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Author for correspondence:

Andrea Nobili [e-mail:
andrea.nobili@unimore.it](mailto:andrea.nobili@unimore.it)

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Flexural edge-waves generated by steady-state propagation of a loaded rectilinear crack in an elastically supported thin plate

Andrea Nobili¹, Enrico Radi², Luca
Lanzoni³

¹ Dipartimento di Ingegneria Enzo Ferrari, Università
degli Studi di Modena e Reggio Emilia, via Vivarelli 10,
41125 Modena, Italy

² Dipartimento di Scienze e Metodi dell'Ingegneria,
Università degli Studi di Modena e Reggio Emilia, via
Amendola 2, 42122 Reggio Emilia, Italy

³ Dipartimento di Economia e Tecnologia, Università
degli Studi della Repubblica di San Marino, Via Salita
alla Rocca 44, 47890 San Marino, RSM

A. A conservative integral for steady-state crack propagation in an elastically supported thin plate

We first recall some established results for isotropic thin plates, see [1] for more details and a variational derivation in the static framework. The present approach generalizes to elastically supported thin plates the derivation given in [2] in the context of three-dimensional elasticity. Having denoted with w the plate transverse deflection, the rotation vector and curvature tensors are defined as

$$\phi = -\text{grad } w, \quad \chi = \text{grad } \phi, \quad (\text{A } 1)$$

where $\text{grad} = \partial_{x_1} + \partial_{x_2}$ is the 2D gradient operator. The internal generalized stress in the plate is expressed by the moment tensor \mathbf{M} which is symmetric. Besides, for linear elastic materials, it is connected to the curvature through a rank-4 tensor \mathcal{D} of elastic moduli [1, §5]

$$\mathbf{M} = \mathcal{D}\chi.$$

We define the shearing force vector

$$\mathbf{q} = \text{div } \mathbf{M}, \quad (\text{A } 2)$$

and with this the equilibrium condition in the absence of distributed loads (??) reads

$$\text{div } \mathbf{q} = kw + \rho h \partial_{tt} w. \quad (\text{A } 3)$$

Let E_B and E_W denote the elastic strain energy stored per unit area in the plate and in the Winkler soil, respectively, while T is the kinetic energy density,

$$E_B = \frac{1}{2} \mathbf{M} : \chi, \quad E_W = \frac{1}{2} kw^2, \quad T = \frac{1}{2} \rho h (\partial_t w)^2, \quad (\text{A } 4)$$

where $:$ stands for the scalar product between tensors, i.e. it implies a double index contraction, as opposed to \cdot which indicates scalar product between vectors. The total energy density is denoted by H ,

$$H = E_B + E_W + T. \quad (\text{A } 5)$$

Now, let the closed contour Γ surround the area A , which is assumed free of singularities, and \mathbf{n} denote the unit vector normal attached to Γ . We shall prove that the following line integral is conservative

$$I = \oint_{\Gamma} [Hn_1 - \partial_{x_1} \phi \cdot \mathbf{M}\mathbf{n} - \mathbf{q} \cdot \mathbf{n} \partial_{x_1} w] ds, \quad (\text{A } 6)$$

for it vanishes for any closed contour Γ free of singularities. Essential to the derivation is the steady-state assumption, which provides a connection between time and space derivatives, namely

$$\partial_t = -c \partial_{x_1},$$

where c is the constant speed of crack propagation. Thus, we have

$$I_1 = \oint_{\Gamma} Hn_1 ds = \int_A \partial_{x_1} H dA = \int_A [\mathbf{M} : \partial_{x_1} \chi + (kw + \rho h \partial_{tt} w) \partial_{x_1} w] dA,$$

having used Green's theorem, the definitions (A 4, A 5) and Euler's theorem for the derivative of an homogeneous function. Then, using Eqs. (A 1), the first term in the integrand may be rewritten according to the integral identity (here the divergence of a rank-2 tensor is assumed to apply on

the second index)

$$\mathbf{M} : \text{grad } \phi = \text{div} (\mathbf{M}^T \phi) - (\text{div } \mathbf{M}) \cdot \phi,$$

thus, by Eqs.(A 3) and (A 2),

$$\begin{aligned} I_1 &= \int_A \left[\text{div} (\mathbf{M}^T \partial_{x_1} \phi) - (\text{div } \mathbf{M}) \cdot \partial_{x_1} \phi + (\partial_{x_1} w) \text{div } \mathbf{q} \right] dA \\ &= \int_A \left[\text{div} (\mathbf{M}^T \partial_{x_1} \phi) - \mathbf{q} \cdot \partial_{x_1} \phi + \text{div } \mathbf{q} \partial_{x_1} w \right] dA \\ &= \int_{\Gamma} [\partial_{x_1} \phi \cdot \mathbf{M} \mathbf{n} + (\mathbf{q} \cdot \mathbf{n}) \partial_{x_1} w] ds, \end{aligned}$$

for

$$\mathbf{q} \cdot \partial_{x_1} \phi = -\mathbf{q} \cdot \partial_{x_1} \text{grad } w = -\text{div} [(\partial_{x_1} w) \mathbf{q}] + \text{div } \mathbf{q} \partial_{x_1} w.$$

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