

ON CORRELATION AND CAUSALITY IN STRUCTURAL DYNAMICS

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ABSTRACT. There has long been interest in the concept of causality in systems theory and structural dynamics. In the context of nonlinear system identification, the Hilbert transform method of detecting nonlinear behaviour depends crucially on the concept of causality. One might argue that there are actually two interrelated definitions, or principles, of causality current and important in structural dynamics, and more widely in general time series analysis. The first ‘definition’ of causality, one might call the responsibility definition, is based on the idea that an event B (the effect) only occurs if an event A (the cause) occurs. The second definition – the priority – definition, asserts that cause must precede effect (in time) in any classical dynamic situation. The responsibility concept of causality is already embedded in structural dynamics, as it largely maps onto the idea of correlation; the coherence function associated with a Frequency Response Function (FRF) is essentially a representation of the idea. The priority concept is the crucial ingredient in the Hilbert transform test mentioned earlier. The objective of the current paper is to discuss where causality can illuminate processes in linear and structural dynamics and to provide illustrations based on transmissibilities and FRFs for Multi-Degree-of-Freedom (MDOF) systems.

KEYWORDS: Causality; correlation; Hilbert transform.

1 INTRODUCTION

Causality has long been a subject of interest to philosophers and physicists; in more recent years, the issues it raises have also come to the attention of economists/econometricians and engineers.

In Physics, it is taken as a fundamental principle of nature that *effect* cannot precede *cause*. The development of Special Relativity appeared to introduce a problem in the sense that the order (in time) of two events *A* and *B* can depend on the state of motion of the observer [1]. This issue is resolved by the fact that such pairs of events only occur if they would need to exchange information at speeds greater than the speed of light. Another interesting problem is raised by the Abraham-Lorenz-Dirac equation for an electron in an electromagnetic field; the equation supports solutions in which the electron can accelerate *before* an electromagnetic wave is incident upon it [2]. The latter issue is a mathematical problem i.e. concerned with the existence of pathological solutions to equations; however, it does raise questions as to the physics underlying its derivation. These matters are concerned with the aforementioned *priority* issue of causality i.e. effect cannot precede cause¹.

¹While the issues of causality mentioned are undeniably interesting in terms of dynamics, they are not usually a cause for

Economics and Engineering are arguably much more concerned with causality in terms of *responsibility* i.e. the question of establishing if an event A truly *is* the cause of an event B . This might also be called the issue of *dependence* and is of great importance if one needs or wishes to build predictive models. The responsibility element of causation is clearly the primary one; if there is no responsibility, the priority question is meaningless. The problem of establishing causality in the responsibility sense is a central problem in Econometrics. Economies or financial processes, in general, are extremely complex nonlinear and nonstationary dynamical systems [3], and their internal dynamics is not at all well understood. While it is clearly desirable to have predictive models for economic time series (share prices, exchange rates, interest rates etc.), first-principle models are usually impossible to construct except in very restricted cases. In the absence of first-principles models, the obvious approach is to apply machine learning or system identification to observed data and thus establish *data-driven* models. The basic problem is to infer a mathematical relationship $y(t) = S[x(t)]$ between some input variable $x(t)$ (cause) and some output variable $y(t)$ (effect). The benefits of establishing such a relationship are clear, given observations of $x(t)$ and some idea of how it will evolve in the near future, one can predict $y(t)$. If $x(t)$ is a variable which can be influenced by policy, the possibility of controlling $y(t)$ arises.

One of the major problems in econometrics is actually establishing that two variables $x(t)$ and $y(t)$ are, in fact, causally related. It is usually much more straightforward to show *correlation* between variables; however, it is well-known (and is actually the subject of humour - see Figure 1) that correlation *does not imply* causation.

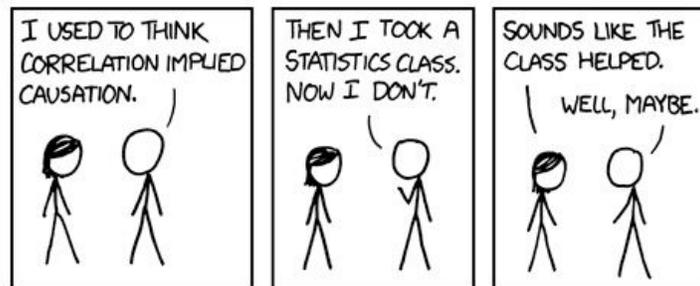


Figure 1: <https://xkcd.com/552/>.

One of the main issues in establishing responsibility between variables is the phenomenon of *spurious correlation* or *spurious regression*. Suppose two unrelated economic variables are dominated by monotonic trends over the length of the time series recorded. If these variables are graphed together, a spurious regression relationship can appear. Consider the following example: Figure 2 shows a graph of the UK unemployment figures from February 2013 to March 2016² plotted against the age of the first

concern in *engineering dynamics*, as they only arise in the relativistic limit (i.e. at velocities close to the speed of light), and most of engineering is firmly Newtonian.

²Taken from <https://www.ons.gov.uk/employmentandlabourmarket/peoplenotinwork/unemployment/timeseries/mgsx>.

author's cat (Alice) at the time³. The two variables appear to be co-varying because of the monotonic trends underlying their behaviour; however, it would clearly be absurd to imply a causal relationship.

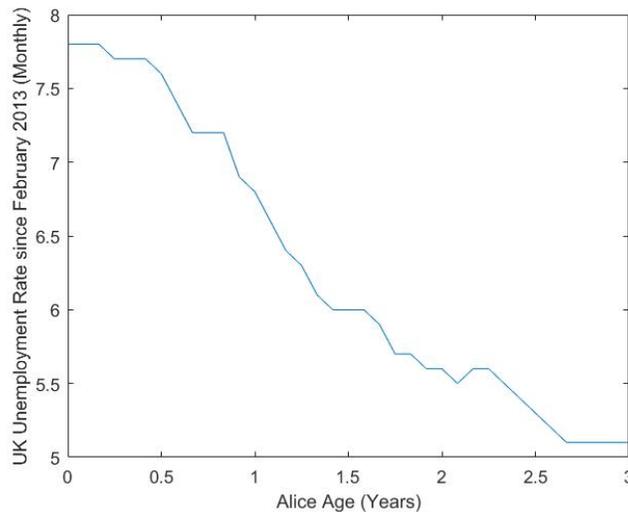


Figure 2: UK unemployment against Alice's age.

Unfortunately, economic problems are much more subtle than this. An interesting example is provided by the *Philips curve*. Philips observed from data that wages increased at a slow rate when unemployment was high and at a faster rate when unemployment was low [4]. On the basis of his observation, he proposed that the variables were (inversely) correlated as illustrated in Figure 3.

The curve began to attract criticism in the 1970's when it became clear that the reality of inflation mechanisms was much more complicated, and unemployment cannot be regarded as the only responsible latent variable; in fact, many argue that it is not the dominant variable [5].

The important point is this: predictive models need correlation or they are empty; they cannot be learned from data. However, they also need *causality* (in the responsibility sense) or they can fall victim to spurious regression. Clearly, continuing to predict unemployment figures from the age of a cat is a recipe for disaster. Something extra is needed to establish causation.

These issues also matter to engineers trying to establish data-based models or to perform system identification (SI). One might argue that there is less of an issue for SI as one tends to know the inputs and outputs from the system of interest; however there are other data-based problems where there are concerns. Causality is interesting in the context of some Structural Health Monitoring (SHM) problems. One of the main issues in data-based SHM, particularly for civil infrastructure, concerns *confounding influences*. Often, when one cannot obtain data from a damaged condition of a structure, one is forced to rely on *novelty detection* i.e. one fixes a data-based model of the undamaged condition and then

³Clearly using Alice's age is simply a rescaling of the time coordinate; however, this serves to make the point.

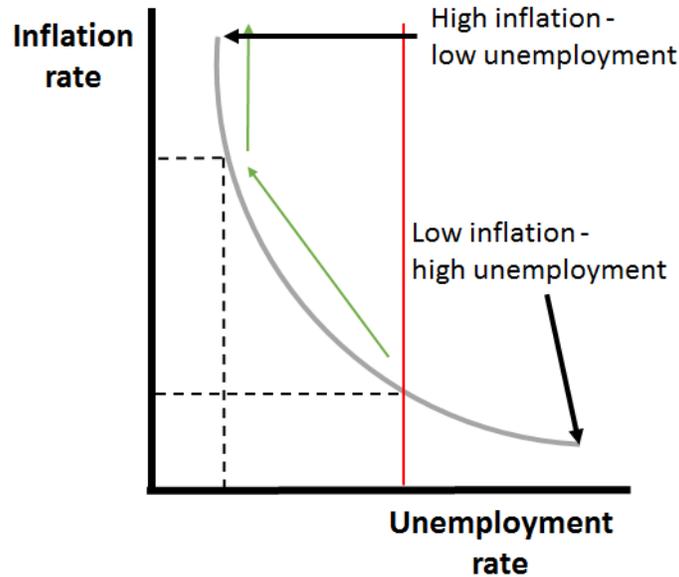


Figure 3: The Phillips curve.

monitors subsequent data to see if it remains consistent with the undamaged model. Any deviations or novelty could then indicate damage. Problems occur if the structure changes its state due to benign environmental variations (e.g. temperature) not accounted for previously; such a change could be falsely interpreted as damage. One technique for removing confounding influences is to learn how they effect the diagnostic data and remove them; however, it is not always straightforward to establish what the causes of benign variations are, a careful study of this issue in the context of bridge SHM is provided in [6].

Although many methods have been proposed in the econometrics literature for detecting causality, none can be considered a panacea. One of the earliest and most persistently used approaches is Granger causality [7] and its variants [8, 9]. Although initially proposed for linear models, Granger causality has also been extended to nonlinear systems [10]. (It should be noted that many of the developments in causality measures at the moment are coming from the neuroscience community; [9, 10] are examples.)

There does not appear to have been a concerted research effort in the area of causality from the structural dynamics community, although the commonly-used coherence function and its more sophisticated variants [11, 12] can be considered as causality measures. The current paper represents the start of a programme of research by the authors; as such it has fairly modest aims. The objective of the paper is to establish some terminology and state the basic problems to be addressed. It will discuss the Hilbert transform (HT), as a method of extracting causality information from time series and use the HT to show how to construct pairs of time series with prescribed causality relationships which can be then be used to benchmark and validate other methods.

The layout of the paper is as follows: the next section discusses the problem of causality, particularly as it relates to time series analysis, and introduces a number of definitions and defines notation. The section also explains how the Hilbert transform can be used as a causality test. In Section 3, a series of benchmark data sets is described and the use of the Hilbert transform is illustrated. The paper ends with some brief conclusions.

2 CORRELATION AND CAUSALITY

2.1 Basic Definitions

It will be useful to establish some definitions, terminology and notation here.

Two events A and B will be said to be *correlated*⁴ if $P(B|A) > 0$. This will be denoted $A \sim B$.

If A and B both have finite probability of occurring, then Bayes' Theorem shows trivially that $A \sim B$ implies $B \sim A$; in fact, a little more effort shows that \sim is an equivalence relation.

A is said to be a *cause* or *weak cause*⁵ of B iff $P(B|\bar{A}) = 0$ and $P(B|A) > 0$. This will be denoted by $A \xrightarrow{c} B$.

A is said to be a *strong cause*⁶ of B iff $P(B|\bar{A}) = 0$ and $P(B|A) = 1$. This will be denoted by $A \xrightarrow{c} B$.

It follows trivially from these definitions that, if A and B are causally related, they are correlated; the converse does not follow.

2.2 Time Series

The extension of the basic definitions to time series immediately present problems; however, the notation will prove useful. The 'events' of interest here are pairs of discretely observed/sampled time series $X = \{x_1, \dots, x_N\}$ and $Y = \{y_1, \dots, y_N\}$. Framing the problem in terms of probabilities immediately presents issues, so for now, this approach will be placed on hold. It will be assumed that a causal relation between $x(t)$ and $y(t)$ means that $x(t)$ can be considered as an input to some system S , for which $y(t)$ is a corresponding output i.e. $y(t) = S[x(t)]$ for some functional S . For the remainder of the paper, it will be assumed that S is a linear functional.

In a system identification problem, the engineer usually knows what the 'system' is, and may well have some *a priori* knowledge that will help to form a model. In contrast, the econometrician will not usually know what (if any) 'system' relates two time series, so options are more limited. One thing which can be established straightforwardly though, is if the two time series are correlated. For precision, $x(t)$

⁴This is a very weak definition of correlation; in fact if $P(B) \neq 0$, it amounts to saying that $P(A, B) > 0$ which asserts little more than the fact that A and B can co-occur. However, this weak definition will be enough to establish that correlation is an equivalence relation. A more meaningful definition of correlation may be to say that A and B are correlated if they are *not* independent i.e. $P(A, B) \neq P(A)P(B)$ or $P(B|A) \neq P(B)$.

⁵This is accounting for *inefficiency* i.e. A may not always cause B .

⁶For the moment, this is not accounting for *counteractions* i.e. the event A may be followed by an event C that prevents the occurrence of B .

and $y(t)$ will be said to be correlated ($x(t) \sim y(t)$) iff $\phi_{xy}(\tau) = E[x(t)y(t + \tau)] \neq 0$. (By 'non-zero', the definition means 'non-zero with statistical confidence'.)

Assuming that $x(t)$ has finite power, this definition amounts to saying that $Y(\omega) = H(\omega)X(\omega)$, where $H(\omega) = S_{xy}(\omega)/S_{xx}(\omega)$ and $X(\omega)$ is the Fourier transform of $x(t)$ etc; $S_{xy}(\omega)$ is the cross-power spectrum between x and y , $S_{xx}(\omega)$ is the auto-power spectrum of x and $H(\omega)$ is the *Frequency Response Function* (FRF) of the process $x \rightarrow y$. The definition does not in itself imply any *causal* relation between x and y .

Let z , x and y be some functions of time such that $x(t) = S_1[z(t)]$ and $y(t) = S_2[z(t)]$ and let the FRFs of the corresponding processes be $H_1(\omega)$ and $H_2(\omega)$. It is immediately clear that $x(t)$ and $y(t)$ need not have any causal relationship at all; however $Y(\omega) = H_2(\omega)H_1(\omega)^{-1}X(\omega)$, so that $y(t) \sim x(t)$. This argument has basically established transitivity for the relation \sim , and a little more thought shows that \sim is again an equivalence relation, with the definition given here.

The definitions so far are extending the scope of the discrete event definitions in the last subsection, but they are not helping with the issue of causality; however, the notation can still be used, so that $x(t) \stackrel{C}{\Rightarrow} y(t)$ will mean that the time series $x(t)$ is the cause of the time series $y(t)$. Having decided above that $Y(t) = S[x(t)]$, for some system S means that $x(t) \stackrel{C}{\Rightarrow} y(t)$, what has been established is that $x(t) \stackrel{C}{\Rightarrow} y(t)$ implies $x(t) \sim y(t)$, but not the converse.

To make a little more progress, one can make use of the *priority* aspect of causality. Using this idea, one can at least *rule out* the possibility of a causal relationship. One observes that, for a causal relationship $x(t) \rightarrow y(t)$, one cannot have a situation where $y(t)$ depends on values of x *in the future* of t . This would be visible in the cross-correlation function $\phi_{xy}(\tau)$, but the information is also encoded in $H(\omega)$.

If S is a linear functional (system), it follows from the Reisz representation theorem [13] that one has a representation,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \quad (1)$$

(which is immediately recognisable as Duhamel's integral from structural dynamics [11]; however, the context is clearly more general than that). In order that y cannot depend on values of x in its future, one must have that $h(\tau) = 0$ for all $\tau > 0$. This gives a recipe for checking aspects of causality.

Suppose one has measurements from two time series $x(t)$ and $y(t)$, and that one has additionally established that $x(t) \sim y(t)$. The question is whether in fact $x(t) \stackrel{C}{\Rightarrow} y(t)$ or $y(t) \stackrel{C}{\Rightarrow} x(t)$. It may not even be known if a candidate system S exists, but one can compute $H(\omega) = S_{xy}(\omega)/S_{xx}(\omega)$ and invert the Fourier transform to find $h(t)$. If one then sees that $h(t) \neq 0$ for $t < 0$, then the possibility that $x(t) \stackrel{C}{\Rightarrow} y(t)$ is eliminated. One can then carry out the check for $y(t) \stackrel{C}{\Rightarrow} x(t)$.

This may be familiar to readers with an interest in nonlinear structural dynamics. In fact, the fairly well-known Hilbert transform (HT) test for nonlinearity is actually a test for *causality*. The advantage of the HT is that, in the procedure above, one can test $H(\omega)$ directly for causality, without the inversion needed to form $h(t)$. Details are summarised in the next subsection.

2.3 The Hilbert Transform

Only a brief description of the relevant theory for the Hilbert transform (HT) will be given here, the curious reader can consult [11] for more details. In the form in which it operates on an FRF $H(\omega)$, the HT is defined by,

$$\mathcal{H}[H(\omega)] = \tilde{H}(\omega) = -\frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{H(\Omega)}{\Omega - \omega} d\Omega \quad (2)$$

The HT is a linear integral transform. It has a number of interesting properties; for the purposes of the current paper, the most important is summarised in the following. Suppose that the HT operates on a frequency domain function $G(\omega)$; it can be shown that, if $g(t)$, the inverse Fourier transform of $G(\omega)$, has the property $g(t) = 0$ for all $t < 0$, then $\mathcal{H}[H(\omega)] = H(\omega)$ [11]. This is the origin of the HT as a test for causality. The test is usually implemented visually, i.e. the FRF of interest $H(\omega)$ and its HT $\tilde{H}(\omega)$ are plotted on the same axes; if the two graphs overlay, then the relationship $x(t) \rightarrow y(t)$ is causal. (To see how this is usually used as a test for *nonlinearity*, the reader may consult [11].)

In the next section, the HT will be demonstrated as a test for causality on a number of benchmark time series; these time series will be generated as the input and outputs of a linear Multi-Degree-of-Freedom (MDOF) system. This means that one can create a number of different pairs of variables for causality testing. These pairs will fall into one of two classes: either the input force and a displacement response will be taken (in which the FRF is of the classic form), or two displacement responses will be taken (in which case the ‘FRF’ would usually be called a *transmissibility*). In order to deal with this flexibility, one needs to be aware of a subtlety of the HT test. The relationship described above between the HT of an FRF and causality of its inverse Fourier transform relies on the fact that the FRF $H(\omega)$ is *square-integrable*. In simple terms, square-integrability is assured if the FRF tends to zero as ω tends to infinity. As discussed in [11], problems only occur with classic force/response FRFs if the response taken is an acceleration; in this case the FRF may tend to a constant value as ω tends to infinity and the HT test for causality breaks down. Fortunately, a simple correction to the procedure suffices to restore order [11]. Unfortunately, the situation observed in the next section, when transmissibilities are used, can be more severe, in the sense that $H(\omega)$ can tend to *infinity* as ω tends to infinity. In this situation, the simple correction mentioned above fails. However, it will be possible to give some examples where causality is confirmed by using the HT.

There are various numerical methods of implementation of the HT; the one used in this paper is to directly carry out the integral in equation (2) using an algorithm equivalent to the Trapezium rule [11].

3 BENCHMARKS FOR CAUSALITY TESTS

The purpose of this section is to explain a series of benchmark data sets for causality testing and to illustrate how the Hilbert transform can be used to carry out tests. The idea is to generate a number of time series data sets where the causality relationships between pairs of variables are known, so that they can form a basis for algorithm development and validation.

The particular data set described here is composed of the input and outputs of a five-DOF mass-spring-damper system. The five masses (labelled m_1 to m_5) are connected in a chain, with no springs connecting masses that are not immediate neighbours; dampers are only connected to ground. Only the two end masses (m_1 and m_5) are connected by springs to ground. The displacement of mass m_i is denoted $y_i(t)$. A single force $x(t)$ is applied to mass m_2 . In the simulations presented here, all masses have been set equal to 1 kg, all dampers have constants 10 Ns/m and all springs have stiffness 10^4 N/m. The result of the simulation is six time series: the force $x(t)$ and the five displacements $y_i(t)$; the test for causality can be applied to any pair from this set.

In the simulation reported here, a fixed-step 4th-order Runge-Kutta scheme was used to integrate the equations of motion forward in time. $x(t)$ was a zero-mean unit-variance Gaussian white noise sequence. A sampling frequency of 10000 Hz was adopted and 3×10^6 points were sampled for each variable. In order to compute the FRFs for the Hilbert transform tests, a Welch approach was taken, averaging over non-overlapping segments of length 131072 points; a Hanning window was applied to each segment before the Fast Fourier Transform was applied.

Results for selected pairs of time series will now be presented. In terms of causality, the ultimate cause of the motion is the force $x(t)$; so one would expect that any test of a process $x(t) \rightarrow y_i(t)$ should show that it is causal, i.e. that $x(t) \stackrel{C}{\Rightarrow} y_i(t)$. To illustrate this, the pair $\{x(t), y_5(t)\}$ was chosen; this contains the response at the mass furthest from the forcing. The FRF for the process $x(t) \rightarrow y_5(t)$ is shown in Figure 4 together with its HT; as one would expect, the FRF and HT overlay almost perfectly indicating that the associated system/process is causal.

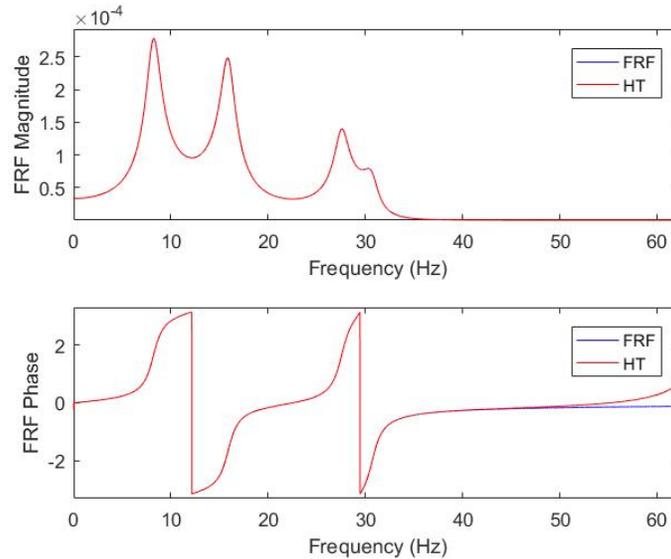


Figure 4: FRF and corresponding Hilbert transform for process $x(t) \rightarrow y_5(t)$.

The next process considered is $y_3(t) \rightarrow y_5(t)$. This is interesting because it is the first occurrence of a transmissibility. As mass m_3 is directly between the forced mass m_2 and the mass m_5 , all information from the force is channelled through it, and one would therefore expect that y_3 would be perceived as causing the vibrations as far as m_5 is concerned. Figure 5 shows the FRF (transmissibility) and corresponding HT for the process $y_3(t) \rightarrow y_5(t)$. The first observation is that the FRF and HT again overlay almost perfectly, indicating that $y_3(t) \stackrel{C}{\Rightarrow} y_5(t)$, as anticipated. The second observation is that the FRF appears to tend to zero at high frequencies, so no correction term was needed.

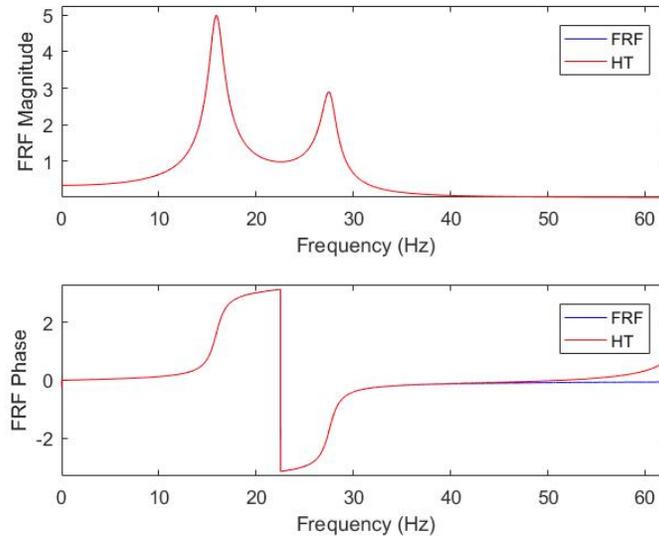


Figure 5: FRF and corresponding Hilbert transform for process $y_3(t) \rightarrow y_5(t)$.

The next illustration is for the process $y_1(t) \rightarrow y_3(t)$. Masses m_1 and m_3 are either side of the forced mass m_2 . This would imply that they ‘receive information’ from the forcing ‘at the same time’ and therefore that neither response is an entirely credible cause for the other. (The interesting point here is that the case is in a sense marginal). Figure 6 shows the FRF and HT for the process; the process fails the HT test. The issue is that the FRF shows ‘bad’ high-frequency behaviour; however, in this case it seems to be the mildest possible i.e. the FRF asymptotes to a constant. As mentioned in the last section, a simple correction can be made by subtracting off the value of the high-frequency constant value before the HT, and then restoring it afterwards [11]. When the correction is made, the HT indicates that the process $y_1(t) \rightarrow y_3(t)$ is in fact, causal (Figure 7).

The next illustration is for the process $y_5(t) \rightarrow y_3(t)$. Based on the second example, one would expect this process to fail the causality test; mass m_5 is the last mass to receive information from the force and so y_5 should not be a credible cause for any other response. Unfortunately, it is not possible to apply the HT in this case, as the asymptotic behaviour is ‘too bad’; Figure 8 shows that the transmissibility tends

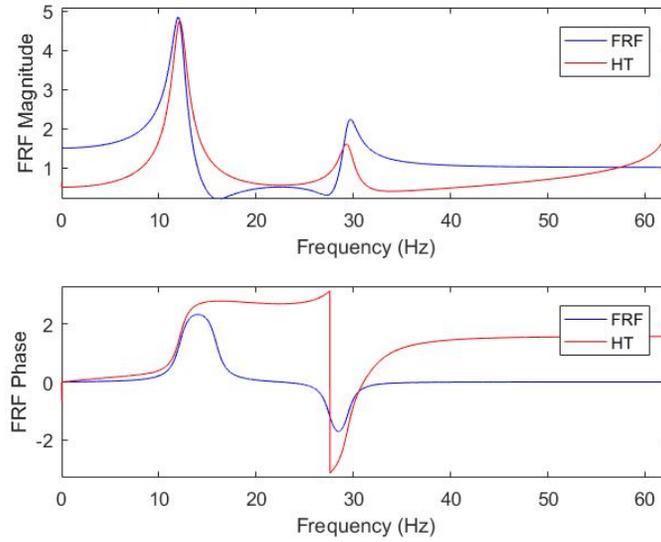


Figure 6: FRF and corresponding Hilbert transform for process $y_1(t) \rightarrow y_3(t)$.

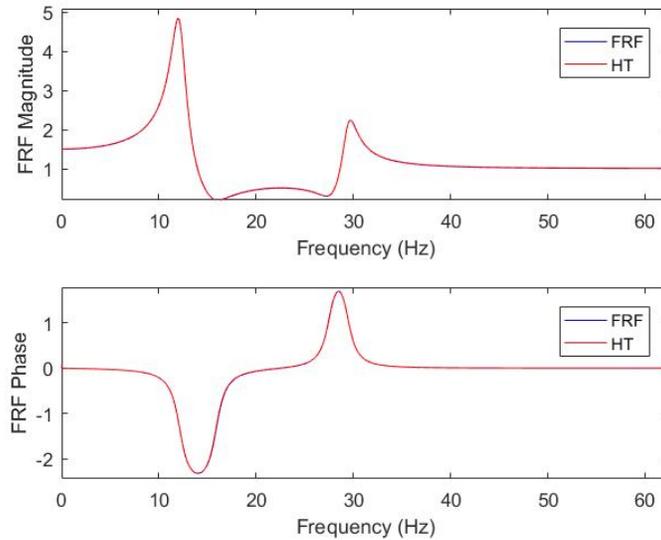


Figure 7: FRF and corresponding Hilbert transform for process $y_1(t) \rightarrow y_3(t)$ (Corrected HT).

to infinity as a polynomial rather than asymptoting to a constant value. Current correction procedures for the HT are inadequate for dealing with this situation.

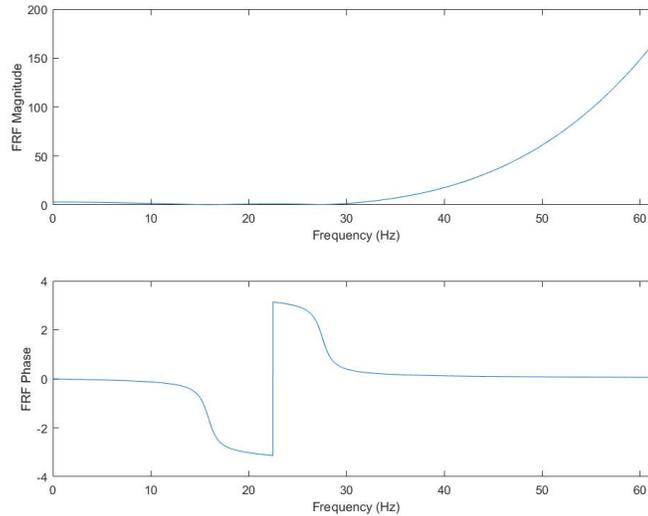


Figure 8: FRF for process $y_5(t) \rightarrow y_3(t)$.

One might speculate that the bad asymptotic behaviour of the FRF is in itself symptomatic of acausality. The transmissibility above that one would expect to result from an acausal process, does indeed diverge at high frequencies. Furthermore, the process above which one might expect to be marginally causal did have bad high frequency behaviour, but the mildest possible, which proved correctable. In support of this one can observe that any FRF for a process $x(t) \rightarrow y_i(t)$ will go to zero at infinity like ω^{-2} . An ‘FRF’ for a process $y_i(t) \rightarrow x(t)$ – which is clearly acausal – will thus diverge at high frequencies as ω^2 . At this stage this observation is no more than a hypothesis and will require further study in order to confirm or refute it.

4 CONCLUSIONS

This paper is intended as the first to report on a programme of work on correlation and causality in structural dynamics; as such, it is mainly concerned with preliminary work. Some definitions, terminology and notation are introduced and the basic motivation for studying causality is given. The main results of the paper are to introduce the Hilbert transform as a causality test in this context and to illustrate how benchmark data sets with controlled ‘degrees’ of causality might be constructed in order to serve as validation data for test methods. The results are encouraging; within the limited context allowed for it, the HT test proves successful. However, further work is needed to overcome issues associated with the

failure of the HT transform in situations where the FRF is not square-integrable; corrections currently only work for the mildest forms of high-frequency pathology.

An immediate further limitation of the HT approach, is that it cannot be used with nonlinear systems. It is well-known that nonlinearity introduces an 'artificial noncausality' into system FRFs; this is the basis for the HT's use as a test for nonlinearity. If a nonlinear system has a representation as a Volterra series, a series of multi-dimensional impulse responses can be defined; if one can find an unbiased estimate of the lowest-order (underlying linear) impulse response, then this should be free of artificial noncausality [11]

The main aims of later work will be to trial other causality metrics on the benchmark data, before moving on to the more challenging problem of designing new metrics. Ultimately, the aim will be to consider econometric data; however, this is expected to raise the most severe difficulties.

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