

# A METHOD FOR COMPUTATION OF REALIZABLE OPTIMAL FEEDBACK FOR SEMI-ACTIVE CONTROLLED STRUCTURES

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**ABSTRACT.** Semi-active control systems provide an attractive solution for the structural control problem. As known, dividing the controller into two components - a damper controller and a system controller simplifies the control law design. The damper controller generates control forces that track some control signal generated by the system controller. A common approach in damper controllers is to use signal clipping whenever the signals generated by the system controller are not realizable due to the physical control limitations. The present study suggests an algorithm for computation of a realizable optimal control feedback, for vibrating structure. The computed control forces are suitable for realization by semi-active dampers controllers without the need for signal clipping. The effectiveness of the suggested method is demonstrated by numerical example.

**KEYWORDS:** optimal control, feedback, semi-active structural control, Krotov's method

## 1 INTRODUCTION

Semi-active control is well known strategy [1] which use inherently dissipative controlled actuators for improving stability of dynamical systems. While semi-active systems are restricted to systematic mechanical energy dissipation [2], semi-active actuators have a more severe restriction. These devices are characterized by local energy dissipation, in other words, the work done by the actuator can only be negative.

A intuitive method for semi-active control law is the clipped optimal control, according to which, the controller design is usually done by some known optimal method and by ignoring the actuator constraints. Later, when the control forces are applied, they are arbitrarily clipped whenever the constraints are violated. The approach was used for semi-active control by many researchers [3, 4, 5, 6, 7, 8]. However, although clipped optimal control has the benefit of being simple, it distorts the control signal and therefore raises a question on the theoretical justification of the clipped signal.

There is a lack of optimization techniques with firm theoretical background that can be used for optimal semi-active control design. This study suggests a method for solving optimal semi-active control problems with suitable theoretical foundation. Bilinear model with a single, constrained control input,

is used for the semi active controlled plant. A quadratic performance index is defined for control evaluation. An algorithm is suggested for the optimal control law computation and proof outlines for the algorithm convergence and the optimality of the solution, is given. Efficiency of the suggested method is demonstrated by numerical example.

## 2 BILINEAR MODELS FOR SEMI-ACTIVE STRUCTURAL CONTROL

Bilinear models are simple and at the same time effective nonlinear dynamic models that appear in many practical modern control problems [9] and in some linearizations of nonlinear systems at a common fixed point [10]. Despite their nonlinearity, they are very “close” to linear systems and therefore they can be treated by several techniques and procedures from linear systems theory [11]. This type of models can be used for semi-active controlled plants, as follow.

The state space model for a free vibrating linear structure, equipped with a set of actuators is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t); \quad \mathbf{x}(0); \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} \quad (1a)$$

$$\mathbf{A} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}_d \end{bmatrix} \in \mathbb{R}^{n \times n}; \quad \mathbf{B} = [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_{n_w}] \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{\Phi} \end{bmatrix} \in \mathbb{R}^{n \times n_w} \quad (1b)$$

where  $\mathbf{M} > 0$ ,  $\mathbf{C}_d \geq 0$  and  $\mathbf{K} > 0$  are  $l \times l$  mass, damping and stiffness matrices, respectively;  $l$  is number of dynamic degrees of freedom (DOF);  $\mathbf{z}$  is a vector function, which represents the displacements in the dynamic degrees of freedom;  $\mathbf{w}$  is a vector function of the control forces;  $n_w$  is the number of control forces and  $\mathbf{\Phi}$  is an  $l \times n_w$  input matrix that describes how the force inputs affects the structure's DOFs.

For several types of semi-active actuators, such as magneto-rheological dampers (MRD), pneumatic actuators [12], electro-magnetic friction dampers [13], semi-active viscous dampers [14], etc. the generated force is always opposed to some linear combination of the plant states [15].

For example, the variable viscous damper [14] is a type of semi-active actuator, in which the damping gain is controlled by real-time adjustment of the mounting angle between two passive viscous dampers. The produced force is governed by

$$w(t) = -u(t)\dot{z}_d(t)$$

where the viscous damping gain  $u$  is a nonnegative, continuous function and  $\dot{z}_d$  is the damper velocity. Since  $\dot{z}_d$  can be written as a linear combination of the states variables, the actuator control force can be written in the bilinear form -  $w(t) = -u(t)\mathbf{c}\mathbf{x}(t)$ , where  $\mathbf{c}$  is a row vector.

Another example for bilinear force representation is MRDs. According Bingham's constitutive model, the MRD force is the sum of two components [16] - viscous damping force and a controllable

yield force,  $w^y$ . While the damping coefficient can be designed by linear viscous design methods [17], a control law should be found for  $w^y$ . Since  $w^y$  is always opposed to the damper velocity, it can also be represented in the bilinear form  $w^y(t) = -u(t)\mathbf{c}\mathbf{x}(t)$ , only that here  $u$  is a nonnegative, unnecessarily continuous, function. From physical viewpoint, that representation can be interpreted as an equivalent variable viscous damper which is equivalent to the MRD. Hence, the control forces of variable viscous dampers and MRD control force can be represented as a bilinear feedback:

$$w(u(t), \mathbf{x}(t)) = -u(t)\mathbf{c}\mathbf{x}(t) \quad (2)$$

By virtue of the above, and assuming that each actuator is controlled by a single control signal, i.e.  $n_u = n_w$ , the state-space equation of a plant configured with  $n_w$  semi-active actuators, can be written as:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{n_w} \mathbf{b}_i w_i(u_i(t), \mathbf{c}_i \mathbf{x}(t)) = \mathbf{A}\mathbf{x}(t) - \sum_{i=1}^{n_u} u_i(t) \mathbf{b}_i \mathbf{c}_i \mathbf{x}(t) \quad (3)$$

where  $\{\mathbf{c}_i\}$  are defined according to the setup of the semi-active actuator and  $\{u_i\}$  is a set of the corresponding control input signals. That allows for a semi-active control problem be defined as a constrained bilinear quadratic regulator problem, defined hereinafter.

### 3 OPTIMAL CONTROL PROBLEM AND SOLUTION - MAIN RESULTS

The constrained bilinear quadratic regulator (CBQR) control problem is the look for an optimal and admissible process, denoted by  $(\mathbf{x}^*, u^*)$ , that satisfies the bilinear states equation:

$$\dot{\mathbf{x}}(t) = [\mathbf{A} - u(t)\mathbf{b}\mathbf{c}]\mathbf{x}(t); \quad \mathbf{x}(0) \quad (4)$$

and minimizes the performance index:

$$J(\mathbf{x}, u) = \frac{1}{2} \int_0^{t_f} \mathbf{x}(t)^T \mathbf{Q}\mathbf{x}(t) + ru(t)^2 dt$$

while obeying the constraint  $u(t) \geq 0$  for all  $t \in [0, t_f]$ . Here  $\mathbf{A}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are defined in Eq. (1b);  $\mathbf{Q} \geq 0$  and  $r > 0$ .

A common approach for solving many optimal control problems is to use first order variational calculus or Pontryagin's minimum principle for deriving a suitable solution methods. However, for many problems these theorems provide merely necessary conditions [18] and as a result, the calculated solution is at-most a candidate local optimum. Starting in the sixties, new results on sufficient conditions for global optimum of optimal control problems were published by V. F. Krotov [19]. These results were successfully used for solving problems in quantum-mechanics [9] as well as oscillation damping

of a simple beam [20]. This approach requires the formulation of a solving function, which is related to the addressed optimal control problem. This function is usually denoted as 'Krotov function'. If such function can be found, it allows for a global optimum be computed by an algorithm which is known as "global method of successive improvements of control" or "Krotov's method" [9].

Some definitions should be made before Krotov's method can be used for solving the CBQR problem. Let the term 'admissible process' refer to an  $\mathbf{x}$  and  $u$  that satisfy Eq. (4) as well as  $u(t) \geq 0$  for all  $0 \leq t \leq t_f$ . It will be denoted by  $(\mathbf{x}, u)$ . A sequence of processes  $\{(\mathbf{x}_k, u_k)\}$  is called optimizing sequence if  $J(\mathbf{x}_k, u_k) \geq J(\mathbf{x}_{k+1}, u_{k+1})$  is satisfied. Let:

$$s(\mathbf{x}(t), u(t), t) = q_t(\mathbf{x}(t), t) + q_x(\mathbf{x}(t), t)[\mathbf{A} - u(t)\mathbf{bc}]\mathbf{x}(t) + \mathbf{x}(t)^T \mathbf{Q}\mathbf{x}(t) + ru(t)^2 \quad (5)$$

$$s_f(\mathbf{x}_f, t_f) = -q(\mathbf{x}_f, t_f) \quad (6)$$

where  $q$  is a Krotov function. Krotov showed that if one can find  $q$  such that for a given process  $(\mathbf{x}_k, u_k)$ :

$$s(\mathbf{x}_k(t), u_k(t), t) = \max_{\mathbf{x}(t)} s(\mathbf{x}(t), u_k(t), t); \quad s_f(\mathbf{x}_{kf}, t_f) = \max_{\mathbf{x}_f} s(\mathbf{x}_f, t_f) \quad (7)$$

is satisfied for all  $0 \leq t \leq t_f$ , than an improved process,  $(\mathbf{x}_{k+1}, u_{k+1})$ , can be computed by carrying out the steps described in [19].

In order to use this approach for the CBQR problem, consider a known admissible process  $(\mathbf{x}_k, u_k)$  and a corresponding terminal value, differential Lyapunov equation:

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)[\mathbf{A} - u_k(t)\mathbf{bc}] - [\mathbf{A}^T - u_k(t)\mathbf{c}^T \mathbf{b}^T]\mathbf{P}(t) - \mathbf{Q}; \quad \mathbf{P}(t_f) = \mathbf{0} \quad (8)$$

It can be proved that a Krotov function defined by:

$$q(\mathbf{x}(t), t) = 0.5\mathbf{x}(t)^T \mathbf{P}(t)\mathbf{x}(t)$$

satisfies Eq. (7) and therefore allows the propagation to an improved process-  $(\mathbf{x}_{k+1}, u_{k+1})$ . The algorithm produced by these results is summarized in Fig. (1). Its output is an arbitrary approximation for the optimal  $\mathbf{P}$ , which defines the optimal control law, which is given by:

$$\hat{u}(\mathbf{x}(t), t) = \max \left[ \frac{\mathbf{x}^T(t)\mathbf{P}(t)\mathbf{bcx}(t)}{r}, 0 \right]$$

The difference in notation between  $u$  and  $\hat{u}$  is meant to distinguish between control signal ( $u$ ) and control law ( $\hat{u}$ ).

Since  $J$  has an infimum, the convergence of  $\{(\mathbf{x}_k, u_k)\}$  to the optimal process, is guaranteed. It should be noted that the use of absolute value in step (4) of the iterations stage is theoretically unnecessary. Though, due to numerical computation errors, the algorithm might lose its monotonicity when it gets closer to the optimum.

#### 4 NUMERICAL EXAMPLE

To demonstrate the efficiency of the CBQR control design dynamic model of a 3 floor free vibrating reinforced concrete structure was analyzed. The control force is applied through a single variable viscous damper, which is installed at the first floor. The model is represented as a plane frame and its dynamic scheme is presented in Fig. (2). All computations were carried out using original routines written in MATLAB framework.

The mass, damping and stiffness matrices as well as the control force distribution vector are:

$$\mathbf{M} = \begin{bmatrix} 25.2 & 0.246 & -0.0591 \\ 0.246 & 25.2 & 0.168 \\ -0.0591 & 0.168 & 25.5 \end{bmatrix} \times 10^3; \quad \mathbf{C}_d = \begin{bmatrix} 160 & -68.6 & 6.75 \\ -68.6 & 152 & -60.3 \\ 6.75 & -60.3 & 82.7 \end{bmatrix} \times 10^3$$

$$\mathbf{K} = \begin{bmatrix} 80.2 & -42.1 & 4.17 \\ -42.1 & 75.5 & -37 \\ 4.17 & -37 & 32.6 \end{bmatrix} \times 10^6; \quad \boldsymbol{\phi} = \begin{bmatrix} 0.982 \\ 0.0092 \\ -0.000875 \end{bmatrix}$$

$\mathbf{K}$ ,  $\mathbf{M}$  and  $\boldsymbol{\phi}$  were computed by suitable Ritz transformations.  $\mathbf{C}_d$  is a Rayleigh proportional damping matrix. The row vector that project the states on the damper velocity is  $\mathbf{c} = [\mathbf{0} \quad \boldsymbol{\phi}^T]^T$ . The states and control weighting matrices are:

$$\mathbf{Q} = \begin{bmatrix} 6 & -3 & 0 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^{15}; \quad r = 1 \quad (9)$$

The initial condition was chosen to be-  $\mathbf{x}(0) = [0 \quad 0 \quad 0 \quad 0.4 \quad 0.4 \quad 0.4]^T$ .

The following cases were considered:

**Case A:** Uncontrolled structure.

**Case B:** LQR controlled structure.

**Case C:** BQR controlled structure.

The LQR control weights were chosen such that the mechanical energy dissipated in each actuator, for cases B and C, would be equal. 10 iterations were carried out. The performance index improvement for each iteration is illustrated in Fig. (3). As it is evident from the figure, convergence was achieved after two iterations and the change in  $J$  is monotonic.

The control signal is given in Fig. (4). Its continuous form should be noted. For many semi-active control design approaches, e.g. bang-bang control [11] or clipped optimal control [4],  $u$  is not continuous and rapidly varying. From practical viewpoint it is an advantage of the CBQR as rapid changes might be unwanted and sometimes impossible for control realization.

Fig. (5) illustrates the correspondence between the actuator control force and velocity for case B. Fig. (6) does the same for case C. It is observed that for case B there are intervals in which the computed control force is required to have the same sign as the actuator velocity. This is physically impossible for many semi-active actuators. On the other hand, the control force for case C is always opposed to the actuator velocity. The roof displacements for cases A to C are presented in Fig. (7).

## 5 CONCLUSIONS

In this study a semi-active control design problem was formulated as a single input CBQR, i.e. an optimal control problem for a free vibrating bilinear system with a constrained single control signal and a quadratic performance index. The constraint on the control signal was defined such that the computed signal would satisfy the physical constraints of the semi-active actuator without the need of any arbitrary signal clipping.

The problem was solved by Krotov's method, which is also known as "global method of successive improvements of control". Since the use of Krotov's method requires the formulation of Krotov function that corresponds to the properties of the addressed optimal control problem, a Krotov function that suites the CBQR problem had to be found.

After the suitable function was derived, the solution was organized as an algorithm, which requires the solving of the states equation and a differential Lyapunov equation at each iteration. A proof outline for the algorithm convergence was provided.

The proposed method is suitable for semi-active control design of stable, free vibrating plants with a single control input. Its efficiency is demonstrated in a numerical example. It was demonstrated that the CBQR control signal is continuous, which is an advantage since from practical viewpoint, rapid changes might be unwanted (and sometimes impossible) for control realization. The CBQR control force is also always opposed to the actuator velocity and therefore is suitable for realization of semi-active control systems, without the need of any signal clipping. The improvement in the controlled plant response is evident from the simulations results.

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**Input:**  $\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{Q}, r, \mathbf{x}(0)$ .

**Initialization:**

- (1) Select a convergence tolerance -  $\epsilon > 0$ .
- (2) Set  $u_0 = 0$  and solve:

$$\begin{aligned}\dot{\mathbf{x}}_0(t) &= \mathbf{A}\mathbf{x}_0(t); & \mathbf{x}(0) \\ \dot{\mathbf{P}}_0(t) &= -\mathbf{P}_0(t)\mathbf{A} - \mathbf{A}^T\mathbf{P}_0(t) - \mathbf{Q}; & \mathbf{P}_0(t_f) = \mathbf{0}\end{aligned}$$

- (3) Compute:

$$J_0(\mathbf{x}_0, u_0) = \frac{1}{2} \int_0^{t_f} \mathbf{x}_0(t)^T \mathbf{Q} \mathbf{x}_0(t) dt$$

**Iterations:** For  $k = \{0, 1, 2, 3, 4, \dots\}$ :

- (1) Propagate to the improved process by solving:

$$\dot{\mathbf{x}}_{k+1}(t) = [\mathbf{A} - \hat{u}_{k+1}(t, \mathbf{x}_{k+1}(t))\mathbf{b}\mathbf{c}]\mathbf{x}_{k+1}(t); \quad \mathbf{x}_{k+1}(0) = \mathbf{x}(0)$$

where

$$\hat{u}_{k+1}(t, \mathbf{x}_{k+1}(t)) = \max \left[ \frac{\mathbf{x}_{k+1}^T(t)\mathbf{P}_k(t)\mathbf{b}\mathbf{c}\mathbf{x}_{k+1}(t)}{r}, 0 \right]$$

and set  $u_{k+1}(t) = \hat{u}_{k+1}(t, \mathbf{x}_{k+1}(t))$ .

- (2) Solve:

$$\begin{aligned}\dot{\mathbf{P}}_{k+1}(t) &= -\mathbf{P}_{k+1}(t)[\mathbf{A} - u_{k+1}(t)\mathbf{b}\mathbf{c}] - [\mathbf{A}^T - u_{k+1}(t)\mathbf{c}^T\mathbf{b}^T]\mathbf{P}_{k+1}(t) - \mathbf{Q} \\ \mathbf{P}_{k+1}(t_f) &= \mathbf{0}\end{aligned}$$

- (3) Compute:

$$J(\mathbf{x}_{k+1}, u_{k+1}) = \frac{1}{2} \int_0^{t_f} \mathbf{x}_{k+1}(t)^T \mathbf{Q} \mathbf{x}_{k+1}(t) + r u_{k+1}^2(t) dt$$

- (4) If  $|J(\mathbf{x}_k, u_k) - J(\mathbf{x}_{k+1}, u_{k+1})| < \epsilon$ , stop iterating, otherwise - continue.

**Output:**  $\mathbf{P}_{k+1}$ .

Figure 1: CBQR - Successive improvement of control process.

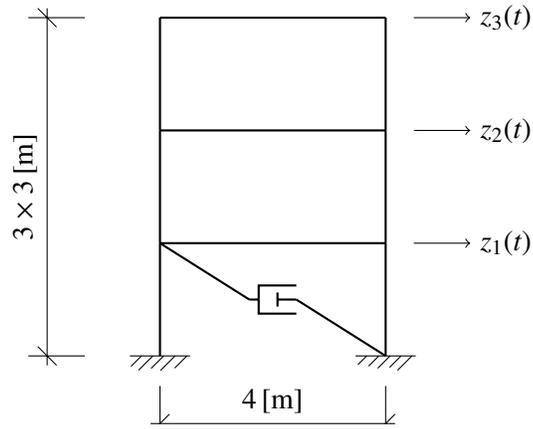


Figure 2: Dynamic scheme of the evaluated model.

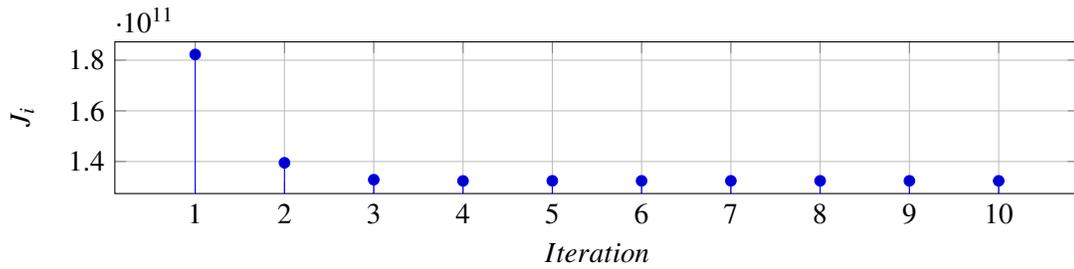


Figure 3: Performance index values for each iteration.

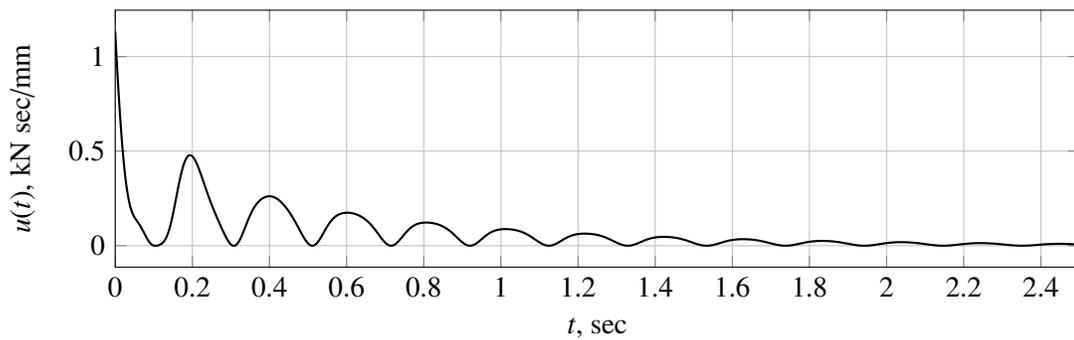


Figure 4: CBQR control signal.

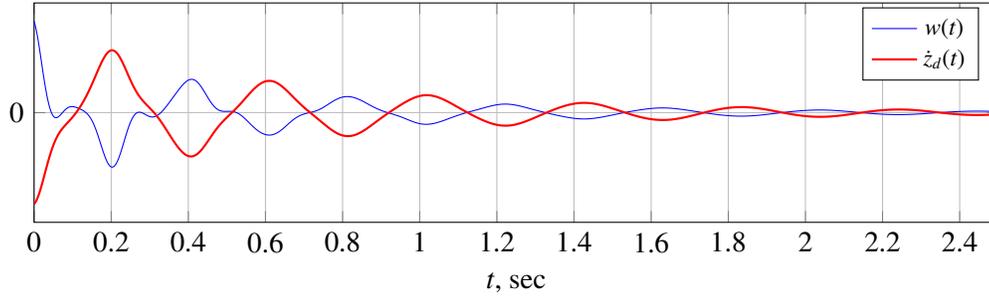


Figure 5: Case B - Normalized control force vs. actuator velocity .

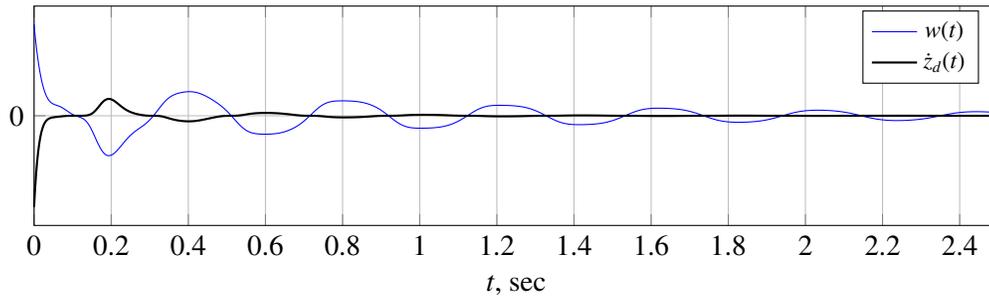


Figure 6: Case C - Normalized control force vs. actuator velocity.

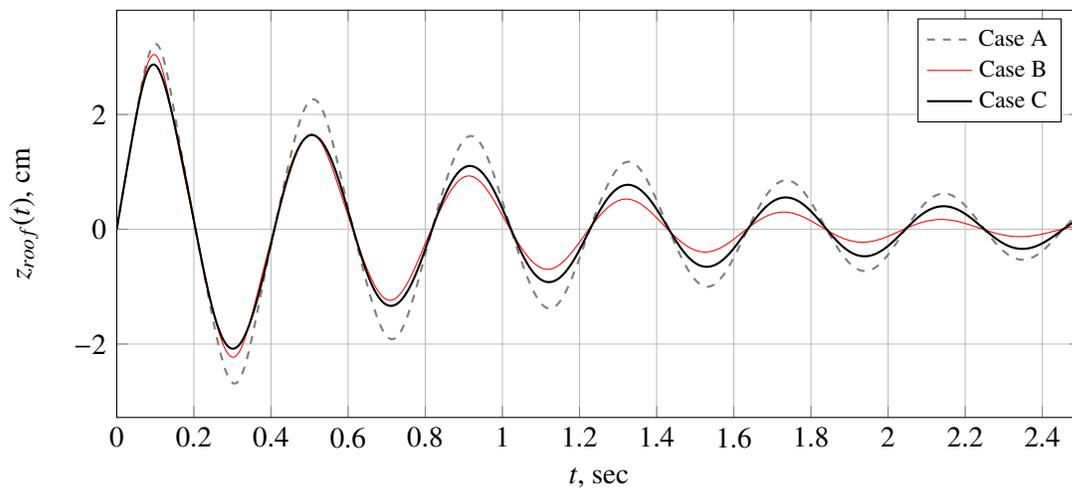


Figure 7: Roof displacements.