

# Supplementary Material for Analysing causal structures with entropy

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## 1 Entropy inequalities for Example 10

In the following we provide the basic inequalities for the quantum instrumental scenario,  $IC^Q$ , i.e., the constraints making up the matrix  $M_B(IC^Q)$ .

$$I(A_Y : A_Z) \geq 0 \tag{1}$$

$$I(A_Y : X) \geq 0 \tag{2}$$

$$I(A_Z : X) \geq 0 \tag{3}$$

$$I(A_Y : A_Z|X) \geq 0 \tag{4}$$

$$I(A_Y : X|A_Z) \geq 0 \tag{5}$$

$$I(A_Z : X|A_Y) \geq 0 \tag{6}$$

$$I(A_Y : Z) \geq 0 \tag{7}$$

$$I(X : Z) \geq 0 \tag{8}$$

$$I(X : Z|A_Y) \geq 0 \tag{9}$$

$$I(A_Y : Z|X) \geq 0 \tag{10}$$

$$I(A_Y : X|Z) \geq 0 \tag{11}$$

$$I(X : Y) \geq 0 \tag{12}$$

$$I(Y : Z) \geq 0 \tag{13}$$

$$I(Y : Z|X) \geq 0 \tag{14}$$

$$I(X : Z|Y) \geq 0 \tag{15}$$

$$I(X : Y|Z) \geq 0 \tag{16}$$

$$H(A_Z|X) \geq 0 \tag{17}$$

$$H(A_Y A_Z|X) \geq 0 \tag{18}$$

$$H(X|A_Y A_Z) \geq 0 \tag{19}$$

$$H(A_Y|XZ) \geq 0 \tag{20}$$

$$H(X|A_Y Z) \geq 0 \tag{21}$$

$$H(Z|A_Y X) \geq 0 \tag{22}$$

$$H(X|YZ) \geq 0 \tag{23}$$

$$H(Y|XZ) \geq 0 \tag{24}$$

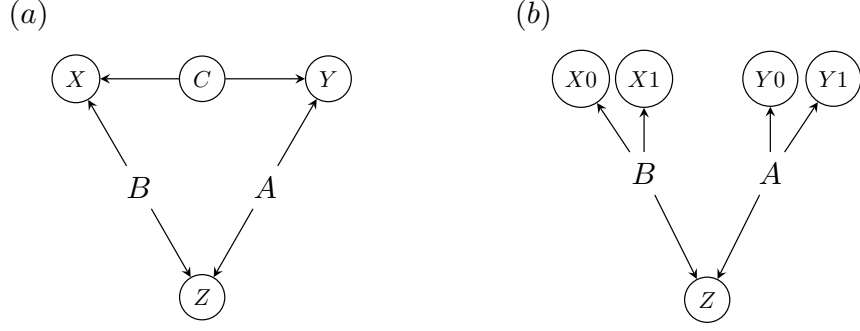
$$H(Z|XY) \geq 0 \tag{25}$$

$$H(A_Z|A_Y) + H(A_Z|X) \geq 0 \tag{26}$$

$$H(A_Y|A_Z) + H(A_Y|X) \geq 0 \tag{27}$$

$$H(A_Z|A_Y X) + H(A_Z) \geq 0 \tag{28}$$

$$H(A_Y|A_Z X) + H(A_Y) \geq 0 \tag{29}$$



**Figure 1:** Post-selecting (a) on a binary observed variable  $C$  leads to the causal structure (b).

Independence constraints and data processing inequalities are provided in the main text. If we include these and remove redundant inequalities we obtain the following set of constraints, which for convenience we give in matrix form (such that  $\Gamma(IC^Q) = \{v \in \mathbb{R}_{\geq 0}^{15} \mid M \cdot v \geq 0\}$ ):

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

## 2 Entropy inequalities for Example 16

The causal structure of Figure 1, previously considered in [1, 2] is analysed entropically by means of the entropic post-selection technique. The outer approximation to the entropic cone of the causal structure of Figure 1 is computed and marginalised to vectors

$$\begin{aligned} & (H(Z), H(X|_{C=0}), H(X|_{C=1}), H(Y|_{C=0}), H(Y|_{C=1}), H(X|_{C=0}Z), H(X|_{C=1}Z), H(Y|_{C=0}Z), \\ & H(Y|_{C=1}Z), H(X|_{C=0}Y|_{C=0}), H(X|_{C=1}Y|_{C=1}), H(X|_{C=0}Y|_{C=0}Z), H(X|_{C=1}Y|_{C=1}Z)). \end{aligned}$$

The following 14 extremal rays are obtained from this computation, where each ray is represented by one particular vector on it. The tip of this pointed polyhedral cone is the zero-vector.

- (1) 0000100010101
- (2) 0001000101010
- (3) 0010001000101
- (4) 0100010001010
- (5) 1111122222222
- (6) 1010112120212
- (7) 1101021212021
- (8) 1000011110011
- (9) 1000111110111
- (10) 1001011111011
- (11) 1010011110111
- (12) 1100011111011
- (13) 1001111111111
- (14) 1110011111111

The corresponding inequality description is given by the 2 equalities and 18 inequalities (or equivalently 22 inequalities if each equality is written as two inequalities).

$$H(X_{|C=0}) + H(Y_{|C=0}) = H(X_{|C=0}Y_{|C=0}) \quad (1)$$

$$H(X_{|C=1}) + H(Y_{|C=1}) = H(X_{|C=1}Y_{|C=1}) \quad (2)$$

$$H(X_{|C=1}Y_{|C=1}) \leq H(X_{|C=1}Y_{|C=1}Z) \quad (3)$$

$$H(X_{|C=0}Y_{|C=0}) \leq H(X_{|C=0}Y_{|C=0}Z) \quad (4)$$

$$H(Y_{|C=1}Z) \leq H(X_{|C=1}Y_{|C=1}Z) \quad (5)$$

$$H(Y_{|C=0}Z) \leq H(X_{|C=0}Y_{|C=0}Z) \quad (6)$$

$$H(X_{|C=1}Z) \leq H(X_{|C=1}Y_{|C=1}Z) \quad (7)$$

$$H(X_{|C=0}Z) \leq H(X_{|C=0}Y_{|C=0}Z) \quad (8)$$

$$H(Y_{|C=0}Z) \leq H(Z) + H(Y_{|C=0}) \quad (9)$$

$$H(Y_{|C=1}Z) \leq H(Z) + H(Y_{|C=1}) \quad (10)$$

$$H(Y_{|C=1}) + H(X_{|C=1}Z) \leq H(Z) + H(X_{|C=1}Y_{|C=1}) \quad (11)$$

$$H(Y_{|C=0}) + H(X_{|C=0}Z) \leq H(Z) + H(X_{|C=0}Y_{|C=0}) \quad (12)$$

$$H(Z) + H(X_{|C=0}Y_{|C=0}Z) \leq H(X_{|C=0}Z) + H(Y_{|C=0}Z) \quad (13)$$

$$H(Z) + H(X_{|C=1}Y_{|C=1}Z) \leq H(X_{|C=1}Z) + H(Y_{|C=1}Z) \quad (14)$$

$$H(Y_{|C=1}) + H(X_{|C=1}Z) + H(X_{|C=0}Y_{|C=0}) \leq H(Y_{|C=0}) + H(X_{|C=0}Z) + H(X_{|C=1}Y_{|C=1}Z) \quad (15)$$

$$H(Y_{|C=1}) + H(Y_{|C=0}Z) + H(X_{|C=0}Y_{|C=0}) \leq H(Y_{|C=0}) + H(Y_{|C=1}Z) + H(X_{|C=0}Y_{|C=0}Z) \quad (16)$$

$$H(X_{|C=1}Z) + H(Y_{|C=1}Z) + H(X_{|C=0}Y_{|C=0}) \leq H(X_{|C=0}Z) + H(Y_{|C=0}Z) + H(X_{|C=1}Y_{|C=1}Z) \quad (17)$$

$$H(X_{|C=0}Z) + H(Y_{|C=0}Z) + H(X_{|C=1}Y_{|C=1}) \leq H(X_{|C=1}Z) + H(Y_{|C=1}Z) + H(X_{|C=0}Y_{|C=0}Z) \quad (18)$$

$$H(Y_{|C=0}) + H(Y_{|C=1}Z) + H(X_{|C=1}Y_{|C=1}) \leq H(Y_{|C=1}) + H(Y_{|C=0}Z) + H(X_{|C=1}Y_{|C=1}Z) \quad (19)$$

$$H(Y_{|C=0}) + H(X_{|C=0}Z) + H(X_{|C=1}Y_{|C=1}) \leq H(Y_{|C=1}) + H(X_{|C=1}Z) + H(X_{|C=0}Y_{|C=0}Z). \quad (20)$$

## References

- [1] Henson, J., Lal, R. & Pusey, M. F. Theory-independent limits on correlations from generalized Bayesian networks. *New Journal of Physics* **16**, 113043 (2014).

- [2] Pienaar, J. Which causal structures might support a quantum–classical gap? *New Journal of Physics* **19**, 043021 (2017). URL <http://stacks.iop.org/1367-2630/19/i=4/a=043021>.