

**Comparing the constructions of
Goldberg, Fuller, Caspar, Klug and Coxeter
and a general approach to
local symmetry-preserving operations**

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Supplementary material

A first step towards general local operations preserving orientation-preserving symmetries but not necessarily orientation-reversing automorphisms

Although the emphasis of this paper is on the work of Goldberg, Fuller, Caspar and Klug and on operations preserving all symmetries, we want to make a first step – details are postponed to a later paper – towards operations that preserve orientation-preserving automorphisms but not necessarily orientation-reversing automorphisms. In the case of Goldberg, these operations could be described by two subdivisions – one of a black and one of a white triangle. In general, such a description is not possible, and it is necessary to describe how a quadrangle containing a white and a black triangle that share an edge must be subdivided. See Figure 3 and Figure 4 from the main text for an illustration of the following definition.

Definition 1 *Let T be a periodic 3-connected tiling of the Euclidean plane with chamber system C_T , and let v_2, v_0, v_1, v'_0 be points in the Euclidean plane so that:*

- (i) v_2 is the center of a rotation ρ_{v_2} by 60 degrees in counterclockwise direction that is a symmetry of the tiling.
- (ii) $v'_0 = \rho_{v_2}(v_0)$
- (iii) v_1 is the midpoint between v'_0 and v_0 and the center of a rotation ρ_{v_1} by 180 degrees that is a symmetry of the tiling (so also $v'_0 = \rho_{v_1}(v_0)$).

Let Q be a simple cycle in C_T through v_2, v_0, v_1, v'_0 (in this order). For $\{x, y\} \in \{\{v_2, v_0\}, \{v_2, v'_0\}, \{v_1, v'_0\}, \{v_1, v_0\}\}$ let $P_{x,y}$ denote the path on Q from x to y not containing any other vertex of $\{v_2, v_0, v_1, v'_0\}$.

If $P_{v_2, v'_0} = \rho_{v_2}(P_{v_2, v_0})$ and $P_{v_1, v'_0} = \rho_{v_1}(P_{v_1, v_0})$, then we call the quadrangle Q with corner points v_2, v_0, v_1, v'_0 labelled with the names v_2, v_0, v_1, v'_0 and subdivided into chambers as given by C_T , a local operation that preserves orientation-preserving symmetries, lops operation for short.

Let P be a polyhedron or tiling given as a chamber system. We call a pair of (black and white) chambers sharing a 1-vertex and a 2-vertex a double chamber. Each chamber is contained in exactly one double chamber. The result of applying a lops operation O to a polyhedron or tiling P is given by subdividing each double chamber of P with O by identifying v_2 with the 2-vertex of the double chamber, v_0 with the 0-vertex of the white chamber, v_1 with the 1-vertex of the double chamber, and v'_0 with the 0-vertex of the black chamber. Note that this operation is purely combinatorial and that the angles of O are not preserved.

As already mentioned in the section about Goldberg's approach where the chiral case is discussed, also here there are various ways to draw the boundaries of the quadrangle Q that lead to equivalent operations, that is, operations that when applied to a polyhedron give the same result. We will postpone working out the details to a later paper.

It is tempting to conjecture that for each lops operation there is a way to choose the quadrangle so that the subdivision of the double chamber can be expressed by two subdivisions of the chambers – one for the black chambers and one for the white chambers.

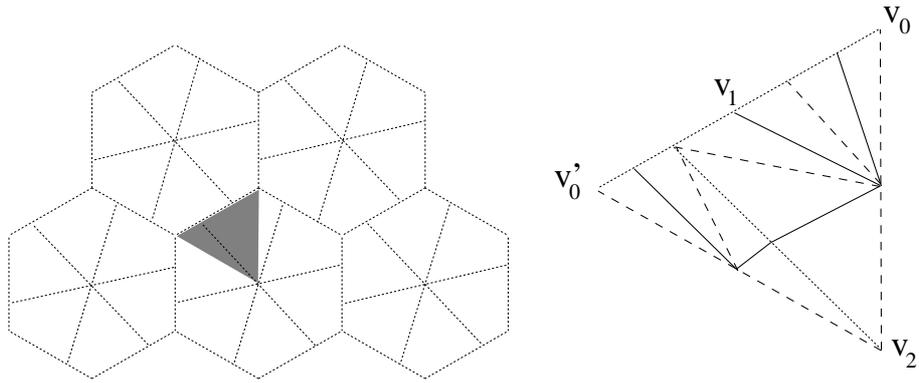


Figure S1: A lops operation that cannot be described by separate subdivisions of black and white chambers.

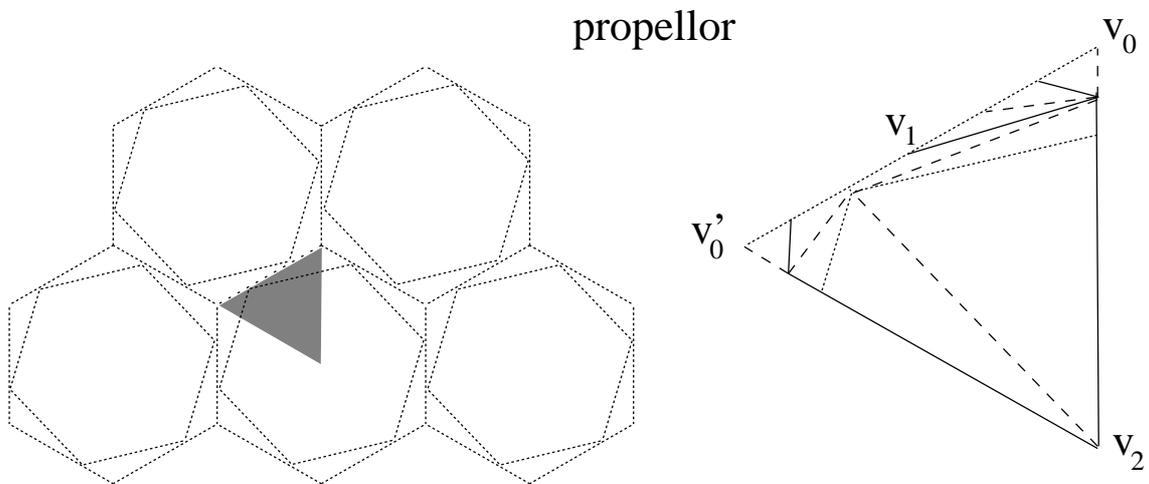


Figure S2: The lops operation *propellor* and the periodic tiling from which it originates.

Figure S1 gives an example of an operation where this is not the case. In order to avoid misunderstandings, we mention the fact already here, but a proof has to be postponed as first the necessary prerequisites (e.g. results about different ways to choose the edges of the quadrangles that lead to equivalent operations) have to be formally introduced and proven.

Again, it must be shown that operations used in the literature are covered by this definition. The mirror image plays a special role as an operation as it does not change the combinatorial structure of a polyhedron. It is neither an lsp operation nor a lops operation. On the level of chamber systems it can be described as changing the orientation of the chambers, so black ones become white and the other way around. We will show that the well known operations *propellor*, *snub* and *whirl* can be obtained from tilings. Other operations (e.g. *gyro*) can be obtained by a combination of lsp operations and these lops operations. These operations are basic to the field. People who are not familiar with these operations (or at least their names) can take their representations as lops operations as the definition. In Figure S2, Figure S3, and Figure S4 the quadrangles v_2, v_0, v_1, v'_0 are shown next to the tiling for better visibility. All chiral Caspar-Klug operations can be represented this way.

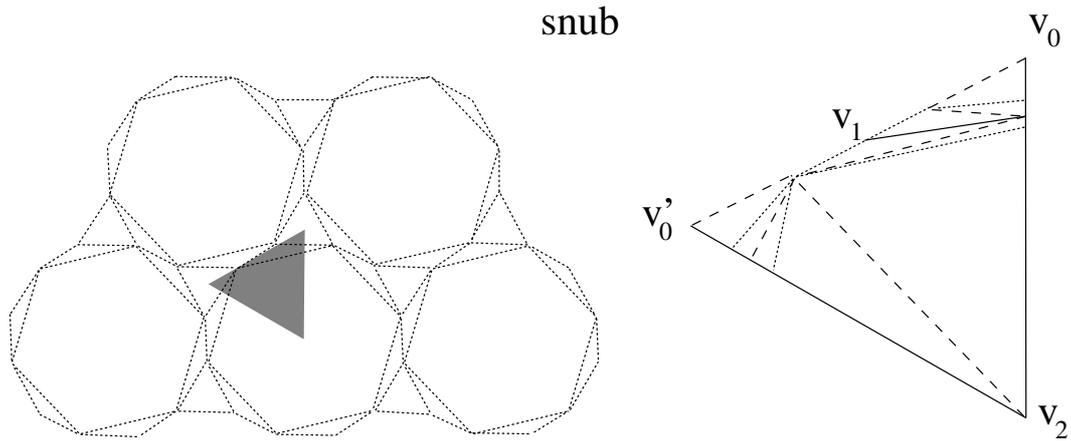


Figure S3: The losp operation *snub* and the periodic tiling from which it originates.

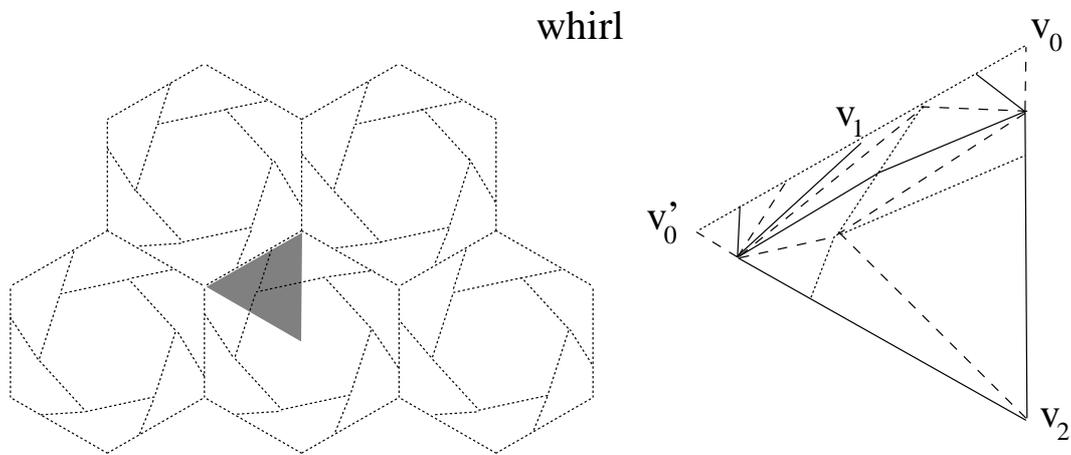


Figure S4: The losp operation *whirl* and the periodic tiling from which it originates.

Although we could not find publications where decorations of double chambers are used to represent losp operations like propellor, snub or whirl, it may be considered common knowledge among people who work with chamber systems that such a representation is possible. Again, Definition 1 establishes the inverse direction and gives a criterion for which decompositions of double chambers define an operation.