Curvature, metric and parametrization of origami tessellations: Theory and application to the eggbox pattern - Electronic Supplementary Material

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1. Christoffel symbols of a diagonal metric

Let $\mathscr{S}$ be a smooth surface of ϕ. Then the family (ϕ_x, ϕ_y, N) is a basis of the space $\mathbb{R}^3$ in which $\mathscr{S}$ is embedded. Hence, the second derivatives ϕ_{xx}, ϕ_{yy} and ϕ_{xy} can be expanded relatively to that basis. For instance, one has

$$\phi_{xx} = \Gamma^1_{11}\phi_x + \Gamma^2_{11}\phi_y + L\hat{n}$$

where the Γ coefficients are known as Christoffel symbols and L is the first coefficient of the second fundamental form II.

It is known that the Christoffel symbols can be written in terms of the metric I of $\mathscr{S}$ and of its first derivatives. When I is diagonal, i.e., ϕ_x and ϕ_y are orthogonal, these expressions are particularly simple. Here, we calculate Γ^1_{11} as an example. Indeed, one has

$$\Gamma^1_{11} = \frac{\langle \phi_{xx}, \phi_x \rangle}{\langle \phi_x, \phi_x \rangle} = \frac{1}{2}\frac{\langle \phi_x, \phi_x \rangle_x}{\langle \phi_x, \phi_x \rangle}.$$

As for the other Christoffel symbols, denoting

$$I = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

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one has

$$\phi_{xx} = \frac{a_x}{2a}\phi_x - \frac{a_y}{2b}\phi_y + L\hat{n}$$
$$\phi_{yy} = -\frac{b_x}{2a}\phi_x + \frac{b_y}{2b}\phi_y + M\hat{n}$$
$$\phi_{xy} = \frac{a_y}{2a}\phi_x + \frac{b_x}{2b}\phi_y + N\hat{n}.$$

These are sometimes referred to as the frame equations.

Finally, the in-plane and out-of-plane Poisson's ratios being equal and of opposite signs reads

$$\frac{\mathrm{d}\sqrt{a}/\,\mathrm{d}\sqrt{b}}{\sqrt{a}/\sqrt{b}} = -\frac{L/a}{N/b},$$

which, upon simplification, transforms into

$$\frac{\mathrm{d}a}{\mathrm{d}b} = -\frac{L}{N}.$$

It is then remarkable that, due to the first two frame equations together with the chain rule, the above scalar equation translates into the vectorial partial differential equation

$$\phi_{xx}/L - \phi_{yy}/N = 0.$$

2. Out-of-plane Poisson's ratio

Consider the set of vertices depicted in Figure 1. Note that vertices O, A, B, C and D are equivalent so that we have

$$\phi_{xx} = \frac{A + C - 2O}{r^2} + o(1) \quad \text{and} \quad \phi_{yy} = \frac{B + D - 2O}{r^2} + o(1).$$

Therein and hereafter dependency over M was dropped to simplify notations. These vertices being initially in a periodic state corresponding to S^0, the above relations transform into

$$\phi_{xx} = \delta A + \delta C - 2\delta O \quad \text{and} \quad \phi_{yy} = \delta B + \delta D - 2\delta O,$$

where the δ quantities are corrections to the location of vertices dictated by $\delta\theta$, δu and δv. However, $\delta\theta$ having no impact on second derivatives, it

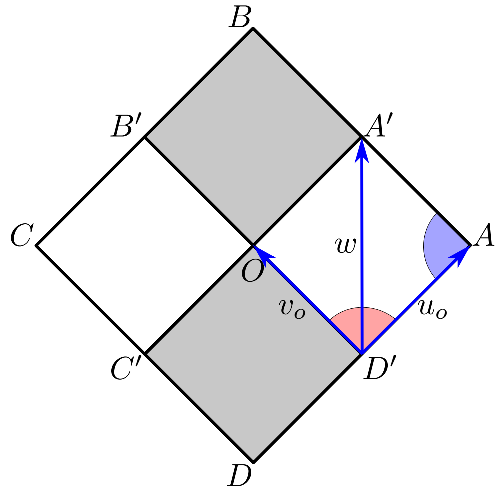

Figure 1. A set of vertices, depicted prior to perturbation, allowing to estimate the second derivatives of ϕ. Vectors u_o and v_o are unitary and here, along with vector w, they are scaled by a factor r.

is legitimate to assume that one of the 4 inspected pyramids, say $OC'DD'$, remains in its initial state. Correspondingly,

$$\delta O = \delta D' = \delta D = \delta C' = 0,$$

and

$$\phi_{xx} = \delta A + \delta C \quad \text{and} \quad \phi_{yy} = \delta B.$$

Now let us calculate the following scalar products.

1. The correction δA is orthogonal to u_o so that

$$\langle \phi_{xx}, u_o \rangle = \langle \delta C, u_o \rangle .$$

Also,

$$\langle \phi_{xx}, v_o \rangle = \langle \delta A, v_o \rangle .$$

2. The correction $\delta B - \delta B'$ is orthogonal to $\overrightarrow{B'B}$ which is collinear to u_o by periodicity of the initial state. Hence,

$$\langle \phi_{yy}, u_o \rangle = \langle \delta B, u_o \rangle = \langle \delta B', u_o \rangle .$$

Also,

$$\langle \phi_{yy}, v_o \rangle = \langle \delta B, v_o \rangle = \langle \delta A', v_o \rangle .$$

3. We already know that vector w can be written in terms of u_o and v_o. Furthermore, by symmetry, w is a linear combination of $u_o + v_o$ and $u_o \wedge v_o$. Therefore, there exist two functions a and b satisfying

$$\begin{aligned} w = w(u_o, v_o) &= a(\langle u_o, v_o \rangle)(u_o + v_o) + b(\langle u_o, v_o \rangle) u_o \wedge v_o \\ &= a(\cos\theta)(u_o + v_o) + b(\cos\theta) u_o \wedge v_o. \end{aligned}$$

Thus,

$$\begin{aligned} \langle \delta A', v_o \rangle &= \left\langle \frac{w(u_o + r\delta A, v_o) - w(u_o, v_o)}{r}, v_o \right\rangle + o(1) \\ &= \langle a(\cos\theta)\delta A + a'(\cos\theta) \langle \delta A, v_o \rangle (u_o + v_o), v_o \rangle \\ &= [a(\cos\theta) + a'(\cos\theta)(1 + \cos\theta)] \langle \delta A, v_o \rangle, \end{aligned}$$

with $a' = \mathrm{d}a/\mathrm{d}\cos\theta$. Similarly,

$$\langle \delta B', u_o \rangle = [a(\cos\theta) + a'(\cos\theta)(1 + \cos\theta)] \langle \delta C, u_o \rangle.$$

Recalling the definition of θ^*, it is easy to check that both

$$a(\cos\theta) = \frac{\langle w(u_o, v_o), u_o + v_o \rangle}{\langle u_o + v_o, u_o + v_o \rangle} = \sin^2(\theta^*)$$

and

$$a(\cos\theta) + a'(\cos\theta)(1 + \cos\theta) = \frac{\cos^2(\theta^*/2)}{\cos^2(\theta/2)}$$

hold. Consequently, we have proven the identity

$$\langle \phi_{yy}, u_o + v_o \rangle = \frac{\cos^2(\theta^*/2)}{\cos^2(\theta/2)} \langle \phi_{xx}, u_o + v_o \rangle.$$

Decomposing $u_o + v_o$ into tangential and normal components, we finally derive the compatibility relation

$$\langle \phi_{yy}, \hat{n} \rangle = \frac{\cos^2(\theta^*/2)}{\cos^2(\theta/2)} \langle \phi_{xx}, \hat{n} \rangle$$

implying the equality between the in-plane and out-of-plane Poisson's ratios:

$$\nu_{\mathrm{out}} = -\frac{\tan^2(\theta^*/2)}{\tan^2(\theta/2)} = -\nu.$$

The other 4 compatibility relations involving the tangential components can be derived directly from the metric. The first two are obtained by differentiating

$$\langle \phi_x, \phi_y \rangle = 0$$

with respect to x and y and respectively read

$$\begin{aligned} \langle \phi_{xx}, \phi_y \rangle + \langle \phi_x, \phi_{xy} \rangle &= 0, \\ \langle \phi_{xy}, \phi_y \rangle + \langle \phi_x, \phi_{yy} \rangle &= 0. \end{aligned}$$

The second two are obtained by differentiating Equation (2.3), i.e.,

$$4(1 - \langle \phi_x, \phi_x \rangle /4)(1 - \langle \phi_y, \phi_y \rangle /4) = 1,$$

with respect to x and y and respectively read

$$\begin{aligned} -c^{*2} \langle \phi_{xx}, \phi_x \rangle - c^2 \langle \phi_{xy}, \phi_y \rangle &= 0, \\ -c^{*2} \langle \phi_{xy}, \phi_x \rangle - c^2 \langle \phi_{yy}, \phi_y \rangle &= 0. \end{aligned}$$