Functional difference equations and eigenfunctions of a Schrödinger operator with $\delta'-$interaction on a circular conical surface

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Abstract

Eigenfunctions and their asymptotic behaviour at large distances for the Laplace operator with singular potential, the support of which is on a circular conical surface in three-dimensional space, are studied. Within the framework of incomplete separation of variables an integral representation of the Kontorovich-Lebedev type for the eigenfunctions is obtained in terms of solution of an auxiliary functional difference equation with a meromorphic potential. Solutions of the functional difference equation are studied by reducing it to an integral equation with a bounded selfadjoint integral operator. To calculate the leading term of the asymptotics of eigenfunctions the Kontorovich-Lebedev integral representation is transformed to a Sommerfeld-type integral which is well adapted to application of the saddle point technique. Outside a small angular vicinity of the so called singular directions the asymptotic expression takes on an elementary form of exponent decreasing in distance. However, in an asymptotically small neighbourhood of singular directions the leading term of the asymptotics also depends on a special function closely related to the function of parabolic cylinder (Weber function).

Key words. Functional difference equations, Robin Laplacians, $\delta'-$interaction, eigenfunctions, asymptotics

1 Introduction and motivation

Model problems for a Laplacian with a singular potential having its support on some surfaces arise in quantum physics [1] (approximate atomic Hamiltonians in strong homogeneous magnetic fields), see also [2], in quantum optics and acoustics [3] (photonic crystals with high contrast) as well as in the classical scattering of acoustic or electromagnetic waves on semi-transparent canonical surfaces (cone, wedge), [4],[5], [6] Chapter 4 and [7]. In the works [4], [5], [6], Chapter 4 and [7], solutions are constructed within the framework of incomplete separation of variables and, in essence, are represented with the aid of the decomposition by 'generalized' eigenfunctions of the positive essential (absolutely continuous) spectrum.

Traditionally, spectrum of the Laplacian with singular potential is studied in the framework of general methods [8],[9], [10] for selfadjoint operators in a Hilbert space. In particular, this requires additional meaningful affords and reductions, including splitting of the operator and incomplete separation of variables [8], [11], see also [12] for elasticity. Similar approaches are used to describe spectra of Robin Laplacians in a cone [13] or in an infinite sector (wedge) [14], see also [15]. It is possible to obtain estimates for their eigenfunctions, to prove their exponential decrease [13] or, in some cases, even to determine them explicitly, e.g. for the Robin Laplacian in a wedge [15].

However, if one needs to find out more detailed information about the behavior of eigenfunctions, to calculate their asymptotics and, to this end, to obtain useful integral representations, it is usually possible only for some special cases, for example, for model problems with symmetry. For instance,

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in our case, this is the axial symmetry of a circular cone. However, even in this case, the problem of description of asymptotic behaviour of the eigenfunctions remains cumbersome and requiring use of special approaches. In our case incomplete separation of variables exploits integral representations and reduces the problem to study of a functional difference equation with a meromorphic potential. Calculation of the asymptotics of the Sommerfeld integral representation for an eigenfunction is carried out in the framework of the steepest descent method and its modifications in the situation when a singularity of the integrand may be located near a saddle point.

In the following section we introduce basic notations and describe the Laplacian at hand. The corresponding selfadjoint operator, conventionally denoted $A_\gamma := -\triangle - \gamma\delta'_C$, is uniquely specified by means of a densely defined semibounded sesquilinear form [8] which is closable in a Hilbert space. We also give the classical formulation of the boundary-value problem corresponding to the operator. The corresponding differential equation with the spectral parameter E is supplemented by the boundary conditions consisting of two correlations which are the condition of continuity of the normal derivative of solution across the conical boundary C and a Robin type condition linking the normal derivative with the jump of the unknown solution on the boundary. A positive constant $\gamma > 0$ is a Robin parameter in this boundary condition. **It is worth noticing that our approach, used in this work, can be also applied to the singular potential with $\delta-$interaction.**

Actually the equation for the eigenfunctions U of the operator A_γ can be formally written as

$$A_\gamma U = EU\,,$$

where E is the spectral parameter. Herein we consider negative spectrum ($E < 0$) of the operator. Recall that the spectrum of the operator A_γ consists of the essential spectrum $\sigma_{ess}(A_\gamma) = [-4\gamma^2, \infty)$ and of the infinite discrete part $\sigma_d(A_\gamma)$ that belongs to the interval $(-\infty, -4\gamma^2)$ and accumulates at the end $-4\gamma^2$ of the essential spectrum, see [8], [11].

We then make use of the Kontorovich-Lebedev (KL) integral representation for a solution of the spectral problem which separates radial and angular variables. The unknown function in the integrand solves a problem for the spherical Laplacian on the unit sphere. By means of further separation of the angular variables the unknown function in the integrand is represented as a product of the spherical functions and of a solution of some homogeneous functional difference equation of the second order with a meromorphic potential depending on the spectral parameter E, see also [16]. Analysis of solutions to this equation plays a crucial role for construction of the eigenfunctions and their asymptotics.

It is worth commenting on the functional difference equations that have recently appeared in various applications. Malyuzhinets has solved some functional difference equations of the first order in order to treat the problem of diffraction by an impedance wedge, [17], see also [18] and [19],[20]. He has constructed a solution of the functional equations by means of the special meromorphic function which is now refered to as the Malyuzhinets function. It is worth noticing that, together with our studies in this work, the use of functional difference equations of the first and second order has also become a common place in some problems of spectral theory [21],[22], in diffraction [23],[16],[6],[19],[7] in water-wave problems [24],[18] and in quantum theory [25]. In particular, some analogues of the special Malyuzhinets function have been used in these works, [26].

The abovementioned functional difference equation can be efficiently studied by means of reduction to an integral equation. The corresponding integral operator is bounded and selfadjoint, whereas its spectrum is directly connected with the spectrum of A_γ. It is possible to describe some useful properties of the eigenfunctions of the integral operator, which enables one to use this information in order to describe asymptotics of the eigenfunctions of A_γ.

To this end, it is convenient to reduce the KL integral representation for the eigenfunctions to a Sommerfeld-type integral. The latter integral is well adapted for application of the saddle point technique to asymptotic evaluations. It turns out that the leading singularities of the integrand in the Sommerfeld integral can be efficiently described. For some directions exactly these singularities are responsible for the leading terms of the asymptotics in the framework of asymptotic evaluation of the integral. However, for some other angular directions the singularities can be close to the saddle points of the Sommerfeld integral, i.e. belong to some angular vicinities of the so called singular directions (see also [27]). In this case, the asymptotics has a more complex form and is described by means of a special function being a close relative of the parabolic cylinder functions (Weber functions). Actually, this special function is responsible for switching of asymptotic regims, i.e. of one rate of decay of an eigenfunction, that is valid for some range of directions, to another rate corresponding to a supplementary range of angles. The asymptotics obtained is one of the main results of this work.

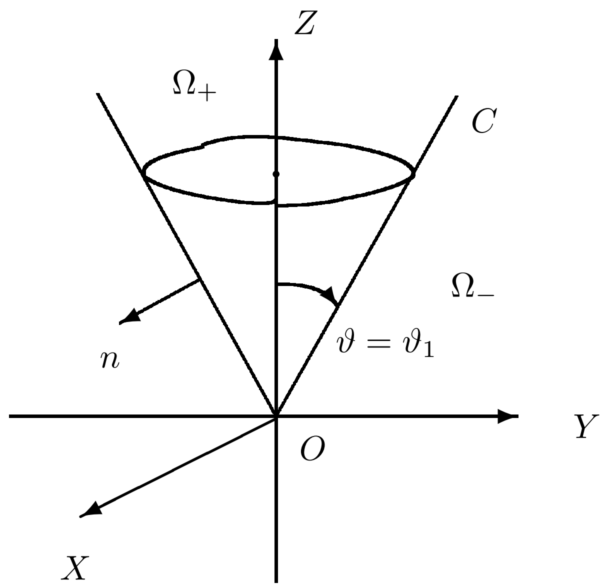

Figure 1: The domains $\Omega_\pm$ and the conical boundary $\mathcal{C}$

2 Formulation and reduction

A circular conical surface $\mathcal{C}$ separates two conical domains Ω_+ and Ω_- of R^3, Fig. 1. The Cartesian coordinates (X, Y, Z) are connected with the spherical ones,

$$X = r\cos\varphi\sin\vartheta, \;\; Y = r\sin\varphi\sin\vartheta, \;\; Z = r\cos\vartheta.$$

The domains $\Omega_\pm$ are defined in the spherical coordinates as $\Omega_+ = \{(r,\vartheta,\varphi):\; r > 0,\; 0 \leqslant \vartheta < \vartheta_1,\; 0 < \varphi \leqslant 2\pi\}$ and $\Omega_- = \{(r,\vartheta,\varphi):\; r > 0,\; \vartheta_1 < \vartheta < \pi,\; 0 < \varphi \leqslant 2\pi\}$, $\mathcal{C}$ is their common boundary, $\mathcal{C} = \{(r,\vartheta,\varphi):\; r > 0,\; \vartheta = \vartheta_1,\; 0 < \varphi \leqslant 2\pi\}$, $0 < \vartheta_1 < \pi/2$.

The symmetric sesquilinear form $(\gamma > 0)$

$$a_\gamma[u,v] = (\nabla u_+, \nabla v_+)_{L_2(\Omega_+)} + (\nabla u_-, \nabla v_-)_{L_2(\Omega_-)} - 2\gamma(u_+|_{\mathcal{C}} - u_-|_{\mathcal{C}}, v_+|_C - v_-|_{\mathcal{C}})_{L_2(\mathcal{C})}, \tag{1}$$

where $\mathrm{Dom}[a_\gamma] = H^1(\Omega_+) \oplus H^1(\Omega_-)$ and $(u,v)_{L_2(B)} = \int_B u(x)\bar{v}(x)\mathrm{d}x$, is closed and semibounded from below so that it uniquely specifies a selfadjoint operator, denoted A_γ (for some additional details see [8]).

In order to study eigenfunctions of this operator it is useful to write down spectral problem $A_\gamma U = EU$ for the operator A_γ defined by (1) in the 'classical' terms of differential equations and boundary conditions,

$$\begin{aligned} -\triangle u^+(r,\omega) - Eu^+(r,\omega) = 0, \;\; (r,\omega) \in \Omega_+\,, \\ -\triangle u^-(r,\omega) - Eu^-(r,\omega) = 0, \;\; (r,\omega) \in \Omega_-, \end{aligned} \tag{2}$$

$\omega := (\vartheta,\varphi)$ and $\triangle = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_\omega$, $\triangle_\omega$ is the Laplacian on the unit sphere S^2. The boundary conditions read

$$\begin{aligned} \left.\frac{\partial u^+}{\partial n}\right|_{\mathcal{C}} &= \gamma(u^+|_{\mathcal{C}} - u^-|_{\mathcal{C}})\,, \\ \left.\frac{\partial u^+}{\partial n}\right|_{\mathcal{C}} &= \left.\frac{\partial u^-}{\partial n}\right|_{\mathcal{C}}\,, \end{aligned} \tag{3}$$

where the normal n is directed from Ω_+ to Ω_- and $\frac{\partial}{\partial n} = \frac{1}{r}\frac{\partial}{\partial \vartheta}$. It is worth commenting on that, in view of the elliptic regularity of solutions and smoothness of the surface $\mathcal{C}$ as $r \neq 0$, the equations (2) and boundary conditions (3) can be understood in classical sense. In physics of wave phenomena the conditions (3) are traditionally called conditions of semitransparency [4]. **Actually, applications of the**

boundary conditions of semitransparency in physics motivated our study of $\delta'-$interaction in this work, see [4], [5]. It should be noticed, however, that the same approach with appropriate modifications enables one to consider singular potential with $\delta-$interaction.

We consider negative spectrum $E < 0$ of A_γ and construct classical solutions of (2), (3) which additionally satisfy the so called Meixner's condition at the vertex O, i.e. as $r \to 0$,

$$
\begin{aligned}
&|u^\pm(r,\omega)| < \frac{c_\pm}{r^{\delta_0^\pm}},\\
&|r\nabla u^\pm(r,\omega)| < \frac{c_\pm}{r^{\delta_0^\pm}} \quad ,
\end{aligned}
\tag{4}
$$

($\delta_0^\pm > -1/2$) uniformly[1] w.r.t. (with respect to) the direction $\omega \in \Sigma_\pm$, where $\Sigma_+ = \Omega_+ \cap S^2$, $\Sigma_- = \Omega_- \cap S^2$ and $\sigma = \mathcal{C} \cap S^2$. The conditions (4) are sufficient for local summability of $u^\pm$ and also for $u^\pm \in H^1_{loc}(\Omega_\pm)$.

Provided $E < -4\gamma^2$ the spectrum of the operator A_γ is discrete, so that in order to find the corresponding eigenfunctions and to prove their exponential decreasing we require the following conditions: let there exist such $\delta_\pm > 0$ that the estimates hold

$$
\int\limits_{\Omega_\pm} |u^\pm(r,\omega)|^2 \exp(2\delta_\pm r) \mathrm{d}x < \infty, \tag{5}
$$

$\mathrm{d}x = r^2 \mathrm{d}r \mathrm{d}\omega$, $\mathrm{d}\omega = \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$. However, provided $0 > E \geqslant -4\gamma^2$, i.e E belongs to the negative essential spectrum, the corresponding 'generalized' eigenfunctions do not vanish at infinity and the conditions (5) fail for them.

2.1 Kontorovich-Lebedev integral representation and separation of the radial variable

Consider negative E and introduce $\kappa = \sqrt{-E} > 0$. Solutions $u^\pm$ in $\Omega_\pm$ are sought in the form of KL integral representation[2]

$$
u^\pm(r,\omega) = \frac{4}{\mathrm{i}\sqrt{2\pi}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{K_\nu(\kappa r)}{\sqrt{\kappa r}}\, \nu \sin(\pi\nu)\, u_\nu^\pm(\omega) \mathrm{d}\nu\,, \tag{6}
$$

where $K_\nu(z)$ is the Macdonald (modified Bessel) function and the unknown $u_\nu^\pm(\omega)$ $(\omega \in \Sigma_\pm)$ should be chosen from an appropriate class of functions and such that the representations (6) would satisfy (2),(3) in classical sense, see also [31]. **(Some background information about the complex form of the KL transform can be found in Sect. 1.4.4 of [6].)**

To this end, consider meromorphic w.r.t. ν functions $u_\nu^\pm(\omega)$ of the complex variable ν that take on their values in Banach spaces $C^2(\Sigma^\pm)$ as functions of ω, (see [29], where meromorphic functions taking their values in a Banach are defined). Additionally, we require that these functions meet the following conditions

- $u_\nu^\pm(\omega) = u_{-\nu}^\pm(\omega)$ is even.
- There exists such $\epsilon > 0$ that $u_\nu^\pm(\omega)$ are holomorphic in the strip

  $$\Pi_\epsilon = \{\nu \in \mathrm{C} : \ \ |\Re(\nu)| < \epsilon\}$$

  with the values in the space $C^2(\Sigma_\pm)$.
- $\left.\frac{\partial u_\nu^\pm(\omega)}{\partial\vartheta}\right|_\sigma$ are meromorphic w.r.t. ν functions with the values in **the space of continuous functions** $C(\sigma)$ and are holomorphic in $\Pi_{1+\epsilon}$.
- As $\nu \to \mathrm{i}\infty$ and $\Re(\nu) = 0$ the functions $\sqrt{\nu}\cos(\pi\nu/2) u_\nu^\pm(\omega)$ of ν are absolutely integrable uniformly w.r.t. $\omega \in \overline{\Sigma^\pm}$.

[1] The upper or lower signs in (4) and further are only simultaneously taken.

[2] **We cannot assert that any eigenfunction is represented this way, however, we believe in this.**

The last condition is sufficient to ensure absolute and uniform convergence of the integral in (6) because the Macdonald function can be estimated as

$$K_\nu(z) \sim \mathrm{Const}\,\frac{\nu^{-1/2}\cos(\nu[\pi/2 + |\arg(z)|])}{\sin(\pi\nu)},$$

$\nu \to \mathrm{i}\infty$ and $\Re(\nu) = 0$ for $|\arg z| \leqslant \pi/2$, $|z|$ is arbitrarily fixed. More precise estimates for $u^\pm_\nu(\omega)$ as $\nu \to \mathrm{i}\infty$ will be discussed below. Let $\mathcal{M}$ be the class of functions $u^\pm_\nu(\omega)$ described above. Usage of this class is motivated by verification of the following Lemma.

Lemma 2.1 *Let the functions $u^\pm_\nu(\omega)$ belong to $\mathcal{M}$ and satisfy the equations*

$$(\triangle_\omega + (\nu^2 - 1/4))u^\pm_\nu(\omega) = 0\,, \quad \omega \in \Sigma_\pm, \tag{7}$$

and the boundary conditions

$$\begin{aligned} &\left.\frac{\partial u^+_{\nu+1}}{\partial\vartheta}\right|_\sigma - \left.\frac{\partial u^+_{\nu-1}}{\partial\vartheta}\right|_\sigma - \frac{2\gamma\nu}{\kappa}\left(u^+_\nu|_\sigma - u^-_\nu|_\sigma\right) = 0\,, \\ &\left.\frac{\partial u^+_\nu}{\partial\vartheta}\right|_\sigma = \left.\frac{\partial u^-_\nu}{\partial\vartheta}\right|_\sigma . \end{aligned} \tag{8}$$

Then the integral representations (6) satisfy the equations (2) and the boundary conditions (3) in classical sense.

Remark that the first boundary condition in (8) is nonlocal (w.r.t. ν), which is a manifestation of the incomplete separation of the radial variable in the first boundary condition in (3).

The equations (2) are verified from (7) by the direct substitution and differentiation taking into account that $\left\{\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)\right\}K_\nu(z) = 0$. We make use of the **identity** $\frac{K_\nu(z)}{z} = \frac{K_{\nu+1}(z) - K_{\nu-1}(z)}{2\nu}$ from 8.486(10) in [30] and from the boundary conditions (8) thus have

$$\begin{aligned} &\frac{1}{\kappa r}\left.\frac{\partial u^+}{\partial\vartheta}\right|_{\mathcal{C}} - \frac{\gamma}{\kappa}(u^+|_{\mathcal{C}} - u^-|_{\mathcal{C}}) = \\ &\frac{4}{\mathrm{i}\sqrt{2\pi}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(\frac{K_{\nu+1}(\kappa r) - K_{\nu-1}(\kappa r)}{2\sqrt{\kappa r}}\,\sin(\pi\nu)\,\left.\frac{\partial u^+_\nu}{\partial\vartheta}\right|_\sigma - \frac{\gamma}{\kappa}\frac{K_\nu(\kappa r)}{\sqrt{\kappa r}}\,\nu\sin(\pi\nu)\,(u^+_\nu|_\sigma - u^-_\nu|_\sigma)\right)\mathrm{d}\nu = \\ &\frac{4}{\mathrm{i}\sqrt{2\pi}}\int\limits_{-\mathrm{i}\infty+1}^{\mathrm{i}\infty+1}\frac{K_\nu(\kappa r)}{2\sqrt{\kappa r}}\,\sin(\pi[\nu-1])\,\left.\frac{\partial u^+_{\nu-1}}{\partial\vartheta}\right|_\sigma\mathrm{d}\nu - \frac{4}{\mathrm{i}\sqrt{2\pi}}\int\limits_{-\mathrm{i}\infty-1}^{\mathrm{i}\infty-1}\frac{K_\nu(\kappa r)}{2\sqrt{\kappa r}}\,\sin(\pi[\nu+1])\,\left.\frac{\partial u^+_{\nu+1}}{\partial\vartheta}\right|_\sigma\mathrm{d}\nu - \\ &\frac{4}{\mathrm{i}\sqrt{2\pi}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\nu\sin(\pi\nu)\,\frac{\gamma}{\kappa}\,\frac{K_\nu(\kappa r)}{\sqrt{\kappa r}}\,(u^+_\nu|_\sigma - u^-_\nu|_\sigma)\mathrm{d}\nu = \\ &\frac{4}{\mathrm{i}\sqrt{2\pi}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{K_\nu(\kappa r)}{2\sqrt{\kappa r}}\,\sin(\pi\nu)\left(\left.\frac{\partial}{\partial\vartheta}\left(u^+_{\nu+1} - u^+_{\nu-1}\right)\right|_\sigma - \frac{2\gamma\nu}{\kappa}\left(u^+_\nu|_\sigma - u^-_\nu|_\sigma\right)\right)\mathrm{d}\nu = 0\,. \end{aligned}$$

It is obvious that the formal reductions above are easily justified for the functions $u^\pm_\nu(\omega)$ from $\mathcal{M}$. The second boundary condition in (3) is similarly verified.

Remark 2.1 *Together with the KL integral representation (6) it is also useful to exploit other repesentations for solutions, e.g. so called Watson-Bessel (WB) integral,*

$$u^\pm(r,\omega) = 2\mathrm{i}\sqrt{2\pi}\int\limits_{\Gamma_\Phi}\frac{J_\nu(\mathrm{i}\kappa r)}{\sqrt{\kappa r}}\,\nu\exp(-\mathrm{i}\pi\nu/2)\,u^\pm_\nu(\omega)\mathrm{d}\nu\,, \tag{9}$$

where the contour $\Gamma_\phi = (\infty\,\mathrm{e}^{-\mathrm{i}\Phi}, \infty\,\mathrm{e}^{\mathrm{i}\Phi})$, $\Phi \in [0, \pi/2]$ so that the contour of integration (see also [31] for details) comprises the positive part of real axis, in particular, as $\Phi = \pi/2$ and it coincides with $\mathrm{i}\mathbb{R}$ *as $\Phi = 0$.* **The functions $u^\pm_\nu(\omega)$ are also proved to be holomorphic w.r.t. ν outside some**

vicinity of the real axis for any ω. The latter means, in particular, that in the process of deformation of the contour Γ_Φ, as Φ varies from $\pi/2$ to 0, no singularities of the integrand are crossed.

The WB integral representation (9) can be applied in order to verify the estimates (4) in a similar way as it is shown in [6], Sect. 5.2.2. It is assumed **(and can be shown)** *that*

$$|u_\nu^\pm(\omega)| \leqslant c_\pm \frac{|\exp(\mathrm{i}\nu\widehat{\theta}^\pm(\omega))|}{\sqrt{|\nu|}}$$

for some $\widehat{\theta}^\pm(\omega)$ and $\nu \in \Gamma_\Phi$ as $\nu = |\nu|\,\mathrm{e}^{\pm\mathrm{i}\Phi} \to \infty\,\mathrm{e}^{\pm\mathrm{i}\Phi}$. As a result, the integrand is then estimated as

$$|J_\nu(\mathrm{i}\kappa r)\,\nu\exp(-\mathrm{i}\pi\nu/2)\,u_\nu^\pm(\omega)| \leqslant$$

$$c\sqrt{|\nu|}\,\exp\{-|\nu|\log|\nu|\cos\Phi - |\nu|(\sin\Phi[\arg(\kappa r) - \Phi] + |\sin\Phi|\widehat{\theta}^\pm(\omega) - \cos\Phi[1 + \log(\kappa r/2)])\}\,,$$

where the asymptotic formula for $J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$ and Stirling asymptotics for $\Gamma(\nu+1)$ are used. **Some additional information about use of the W.-B. intgeral representation can be found in [31].**

The homogeneous problem (7),(8) for the Laplacian $\triangle_\omega$ on the unit sphere should be solved by further separation of the angular variables $\omega = (\vartheta, \varphi)$ by means of spherical functions.

3 Spherical functions and a functional difference equation

We are looking for solutions of the equations (7) in the form of spherical functions

$$\begin{aligned} u_\nu^+(\omega) &= C_n^+(\nu)\,\mathrm{e}^{-\mathrm{i}n\varphi}\,\frac{\mathrm{P}_{\nu-1/2}^{-|n|}(\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(\cos\vartheta_1)}\,, \quad 0 \leqslant \vartheta \leqslant \vartheta_1 \\ u_\nu^-(\omega) &= C_n^-(\nu)\,\mathrm{e}^{-\mathrm{i}n\varphi}\,\frac{\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta_1)}\,, \quad \pi \geqslant \vartheta \geqslant \vartheta_1\,, \end{aligned} \tag{10}$$

where $n = 0, \pm1, \pm2, \ldots$, $\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta)$ is the associated Legendre function [30] and $C_n^\pm$ are still unknown, $d_{\vartheta_1} := \frac{\partial}{\partial\vartheta_1}$. It is worth mentioning that we could also take some linear combinations of spherical functions in the right-hand side of (10).

Substitute (10) into the boundary conditions (9), thus obtain

$$\begin{aligned} &C_n^+(\nu+1) - C_n^+(\nu-1) - \frac{2\nu\gamma}{\kappa}\left(\frac{\mathrm{P}_{\nu-1/2}^{-|n|}(\cos\vartheta_1)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(\cos\vartheta_1)}\,C_n^+(\nu) - \frac{\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta_1)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta_1)}\,C_n^-(\nu)\right) = 0\,, \\ &C_n^+(\nu) = C_n^-(\nu). \end{aligned} \tag{11}$$

We can eliminate $C_n^-(\nu)$ from the first equation in (11) and arrive at a key object of our study, i.e. to the functional difference equation of the second order

$$C_n^+(\nu+1) - C_n^+(\nu-1) - 2\mathrm{i}\,\Lambda\,W_n(\nu)\,C_n^+(\nu) = 0\,, \tag{12}$$

where

$$\Lambda = \frac{2\gamma}{\kappa},$$

$$W_n(\nu) = \frac{1}{2}[w_n(\nu) - \mathcal{W}_n(\nu)]\,,$$

with

$$w_n(\nu) = -\mathrm{i}\nu\,\frac{\mathrm{P}_{\nu-1/2}^{-|n|}(\cos\vartheta_1)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(\cos\vartheta_1)}, \quad \mathcal{W}_n(\nu) = -\mathrm{i}\nu\,\frac{\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta_1)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(-\cos\vartheta_1)}.$$

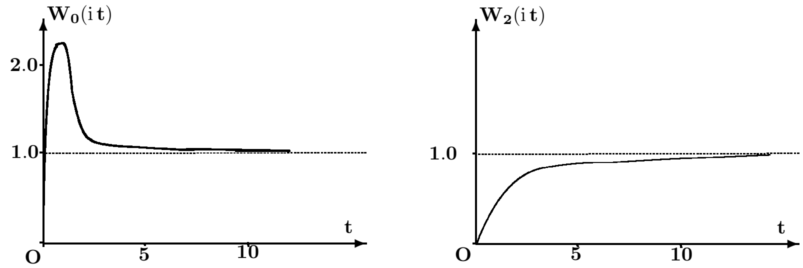

Figure 2: **Characteristic behaviour of the potentials $W_n(\nu)$ on the imaginary axis, $\nu = \mathrm{i}t$, as $n = 0$ and $n = 2$, $\vartheta_1 = 150°$**

$W_n(\nu)$ is the meromorphic coefficient (potential) in (12). Difference equations of the second order with entire or meromorphic potentials have been recently considered in various applications [21],[16],[23],[22], see also [20].

For any fixed n and ϑ_1 the potential $W_n(\nu)$ is meromorphic and is holomorphic in some strip Π_ϵ, $\epsilon > 0$. It is odd, $W_n(\nu) > 0$ as $\nu \in \mathrm{iR}_+$ **(see also appeindix D in [16])**, has the asymptotics

$$W_n(\nu) = 1 + O(1/\nu), \quad \nu \in \Pi_\epsilon,\ \nu \to \mathrm{i}\infty.$$

The latter asymptotics follows from 8.721(3) in [30]. The poles of the potential are on the real axis **because the zeros of $\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}^{-|n|}(\pm\cos\vartheta_1)$ are on the real axis**. These properties are actually verified by means of known results dealing with the associated Legendre function [30]. It is then it is verified that for $n = 0$ the potential $W_0(\nu)$ is such that $W_0^* = \sup_{\nu\in \mathrm{i}R_+} W_0(\nu) > 1$ and there exists such $N_* > 0$ that $W_0(\nu) > 1$ as $\nu \in \mathrm{i}R_+$ and $|\nu| > N_*$. On the other hand, $0 \leqslant W_n(\nu) < 1$ for $n = \pm1, \pm2, \dots$. **Fig. 2 shows characteristic behaviour of the potentail W_n. It is important to notice that only for $n = 0$ the value of the potential on the imaginary axis can exceed unity.**[3]

The following section is devoted to study of solutions for the functional difference equations having potentials from a class that includes $W_0(\nu)$. Reduction to an integral equation with a bounded selfadjoint operator forms a basis of our study there. The equation with the potentials $W_n(\nu)$ for $n = 1, 2, \dots$ are studied similarly. However, contrary to the equation with the potential $W_0(\nu)$, they have no 'discrete spectrum' solutions i.e those which correspond to the eigenfunctions of the operator A_γ.

4 Solutions of an auxiliary functional difference equation with meromorphic potential

Consider a class $\mathcal{V}$ of meromorphic potentials V such that each of them is holomorphic in the strip Π_ϵ, for some small $\epsilon > 0$, $V(\nu) = -V(-\nu)$, $V(\nu) > 0$ as $\nu \in \mathrm{iR}_+$. The asymptotics of the potentials reads

$$V(\nu) = 1 + O(1/\nu), \quad \nu \in \Pi_\epsilon,\ \nu \to \mathrm{i}\infty\,.$$

We also assume that the potentials are such that $V^* = \sup V(\nu) > 1$ as $\nu \in \mathrm{iR}_+$ and for any small $\delta_* > 0$ there exists such positive $N(\delta_*)$ that for $|\nu| > N(\delta_*)$ ($\nu \in \mathrm{iR}_+$) one has $V(\nu) > 1 + \delta_*$. Poles of each function V are located in some strip parallel to the real axis. **Zero of V at $\nu = 0$ is assumed to be simple**. We study meromorphic even solutions χ of the homogeneous equation

$$\chi(\nu+1) - \chi(\nu-1) \,-\, 2\mathrm{i}\,\Lambda\, V(\nu)\,\chi(\nu) \,= 0 \tag{13}$$

as $\Lambda \in [0, \infty)$. The desired solutions are to be holomorphic in a vicinity of the imaginary axis and exponentially vanish there at infinity, $\chi(\nu) = \chi(-\nu)$.

[3]The numerical tests have been carried out by Dr. Eng. Ning Yan Zhu from Stuttgart University.

In order to motivate and prove a main result of this section we, first, reduce the equation (13) to an integral equation. We apply Fourier transform with integration along the imaginary axis (see e.g. Sect. 7.3 in [20]). **We exploit a simple Lemma which follows from the known technique developed for a class of functional equations [20], Chapter 7. Namely, we make use of**

Lemma 4.1 ***Let*** $H(\nu)$ ***be holomorphic as*** $\nu \in \Pi_\delta$ ***and*** $|H(\nu)| \leqslant c_H \mathrm{e}^{-\varkappa|\nu|}$, $|\nu| \to \infty$, $\varkappa > 0$ ***in this strip,*** $H(\nu) = -H(-\nu)$***. Then an even solution*** $s(\nu)$ ***of the equations***

$$s(\nu \pm 1) - s(-\nu \pm 1) = \mp 2\mathrm{i} H(\nu)\,,$$

which is regular (holomorphic) in the strip $\nu \in \Pi_{1+\delta}$ ***and exponentially vanishes as*** $|\nu| \to \infty$ ***there, is given by***

$$s(\nu) = \frac{1}{4} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, H(\tau) \left(\frac{\sin(\pi\tau/2)}{\cos(\pi\tau/2) - \sin(\pi\nu/2)} + \frac{\sin(\pi\tau/2)}{\cos(\pi\tau/2) + \sin(\pi\nu/2)} \right) =$$

$$\frac{1}{4} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, H(\tau)\, \frac{\sin \pi\tau}{\cos \pi\tau + \cos \pi\nu}\,, \qquad \nu \in \Pi_{1+\delta}\,.$$

Actually, proof of this Lemma is a direct consequence of application of the so-called $S-$**integrals (see some details in Sect. 7.3.2 in [20], formula (7.24)), which is equivalent to the Fourier transform along the imaginary axis. The Lemma describes 'inversion' of the difference operator in the left-hand side of the functional difference equation.**

As a result, from Lemma 4.1 and (13) we arrive at the integral representation

$$\chi(\nu) = -\frac{\Lambda}{2} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}t\, \frac{\sin(\pi t) V(t)}{\cos(\pi t) + \cos(\pi\nu)}\, \chi(t) \tag{14}$$

as $\nu \in \Pi_{1+\epsilon}$. Provided $\chi(\cdot)$ is known and holomorphic in Π_ϵ in the integrand of the right-hand side of (14), the left-hand side is then defined by the integral and is holomorphic in $\Pi_{1+\epsilon}$. Remark that, having specified $\chi(\cdot)$ in the strip $\Pi_{1+\epsilon}$, one can continue $\chi(\cdot)$ onto the whole complex plane as a meromorphic function by means of the functional equation (13).

However, if $\nu \in \mathrm{iR}$, the representation (14) becomes an integral equation to determine $\chi(\cdot)$ on the imaginary axis, written as

$$\chi(\nu) = -\Lambda \int_{0}^{\mathrm{i}\infty} \mathrm{d}t\, \frac{\sin(\pi t) V(t)}{\cos(\pi t) + \cos(\pi\nu)}\, \chi(t)\,, \tag{15}$$

$\nu \in \mathrm{iR}_+$. We reduce the equation (15) to a more convenient form by means of introduction of new variables

$$x = \frac{1}{\cos \pi\nu}\,, \quad y = \frac{1}{\cos \pi t}, \quad \frac{\mathrm{d}y}{\pi} = \frac{\sin \pi t}{\cos^2 \pi t}\mathrm{d}t,$$

and new unknown

$$h(x) = \cos \pi\nu\, \chi(\nu)|_{x=\frac{1}{\cos \pi\nu}}\,,$$

$x, y \in [0, 1]$,

$$h(x) - \frac{\Lambda}{\pi} \int_0^1 \mathrm{d}y\, \frac{v(y)}{x+y}\, h(y) \, = 0\,, \tag{16}$$

where $v(y) = V(t)|_{y=\frac{1}{\cos \pi t}}$ and $v(y) = 1 + O(\frac{1}{\log(2/y)})$ as $y \to 0$. Finally, from (16) the desired form of the integral equation with a symmetric integral operator reads

$$\rho(x) - \frac{\Lambda}{\pi} \int_0^1 \mathrm{d}y\, \frac{\sqrt{v(x)v(y)}}{x+y}\, \rho(y) \, = 0\,, \tag{17}$$

where $\rho(x) = \sqrt{v(x)}h(x)$.

4.1 Study of the integral operator in (17)

The positive operator $K:\ L_2(0,1)\to L_2(0,1)$ is defined by[4]

$$(K\rho)(x)=\frac{1}{\pi}\int\limits_0^1 \mathrm{d}y\,\frac{\sqrt{v(x)v(y)}}{x+y}\,\rho(y)$$

and the integral equation in (17)[5] is written in the form

$$(K\rho)(x)=\lambda\rho(x), \tag{18}$$

$\lambda=\Lambda^{-1}$. The operator K is selfadjoint and bounded (compare the estimate with that analogous in Sect. 2.10, [9]),

$$|(Kh,g)_{L_2(0,1)}|\leqslant V^*\|h\|\|g\|\,,$$

$V^*=\sup_{[0,1]}v(y)>1$.

We turn to study of the spectrum $\sigma(K)\subset[0,V^*]$ of the operator K in (18). To this end, it is useful to represent it in the form

$$K=D+[K-D],$$

where

$$(D\rho)(x)=\frac{1}{\pi}\int\limits_0^1\frac{\mathrm{d}y}{x+y}\,\rho(y)$$

is the so called Dixon's operator (see Sect. 11.18 in [32]) and

$$([K-D]\rho)(x)=\frac{1}{\pi}\int\limits_0^1 \mathrm{d}y\,\frac{\sqrt{v(x)v(y)}-1}{x+y}\,\rho(y)$$

is of the Hilbert-Schmidt class in view of the estimate $|v(x)v(y)-1|\leqslant\ \mathrm{const}\left(\frac{1}{\log(2/y)}+\frac{1}{\log(2/x)}\right)$.

The selfadjoint Dixon's operator is bounded and admits explicit characterization. In particular, its spectrum is purely essential coinciding with the segment $[0,1]$, which follows, e.g. from Sect. 11.17 in [32], see also the Appendix.[6] Indeed, it is possible to reduce the equation $(D\rho)(x)=\lambda\rho(x)$ to that specified by a convolution-type operator on the semi-axis. The latter equation has nontrivial solutions for each $\lambda\in[0,1]$. The operator $K=D+[K-D]$ is a compact pertubation of the Dixon's operator so that its essential spectrum $\sigma_{ess}(K)$ also coincides with $[0,1]$.

However, the operator K **may also have** the discrete component $\sigma_d(K)$ of the spectrum. Indeed, the operator K is positive so that its discrete spectrum, if exists, is located on the interval $(1,V^*]$, i.e. $\sigma_d(K)\subset(1,V^*]$. We can also expect that $\sigma_d(K)$ is not empty in view of the properties of the potential v which follow from those of the potentials $V\in\mathcal{V}$. In order to prove this it is sufficient to find a nontrivial $\rho\in L_2(0,1)$ and a positive d such that

$$\frac{(K\rho,\rho)}{(\rho,\rho)}\geqslant 1+d,$$

which follows from the variational (minimaximal) principle [9].

Indeed, in order to get the latter estimate we take the following testing function $h(x)=[v(x)]^{-1/2}$. **Then we consider the normalized function**

$$\rho(x)=\frac{h(x)}{\|h(x)\|},\qquad \text{where}\ \ \|h(x)\|^2=\int\limits_0^1\frac{\mathrm{d}x}{v(x)}\,.$$

[4]Remark that $x=0$ plays the role of the singular end of the interval $[0,1]$ for the operator K by analogy with the differential operators.

[5] This operator belongs to a class of weighted Hankel operators that are considered in the corresponding literature.

[6] **Making use of the spectral measure for the Dixon's operator discussed in the Appendix, one could compute singular sequences corresponding to any point from the segment** $[0,1]$, **i.e. from the essential spectrum.**

(Remark that, in view of our assumptions for $V(\nu)$, one has $v(x) = O(\sqrt{1-x})$ as $x \to 1-0$ and then $h \in L_2(0,1)$.) We have

$$(K\rho,\rho) = \frac{1}{\pi}\int\limits_0^1 \mathrm{d}x \int\limits_0^1 \mathrm{d}y\, \frac{\sqrt{v(x)v(y)}}{x+y}\,\rho(y)\,\overline{\rho(x)} =$$

$$\frac{1}{\pi\|h(x)\|^2}\int\limits_0^1 \mathrm{d}x \int\limits_0^1 \mathrm{d}y\, \frac{1}{x+y} = \frac{2\log 2}{\pi\int_0^1 \frac{\mathrm{d}x}{v(x)}}\,.$$

Now, impose an additional restriction to the potential v assuming that

$$\frac{2\log 2}{\pi} > \int\limits_0^1 \frac{\mathrm{d}x}{v(x)} \tag{19}$$

and conclude

$$(K\rho,\rho) > 1.$$

The latter inequality means that the discrete spectrum of K is then nonempty. The sufficient condition (19) can be written in terms of the potential $V(\nu)$ of the functional equation (13)

$$\int\limits_0^\infty \frac{\mathrm{d}t}{\cosh^2(\pi t)}\,\frac{\pi\sinh(\pi t)}{V(\mathrm{i}t)} < \frac{2\log 2}{\pi}\,. \tag{20}$$

Below we consider validity of this sufficient condition as $V(\nu) = W_n(\nu)$, i.e. for the problem at hand.

Remark 4.1 ***It is worth remarking that the condition (19) has simple geometrical interpretation. Namely, the square under the curve $y = \frac{1}{\pi}\log(1+1/x)$ and above $y=0$ on the segment $(0,1]$, which is equal to $\frac{2\log 2}{\pi}$, must be greater than the square under the curve $y = 1/v(x)$ and above $y=0$ on the same interval.***

It is also possible that finitely or infinitely many members of a sequence $\{h_n\} \subset L_2(0,1)$ satisfy the estimate $(K\rho,\rho) > (\rho,\rho)$ as $\rho = h_n$. In our case we can take $\rho_n = h_n/\|h_n\|$, where

$$h_n(x) = \begin{cases} [v(x)]^{-1/2}\,, x \in [\frac{1}{n},1] \\ 0\,,\quad x \in [0,\frac{1}{n}) \end{cases}\,.$$

It is assumed that the inequality (19) is valid, then $(K\rho_n,\rho_n) > 1$ is also true for any sufficiently large n because

$$(K\rho_n,\rho_n) = \frac{2\log 2 + \frac{2}{n}\log\frac{2}{n} - 2\left(1+\frac{1}{n}\right)\log\left(1+\frac{1}{n}\right)}{\pi\int_{1/n}^1 \frac{\mathrm{d}x}{v(x)}} \;\to\; \frac{2\log 2}{\pi\int_0^1 \frac{\mathrm{d}x}{v(x)}} > 1$$

as $n \to \infty$. (Remark that $\{\rho_n\}$ are linear independent.) In this case, i.e. for $V \in \mathcal{V}$ satisfying (20), the discrete spectrum is nonempty and infinite.

Let us denote the correspoding eigenvalues λ_m, $m \in M = \{1,2,3\ldots\}$ counting[7] them in the order of their decreasing and taking into account their multiplicity then

$$\sigma_d(K) = \cup_{m\in M}\,\lambda_m\,.$$

The corresponding eigenfunctions are denoted ρ_m. These eigenfunctions are not only from $L_2(0,1)$ but are also continuous on $(0+,1]$ in view of the continuity of the kernel in (17).

It worth commenting that in this work we are interested in behaviour of some 'individual' eigenfinction $u_m^\pm$ i.e. assume that m is fixed. On the other hand, complete description of the negative spectrum of the operator A_γ is closely related with the following Lemma.

Lemma 4.2 *The spectrum $\sigma(K)$ of the operator K consists of the essential spectrum $\sigma_{ess}(K) = [0,1]$ and,* **provided (19) is valid***, of the* **infinite** *discrete part $\sigma_d(K) = \cup_{m\in M}\lambda_m$.*

Now we return to the equation (13).

[7] We notice that the set M of natural numbers may be also finite for some other classes of potentials.

4.2 Existence and asymptotic behaviour of the eigensolutions for the functional difference equation

In this section we prove the following statement

Theorem 4.1 *Let the potential V in the equation (13) belong to the class $\mathcal{V}$* **and satisfy the condition (20)**. *Then there exists* **an infinite** *number of even solutions χ_m (see (21)) of the equation (13) corresponding to $\Lambda = \Lambda_m = \lambda_m^{-1}$, $m \in M$ such that these solutions are meromorphic, holomorphic in a vicinity of the imaginary axis and exponentially vanish at infinity,*

$$\chi_m(\nu) = O\left(\frac{1}{\cos(\nu[\pi - \tau_m])}\right), \quad \nu \to \mathrm{i}\infty\,, \quad \nu \in \Pi_\epsilon\,, \quad \tau_m \in (0, \pi/2)$$

so that the integral in (22) converges.

The proof makes use of the Lemma 4.2. We reduce the functional equation (13) to the integral equation. We can assert that this integral equation (15) has a number of continuous (on $(0,1]$) solutions $\chi_m(\nu)$,

$$\chi_m(\nu) = \frac{\rho_m(x)|_{x=1/\cos\pi\nu}}{\cos\pi\nu}\,, \quad \rho_m \in L_2(0,1)\,, \tag{21}$$

corresponding to $\Lambda_m,\ m \in M$,

$$\Lambda_m = \frac{1}{\lambda_m},$$

where λ_m is an eigenvalue of the operator K and $\rho_m(x)$ is an eigenfunction. The solutions $\chi_m(\nu)$ are continuous and such that the integral

$$\int_0^{\mathrm{i}\infty} |\chi_m(\nu)|^2 \ |\sin(\pi\nu)|\,\mathrm{d}\nu| \ < \ \infty \tag{22}$$

converges, which is equivalent to the estimate $\int_0^1 |\rho_m(x)|^2 \mathrm{d}x < \infty$.

Taking into account that $\chi_m(\nu)$ is continued onto the whole imaginary axis due to evenness, we can assert that, in view of the representation (14), the left-hand side in (14) is holomorphic as $\nu \in \Pi_{1+\epsilon}$. Meromorphic continuation of $\chi_m(\nu)$ from this strip onto the whole complex plane is performed by use of the functional equation.

It is **now** convenient to introduce the parameter τ in accordace with

$$\sin\tau = \Lambda$$

and $\tau \in (0, \pi/2)$ as $\Lambda \in (0,1)$ and, otherwise, $\tau = \pi/2 + \mathrm{i}t,\ t \in [0, \infty)$ as $\Lambda \in [1, \infty)$, i.e. $\Lambda^{-1} \in \sigma_{ess}(K)$. In particular,

$$\sin\tau_m = \Lambda_m\,,$$

where $\Lambda_m = \lambda_m^{-1},\ \lambda_m \in \sigma_d(K)$. (Recall that $\lambda^{-1} = \Lambda = \frac{2\gamma}{\kappa}$, see (12).)

Actually, we have already shown existence of the **desired** solutions and now we turn to discussion of the estimate in the Theorem 4.1 as $\nu \to \mathrm{i}\infty\,,\ \ \nu \in \Pi_\epsilon$. The idea of such verification is rather natural. It is well established in the Fourier analysis of functions on the axis that the behaviour of a function at infinity is closely related to the regularity domain of its Fourier transform. As a result, it is sometimes more efficient to identify positions of singularities of the Fourier transform on the complex plane in order to terminate its regularity domain. Then the asymptotics of the function at infinity is specified by the position of singularities. **Indeed, consider the Fourier transform of χ_m along the imaginary axis**

$$F_m(\alpha) = \frac{1}{\mathrm{i}} \int_{\mathrm{i}R} \chi_m(\nu) \mathrm{e}^{\mathrm{i}\nu\alpha}\,\mathrm{d}\nu\,,$$

where, in view of the estimate (22), $F_m(\alpha)$ is holomorphic in the strip $\Pi_{\pi/2}$ and is even. We apply the transform to the functional equation (13) with $\Lambda_m = \sin\tau_m$ and after some calculations obtain

$$F_m(\alpha) = \frac{(-2\mathrm{i})\sin\tau_m}{\sin\alpha + \sin\tau_m}\,\frac{1}{\mathrm{i}} \int_{\mathrm{i}R} [V(\nu) - 1]\chi_m(\nu)\mathrm{e}^{\mathrm{i}\nu\alpha}\,\mathrm{d}\nu\,.$$

ϑ_1	$H(\vartheta_1)$
$29\pi/30$	0.294404
$28\pi/30$	0.153376
$9\pi/10$	0.020672
$26\pi/30$	-.102446
$25\pi/30$	-.215296
$8\pi/10$	-.317517
$23\pi/30$	-.408935
$\pi/2$	-.751230

Table 1: The values of $H(\vartheta_1)$. Positive values of $H(\vartheta_1)$ correspond to that the sufficient condition for existence of the discrete spectrum is satisfied.

Simple analysis of the latter representation enables us to assert that $F_m(\alpha)$ is holomorphic as $\alpha \in \Pi_{\pi-\tau_m}$ and the nearest to the imaginary axis singularities are on the real axis at $\alpha = -(\pi - \tau_m)$ and also at $\alpha = \pi - \tau_m$ by parity. Taking into account the inverse Fourier transform and position of the leading singularities, we arrive at the estimate $\chi_m(\nu) = O\left(\frac{1}{\cos(\nu[\pi - \tau_m])}\right), \quad \nu \to \mathrm{i}\infty$. We omitted some simple calculations.

Remark 4.2 *It is worth commenting on the case when $\Lambda^{-1} = \lambda \in \sigma_{ess}(K)$ and then $\tau = \pi/2 + \mathrm{i}t, \ t \in [0, \infty)$. The corresponding solutions χ of the equation (13) still exist, however, the estimate (22) for them fails. Nevertheless, one has $\chi(\nu) = O\left(\frac{1}{\cos(\nu[\pi - \tau])}\right), \quad \nu \to \mathrm{i}\infty$ on the imaginary axis.*

5 The eigenfunctions of the operator A_γ

We can use the Theorem 4.1 for the functional equation (13) taking $V(\nu) = W_n(\nu)$. It is verified that **for $n = 0$ the conditions $W_n \in \mathcal{V}$ and (20) can be satisfied. Indeed, in view of the explicit formula for $W_0(\nu)$, since**

$$\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}(\cos\vartheta_1) = -\sin\vartheta_1 \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}_{\nu-1/2}(x)\Big|_{x=\cos\vartheta_1},$$

we find that $(W_0(\nu))^{-1} = \sin\vartheta_1\psi(\nu,\vartheta_1)$ with some meromorphic ψ which is bounded as $\sin\vartheta_1$ and $\psi(\cdot,\vartheta_1)$ is integrable. As a result, at least for a sufficiently small $\sin\vartheta_1$ the condition (20) is satisfied as $V(\nu) = W_0(\nu)$. However, it is not the case for $V(\nu) = W_0(\nu)$ as $\vartheta_1 = \pi/2$.[8] The Table 1 shows dependence of

$$H(\vartheta_1) = \frac{2\log 2}{\pi} - \int_0^\infty \frac{\mathrm{d}t}{\cosh^2(\pi t)} \frac{\pi\sinh(\pi t)}{w_0(t,\vartheta_1)}$$

on the cone's opening ϑ_1, $w_0(t,\vartheta_1) := W_0(\nu), \quad \nu = \mathrm{i}t$,

$$w_0(t,\vartheta_1) = \frac{t}{2}\left(\frac{\mathrm{P}_{\mathrm{i}t-1/2}(\cos\vartheta_1)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\mathrm{i}t-1/2}(\cos\vartheta_1)} - \frac{\mathrm{P}_{\mathrm{i}t-1/2}(-\cos\vartheta_1)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\mathrm{i}t-1/2}(-\cos\vartheta_1)}\right).$$

We can verify that, as $|n| \neq 0$, the sufficient condition (20) fails for $V(\nu) = W_n(\nu)$ including also small $\sin\vartheta_1$. This can be expected considering the graphs for $W_n(\nu)$ (see e.g. Fig. 2) and taking into account the remark 4.1.

We can assert that for some range of parameters $\vartheta_1 \in (\pi/2, \pi)$ and $n = 0$ the equation (12) has the nontrivial solution $C_0(\nu,\tau)$ of (12) (the lower index 0 is omitted below) as $\Lambda = \Lambda_m$, $\tau = \tau_m$ in (12). For these solutions we shall use notations $C(\nu,\tau_m)$ with $\Lambda_m = \sin\tau_m = \lambda_m^{-1}$. (Recall that $\lambda_m \in \sigma_d(K)$). **One can expect that the discrete spectrum exists for all $\vartheta_1 \in (\pi/2, \pi)$, however, the sufficient condition (20) is too rough in order to prove this.**

[8] It is known that for the degenerate cone i.e as $\vartheta_1 = \pi/2$ the dicrete spectrum of the operator A_γ is empty.

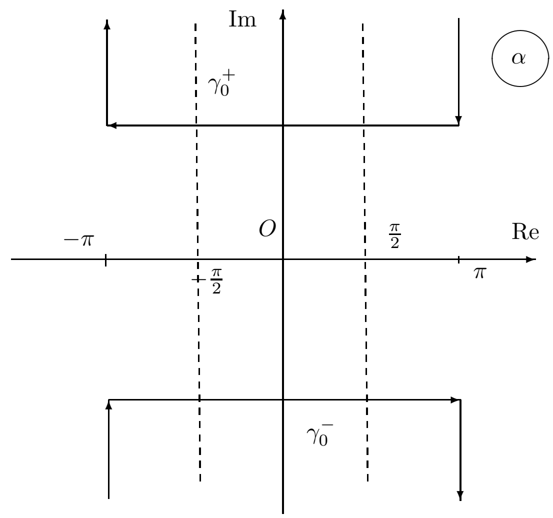

Figure 3: The Sommerfeld double-loop contour $\gamma_0 = \gamma_0^+ \cup \gamma_0^-$.

The KL intergal representations for the eigenfunctions take on the form[9]

$$u_m^{\pm}(r,\omega) = \frac{4}{\mathrm{i}\sqrt{2\pi}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\nu\, \nu \sin(\pi\nu)\, C(\nu,\tau_m)\, \frac{K_\nu(\kappa_m r)}{\sqrt{\kappa_m r}}\, \frac{\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta_1)}\,, \tag{23}$$

where we wrote $C(\nu,\tau_m)$ instead of $C_0^{\pm}(\nu)$ because $n=0$, $\kappa_m = 2\gamma\lambda_m$. We take the upper signs $+$ as $\omega = (\vartheta,\varphi) \in \Sigma_+$ i.e. $0 \leqslant \vartheta \leqslant \vartheta_1$ the lower signs $-$ as $\omega = (\vartheta,\varphi) \in \Sigma_-$ i.e. $\vartheta_1 \leqslant \vartheta \leqslant \pi$ in (23).

5.1 Sommerfeld integral representations for the eigenfunctions and their asymptotics, $r \to \infty$

It is worth remarking that, in order to obtain the asymptotics, it is not possible to make use of $K_\nu(\kappa r) = \sqrt{\frac{\pi}{2}}\,\frac{\exp(-\kappa r)}{\sqrt{\kappa r}}\,(1 + O(1/[\kappa r]))$ in (23) because, otherwise, the integrals in (23) would diverge. However, it is useful to transform the KL integral representation to that of the Sommerfeld type (see Chapter 5 in [6], [31]) which is well adapted to calculation of the asymptotics as $\kappa r \to \infty$. To this end, we exploit Sommerfeld integral representation for the Macdonald function that takes the form

$$K_\nu(z) = \frac{1}{4\mathrm{i}} \int\limits_{\gamma_0} \mathrm{e}^{z\cos\alpha}\, \frac{\sin(\nu\alpha)}{\sin(\pi\nu)}\, \mathrm{d}\alpha$$

as $\mathrm{Re}z > 0$, whereas the integration contour γ_0 is shown in Fig. 3. The latter integral rapidly converges due to vanishing of the exponent in the integrand at the ends of γ. We substitute the latter representation of the Macdonald function into (23), change the orders of integration, which is justified (see also Sect. 5.6 in [6] and [31]), and arrive at

$$u_m^{\pm}(r,\omega) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma_0} \mathrm{d}\alpha\, \frac{\mathrm{e}^{\kappa_m r\cos\alpha}}{\sqrt{\kappa_m r}}\, \Phi_m^{\pm}(\alpha,\vartheta) \tag{24}$$

with

$$\Phi^{\pm}(\alpha,\vartheta) := \Phi_m^{\pm}(\alpha,\vartheta) = \frac{\sqrt{2\pi}}{\mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\nu\, \nu \sin(\alpha\nu)\, \frac{\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta_1)}\, C^{\pm}(\nu,\tau_m)\,. \tag{25}$$

For breivity we omit the lower index m for $\Phi_m^{\pm}(\alpha,\vartheta)$. From the asymptotic behaviour of $C^{\pm}$ and of the Legendre functions (as $\nu \to \mathrm{i}\infty$) **we have**

$$\left| \frac{\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta_1)}\, C^{\pm}(\nu,\tau_m) \right| \leqslant \frac{\mathrm{Const}}{|\cos(\nu[\pi - \tau_m \pm (\vartheta_1 - \vartheta)])|}$$

[9] The corresponding eigenvalues $E = E_m$ are specified as $E_m = -\frac{4\gamma^2}{\sin^2\tau_m}$ and the eigenfunctions are rotationally symmetric.

as $\nu \to \mathrm{i}\infty$ **in the integrand of (25). From this estimate** one concludes that $\Phi^{\pm}(\cdot, \vartheta)$ (Sommerfeld transformants) are holomorphic in the strips $\Pi_{\widehat{\vartheta}^{\pm}}$ correspondingly, where

$$\widehat{\vartheta}^{\pm}(\vartheta) := \widehat{\vartheta}^{\pm}_{m}(\vartheta) = \pi - \tau_m \pm (\vartheta_1 - \vartheta)$$

assuming that $0 \leqslant \vartheta \leqslant \vartheta_1$ for $\widehat{\vartheta}^{+}$ and $\vartheta_1 \leqslant \vartheta \leqslant \pi$ for $\widehat{\vartheta}^{-}$. **The functions $\Phi^{\pm}(\cdot, \vartheta)$ are odd. So, provided one could prove that these functions are holomorphic in upper P_+ halfplane, then the same is valid for the lower P_- halfplane of the complex variable α. It can be verified that the functions $\Phi^{\pm}(\cdot, \vartheta)$ are also continued as holomorphic functions into upper P_+ halfplane (see, e.g., [31]). The corresponding proof requires some work, however, it is conducted in the line of derivations described in Sect. 6.2 in [31], see also Sect. 6.6.2 in [6]. As a result,** the following Lemma is true.

Lemma 5.1 *The Sommerfeld transformants $\Phi^{\pm}(\cdot, \vartheta)$ are odd and holomorphic in $P_+ \cup P_- \cup \Pi_{\widehat{\vartheta}^{\pm}}$ correspondingly as $\tau_m \in (0, \pi/2)$, exponentially vanish as $\alpha \to \pm\mathrm{i}\infty$ on the imaginary axis.*

In particular, the Lemma 5.1 implies that singularities of the analytic functions $\Phi^{\pm}(\cdot, \vartheta)$ can be located on the real axis in the exterior of the strip $\Pi_{\widehat{\vartheta}^{\pm}}$. **The singularities at $\alpha = \widehat{\vartheta}^{\pm}$ are located on the boundary of this strip.** In calculation of the asymptotics for eigenfunctions study of $\Phi^{\pm}(\cdot, \vartheta)$ **at** these singularities plays a crucial role.

It is also useful to introduce the analytic function $\Psi^{\pm}(\cdot, \vartheta)$ by the equalities

$$\Phi^{\pm}(\alpha, \vartheta) = \frac{\partial}{\partial \alpha} \Psi^{\pm}(\alpha, \vartheta)\,,$$

implying that

$$\Psi^{\pm}(\alpha, \vartheta) \;=\; -\frac{\sqrt{2\pi}}{\mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\nu\; \cos(\alpha\nu)\, \frac{\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta_1)}\, C^{\pm}(\nu, \tau_m) \tag{26}$$

and

$$u^{\pm}(r, \omega) = \frac{\sqrt{\kappa r}}{2\pi \mathrm{i}} \int\limits_{\gamma_0} \mathrm{d}\alpha\, \mathrm{e}^{\kappa r \cos\alpha}\, \sin\alpha\, \Psi^{\pm}(\alpha, \vartheta)\,. \tag{27}$$

The transformants $\Psi^{\pm}(\alpha, \vartheta)$ have similar analytic propreties as those of $\Phi^{\pm}(\alpha, \vartheta)$ which are described by the Lemma 5.1, however, $\Psi^{\pm}(-\alpha, \vartheta) = \Psi^{\pm}(\alpha, \vartheta)$ are even.

5.2 Singularities of the Sommerfeld transformants and singular expansions

In order to account for position and character of the leading singularities[10] of the Sommerfeld transformants $\Phi^{\pm}(\alpha, \vartheta)$ and $\Psi^{\pm}(\alpha, \vartheta)$ we, first, consider behaviour of the integrand in (26),

$$\frac{\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta)}{\mathrm{d}_{\vartheta_1}\mathrm{P}_{\nu-1/2}(\pm\cos\vartheta_1)}\, C^{\pm}(\nu, \tau_m) \;=\; O\left(\frac{\mathrm{P}_{\nu-1/2}(-\cos(\nu\widehat{\vartheta}^{\pm}))}{\cos(\pi\nu)}\right), \quad \nu \to \mathrm{i}\infty,$$

noticing that the right-hand side is of $O\left(\frac{1}{\cos[\nu\widehat{\vartheta}^{\pm}]}\right)$ on the imaginary axis. We take into account the known from [30] (formula 7.216) result (and its analytic continuation if necessary)

$$\frac{1}{4\sqrt{2}} \int\limits_{0}^{\infty} \cos(\mathrm{i}t\alpha)\, \frac{\mathrm{P}_{\mathrm{i}t-1/2}(-\cos\theta)}{\cos(\mathrm{i}\pi t)}\, \mathrm{d}t \;=\; \frac{1}{\sqrt{\cos\alpha - \cos\theta}}$$

where $\mathrm{i}\alpha > 0$ and $\sqrt{\cos\alpha - \cos\theta} > 0$ as $-\theta < \alpha < \theta$.

The latter formula enables us to look for the local 'singular' expansion (w.r.t. smoothness) of the Sommerfeld transformants

$$\Psi^{\pm}(\alpha, \vartheta) \;=\; \frac{A_0^{\pm}(\vartheta)}{\sqrt{\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)}} \;+\; A_1^{\pm}(\vartheta)\sqrt{\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)} \;+\; \dots\,, \tag{28}$$

[10] Recall that all singularities are on the real axis as $\tau = \tau_m \in (0, \pi/2)$.

where the dots denote less singular terms including a regular components in some neighbourhood of the point $\alpha = \widehat{\vartheta}^{\pm}(\vartheta)$. (Due to evenness of $\Psi^{\pm}(\alpha,\vartheta)$ the same expansion is valid near $\alpha = -\widehat{\vartheta}^{\pm}(\vartheta)$.) The branch cuts in (28) are correspondingly conducted from $\pm\widehat{\vartheta}^{\pm}$ to $\pm\infty$ along the imaginary axis and $\sqrt{\cos\alpha - \cos\widehat{\vartheta}^{\pm}} > 0$ as $-\widehat{\vartheta}^{\pm} < \alpha < \widehat{\vartheta}^{\pm}$. Recall that

$$\Phi^{\pm}(\alpha,\vartheta) = \frac{\partial}{\partial\alpha}\Psi^{\pm}(\alpha,\vartheta) \ = \ \frac{\frac{1}{2}A_0^{\pm}(\vartheta)\,\sin\alpha}{[\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)]^{3/2}} \ + \ldots . \tag{29}$$

In order to verify that the expansions in (28) and (29) really describe the behaviour of the transformants near the singularities and to determine the unknown $A_j^{\pm}(\vartheta),\ j = 0,1,\ldots$ we formulate equations and boundary conditions for the Sommerfeld transformants, see Sect. 5.6 in [6] and [31]. It is worth mentioning that the Sommerfeld transformants are actually connected with $u_\nu^{\pm}$ by means of the Fourier transform along the imaginary axis, see (10),(25) and (26). We then easily obtain, from (7), that

$$(\triangle_\omega - \partial_\alpha^2 - 1/4)\Phi^{\pm}(\alpha,\vartheta) \ = 0\,,\quad \omega\in\Sigma_{\pm}, \tag{30}$$

and

$$(\triangle_\omega - \partial_\alpha^2 - 1/4)\Psi^{\pm}(\alpha,\vartheta) \ = 0\,,\quad \omega\in\Sigma_{\pm}. \tag{31}$$

For real α the latter equations are of the hyperbolic type, however, we consider them also for the complex-valued α. Consider the following formal computation, assuming that

$$\sin\alpha\frac{\partial\Psi^{+}(\alpha,\vartheta)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} \ = \ \frac{2\gamma}{\kappa}\ \frac{\partial}{\partial\alpha}\left(\frac{\Psi^{+}(\alpha,\vartheta)-\Psi^{-}(\alpha,\vartheta)}{2}\right)\bigg|_{\vartheta=\vartheta_1} \tag{32}$$

and integrating by parts. We verify ($\kappa=\kappa_m$)

$$\frac{1}{\kappa r}\frac{\partial u^{+}(r,\omega)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} \ = \ \frac{1}{\sqrt{\kappa r}}\ \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma_0} \mathrm{d}\alpha\,\mathrm{e}^{\kappa r\cos\alpha}\ \sin\alpha\,\frac{\partial}{\partial\vartheta}\Psi^{+}(\alpha,\vartheta)\bigg|_{\vartheta=\vartheta_1} \ =$$

$$\frac{2\gamma}{\kappa}\ \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma_0}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\kappa r\cos\alpha}}{\sqrt{\kappa r}}\ \frac{\partial}{\partial\alpha}\frac{(\Psi^{+}(\alpha,\vartheta)-\Psi^{-}(\alpha,\vartheta))}{2}\Bigg|_{\vartheta=\vartheta_1} \ =$$

$$\frac{2\gamma}{\kappa}\ \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma_0}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\kappa r\cos\alpha}}{\sqrt{\kappa r}}\ \frac{(\Phi^{+}(\alpha,\vartheta)-\Phi^{-}(\alpha,\vartheta))}{2}\Bigg|_{\vartheta=\vartheta_1} \ = \ \frac{\gamma}{\kappa}\ \left(u^{+}(r,\omega)-u^{-}(r,\omega)\right)\Big|_{\vartheta=\vartheta_1}\ ,$$

which means that the first boundary condition in (3) holds. The second condition in (3) is verified on a similar way provided

$$\frac{\partial\Psi^{+}(\alpha,\vartheta)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} \ = \ \frac{\partial\Psi^{-}(\alpha,\vartheta)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1}\ . \tag{33}$$

From boundary conditions (32) and (33) we also obtain

$$\frac{\partial\Phi^{+}(\alpha,\vartheta)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} \ = \ \frac{2\gamma}{\kappa_m}\ \frac{\partial}{\partial\alpha}\left(\frac{\Phi^{+}(\alpha,\vartheta)-\Phi^{-}(\alpha,\vartheta)}{2\sin\alpha}\right)\bigg|_{\vartheta=\vartheta_1} \tag{34}$$

and

$$\frac{\partial\Phi^{+}(\alpha,\vartheta)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} \ = \ \frac{\partial\Phi^{-}(\alpha,\vartheta)}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1}\ . \tag{35}$$

The initial conditions are specified from (25) as $0\leqslant\alpha<\vartheta^{\pm}$ because the integral rapidly converges and the integrand is known.

Taking into account properties of the transformants described by Lemma 5.1, we arrive at

Lemma 5.2 *The Sommerfeld transformants $\Phi_m^{\pm}(\alpha,\vartheta)$ and $\Psi_m^{\pm}(\alpha,\vartheta)$ solve the equations (30) and (31) and satify the boundary conditions (32)–(35) correspondingly.*

We now substitute the expansion (28) into the equations (31) and equate coefficients at two leading singularities as $\alpha \sim \widehat{\vartheta}^{\pm}$ thus obtain

$$\left(\frac{\mathrm{d}\widehat{\vartheta}^{\pm}(\vartheta)}{\mathrm{d}\vartheta}\right)^2 = 1\,,$$

and

$$2\frac{\mathrm{d}\widehat{\vartheta}^{\pm}(\vartheta)}{\mathrm{d}\vartheta}\frac{\mathrm{d}A_0^{\pm}(\vartheta)}{\mathrm{d}\vartheta} + \left(\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\sin\vartheta\frac{\mathrm{d}\widehat{\vartheta}^{\pm}(\vartheta)}{\mathrm{d}\vartheta} - \cot\widehat{\vartheta}^{\pm}(\vartheta)\right)A_0^{\pm}(\vartheta) = 0\,.$$

The first of the latter two equations is obviously satisfied, whereas solutions of the second one is explicitly found by quadrature. Taking into account the boundary conditions (32),(33), we consider their leading singular terms and arrive at

$$\left.\frac{\mathrm{d}A_0^{+}(\vartheta)}{\mathrm{d}\vartheta}\right|_{\vartheta=\vartheta_1} = \left.\frac{\mathrm{d}A_0^{-}(\vartheta)}{\mathrm{d}\vartheta}\right|_{\vartheta=\vartheta_1}\,.$$

From the equations and boundary condition $A_0^{\pm}(\vartheta)$ are explicitly determined up to a common contant factor. The coefficients $A_j^{\pm}$ are recurrently determined in a similar manner. It is worth mentioning that $A_0^{\pm}(\vartheta) := A_{0m}^{\pm}(\vartheta)$ by definition.

The other singularities of $\Phi^{\pm}(\cdot,\vartheta)$ and $\Psi^{\pm}(\cdot,\vartheta)$ are isolated from $\pm\widehat{\vartheta}^{\pm}$ and located on the real axis outside the strip $\overline{\Pi_{\widehat{\vartheta}^{\pm}}}$.

5.3 Asymptotics of the Sommerfeld integral representations for eigenfunctions

We make use of the Sommerfled integral representation (24) in order to determine the asymptotic behaviour of an eigenfunction as $\kappa r \to \infty$ implying that the method of the steepest descent is to be applied. To this end, we exploit the analytic properties of the Sommerfeld transformants and their behaviour (29) near the leading singularities at $\alpha = \pm\widehat{\vartheta}^{\pm}$. We intend to deform the integration conotur into the steepest descent (SD) paths (see Fig. 4) that conduct through the saddle point $\alpha = \pi$ and $\alpha = -\pi$ which are zeros of the function $S'(\alpha) = -\sin\alpha = 0$, where $S(\alpha) = \cos\alpha$ is the 'phase' function of the rapidly varying exponent $\exp\left(\kappa r S(\alpha)\right)$, in the integrand of (24). The contour of integration γ_0 is deformed into the steepest descent paths $\mathrm{Re}\alpha = \pm\pi$ denoted $\gamma_0^{\pm\pi}$. However, in the process of such deformation the singularities on the real eaxis in the exterior of strip $\Pi_{\widehat{\vartheta}^{\pm}}$ can be captured and also contribute to the asymptotics. Due to parity of the integrand it is sufficient to consider the contribution of only one saddle point (say $\alpha = \pi$) and of the corresponding positive leading singularity at $\alpha = \widehat{\vartheta}^{+}$.

It is natural to distinguish two situations, namely, the first case (I) corresponds to the situation when singularities are not in some close neighbourhoods of the corresponding saddle points. The second one (II) is related to the situation when the singularity at $\alpha = \widehat{\vartheta}^{+}$ is in some close vicinity of the saddle point $\alpha = \pi$ (then $\alpha = -\widehat{\vartheta}^{+}$ is close to $\alpha = -\pi$). To this end, we introduce some useful terminology. Let $\varepsilon > 0$ be some sufficiently small number. An angular neighbourhood of directions for the solution u^{+}, described by the inequality $|\widehat{\vartheta}^{+}(\vartheta) - \pi| \leqslant O(1/[\kappa r]^{1/2-\varepsilon})$, is called *vicinity of the surface of singular directions* [27], [6]. The surface is defined by $\widehat{\vartheta}^{+}(\vartheta) = \pi,\ r > 0$ implying that $\kappa r \to \infty$. In the same manner, angular vicinity of the surface of singular directions is defined for for u^{-}, $|\widehat{\vartheta}^{-}(\vartheta) - \pi| \leqslant O(1/[\kappa r]^{1/2-\varepsilon})$. It is obvious that these surfaces, if exist, are some conical surfaces and their angular vicinities are asymptotically small. The term 'singular' is justified by the fact that for the scattering problems in a cone the diffraction coefficient (see [27], [6]) of the spherical wave from the vertex is singular in these directions. As a result, provided ϑ is not close to singular directions, i.e. outside an angular vicinity of singular directions (case I), the conventional saddle point technique is applied to the asymptotic evaluation of the Sommerfeld integrals [33]. Otherwise, an appropriate modification of the method is exploited (case II).

It is also worth commenting that a surface of singular directions depends on two parameters ϑ_1 and τ_m for a fixed m because its equation is $\widehat{\vartheta}_m^{\pm}(\vartheta) = \pi,\ \ r > 0$ or $\tau_m = \pm(\vartheta_1 - \vartheta),\ \ r > 0$. Additionally, for the upper signs one has, as $\vartheta = \vartheta_1 - \tau_m > 0$, the surface of the singular directions is in Ω_{+}, whereas for the lower signs, as $\vartheta = \tau_m + \vartheta_1 < \pi$, the surface is in Ω_{-}.

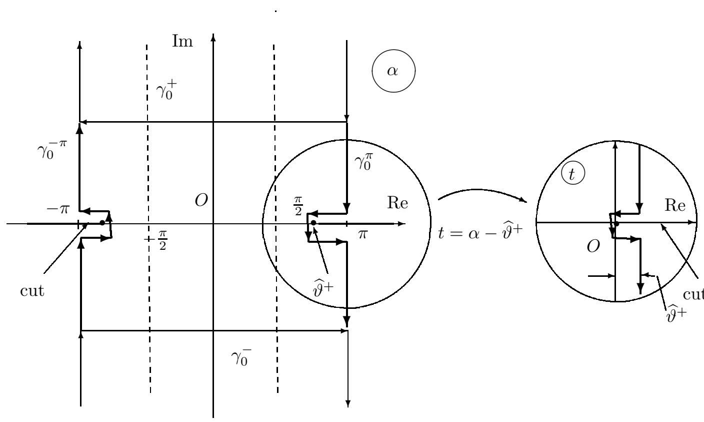

Figure 4: Contours for the asymptotic evaluation of the Sommerfeld integral.

5.3.1 Asymptotics outside close angular vicinity of singular directions

We turn to the case I and consider contribution due to the integration over the countour γ_0^π. Taking into account that in some $O(1/[\kappa r]^{1/2-\epsilon})$-vicinity $(0<\epsilon<1/2)$ of the branch point $\alpha=\widehat{\vartheta}^\pm$, $(\widehat{\vartheta}^\pm<\pi)$ the expression (29) is valid for the Sommerfeld transfomants, one has[11]

$$u^\pm(r,\omega)=\frac{\mathrm{e}^{\kappa r\cos\widehat{\vartheta}^\pm(\vartheta)}}{\pi\mathrm{i}}\frac{A_0^\pm(\vartheta)}{2}\int\limits_{l_\epsilon(\kappa r)}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\kappa r[\cos\alpha-\cos\widehat{\vartheta}^\pm(\vartheta)]}}{\sqrt{\kappa r}}\left(\frac{\sin\alpha}{[\cos\alpha-\cos\widehat{\vartheta}^\pm(\vartheta)]^{3/2}}+\dots\right)+ \tag{36}$$

$$+O\left(\frac{\mathrm{e}^{-\kappa r}}{\kappa r}\right),\quad \kappa r\to\infty\,,$$

where integration in the first term is conducted along the part of the contour that comprises the cut from $\widehat{\vartheta}^\pm$ to ∞ and is contained in the $O(1/[\kappa r]^{1/2-\epsilon})$ circular vicinity of the point $\widehat{\vartheta}^\pm$, $\pi/2<\widehat{\vartheta}^\pm(\vartheta)<\pi$. The remainder term $O\left(\frac{\mathrm{e}^{-\kappa r}}{\kappa r}\right)$ is due to the contribution of the saddle point π which can be separated from the first summand in (36). Remark that we used symmetry of the integration contours and parity of the integrand. In the integral term of (36) we exploit a nondgenerate change of the integration variable

$$\zeta=-[\cos\alpha-\cos\widehat{\vartheta}^\pm(\vartheta)]\,,$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\alpha}=\sin\alpha=\sin\widehat{\vartheta}^\pm(\vartheta)+O(1/[\kappa r]^{1/2-\epsilon})\,,$$

where $\vartheta^\pm(\vartheta)$ is not close to π. From (36) we arrive at

$$u_m^\pm(r,\omega)=\frac{1}{\pi\mathrm{i}}\frac{\mathrm{e}^{\kappa_m r\cos\widehat{\vartheta}_m^\pm(\vartheta)}}{\sqrt{\kappa_m r}}\frac{A_{0\,m}^\pm(\vartheta)}{2}\int\limits_l\mathrm{d}\zeta\,\frac{\mathrm{e}^{-\kappa_m r\zeta}}{(\sqrt{-\zeta})^3}\left(1+O\left(\frac{1}{\kappa_m r}\right)\right)\,, \tag{37}$$

where the contour l goes from $+\infty$ along the upper side of the branch cut from 0 to $+\infty$ then comprises 0 from the left and outgoes to $+\infty$ along the lower side of the branch cut. The branch of the root is fixed by the condition: $\arg\zeta=0$ on the upper side of the branch cut and $2\pi>\arg\zeta>0$. The integral in (37) is explicitly computed so that the asymptotics of the eigenfunction is

$$u_m^\pm(r,\omega)=\frac{2}{\sqrt{\pi}}A_{0\,m}^\pm(\vartheta)\,\mathrm{e}^{-\kappa r\cos[\tau_m\mp(\vartheta_1-\vartheta)]}\left(1+O\left(\frac{1}{\kappa_m r}\right)\right) \tag{38}$$

as $\pi/2<\widehat{\vartheta}_m^\pm(\vartheta)<\pi$, $|\widehat{\vartheta}_m^\pm(\vartheta)-\pi|\geqslant O(1/[\kappa r]^{1/2-\epsilon})$, $m\in M$, ($E_m=-\frac{4\gamma^2}{\sin^2\tau_m}$ is the eigenvalue). It is obvious from (38) that in the nonsingular directions the asymptotics has an elementary expression in

[11] Dependence on m is omitted, e.g. $\kappa_m\to\kappa$, , $A_{0m}^\pm\to A_0^\pm$.

terms of the exponent rapidly vanishing as $r \to \infty$. Contribution of the saddle points in this case is asymptotically negligible in comparison with the leading approximation (38).

It is useful to notice that, provided $\widehat{\vartheta}^{\pm}_m(\vartheta) > \pi$ and $|\widehat{\vartheta}^{\pm}_m(\vartheta) - \pi| \geqslant O(1/[\kappa_m r]^{1/2-\varepsilon})$, which means that the branch points $\pm\widehat{\vartheta}^{\pm}$ are in the exterior of the strip $\overline{\Pi_\pi}$, the saddle points $\pm\pi$ are responsible for the leading asymptotic contributions. In this case the SD paths $\gamma_0^{\pm}$ are conducted across the saddle points $\pm\pi$ correspodingly then, from (24), by means of the conventional steepest descent technique we have

$$u^{\pm}_m(r,\omega) = -\sqrt{\frac{2}{\pi}}\,\Phi^{\pm}_m(\pi,\vartheta)\,\frac{\mathrm{e}^{-\kappa_m r}}{\kappa_m r}\,\left(1 + O\left(\frac{1}{\kappa_m r}\right)\right). \tag{39}$$

In the asymptotics (39) the factor $\Phi^{\pm}_n(\pi,\vartheta)$ is computed by means of the representation (25) which is valid in the strip $\Pi_{\widehat{\vartheta}^{\pm}}$, where $\widehat{\vartheta}^{\pm}_m(\vartheta) > \pi$.

5.3.2 Asymptotics near the singular directions

Now we turn to the case II when the singularies at $\pm\vartheta^{\pm}(\vartheta)$ can be located in $O(1/[\kappa_m r]^{1/2-\varepsilon})$-vicinities of the saddle points $\pm\pi$ correspodingly. In these conditions we represents the expression (24) as (dependence on m is skipped)

$$u^{\pm}(r,\omega) = \frac{\mathrm{e}^{\kappa r\cos\widehat{\vartheta}^{\pm}(\vartheta)}}{\pi\mathrm{i}}\,\frac{A_0^{\pm}(\vartheta)}{2}\int\limits_{\gamma_0^{\pi}\cap B_\pi([\kappa r]^{-1/2+\varepsilon})}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\kappa r[\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)]}}{\sqrt{\kappa r}}\,\frac{\sin\alpha}{[\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)]^{3/2}} + \tag{40}$$
$$+\,\delta u^{\pm}(r,\omega)\,,$$

where

$$\delta u^{\pm}(r,\omega) \;=\; \frac{\mathrm{e}^{\kappa r\cos\widehat{\vartheta}^{\pm}(\vartheta)}}{\pi\mathrm{i}}\int\limits_{\gamma_0^{\pi}}\mathrm{d}\alpha\,\frac{\mathrm{e}^{\kappa r[\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)]}}{\sqrt{\kappa r}}\delta\Phi^{\pm}(\alpha,\vartheta)$$

and

$$\delta\Phi^{\pm}(\alpha,\vartheta) \;:=\; \Phi^{\pm}(\alpha,\vartheta) \;-\; \frac{A_0^{\pm}(\vartheta)}{2}\,\frac{\sin\alpha}{[\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)]^{3/2}} \;=\; O\left([\cos\alpha - \cos\widehat{\vartheta}^{\pm}(\vartheta)]^{1/2}\right)$$

as $\alpha \sim \widehat{\vartheta}^{\pm}(\vartheta)$. In the representation (40) we integrate along the part of the SD contour γ_0^{π} that is contained in the circle $B_\pi([\kappa r]^{-1/2+\varepsilon})$ (see Fig. 4) centered at the point π and having small radius of $O(([\kappa r]^{-1/2+\varepsilon})$. The term $\delta u^{\pm}(r,\omega)$ in (40) plays the role of the asymptotic correction in comparison with the first summand so that we pay our principal attention to reductions of the first term. We make use of change of the variable $\alpha = \pi + \vartheta^{\pm}_*(\vartheta) + t$ and write (40) as

$$u^{\pm}(r,\omega) =$$
$$\frac{\mathrm{e}^{-\kappa r\cos\vartheta^{\pm}_*(\vartheta)}}{\pi\mathrm{i}}\,\frac{A_0^{\pm}(\vartheta)}{2}\int\limits_{\gamma_0^{\pi}\cap B_0([r]^{-1/2+\varepsilon})}\mathrm{d}t\,\frac{\mathrm{e}^{-\kappa r[\cos(t+\vartheta^{\pm}_*(\vartheta)(\vartheta)) - \cos\vartheta^{\pm}_*(\vartheta)]}}{\sqrt{\kappa r}}\,\frac{-\sin(t+\vartheta^{\pm}_*(\vartheta))}{(-[\cos t - \cos\vartheta^{\pm}_*(\vartheta)])^{3/2}} \tag{41}$$
$$+\,\delta u^{\pm}(r,\omega)\,,$$

where $\vartheta^{\pm}_*(\vartheta) = -\tau_m \pm (\vartheta_1 - \vartheta)$ and the remainder is $\delta u^{\pm}(r,\omega)$.

We asymptotically reduce the integral in (41). The contour of integration (shown in Fig. 4, right) can be supplemented by its infinite parts then deformed into the imaginary axis comprising the branch point from the left. The resulting contour is denoted S_0. This procedure contributes some exponentially small relative error. Then we introduce new variable of integration z in accordance with

$$\cos(t + \vartheta^{\pm}_*(\vartheta)) - \cos\vartheta^{\pm}_*(\vartheta) = z^2/2 + az\,,$$

where $a = (-2\mathrm{i})\sin\frac{\vartheta^{\pm}_*(\vartheta)}{2}$,

$$-\sin(t + \vartheta^{\pm}_*(\vartheta))\mathrm{d}t \;=\; (z + a)\mathrm{d}z\,.$$

The change of the variable is regular provided $z = 0$ corresponds to $t = 0$ and $z = -a$ corresponds to $t = -\vartheta^{\pm}_*(\vartheta)$. It is easily computed that

$$z = 2\mathrm{i}\sin\frac{\vartheta^{\pm}_*(\vartheta)}{2} - 2\mathrm{i}\sin\left(\frac{t + \vartheta^{\pm}_*(\vartheta)}{2}\right).$$

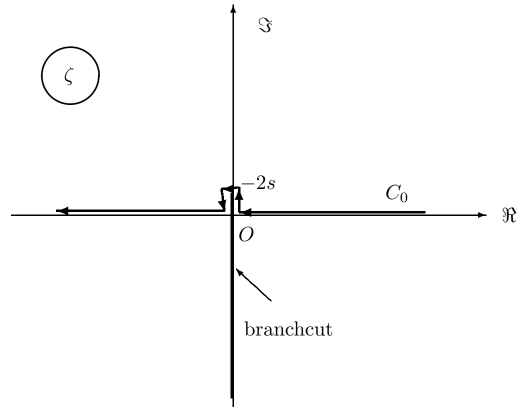

Figure 5: The integration contour C_0.

We make use of the notation $s = a\sqrt{\kappa r}$ and write

$$W(s) := \int\limits_{C_0} \mathrm{d}z\, \frac{\mathrm{e}^{-\kappa r[z^2/2+az\,]}}{\sqrt{\kappa r}}\, \frac{(z+a)}{(-[z^2/2+az\,])^{3/2}} \;=\; \int\limits_{C_0} \mathrm{d}\zeta\, \frac{(\zeta+s)\,\mathrm{e}^{-[\zeta^2/2+s\zeta\,]}}{(-[\zeta^2/2+s\zeta\,])^{3/2}}\,,$$

$\zeta = z\sqrt{\kappa r}$.

The contour of integration S_0 is transformed into C_0 (Fig 5). As a result, the integration contour C_0 goes from ∞ along the real axis to $0+$ then comprises the branch cuts in the upper halfplane and outgoes from $0-$ to $-\infty$. It is worth commenting on the choice of the branch in the denominator of the integrand. The cuts are conducted from $-2s$ to $-\mathrm{i}\infty$ and from 0 to $-\mathrm{i}\infty$ along the imaginary axis. We choose the branches as follows: $\arg(\zeta) = -3\pi/2$ and $\arg(\zeta+2s) = -3\pi/2$ on the right side of the corresponding cut. Such a choice is motivated by the condition $\sqrt{\cos\alpha - \cos\theta} > 0$ as $-\theta < \alpha < \theta$ mentioned above. As a result of the asymptotic evaluation, we arrive at

$$u_m^{\pm}(r,\omega) = \frac{\mathrm{e}^{-\kappa_m r\cos[\tau_m \mp(\vartheta_1-\vartheta)]}}{2\pi\mathrm{i}}\, A_{0m}^{\pm}(\vartheta)\, W\left(-2\mathrm{i}\sqrt{\kappa_m r}\,\sin\frac{[\tau_m\mp(\vartheta_1-\vartheta)]}{2}\right) + \delta u_m^{\pm}(r,\omega)\,, \tag{42}$$

where only the leading term of the asymptotics is explicitly represented, as $|\widehat{\vartheta}^{\pm}(\vartheta)-\pi| \leqslant O(1/[\kappa r]^{1/2-\varepsilon})$,

$$\delta u_m^{\pm}(r,\omega) \;=\; O\left(\frac{W\left(a\sqrt{\kappa_m r}\right)}{\kappa_m r}\;\mathrm{e}^{-\kappa r\cos[\tau_m\mp(\vartheta_1-\vartheta)]}\right).$$

The special function W in (42) has the same exponential factor as that in the integral representation for the parabolic cylinder (Weber) functions [30]. It is reasonable to expect that W has also similar properties.

The main result of this section is then given by

Theorem 5.1 *For any m the eigenfunction $u_m^{\pm}(r,\omega)$ (corresponding to the eigenvalue $E_m = -\frac{4\gamma^2}{\sin^2\tau_m}$) has the asymptotics described by the expressions (38) and (39) for the case I, i.e. for nonsingular directions, and by the expression (42) for the vicinity of singular directions, case II.*

It is worth remarking that the eigenfunctions exponentially vanish as $r\to\infty$, however, the rate of the corresponding decreasing is described by the Theorem 5.1.

6 Conclusion

In this work we studied a particular problem for the Schrödinger equation with a singular potential supported on a right-circular conical surface. The symmetry of the support enabled us to make use of the incomplete separation of the spherical variables looking for the eigenfunctions in the form of the Kontorovich-Lebedev (KL) integrals and to reduce study of the eigenfunctions to an auxiliary functional difference equation with meromorphic potential from a special class. We investigated solutions of the functional difference equation. To this end, a weighted Hankel integral operator attributed to the functional difference equation was considered. In particular, its discrete and essential[12] spectrum were described. **A simple sufficient condition for existence of the infinite discrete part of the spectrum was obtained. For the problem under consideration this condition ensures existence of the infinite discrete spectrum only for some range of ϑ_1, however, we belive that the discrete spectrum exists for all $\vartheta_1 \in (\pi/2, \pi)$.**

In order to determine the asymptotics the KL integral representation was reduced to that of the Sommerfeld type. On this way, we obtained asymptotic expressions for the corresponding eigenfunctions by means of application of the saddle point technique (or its appropriate modification) to the Sommerfeld integral. In particular, the asymptotics is described by a special function of Weber (parabolic cylinder) type in the so called singular directions, that is responsible for switching the regims of decay of an eigenfunction. **Each surface of singular directions is a circular conical surface which is specified by the corresponding eigenvalue and by the cone's opening ϑ_1.** In the nonsingular directions the asymptotics is represented by elementary expressions, **by exponents** vanishing at infinity. **From the physical point of view our asymptotic result for an eigenfunction can be interpreted as follows. The energy corresponding to the eigenfinction is mainly concentrated in a neighbourhood of the conical singularity of the boundary and exponentially decays with the distance. However, the rate of the decay depends on the direction of observation, i.e. is it contained inside or outside the cone of singular directions. The regim of the energy decay is switched near the cone of singular directions and is governed by the Weber-type special function.**

It is natural to expect that the KL integral representation can be also applied in 2D, 3D and higher dimensions. For the rotational symmetry the problem is reduced to the corresponding functional difference equation and then to 1D integral equation. It seems, that the spectral properties of the latter can be studied in a similar manner. However, for the conical surfaces with more general cross-sections the problem at hand is reduced to some complex problem for the Laplacian on the unit sphere with nonlocal w.r.t. ν boundary conditions, where ν is the variable of separation. It seems, however, that this problem for the spherical Laplacian is not much simpler than the original problem.

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[12] Actually, this part of the spectrum is absolutely continuous.

9 Appendix. Dixon's operator and some of its spectral properties.

In this Appendix we consider some spectral properties of the Dixon's integral operator (see Sect. 11.18 in [32], [16])

$$(D\rho)(x) = \frac{1}{\pi}\int_0^1 \frac{\mathrm{d}y}{x+y}\,\rho(y)\,.$$

The equation

$$[D-\mu]f(x) = G(x), \quad f, G \in L_2(0,1)$$

is an explicitly solvable model so that its solutions can be computed in an explicit form. The spectrum $\sigma[D]$ of the operator D coincides with the segment $[0,1]$, it is essential, $\sigma[D] = \sigma_{ess}[D] = [0,1]$.The resolvent $[D-\mu]^{-1}$ of the Dixon's operator takes on the form

$$f(x) = [D-\mu]^{-1}G = -\frac{1}{\mu}\left\{ G(x) + \frac{1}{\pi\mu}\int_0^1 \mathrm{d}z\, \chi_D(x,z;\mu)G(z) \right\}, \quad \mu \notin \sigma[D]\,,$$

where the kernel

$$\chi_D(x,z;\mu) \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}p\, \frac{\pi p \tanh \pi p}{[\mu \cosh \pi p - 1]}\, \frac{1}{x}\mathrm{P}_{\mathrm{i}p-\frac{1}{2}}\left(\frac{1}{x}\right) \frac{1}{z}\mathrm{P}_{\mathrm{i}p-\frac{1}{2}}\left(\frac{1}{z}\right)$$

solves the equation

$$\chi_D(x,y;\mu) = \frac{1}{x+y} + \frac{1}{\pi\mu}\int_0^1 \mathrm{d}z\, \frac{\chi_D(y,z;\mu)}{x+z}\,.$$

The spectral measure E_t is completely specified by the operator D and is computed explicitly by means of the known formula

$$E_t G = \lim_{\epsilon\to 0+} \frac{1}{2\pi\mathrm{i}} \int_{S_t^\epsilon} [D-\mu]^{-1} G\, \mathrm{d}\mu\,,$$

where the strong limit exists and the contour S_t^ϵ connects the points $t-\mathrm{i}\epsilon$ and $t+\mathrm{i}\epsilon$ ($t \geqslant 0$) comprising the cut conducted along the spectrum $\sigma[D] = [0,1]$ from the left. We find

$$E_t G = G(x) + \frac{1}{2\pi\mathrm{i}}\int_{S_t^0} \frac{\mathrm{d}\mu}{\mu}\left(\frac{1}{\pi}\int_0^1 \chi_D(x,y;\mu)G(y)\,\mathrm{d}y \right), \quad G \in L_2(0,1).$$

References

[1] Brummelhuis R and Duclos P. 2206. Effective Hamiltonians for atoms in very strong magnetic fields. *J. Math. Phys.* **47**(3).

[2] Exner P. Leaky quantum graphs: a review. 2008. In: *Analysis on graphs and its applications, volume 77 of Proc. Sympos. Pure Math.* Amer. Math. Soc.: Providence, RI.

[3] Figotin A and Kuchment P. 1996. Band-gap structure of spectra of periodic dielectric and acoustic media. II: Two-dimensional photonic crystals *SIAM J. Appl. Math.* **56**(6).

[4] Lyalinov MA, Zhu NY. 2007. Acoutic scattering by a circular semi-transparent conical surface. *J. Eng. Math.* **59**.

[5] Lyalinov MA, Zhu NY and Smyshlyaev VP. 2010. Scattering of a plane electromagnetic wave by a hollow circular cone with thin semi-transparent walls. *IMA Journ. Appl. Math.* 75. (See doi:10.1093/imamat/hxq030)

[6] Lyalinov MA and Zhu NY. 2012 *Scattering of Waves by Wedges and Cones with Impedance Boundary Conditions* (Mario Boella Series on Electromagnetism in Information & Communication). Edison, NJ: SciTech-IET.

[7] Grikurov VE, Lyalinov MA. 2008. Diffraction of the surface H-polarized wave by an angular break of a thin dielectric slab. *J. of Math. Sciences* **155**(3).

[8] Behrndt B, Exner P, Lotoreichik V. 2013. Schrödinger operators with δ- and δ'-interactions on Lipschitz surfaces and chromatic numbers of associated partitions. *Reviews in Mathematical Physics* **26**(08). See DOI: 10.1142/S0129055X14500159.

[9] Birman MSh, Solomjak MZ. 1987. *Spectral theory of selfadjoint operators in Hilbert spaces.* Dordrecht, Holland.

[10] Kato T. 1995. *Perturbation theory for linear operators.* Springer-Verlag, Berlin.

[11] Lotoreichik V, Ourmiéres-Bonafos Th. 2016. On the Bound States of Schrödinger Operators with δ-interactions on Conical Surfaces. *Comm. Partial Differential Equations*, **41**. (See also https://arxiv.org/abs/1510.05623.)

[12] Babich VM. 2011. A class of topographical waveguides, *St. Petersburg Math. J.* **22**.

[13] Pankrashkin K. 2016. On the discrete spectrum of Robin Laplacian in conical domains. *Math. Model. Nat. Phenom.* **11**. (See also https://arxiv.org/pdf/1507.08980.pdf.)

[14] Khalile M & Pankrashkin K. 2018. Eigenvalues of Robin Laplacians in infinite sectors, *Mathematische Nachrichten.* **291**. See DOI: 10.1002/mana.201600314.

[15] Lyalinov MA. 2019. A comment on eigenfunctions and eigenvalues of the Laplace operator in an angle with Robin boundary conditions. *Zapiski Nauch. Sem POMI Steklov, RAN.* **483**. (In Russian)

[16] Bernard JML. 1997 *Méthode analytique et transformées fonctionnelles pour la diffraction d'ondes par une singularité conique: équation intégrale de noyau non oscillant pour le cas d'impédance constante*, rapport CEA-R-5764. Editions Dist-Saclay. (erratum in J. Phys. A, vol.32, p.L45), an extended version in Bernard JML. 2014. *Advanced Theory of Diffraction by a Semi-infinite Impedance Cone.* Oxford, UK: Alpha Science Ser. Wave Phenom, Alpha Science.

[17] Maliuzhinets [Malyuzhinets] GD. 1958. Excitation, reflection and emission of surface waves from a wedge with given face impedances. *Soviet Physics: Doklady.* **3**(4).

[18] Roseau M. 1958. Short waves parallel to the shore over a sloping beach. *Comm. Pure Appl. Math.* **11 A958**.

[19] Williams WE. 1959. Diffraction of an $E-$polarised wave by an imperfectly conducting wedge. *Proc. R. Soc. Lond.* **A252**.

[20] Babich VM, Lyalinov MA, Grikurov VE. 2008. *Diffraction Theory. The Sommerfeld-Malyuzhinets Technique.* Oxford, UK: Alpha Science Ser. Wave Phenom. Alpha Science.

[21] Buslaev VS & Fedotov AA. 1997. The Harper equation: monodromization without semiclassics. *St. Petersburg Math. J.* **8**.

[22] Fedotov AA, & Sandomirskiy F. 2015. An exact renormalization formula for the Mariland model. *Commun. Math. Phys.* **334**.

[23] Lyalinov MA & Zhu NY. 2003. A solution procedure for second-order difference equations and its application to electromagnetic-wave diffraction in a wedge-shaped region. *Proc. R. Soc. Lond.* **A2040**.

[24] Lawrie JB & King AC. 1994. Exact solution to a class of the functional difference equations with application to a moving contact line flow. *Euro. J. Appl. Math.* **5**.

[25] Faddeev LD, Kashaev R & Volkov A. 2001. Strongly coupled quantum discrete Liouville theory: algebraic approach and duality. *Commun. Math. Phys.* **219**.

[26] Fedotov AA. 2016. Quasiclassical asymptotics of Malyuzhinets functions. *Zapiski Nauch. Semin POMI RAN.* **451**.

[27] Babich VM, Dement'ev DB, Samokish BA and Smyshlyaev VP. 2000. On Evaluation of the Diffraction Coefficients for Arbitrary 'Nonsingular' Directions of a Smooth Convex Cone. *SIAM J. Appl. Math.* **60**(2).

[28] Bernard JML and Lyalinov MA. 2001. Diffraction of acoustic waves by an impedance cone of an arbitrary cross-section. *Wave Motion.* **33**.

[29] Dieudonne J. 1960 *Foundations of modern analysis.* Academic Press, New York and London.

[30] Gradstein IS, Ryzhik IM. 1980 *Tables of Integrals, Series and Products.* 4th ed., Academic Press, Orlando.

[31] Lyalinov MA. 2018. Acoustic scattering by a semi-infinite angular sector with impedance boundary conditions. *IMA Journal on Applied Mathematics* **83**.

[32] Titchmarsh EC. 1937 *Introduction to the theory of Fourier integrals.* Oxford Press.

[33] Fedoryuk MV. 1987 *Asymptotics: Integrals and Series* Nauka, Moscow. (In Russian).