Further details and equations of a compartmental model for describing mixing in manholes

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INTRODUCTION

This document outlines the equations and solution method of a compartmental model for describing mixing in manholes. The model is fully described in the journal article "Predicting manhole mixing using a compartmental model" (Sonnenwald et al., submitted). A brief description is provided here. Albertson et al. (1950) described the behavior of a jet of water entering a semi-infinite volume as a cone-shaped jet core of uniform velocity, decreasing in diameter, dissipating its energy into a second, expanding, cone. The compartmental model uses the boundaries of these cones to define model zones, forming the jet core zone (V_1) and the jet diffusion zone (V_3), with the area outside of the jet diffusion zone being defined as the outer storage volume (V_4). An additional high-surcharge storage zone (V_5) has also been defined for surcharges above manhole diameter. These zones are illustrated in Fig. 1 for a circular unbenched manhole. Mass entering the manhole enters V_1. If the manhole inlet is aligned with the outlet, mass from V_1 is either transported to the outlet or goes into V_3. Mass that does not exit from V_1 exits instead from V_3. Any other flow entrained in V_3 that does not exit exchanges with V_4. If V_5 exists, V_4 in turn exchanges with V_5 by the same amount as well.

GEOMETRY

The distance the jet core penetrates the manhole is controlled by the diameter of the manhole inlet, such that $L_j = D_p/(2\alpha_2)$ where L_j is the length of the jet, D_p is the pipe diameter, and α_2 is a coefficient representing the rate of jet core dissipation, given as $\alpha_2 = 1/12.4$ by Albertson et al. (1950). α_2 also describes the slope of the cone representing the jet core zone. Thus, the volume of the jet core zone is

$$V_1 = \frac{\pi}{24\alpha_2} D_p^3 - V_{1b} \tag{1}$$

where V_{1b} is the portion of the cone that extends beyond the manhole diameter when $L_j > D_m$, where D_m is manhole diameter, given by

$$V_{1b} = \frac{-\pi\left(-0.5D_p + \alpha_2 D_m\right)^3}{3\alpha_2} \tag{2}$$

When the manhole diameter is greater than the length of the jet core ($D_m/D_p > 1/2\alpha_2$), then $V_{1b} = 0$. Similarly, the rate the jet diffusion zone cone expands at is 1 in m where $m = 5$ (Albertson et al., 1950) and the volume of the jet diffusion zone, ignoring the curvature of the manhole wall, is thus

$$V_3 = \frac{\pi}{3}\left(\frac{D_p}{2} + \frac{D_m}{m}\right)^2 \left(\frac{mD_p}{2} + D_m\right) - V_{3b} - V_{3c} - V_{3d} - V_1 \tag{3}$$

where V_{3b} is the portion of the zone that extends upstream of the inlet beyond the diameter of the manhole, given by

$$V_{3b} = \frac{m\pi D_p^3}{24} \tag{4}$$

V_{3c} is the portion that extends below the bottom of the manhole, which using Rajpoot (2016) for a sliced cone gives,

$$V_{3c} = \frac{\left(0.5mD_p + D_m\right)}{\left(3\left(0.5D_p + \frac{D_m}{m}\right)\right)} \left[\begin{array}{c} \left(0.5D_p + \frac{D_m}{m}\right)^3 cos^{-1} \frac{0.5D_p + B}{\left(0.5D_p + \frac{D_m}{m}\right)} - 2\left(0.5D_p + \frac{D_m}{m}\right)\left(0.5D_p + B\right)\sqrt{\left(0.5D_p + \frac{D_m}{m}\right)^2 - \left(0.5D_p + B\right)^2} \\ +\left(0.5D_p + B\right)^3 ln\left(\frac{\left(0.5D_p + \frac{D_m}{m}\right) + \sqrt{\left(0.5D_p + \frac{D_m}{m}\right)^2 - \left(0.5D_p + B\right)^2}}{0.5D_p + B}\right) \end{array} \right] \quad (5)$$

V_{3d}, the portion of V_3 that extends above the water surface if surcharge depth above the soffit S is less than D_m/m, is given by

$$V_{3d} = \frac{\left(0.5mD_p + D_m\right)}{\left(3\left(0.5D_p + \frac{D_m}{m}\right)\right)} \left[\begin{array}{c} \left(0.5D_p + \frac{D_m}{m}\right)^3 cos^{-1} \frac{0.5D_p + S}{\left(0.5D_p + \frac{D_m}{m}\right)} - 2\left(0.5D_p + \frac{D_m}{m}\right)\left(0.5D_p + S\right)\sqrt{\left(0.5D_p + \frac{D_m}{m}\right)^2 - \left(0.5D_p + S\right)^2} \\ +\left(0.5D_p + S\right)^3 ln\left(\frac{\left(0.5D_p + \frac{D_m}{m}\right) + \sqrt{\left(0.5D_p + \frac{D_m}{m}\right)^2 - \left(0.5D_p + S\right)^2}}{0.5D_p + S}\right) \end{array} \right] \quad (6)$$

otherwise $V_{3d} = 0$. The volume of the outer mixing zone is given as

$$V_4 = \frac{\pi\left(S + D_p + B\right)D_m^2}{4} - V_3 - V_1 - V_5 \quad (7)$$

where V_5 is the high-surcharge storage zone. V_5 is only included within the model when surcharge depth exceeds manhole diameter, given by

$$V_5 = \frac{\pi(S - D_m)D_m^2}{4} \quad (8)$$

if $S > D_m$ and otherwise $V_5 = 0$. The system volume can be calculated as the sum of the zones plus any length of pipe included on either side of the manhole. This extra length of pipe may be needed for comparing against recorded experimental data if instruments could not be placed precisely at the manhole inlet and outlet.

FLOW

Following Mark and Ilesanmi-Jimoh (2017), the exchange between the zones is controlled by the flow rates estimated using the jet theory given by Albertson et al. (1950). Flow from the jet core to the outlet ($Q_{1,2}$), assuming the inlet and outlet are aligned, is given by

$$\frac{Q_{1,2}}{Q} = 1 - 4\left(\frac{\alpha_2 D_m}{D_p}\right) + 4\left(\frac{\alpha_2 D_m}{D_p}\right)^2 \quad (9)$$

where Q is the flow rate into the manhole. If the jet core does not reach the outlet ($D_m/D_p > 1/2\alpha_2$), then $Q_{1,2} = 0$. If the jet core and outlet are not aligned, i.e., a stepped manhole, $Q_{1,2}$ is reduced by the fraction of the jet core cross-section that does not overlap with the outlet. From continuity, flow from the jet core to the jet diffusion zone ($Q_{1,3}$) is given by

$$Q_{1,3} = Q - Q_{1,2} \tag{10}$$

Flow in the jet diffusion zone (Q_3) at a distance X from the inlet is calculated by integrating the velocity at X and radius r from the jet center giving

$$\frac{Q_3}{Q} = \frac{4}{\pi D_p^2} \int_0^{2\pi} \int_{0.5D_p - \alpha_2 D_m}^{\infty} r \exp\left(\frac{-\left(r + \alpha_2 X - 0.5D_p\right)^2}{2(\alpha_2 X)^2}\right) \mathrm{d}r\, \mathrm{d}\phi \tag{11}$$

where ϕ is an integration variable representing angle. Integrating for $X = D_m$ and assuming the edge of the area integrated is sufficiently far from the jet edge as to capture most of a Gaussian profile, in this case taking the integration limit as D_m instead of ∞ for convenience, gives

$$\frac{Q_3}{Q} = 2\left(\frac{\alpha_2 D_m}{D_p}\right)^2 \left(4 + \frac{\sqrt{2\pi}\left(D_p - 2\alpha_2 D_m\right)}{\alpha_2 D_m}\right) \tag{12}$$

at the wall of the manhole and ignoring the wall's curvature. Similar to calculating V_3, flow contributions from below the bottom of the manhole and above the water surface must be accounted for. Converting Eq. 11 to Cartesian coordinates, integrating, and simplifying gives

$$\frac{Q_{3c}}{Q} \approx \left(\frac{2\alpha_2 D_m}{D_p}\right)^2 \operatorname{erfc}\left(\frac{\alpha_2 D_m + B}{\sqrt{2}\alpha_2 D_m}\right) \tag{13}$$

and

$$\frac{Q_{3d}}{Q} \approx \left(\frac{2\alpha_2 D_m}{D_p}\right)^2 \operatorname{erfc}\left(\frac{\alpha_2 D_m + S}{\sqrt{2}\alpha_2 D_m}\right) \tag{14}$$

If $S > D_m$ then $Q_{3d} = 0$.

If the jet core does not reach the outlet, then the outlet is in the zone of established flow. The form of Eq. 11 for Q_3 changes and based on Albertson et al., (1950) becomes

$$\frac{Q_3}{Q} = \frac{4}{\pi D_p^2} \frac{1}{2\alpha_2} \frac{D_p}{D_m} \int_0^{2\pi} \int_0^{\infty} r \exp\left(\frac{-r^2}{2(\alpha_2 X)^2}\right) \mathrm{d}r\, \mathrm{d}\theta \tag{15}$$

From this, Eqs. 12-14 become

$$\frac{Q_3}{Q} = 4\alpha_2 \frac{D_m}{D_p} \tag{16}$$

$$\frac{Q_{3c}}{Q} \approx 2\alpha_2 \frac{D_m}{D_p} \operatorname{erfc}\left(\frac{D_p + 2B}{2\sqrt{2}\alpha_2 D_m}\right) \tag{17}$$

$$\frac{Q_{3d}}{Q} \approx 2\alpha_2 \frac{D_m}{D_p} \operatorname{erfc}\left(\frac{D_p + 2S}{2\sqrt{2}\alpha_2 D_m}\right) \tag{18}$$

respectively.

If the inlet and outlet are aligned, then all flow through the outlet that does not come from the jet core comes from the jet diffusion zone

$$Q_{3,2} = Q - Q_{1,2} \tag{19}$$

The exchange between the jet diffusion zone and outer mixing zone, therefore, is due to the flow from Q_3 that does not exit through the outlet,

$$Q_{3,4} = (Q_3 - Q_{3c} - Q_{3d}) - Q_{3,2} \tag{20}$$

If there is a step and the outlet touches both the jet diffusion zone and outer mixing zone, then $Q_{3,2}$ is reduced by the fraction of the outlet that does not touch the cross-section of the jet diffusion zone. Flow from the outer mixing zone to the outlet $Q_{4,2}$ is then non-zero and

$$Q_{4,2} = Q - Q_{1,2} - Q_{3,2} \tag{21}$$

From continuity, the exchange between the outer mixing zone and jet diffusion zone is then

$$Q_{4,3} = Q_{3,4} - Q_{4,2} \tag{22}$$

Assuming the high surcharge storage zone V_5 is counter-rotating to the outer mixing zone V_4, it is reasonable to assume that flow from the outer mixing zone to the high-surcharge storage zone would be similar to that of flow from the outer mixing zone to the jet diffusion zone and thus we assume

$$Q_{5,4} = Q_{4,5} = Q_{4,3} \tag{23}$$

ANGLED MANHOLES

For an angled manhole, with an outlet turned through θ degrees, the location of the outlet may be calculated using the rotated projection of the outlet relative to the inlet. The edges of the outlet relative to the manhole centerline Y_o, therefore, can be calculated as

$$Y_o = \frac{D_m}{2} \sin\theta \pm \frac{D_p}{2} \cos -\theta \tag{24}$$

The edges of the jet diffusion zone at the outlet can be calculated as

$$Y_3 = \pm \left[\left(\frac{-D_m - mD_p + \sqrt{-2D_m D_p m^3 + D_m^2 m^4 - D_p^2 m^4}}{2(1+m^2)} \right) \left(\frac{1}{m} \right) + \frac{\mathrm{D_p}}{2} + \frac{D_m}{2m} \right] \tag{25}$$

and the edges of the jet core at the outlet as

$$Y_1 = \pm \left(D_p/2 - D_m \alpha_2 \right) \tag{26}$$

If one outlet edge is greater than the zone edge and the other is not, then the contribution from the outlet on either side of the edge must be calculated, similar to an overlapping outlet and jet core for a stepped manhole. However, the projected outlet is now an ellipse that is being intersected by a circle. An approximation of the overlap can be calculated by treating the outlet as a circle with radius equal to the minor radius of the ellipse of the projected outlet, i.e., treating the outlet as having a radius of $(D_p/2)\cos(-\theta)$.

FLUX EQUATIONS AND DISCRETISATION

Following Mark and Ilesanmi-Jimoh (2017) and the standard textbook approach (Chapra, 1997) the transport (flux) equations between model zones may be written as

$$V_1 \frac{\mathrm{d}C_1}{\mathrm{d}t} = QC_0 - Q_{1,2}C_1 - Q_{1,3}C_1 \tag{27}$$

$$V_3 \frac{\mathrm{d}C_3}{\mathrm{d}t} = Q_{1,3}C_1 - Q_{3,2}C_3 - Q_{3,4}C_3 + Q_{4,3}C_4 \tag{28}$$

$$V_4 \frac{\mathrm{d}C_4}{\mathrm{d}t} = Q_{3,4}C_3 - Q_{4,2}C_4 - Q_{4,3}C_4 - Q_{4,5}C_4 + Q_{5,4}C_5 \quad (29)$$

$$V_5 \frac{\mathrm{d}C_5}{\mathrm{d}t} = Q_{4,5}C_4 - Q_{5,4}C_5 \quad (30)$$

where C_i is the concentration in zone i. In discrete finite difference form with an explicit formulation, Eqs. 27-30 can be solved as

$$C_1^{t+1} = \frac{\Delta t}{V_1}\left(Q_{0,1}C_0^{t-t_1'} - C_1^t\left(Q_{1,2} + Q_{1,3}\right)\right) + C_1^t \quad (31)$$

$$C_3^{t+1} = \frac{\Delta t}{V_3}\left(C_1^t Q_{1,3} - C_3^t\left(Q_{3,2} + Q_{3,4}\right) + C_4^t Q_{4,3}\right) + C_3^t \quad (32)$$

$$C_4^{t+1} = \frac{\Delta t}{V_4}\left(C_3^t Q_{3,4} - C_4^t\left(Q_{4,2} + Q_{4,3} + Q_{4,5}\right) + C_5^t Q_{5,4}\right) + C_4^t \quad (33)$$

$$C_5^{t+1} = \frac{\Delta t}{V_5}\left(C_4^t Q_{4,5} - C_5^t Q_{5,4}\right) + C_5^t \quad (34)$$

where Δt is the time step and t_1' is the time delay in number of time steps. If there are no pipe segments attached to the manhole, t_1' will be zero, otherwise t_1' can be calculated from the product of the time-step size and pipe length divided by peak pipe velocity, rounded to the nearest integer. Finally, the concentration downstream at the outlet can be calculated using

$$C_2 = \frac{C_1 Q_{1,2} + C_3 Q_{3,2} + C_4 Q_{4,2}}{Q} \quad (35)$$

NOTES

In the case of a more complex manhole geometry, the appropriate volumes should be adjusted depending on how the jet core and geometry interact. In the case of benching, it may be suitable to treat any volume below the level of the benching as belonging to the jet core. With a single inlet, if the outlet is of a different diameter than the inlet, the model does not change in steady-state conditions. However, if the model is applied to unsteady flow conditions, then at each time-step as the surcharge level and outflow vary, the geometry and flow between zones should be recalculated taking into consideration that if outflow exceeds $Q_{1,2}$ + $Q_{3,2}$ that the remaining outflow should come from $Q_{4,2}$, i.e., Eq. 21 should be applied.

NOTATION LIST

The following symbols are used in this paper:

B = manhole outlet step size (m);

C = concentration;

C_i = concentration in zone i;

D_m = manhole diameter (m);

D_p = inlet/outlet pipe diameter (m);

L_j = the length of the jet core (m);

m = rate of jet diffusion zone expansion (m/m);

Q = manhole inflow or outflow rate, discharge (m^3/s);

Q_i = flow within zone i (m^3/s);

$Q_{i,j}$ = flow rate from zone i to zone j (m^3/s);

S = surcharge depth above the inlet soffit (m);

t = time (s);

V_0 = upstream inlet zone (m^3);

V_1 = jet core zone (m^3);

V_{1b} = volume of jet core cone extending beyond manhole outlet (m^3);

V_2 = outlet zone (m^3);

V_3 = jet diffusion zone (m^3);

V_{3b} = volume of jet diffusion zone cone extending beyond manhole inlet (m^3);

V_{3c} = volume of jet diffusion zone extending beyond manhole floor (m^3);

V_{3d} = volume of jet diffusion zone extending beyond water surface (m^3);

V_4 = outer mixing zone (m^3);

V_5 = high-surcharge storage zone (m^3);

X = a distance into the manhole from the inlet (m);

Y_1 = distance to edge of jet core from manhole center in horizontal plane (m);

Y_3 = distance to edge of jet diffusion zone from manhole center in horizontal plane (m);

Y_o = distance to edge of outlet from manhole center in horizontal plane (m);

α_2 = the rate of jet core dissipation (m/m); and

θ = manhole outlet angle.

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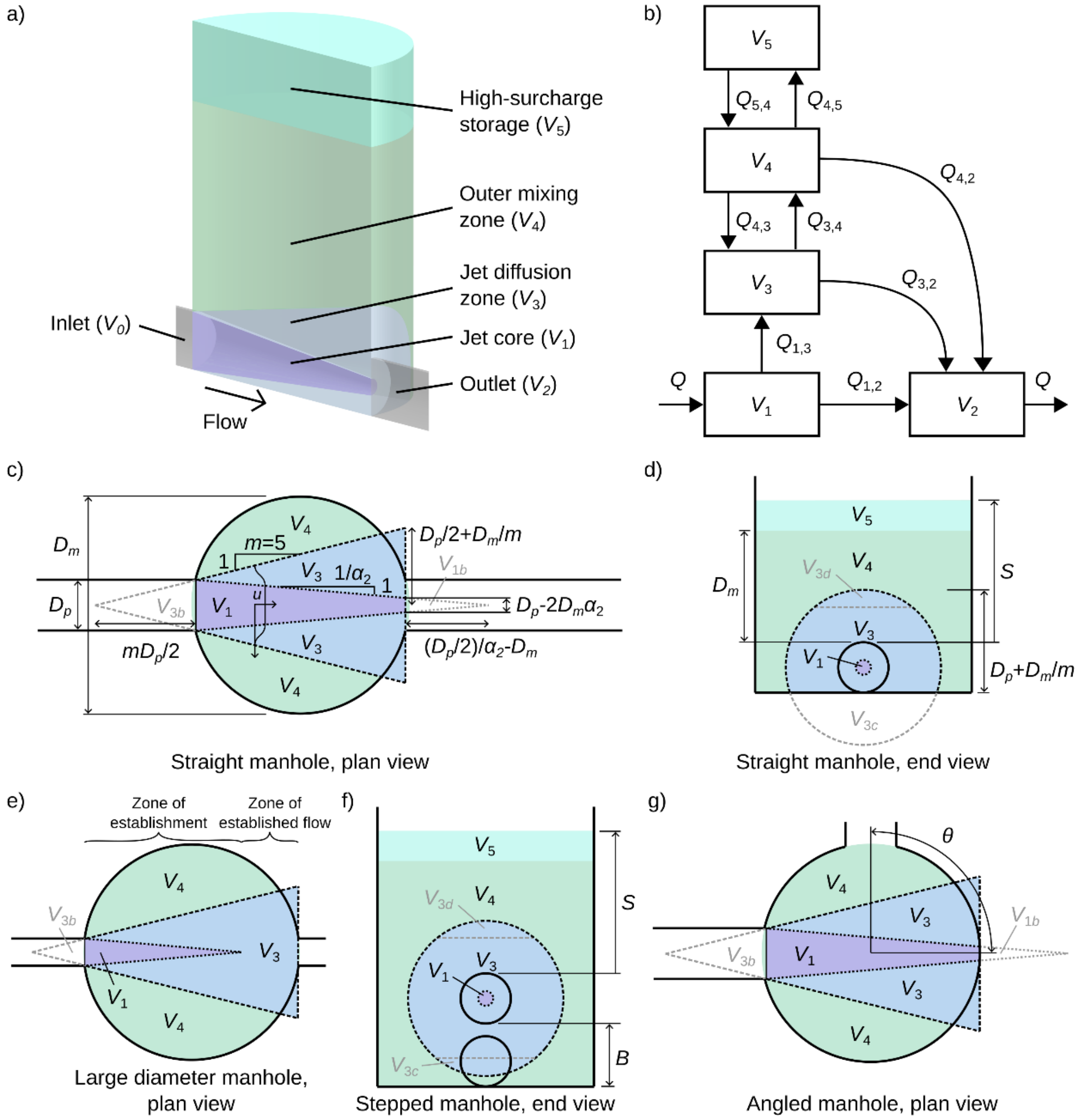

Fig. 1. a) Cross-section indicating main model zones in a simple circular manhole, b) flow relationships between model zones, c) plan view (also illustrating the jet velocity profile) and d) end view of a manhole with $D_m/D_p < L_j$, e) plan view of a large manhole with $D_m/D_p > L_j$, f) end view of a stepped manhole, and g) plan view of an angled manhole, the gray areas indicate subtracted volumes, diagrams not to scale