. S. Ferrara *et al.*), pp. 451–459. New York: Plenum Press.
- (58) 1981 Classical solutions in gauge theories – spherically symmetric monopoles – Lax pairs and Toda lattices. In: *Current Topics in Elementary Particle Physics* (ed. K. H. Mütter & K. Schilling), pp. 199–217. New York: Plenum Press.
- (59) 1981 Magnetic monopoles and other topological objects in quantum field theory. *Phys. Scripta* **24**, 821–826.
- (60) 1981 (With P. Goddard) Charge quantization in theories with an adjoint representation Higgs mechanism. *Nucl. Phys.* **B191**, 511–527.
- (61) 1981 (With P. Goddard) The magnetic charges of stable self-dual monopoles. *Nucl. Phys.* **B191**, 528–548.

- (62) 1982 Self-dual magnetic monopoles. *Czech. J. Phys.* **B32**, 529–536.
- (63) 1982 The structure of self-dual monopoles. In: *Symposium on Particle Physics: Gauge Theories and Lepton-Hadron Interactions* (ed. Z. Horvath *et al.*), pp. 355–395. Budapest, Hungary: Central Research Inst. Physics.
- (64) 1982 Magnetic monopoles and electromagnetic duality conjectures. In: *Monopoles in Quantum Field Theory* (ed. N. S. Craigie *et al.*), pp. 157–191. Singapore: World Scientific.
- (65) 1982 (With N. Ganoulis & P. Goddard) Self-dual monopoles and Toda molecules. *Nucl. Phys.* **B205**, 601–636.
- (66) 1982 (With N. Turok) Z_2 vortex strings in grand unified theories. *Phys. Lett.* **B117**, 193–196.
- (67) 1983 Relations between grand unified and monopole theories. In: *Unification of the Fundamental Particle Interactions II* (ed. J. Ellis & S. Ferrara), pp. 17–28. New York: Plenum Press.
- (68) 1983 Lectures on gauge theories and lie algebras With some applications to spontaneous symmetry breaking and integrable dynamical systems. University of Virginia preprint 83-0529 (unpublished).
- (69) 1983 (With P. C. West) The $N = 4$ supersymmetric E_8 gauge theory and coset space dimensional reduction. *Nucl. Phys.* **B217**, 248–284.
- (70) 1983 (With N. Turok) The symmetries of Dynkin diagrams and the reduction of Toda field equations. *Nucl. Phys.* **B215**, 470–494.
- (71) 1983 (With N. Turok) Algebraic structure of Toda systems. *Nucl. Phys.* **B220**, 491–507.
- (72) 1984 (With P. Goddard) Algebras, lattices and strings. In: *Vertex Operators in Mathematics and Physics* (ed. J. Lepowsky *et al.*), pp. 51–96. New York: Springer-Verlag.
- (73) 1985 (With L. A. Ferreira) Non-compact symmetric spaces and the Toda molecule equations. *Commun. Math. Phys.* **99**, 365–384.
- (74) 1985 (With P. Goddard) Kac-Moody algebras, conformal symmetry and critical exponents. *Nucl. Phys.* **B257**, 226–252.
- (75) 1985 (With N. Turok) Local conserved densities and zero curvature conditions for Toda lattice field theories. *Nucl. Phys.* **B257**, 277–301.
- (76) 1985 (With P. Goddard & A. Kent) Virasoro algebras and coset space models. *Phys. Lett.* **B152**, 88–92.
- (77) 1985 (With P. Goddard & A. Schwimmer) The heterotic string and a fermionic construction of the E_8 Kac-Moody algebra. *Phys. Lett.* **B157**, 393–399.
- (78) 1985 (With P. Goddard & W. Nahm) Symmetric spaces, Sugawara’s energy momentum tensor in two-dimensions and free fermions. *Phys. Lett.* **B160**, 111–116.
- (79) 1985 Kac-Moody algebras: an introduction for physicists. In: *Proceedings of the Winter School Geometry and Physics* (ed. Z. Frolik *et al.*), pp. 177–198. Palermo: Circolo Matematico di Palermo.
- (80) 1986 Kac-Moody and Virasoro Algebras in Local Quantum Physics. In: *Fundamental Problems of Gauge Field Theory* (ed. G. Velo & A. S. Wightman), pp. 51–92. New York: Plenum Press.

- (81) 1986 (With P. Goddard & A. Kent) Unitary representations of the Virasoro and Super-Virasoro algebras. *Commun. Math. Phys.* **103**, 105–119.
- (82) 1986 (With P. Goddard) Kac-Moody and Virasoro algebras in relation to quantum physics. *Int. J. Mod. Phys.* **A1**, 303–414.
- (83) 1986 (With N. Turok) The Toda lattice field theory hierarchies and zero-curvature conditions in Kac-Moody algebras. *Nucl. Phys.* **B265**, 469–484.
- (84) 1986 (With P. Goddard) An introduction to Kac-Moody algebras and their physical applications. In: *Proceedings of the Santa Barbara workshop on Unified String Theories* (ed. M. Green & D. Gross), pp. 214–243. Singapore: World Scientific.
- (85) 1986 Infinite dimensional lie algebras and quantum physics. In: *Fundamental Aspects of Quantum Theory, Como 1985* (ed. V. Gorini & A. Figueredo), pp. 289-293. New York: Plenum Press.
- (86) 1986 (With P. Goddard, W. Nahm & A. Schwimmer) Vertex operators for non-simply-laced algebras. *Commun. Math. Phys.* **107**, 179–212.
- (87) 1986 (With M. D. Freeman) BRS cohomology in string theory and the no ghost theorem. *Phys. Lett.* **B175**, 151–154.
- (88) 1986 (With M. D. Freeman) The calculation of planar one loop diagrams in string theory using the BRS formalism. *Phys. Lett.* **B175**, 155–158.
- (89) 1987 Lectures on algebras, lattices and strings. In: *Supersymmetry, Supergravity, Superstrings '86* (ed. B. de Wit *et al.*), pp. 239–256. Singapore: World Scientific.
- (90) 1987 Infinite dimensional algebras in modern theoretical physics. In: *Proceedings of the VIIIth International Congress on Mathematical Physics, Marseilles, 1986* (ed. M. Mebkhout & R. Sénéor), pp. 242-256. Singapore: World Scientific.
- (91) 1987 (With R. C. Arcuri & J. F. Gomes) Conformal subalgebras and symmetric spaces. *Nucl. Phys.* **B103**, 327–339.
- (92) 1987 (With P. Goddard, W. Nahm, H. Ruegg & A. Schwimmer) Fermions and octonions. *Commun. Math. Phys.* **112**, 385–408.
- (93) 1987 (With P. Goddard & G. Waterson) Superalgebras, symplectic bosons and the Sugawara construction. *Commun. Math. Phys.* **112**, 591–611.
- (94) 1988 The vertex operator construction for non-simply laced Kac-Moody algebras. In: *Infinite-dimensional Lie algebras and their applications, Proceedings, Montreal, 1986* (ed. S. N. Kass), pp. 181-188. Singapore: World Scientific.
- (95) 1988 Loop algebras, QFT and strings. In: *Strings and superstrings: XVIIIth International GIFT Seminar on Theoretical Physics Madrid 1987* (ed. J. R. Mittelbrunn *et al.*), pp. 217-285. Singapore: World Scientific.
- (96) 1988 (ed. with P. Goddard) *Kac-Moody and Virasoro algebras: a reprint volume for physicists*. Singapore: World Scientific.
- (97) 1989 Introduction to string theory: its structure and its uses. *Phil. Trans. R. Soc. London* **A329**, 319-328.
- (98) 1990 Introduction to conformal invariance and infinite dimensional algebras. In: *Physics, geometry, and topology: 1989 Banff NATO ASI* (ed. H.C. Lee), pp. 241-261. New York: Plenum Press.

- (99) 1991 (With M. -F. Chu, P. Goddard, I. Halliday & A. Schwimmer) Quantization of the Wess-Zumino-Witten model on a circle. *Phys. Lett.* **B266**, 71–81.
- (100) 1991 (With A. Fring & H. C. Liao) The Mass spectrum and coupling in affine Toda theories. *Phys. Lett.* **B266**, 82–86.
- (101) 1992 (With A. Fring) The Fusing rule and the scattering matrix of affine Toda theory. *Nucl. Phys.* **B379**, 429–447.
- (102) 1993 (With H. C. Liao & N. Turok) Topological solitons in A_r affine Toda theory. *Phys. Lett.* **B298**, 95–102.
- (103) 1993 (With N. Turok & J. W. R. Underwood) Solitons and the energy momentum tensor for affine Toda theory. *Nucl. Phys.* **B401**, 663–697.
- (104) 1993 (With M. V. Saveliev & J. W. R. Underwood) On a solitonic specialization for the general solutions of some two-dimensional completely integrable systems. *Phys. Lett.* **B311**, 117–122.
- (105) 1993 (With N. Turok & J. W. R. Underwood) Affine Toda solitons and vertex operators. *Nucl. Phys.* **B409**, 509–546.
- (106) 1993 (With M. A. C. Kneipp) Crossing and anti-solitons in affine Toda theories. *Nucl. Phys.* **B408**, 565–573.
- (107) 1994 (With E. Rabinovici & A. Schwimmer) A Class of string backgrounds as a semi-classical limit of WZW models. *Phys. Lett.* **B321**, 361–364.
- (108) 1994 (With A. Fring, P. R. Johnson & M. A. C. Kneipp) Vertex operators and soliton time delays in affine Toda field theory. *Nucl. Phys.* **B430**, 597–614.
- (109) 1995 (With L. Ferreira & M. V. Saveliev) Orthogonal decomposition of some affine Lie algebras in terms of their Heisenberg subalgebras. *Theor. Math. Phys.* **102**, 10–22.
- (110) 1996 (With M. A. C. Kneipp) Solitons and vertex operators in twisted affine Toda field theories. *Commun. Math. Phys.* **177**, 561–582.
- (111) 1996 Exact electromagnetic duality. *Nucl. Phys. Proc. Suppl.* **45A**, 88–102; **46**, 1–15.
- (112) 1997 Lectures on exact electromagnetic duality II. In: *Proceedings of the IX Jorge André e Swieca Summer School Campos de Jordão* (ed. J.C.A. Barata et al.), pp. 166–195. Singapore: World Scientific.
- (113) 1997 Introduction to electromagnetic duality. *Nucl. Phys. Proc. Suppl.* **58**, 43–55.
- (114) 1998 The monopole. In: *Paul Dirac: the man and his work* (ed. P. Goddard), pp. 88–107. Cambridge: Cambridge University Press.
- (115) 1999 Exact electromagnetic duality. In: *Strings, branes and dualities, Cargese, 1997* (ed. L. Baulieu), pp. 3–31. Berlin: Springer-Verlag.
- (116) 1999 (ed. with P. C. West) *Duality and supersymmetric theories*. Cambridge: Cambridge University Press.
- (117) 1999 (With P. C. West) Particle physics and fundamental theory: introduction and guide to duality and supersymmetric theories. In: *Duality and supersymmetric theories* (ed. D. Olive & P. C. West), pp. 1–20. Cambridge: Cambridge University Press.
- (118) 1999 Introduction to duality. In: *Duality and supersymmetric theories* (ed. D. Olive & P. C. West), pp. 62–94. Cambridge: Cambridge University Press.

- (119) 2000 (With M. Alvarez) The Dirac quantization condition for fluxes on four manifolds. *Commun. Math. Phys.* **210**, 13–28.
- (120) 2000 Aspects of electromagnetic duality. In: *Proceedings, Conference on Nonperturbative quantum effects: Paris, 2000* (ed. D. Bernard *et al.*), *JHEP PoS tmr2000 021*, pp. 1–7.
- (121) 2000 Lie algebras, integrability, and particle physics. *Theor. Math. Phys.* **123**, 659–662.
- (122) 2001 (With M. Alvarez) Spin and Abelian electromagnetic duality on four manifolds. *Commun. Math. Phys.* **217**, 331–356.
- (123) 2001 Spin and electromagnetic duality: an outline. Talk at 15th Anniversary Meeting of Dirac Medallists, Trieste, 2000. [hep-th/0104062](#).
- (124) 2001 (With H. Nicolai) The principal $SO(1; 2)$ subalgebra of a hyperbolic Kac-Moody algebra. *Lett. Math. Phys.* **58**, 141–152.
- (125) 2002 The quantization of charges. *Italian Phys. Soc. Proc.* **79**, 173–183.
- (126) 2002 (With M. R. Gaberdiel & P. C. West) A class of Lorentzian Kac-Moody algebras. *Nucl. Phys.* **B645**, 403–437.
- (127) 2002 Duality and Lorentzian Kac-Moody algebras. In: *Proceedings, International Workshop on Integrable Theories, Solitons and Duality UNESP 2002 JHEP PoS un-esp2002 008*, pp. 1–8.
- (128) 2003 Paul Dirac and the pervasiveness of his thinking. Talk at Dirac Centenary Conference, Waco, Texas, 2002. [hep-th/0304133](#).
- (129) 2003 Charges and fluxes for branes. In: *27th Johns Hopkins Workshop on Current Problems Conference. JHEP PoS jhw2003 019*, pp. 1–7.
- (130) 2006 (With M. Alvarez) Charges and fluxes in Maxwell theory on compact manifolds with boundary. *Commun. Math. Phys.* **267** 279–305.
- (131) 2012 From dual fermion to superstring. In: *The Birth of String Theory* (ed. A. Cappelli *et al.*), pp. 346–360.