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The mixed coupled nonlinear Schrödinger equation on the half-line via the Fokas method

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In this paper, we implement the Fokas method to study initial-boundary value (IBV) problems of the mixed coupled nonlinear Schrödinger equation formulated on the half-line with Lax pairs involving 3×3 matrices. The solution can be written in terms of the solution of a 3×3 Riemann-Hilbert problem (RHP). The relevant jump matrices are explicitly expressed in terms of the matrix-value spectral functions $s(k)$ and $S(k)$, which are determined by the initial values and boundary values at $x = 0$, respectively. Some of these boundary values are unknown, however, using the fact that these specific functions satisfy a certain global relation, the unknown boundary values can be expressed in terms of the given initial and boundary data.

Appendix. The asymptotic behaviour of $\{\mu_j(x, t, k)\}_1^3$

Let's introduce the following symbols

$$A = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{I}_{2 \times 2} \end{pmatrix}, \tag{1a}$$

$$\begin{cases} V_1 = \begin{pmatrix} 0 & i\bar{V} \\ iU^T & 0 \end{pmatrix}, V_2^{(1)} = V_1, \\ V_2^{(0)} = \frac{i}{2} \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{I}_{2 \times 2} \end{pmatrix} \left[\begin{pmatrix} 0 & i\bar{V}_x \\ iU_x^T & 0 \end{pmatrix} - \begin{pmatrix} -\bar{V}U^T & 0 \\ 0 & -U^T\bar{V} \end{pmatrix} \right], \end{cases} \tag{1b}$$

where $\mathbb{I}_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $U = (p, q)$, $V = (-p, q)$.

Recalling the Lax pair equations of μ yield

$$\begin{cases} \mu_x + [ik\Lambda, \mu] = V_1\mu, \\ \mu_t + [ik^2\Lambda, \mu] = V_2\mu. \end{cases} \quad (2)$$

Let $\mu(x, t, k)$ be of the form

$$\mu(x, t, k) = D_0 + \frac{D_1}{k} + \frac{D_2}{k^2} + \frac{D_3}{k^3} + \dots \quad (3)$$

Substituting equation (3) into the Lax pair equations (2), and comparing the order of k , one can show that

$$\begin{cases} O(k): [i\Lambda, D_0] = 0, \\ O(1): D_{0x} + [i\Lambda, D_1] = V_1 D_0, \\ O(k^{-1}): D_{1x} + [i\Lambda, D_2] = V_1 D_1, \\ O(k^{-2}): D_{2x} + [i\Lambda, D_3] = V_1 D_2, \end{cases} \quad (4a)$$

$$\text{and } \begin{cases} O(k^2): [i\Lambda, D_0] = 0, \\ O(k): [i\Lambda, D_1] = V_2^{(1)} D_0, \\ O(1): D_{0t} + [i\Lambda, D_2] = V_2^{(1)} D_1 + V_2^{(0)} D_0, \\ O(k^{-1}): D_{1t} + [i\Lambda, D_3] = V_2^{(1)} D_2 + V_2^{(0)} D_1, \\ O(k^{-2}): D_{2t} + [i\Lambda, D_4] = V_2^{(1)} D_3 + V_2^{(0)} D_2, \\ O(k^{-3}): D_{3t} + [i\Lambda, D_5] = V_2^{(1)} D_4 + V_2^{(0)} D_3. \end{cases} \quad (4b)$$

Let D_l be of the following form

$$D_l = \begin{pmatrix} D_{11}^{(l)} & D_{1j}^{(l)} \\ D_{j1}^{(l)} & D_{2 \times 2}^{(l)} \end{pmatrix} \text{ with } D_{1j}^{(1)} = (D_{12}^{(1)}, D_{13}^{(1)}) \text{ and } D_{j1}^{(1)} = (D_{21}^{(1)}, D_{31}^{(1)})^T. \quad (5)$$

Thus, from $O(k^2)$, one has

$$D_{1j}^{(0)} = 0, \quad D_{j1}^{(0)} = 0. \quad (6)$$

From $O(k)$, one has

$$i \begin{pmatrix} 0 & -2D_{1j}^{(1)} \\ 2D_{j1}^{(1)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\bar{\nabla} D_{2 \times 2}^{(0)} \\ i\mathbb{U}^T D_{11}^{(0)} & 0 \end{pmatrix}, \quad (7a)$$

which shows that

$$\begin{cases} D_{1j}^{(1)} = -\frac{1}{2}\bar{\nabla} D_{2 \times 2}^{(0)}, \\ D_{j1}^{(1)} = \frac{1}{2}\mathbb{U}^T D_{11}^{(0)}. \end{cases} \quad (7b)$$

From $O(1)$, one has

$$\begin{aligned} & \begin{pmatrix} D_{11t}^{(0)} & D_{1jt}^{(0)} \\ D_{j1t}^{(0)} & D_{2 \times 2t}^{(0)} \end{pmatrix} + i \begin{pmatrix} 0 & -2D_{1j}^{(2)} \\ 2D_{j1}^{(2)} & 0 \end{pmatrix} \\ &= i \begin{pmatrix} \bar{\nabla} D_{j1}^{(1)} & \bar{\nabla} D_{2 \times 2}^{(1)} \\ \mathbb{U}^T D_{11}^{(1)} & \mathbb{U}^T D_{1j}^{(1)} \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -\bar{\nabla} \mathbb{U}^T D_{11}^{(0)} & -i\bar{\nabla}_x D_{2 \times 2}^{(0)} \\ i\mathbb{U}_x^T D_{11}^{(0)} & \mathbb{U}^T \bar{\nabla} D_{2 \times 2}^{(0)} \end{pmatrix}, \end{aligned} \quad (8a)$$

which shows that

$$\begin{cases} D_{11t}^{(0)} = i\bar{\nabla}D_{j1}^{(1)} - \frac{i}{2}\bar{\nabla}U^T D_{11}^{(0)}, \\ D_{2\times 2t}^{(0)} = iU^T D_{1j}^{(1)} + \frac{i}{2}U^T \bar{\nabla}D_{2\times 2}^{(0)}, \\ D_{1j}^{(2)} = -\frac{1}{2}\bar{\nabla}D_{2\times 2}^{(1)} + \frac{i}{4}\bar{\nabla}_x D_{2\times 2}^{(0)}, \\ D_{j1}^{(2)} = \frac{1}{2}U^T D_{11}^{(1)} + \frac{i}{4}U_x^T D_{11}^{(0)}. \end{cases} \quad (8b)$$

From $O(k^{-1})$, one has

$$\begin{aligned} & \begin{pmatrix} D_{11t}^{(1)} & D_{1jt}^{(1)} \\ D_{j1t}^{(1)} & D_{2\times 2t}^{(1)} \end{pmatrix} + i \begin{pmatrix} 0 & -2D_{1j}^{(3)} \\ 2D_{j1}^{(3)} & 0 \end{pmatrix} \\ &= i \begin{pmatrix} \bar{\nabla}D_{j1}^{(2)} & \bar{\nabla}D_{2\times 2}^{(2)} \\ U^T D_{11}^{(2)} & U^T D_{1j}^{(2)} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -\bar{\nabla}_x D_{j1}^{(1)} & -\bar{\nabla}_x D_{2\times 2}^{(1)} \\ U_x^T D_{11}^{(1)} & U_x^T D_{1j}^{(1)} \end{pmatrix} \\ & \quad - \frac{i}{2} \begin{pmatrix} \bar{\nabla}U^T D_{11}^{(1)} & \bar{\nabla}U^T D_{1j}^{(1)} \\ -U^T \bar{\nabla}D_{j1}^{(1)} & -U^T \bar{\nabla}D_{2\times 2}^{(1)} \end{pmatrix}, \end{aligned} \quad (9a)$$

which shows that

$$\begin{cases} D_{11t}^{(1)} = \frac{1}{4}(\bar{p}p_x - \bar{q}q_x - \bar{p}_x p + \bar{q}_x q) D_{11}^{(0)}, \\ D_{2\times 2t}^{(1)} = \frac{1}{4}(-U^T \bar{\nabla}_x + U_x^T \bar{\nabla}) D_{2\times 2}^{(0)}, \\ D_{1j}^{(3)} = \frac{i}{4}\bar{\nabla}_t D_{2\times 2}^{(0)} - \frac{1}{2}\bar{\nabla}D_{2\times 2}^{(2)} + \frac{i}{4}\bar{\nabla}_x D_{2\times 2}^{(1)} - \frac{1}{8}\bar{\nabla}U^T \bar{\nabla}D_{2\times 2}^{(0)}, \\ D_{j1}^{(3)} = \frac{i}{4}U^T D_{11}^{(0)} + \frac{1}{2}U^T D_{11}^{(2)} + \frac{i}{4}U_x^T D_{11}^{(1)} + \frac{1}{8}U^T \bar{\nabla}U^T D_{11}^{(0)}. \end{cases} \quad (9b)$$

From $O(k^{-2})$, one has

$$\begin{aligned} & \begin{pmatrix} D_{11t}^{(2)} & D_{1jt}^{(2)} \\ D_{j1t}^{(2)} & D_{2\times 2t}^{(2)} \end{pmatrix} + i \begin{pmatrix} 0 & -2D_{1j}^{(4)} \\ 2D_{j1}^{(4)} & 0 \end{pmatrix} \\ &= i \begin{pmatrix} \bar{\nabla}D_{j1}^{(3)} & \bar{\nabla}D_{2\times 2}^{(3)} \\ U^T D_{11}^{(3)} & U^T D_{1j}^{(3)} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -\bar{\nabla}_x D_{j1}^{(2)} & -\bar{\nabla}_x D_{2\times 2}^{(2)} \\ U_x^T D_{11}^{(2)} & U_x^T D_{1j}^{(2)} \end{pmatrix} \\ & \quad - \frac{i}{2} \begin{pmatrix} \bar{\nabla}U^T D_{11}^{(2)} & \bar{\nabla}U^T D_{1j}^{(2)} \\ -U^T \bar{\nabla}D_{j1}^{(2)} & -U^T \bar{\nabla}D_{2\times 2}^{(2)} \end{pmatrix}, \end{aligned} \quad (10a)$$

which shows that

$$\begin{cases} D_{11t}^{(2)} = \frac{1}{4}(-\bar{p}_x p + \bar{q}_x q + \bar{p}p_x - \bar{q}q_x) D_{11}^{(1)} - \frac{1}{4} \left[-\bar{p}p_t + \bar{q}q_t - \frac{i}{2}(-|p|^2 + |q|^2)^2 + \frac{i}{2}|p_x|^2 - \frac{i}{2}|q_x|^2 \right] D_{11}^{(0)}, \\ D_{2\times 2t}^{(2)} = \frac{1}{4}(-U^T \bar{\nabla}_x + U_x^T \bar{\nabla}) D_{2\times 2}^{(1)} - \frac{1}{4} \left(U^T \bar{\nabla}_t + \frac{i}{2}U^T \bar{\nabla}U^T \bar{\nabla} + \frac{i}{2}U_x^T \bar{\nabla}_x \right) D_{2\times 2}^{(0)}, \\ D_{1jt}^{(2)} - 2iD_{1j}^{(4)} = i\bar{\nabla}D_{2\times 2}^{(3)} + \frac{1}{2}\bar{\nabla}_x D_{2\times 2}^{(2)} - \frac{i}{2}\bar{\nabla}U^T D_{1j}^{(2)}, \\ D_{j1t}^{(2)} + 2iD_{j1}^{(4)} = iU^T D_{11}^{(3)} - \frac{1}{2}U_x^T D_{11}^{(2)} + \frac{i}{2}U^T \bar{\nabla}D_{1j}^{(2)}. \end{cases} \quad (10b)$$

Following the same way, the x -part of the Lax pair equation yields the following equations

$$\begin{cases} D_{11x}^{(0)} = 0, \\ D_{2 \times 2x}^{(0)} = 0, \end{cases} \quad (11a)$$

$$\begin{cases} D_{1j}^{(1)} = -\frac{1}{2}\bar{\nabla}D_{2 \times 2}^{(0)}, \\ D_{j1}^{(1)} = \frac{1}{2}U^T D_{11}^{(0)}, \\ D_{11x}^{(1)} = \frac{i}{2}\bar{\nabla}U^T D_{11}^{(0)}, \\ D_{2 \times 2x}^{(1)} = -\frac{i}{2}U^T \bar{\nabla}D_{2 \times 2}^{(0)}, \end{cases} \quad (11b)$$

$$\begin{cases} D_{1j}^{(2)} = \frac{i}{4}\bar{\nabla}_x D_{2 \times 2}^{(0)} - \frac{1}{2}\bar{\nabla}D_{2 \times 2}^{(1)}, \\ D_{j1}^{(2)} = \frac{i}{4}U_x^T D_{11}^{(0)} + \frac{1}{2}U^T D_{11}^{(1)}, \\ D_{11x}^{(2)} = -\frac{1}{4}\bar{\nabla}U_x^T D_{11}^{(0)} + \frac{i}{2}\bar{\nabla}U^T D_{11}^{(1)}, \\ D_{2 \times 2x}^{(2)} = -\frac{1}{4}U^T \bar{\nabla}_x D_{2 \times 2}^{(0)} - \frac{i}{2}U^T \bar{\nabla}D_{2 \times 2}^{(1)}. \end{cases} \quad (11c)$$

Then, from the integral contours γ_j , one can obtain

$$D_{2 \times 2}^{(0)} = \mathbb{I}_{2 \times 2} \text{ and } D_{11}^{(0)} = 1. \quad (12)$$