

## Appendix A

### (a) The Fourier decomposition method for continuous time real function

Let  $x(t)$  be an aperiodic and real function of time, then the FT of  $x(t)$  is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt \quad (\text{A } 1)$$

and inverse FT is defined as

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft) df. \quad (\text{A } 2)$$

It is easy to show that  $\int_{-\infty}^0 X(f) \exp(j2\pi ft) df = \int_0^{\infty} X(-f) \exp(-j2\pi ft) df$ . Using the fact, for real signals,  $X(-f) = X^*(f)$ , we can rewrite (A 2) as

$$x(t) = \int_0^{\infty} [X(f) \exp(j2\pi ft) + X^*(f) \exp(-j2\pi ft)] df. \quad (\text{A } 3)$$

Observe that in this equation, the second term, on the r.h.s., is complex conjugate of the first term. As  $x(t)$  is a real values function, we can write

$$x(t) = \text{Re}\{z(t)\}, \quad (\text{A } 4)$$

where the analytic function  $z(t) = 2 \int_0^{\infty} X(f) \exp(j2\pi ft) df$  and  $\text{Re}\{z(t)\}$  denotes the real part of function  $z(t)$ . Let  $\{A_i = [f_{i-1}, f_i]$  with  $f_0 = 0, f_M = \infty$  and  $\cup A_i = \mathbb{R}_+$  be a partition of positive frequency axis. We write  $z(t)$ , in terms of this partition, as

$$z(t) = 2 \int_0^{\infty} X(f) \exp(j2\pi ft) df = \sum_{i=1}^M a_i(t) \exp(j\phi_i(t)), \quad (\text{A } 5)$$

where (with  $f_0 = 0, f_M = \infty$ )

$$a_i(t) \exp(j\phi_i(t)) \triangleq 2 \int_{f_{i-1}}^{f_i} X(f) \exp(j2\pi ft) df, \quad (\text{A } 6)$$

for  $i = 1, \dots, M$ . In order to obtain the minimum number of AFIBFs in LTH-FS, for each  $i$ , start with  $f_{i-1}$ , increase and select the maximum value of  $f_i$  such that  $f_{i-1} \leq f_i \leq \infty$  and

$$a_i(t) \geq 0, f_i(t) = \frac{1}{2\pi} \frac{d\phi_i(t)}{dt} \geq 0, \forall t. \quad (\text{A } 7)$$

Similarly, in HTL-FS, the lower and upper limits of integration in (A 6) will be from  $f_i$  to  $f_{i-1}$ , respectively, with  $f_0 = \infty, f_M = 0$ , and we can obtain minimum number of AFIBFs by selecting the minimum value of  $f_i$  such that  $0 \leq f_i \leq f_{i-1}$  and (A 7) is satisfied.

From (A 3) and (A 4), it is easy to obtain the relationship between the energy of original signal  $x(t)$  and energy of its analytic signal  $z(t)$  as  $E_x = E_z/2$ , that is, the energy of analytic signal is twice of the energy of original signal.

The IF characterizes a local frequency behaviour as a function of time. In a dual way, the local time behaviour as a function of frequency is characterized by the GD:  $t_i(f) = -(1/2\pi)(d\phi_i(f)/df)$ . The GD measures the average time arrival of the frequency  $f$ . In general, the IF and GD define two different curves in the time–frequency plane. Similar to the process of obtaining the IF, for a signal  $x(t)$ , defined in the interval  $0 \leq t < \infty$ , we obtain

$$a_i(f) \exp(-j\phi_i(f)) = \int_{t_{i-1}}^{t_i} x(t) \exp(-j2\pi ft) dt, \quad (\text{A } 8)$$

with  $t_0 = 0, t_M = \infty$  such that  $a_i(f) \geq 0, t_i(f) = -(1/2\pi)(d\phi_i(f)/df) \geq 0, \forall f$ , for  $i = 1, \dots, M$ . Thus, we can write (A 1) as

$$X(f) = \sum_{i=1}^M a_i(f) \exp(-j\phi_i(f)). \quad (\text{A } 9)$$

## (b) The Fourier decomposition method for discrete time real function

Let  $x[n]$  be an aperiodic, real function of time, then the DTFT of  $x[n]$  is defined as

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] \exp(-j\omega n) \quad (\text{A } 10)$$

and inverse DTFT (IDTFT) is defined as

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \exp(j\omega n) d\omega. \quad (\text{A } 11)$$

It is easy to show that  $\int_{-\pi}^0 X(\omega) \exp(j\omega n) d\omega = \int_0^{\pi} X(-\omega) \exp(-j\omega n) d\omega$ . From (A 10)  $X(-\omega) = X^*(\omega)$  and, hence, we rewrite (A 11) as

$$x[n] = \frac{1}{2\pi} \left[ \int_0^{\pi} X(\omega) \exp(j\omega n) d\omega + \int_0^{\pi} X^*(\omega) \exp(-j\omega n) d\omega \right]. \quad (\text{A } 12)$$

Clearly, in this equation, the second term, on r.h.s., is the complex conjugate of the first term. As  $x[n]$  is real function, we can write

$$x[n] = \text{Re}\{z[n]\}, \quad (\text{A } 13)$$

where analytic signal  $z[n] \triangleq (1/\pi) \int_0^{\pi} X(\omega) \exp(j\omega n) d\omega$  and  $\text{Re}\{z[n]\}$  denotes the real part of function  $z[n]$ . We write  $z[n]$  as follows:

$$z[n] = \frac{1}{\pi} \int_0^{\pi} X(\omega) \exp(j\omega n) d\omega = \sum_{i=1}^M a_i[n] \exp(j\phi_i[n]), \quad (\text{A } 14)$$

where (with  $\omega_0 = 0, \omega_M = \pi$ )

$$a_i[n] \exp(j\phi_i[n]) = \frac{1}{\pi} \int_{\omega_{i-1}}^{\omega_i} X(\omega) \exp(j\omega n) d\omega, \quad (\text{A } 15)$$

for  $i = 1, \dots, M$ . In order to obtain the minimum number of AFIBFs in LTH-FS, for each  $i$ , start with  $\omega_{i-1}$ , increase and select the maximum value of  $\omega_i$  such that  $\omega_{i-1} \leq \omega_i \leq \pi$  and phase  $\phi_i[n]$  is a monotonically increasing function, that is,

$$a_i[n], \omega_i[n] = (\phi_i[n+1] - \phi_i[n]) \geq 0, \quad \forall n. \quad (\text{A } 16)$$

Similarly, in HTL-FS, the lower and upper limits of integration in (A 15) will be from  $\omega_i$  to  $\omega_{i-1}$ , with  $\omega_0 = \pi, \omega_M = 0$ , and we can obtain minimum number of AFIBFs by selecting the minimum value of  $\omega_i$  such that  $0 \leq \omega_i \leq \omega_{i-1}$  and (A 16) is satisfied.