

## Parameter Identification of Hysteresis Using Duffing-Like Model

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### ABSTRACT

This paper discusses a relatively new and standard system identification approach for hysteresis system using the Duffing-like model. As well-known in literature, many structural and mechanical components display the hysteresis behaviour of forward-backward loop on the input-output graph, such as magnetorheological damper and rubber isolator. The existing hysteresis models in literature usually involve non-ideal functions, such as coupled parameters and absolute values. Difficult-to-determine parameters sometimes cause the design, analysis, and compensation work inefficient and ineffective for pragmatic purpose. The proposed Duffing-like model is developed based on the Duffing equation, showing a scenario of continuous nonlinear ordinary differential equation. Therefore, the parameters are decoupled and the identification work can be carried out in a more systematic and deterministic manner. In addition, stability of the Duffing-like model can be proven using the Lyapunov and invariant set theorems, which offer a basis for parameter bounding and control design. Modelling and identification results of a magnetoresistive sensor are shown as an example to demonstrate the proposed techniques; the physical meaning of the parameter is also discussed.

**Keywords:** *Duffing equation, hysteresis, magnetoresistive sensor, system identification*

### 1 INTRODUCTION

Nonlinear hysteresis behaviour commonly occurs in many engineering devices. For example, reference [1] discusses several hysteresis systems, including Schmitt trigger, mechanical backlash, and beam in magnetic field. To be more specific, the author points out the notion that hysteresis is a general property for almost all dynamic systems, but the factor that causes hysteresis may differ in systems. Therefore, in literature, understanding, realization, and compensation of hysteresis dynamics remain an important issue.

To identify the hysteresis behaviour, two well-known mathematical models are proposed in literature, which are the Bingham and Bouc-Wen models, as shown in Table 1. The Bingham model is constructed by combing the terms of coulomb friction and mechanical damping. In the equation,  $y_m$  is the magnitude of coulomb friction,  $c_{po}$  shows the damping coefficient, and  $u$  and  $y$  are the system input and output, respectively; the coulomb friction is represented using a sign function. On the other hand, the Bouc-Wen model [2, 3] is formed by using nonlinear Bouc-Wen, linear damping, and linear

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spring terms. In the equation of Bouc-Wen model,  $c_0$ ,  $k_0$ ,  $\alpha$ ,  $\gamma$ ,  $\beta$ ,  $A$ , are the relevant linear and nonlinear parameters, and  $z$  is the output of nonlinear dynamics. Compared to Bingham model, because the Bouc-Wen model includes more parameters, the hysteresis curve can be fitted with a higher level of accuracy. However, the issue of too many and coupled parameters to be identified degrades the applicability of Bouc-Wen model for pragmatic purpose. In addition, both the Bingham and Bouc-Wen models contain discontinuity of sign and absolute functions, which do not explicitly present physical meaning, and thus render the design, analysis, and compensation work difficult.

To overcome the disadvantages of Bingham and Bouc-Wen models, the continuous Duffing-like model based on Duffing equation is proposed in [4]. Here, the typical form of Duffing equation [5] is introduced as follows:

$$\ddot{x} + \alpha\dot{x} + \beta x + \gamma x^3 = A \cos \omega t \quad (1)$$

In the left-hand side of Eq. (1),  $x$  is the internal state, and  $\alpha$ ,  $\beta$ , and  $\gamma$  are the coefficients of linear damping, linear stiffness, and nonlinear stiffness, respectively. In Eq. (1), the term of  $x^3$  importantly dominates the behaviour of Duffing oscillator. The right-hand side of Eq. (1) shows the external input,  $A$  is the input amplitude, and  $\omega$  is the input frequency. Equation (1) is modified to become the Duffing-like model, which is introduced in Section 2.

Table 1: Bingham and Bouc-Wen models

	Bingham model	Bouc-Wen model
Model		
Equation	$y = y_m \operatorname{sgn}(\dot{u}) + c_{po}\dot{u}$	$y = c_0\dot{u} + k_0(u - u_0) + \alpha z$ $\dot{z} = -\gamma \dot{u} z z ^{n-1} - \beta\dot{u} z ^n + A\dot{u}$

Because the proposed Duffing-like model is continuous and the parameters are related to the system state, it is expected that the parameter identification may be conducted in a more explicit and systematic manner than that of the Bouc-Wen model. Therefore, the objective of this paper focuses on research into the parameter identification. Section 2 introduces the Duffing-like model in more detail, including the state-space representation and stability proof based on the Lyapunov and invariant set theorems. Section 3 performs a series of simulation work in a logical manner to identify the physical meaning of the parameters in relation with the hysteresis dynamics. Numerical and experimental work of a magnetoresistive sensor [6] in literature is discussed in Section 4, to exemplify the proposed concepts. Finally, Section 5 draws the conclusion for this work.

## 2 INTRODUCTION TO DUFFING-LIKE MODEL

This section introduces the mathematical equations of Duffing-like model, and the stability analysis based on the Lyapunov and invariant set theorems are presented. Here, the Duffing oscillator of Eq. (1) is modified to become the Duffing-like model as follows:

$$\ddot{x} + \alpha\dot{x} + \beta x + \gamma\dot{x}^{n_1} + \delta x^{n_2} = c\dot{d} + kd \quad (2)$$

which shows a scenario of continuous nonlinear ordinary differential equation. The left-hand side of Eq. (2) represents the internal system dynamics, and  $x$  is the internal system state. In addition,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are the coefficients of linear damping, linear stiffness, nonlinear damping, and nonlinear stiffness, respectively. The values of  $n_1$  and  $n_2$  reflect the strength of nonlinearity related to  $\dot{x}$  and  $x$ , respectively, which should be odd. The right-hand side of Eq. (2) is associated with the external input dynamics, where  $d$  is the input signal, and  $c$  and  $k$  are the linear damping and linear stiffness, respectively. In many magnetic devices, such as dampers and sensors,  $d$  has explicit physical meaning, such as input displacement or input voltage. However, the internal system state  $x$  may have or have no physical meaning. In comparison with the typical Duffing equation of Eq. (1), the Duffing-like equation of Eq. (2) contains the nonlinear terms related to  $\dot{x}$ , and the order of  $n_1$  and  $n_2$  are not limited to a maximum of three. It is to be noted that, although the physical meaning of the parameters is introduced herein, it will be discussed and verified in more detail in the next section via numerical studies.

The Duffing-like model is a second-order system, which can be represented in state-space form, allowing for the stability analysis and control design to be conducted in a systematic manner. Let  $x = x_1$  and  $\dot{x} = x_2$ , the state equations based on Eq. (2) are expressed by:

$$\dot{x}_1 = x_2 \quad (3)$$

$$\dot{x}_2 = -\beta x_1 - \alpha x_2 - \gamma x_2^{n_1} - \delta x_1^{n_2} + c\dot{d} + kd \quad (4)$$

Thus, Eqs. (3) and (4) can be combined and written in state-space form as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ -\delta x_1^{n_2} - \gamma x_2^{n_1} \end{bmatrix}}_{\mathbf{F}(\mathbf{x})} + \underbrace{\begin{bmatrix} 0 & 0 \\ k & c \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} d \\ \dot{d} \end{bmatrix}}_{\mathbf{d}} \quad (5)$$

$$y = \underbrace{\begin{bmatrix} 0 & A_y \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (6)$$

where  $\mathbf{A}$  is the plant matrix of the hysteresis model, and  $\mathbf{B}$  is the input matrix related to  $d$  and  $\dot{d}$ . In addition,  $\mathbf{F}(\mathbf{x})$  is the matrix including the nonlinear terms, and  $\mathbf{C}$  is the output matrix gives the rate-dependent output  $y$ .

In nonlinear literature, the Lyapunov theory is often used for the stability proof of nonlinear systems, which is also adopted in the following analysis for the Duffing-like model [4]. First, with reference to Eqs.(3)-(6), a scalar Lyapunov-candidate-function  $V$  is chosen as:

$$V = \frac{\beta}{2} x^2 + \frac{1}{2} \dot{x}^2 + \frac{\delta}{n_2 + 1} x^{n_2 + 1} \quad (7)$$

Thus, the corresponding first-order time derivative of  $V$  is given by:

$$\begin{aligned}
\dot{V} &= \dot{x}\ddot{x} + \beta x\dot{x} + \delta x^{n_2}\dot{x} \\
&= \dot{x}(-\beta x - \alpha\dot{x} - \gamma\dot{x}^{n_1} - \delta x^{n_2}) + \beta x\dot{x} + \delta x^{n_2}\dot{x} \\
&= -\alpha\dot{x}^2 - \gamma\dot{x}^{n_1+1}
\end{aligned} \tag{8}$$

To let Eq. (7) be a globally unbounded, positive-definite function,  $\beta$  and  $\delta$  must be positive, and  $n_2$  must be odd. Let  $\alpha$  and  $\gamma$  be positive and  $n_1$  be an odd constant, Eq. (8) therefore shows a semi negative-definite function with the Lyapunov stability guaranteed. Furthermore, to assure the globally asymptotic stability, the invariant set theorem is applied to Eqs. (2) and (8). The proof is not shown here for the sake of brevity; the detail refers to [4]. Accordingly, the proof reaches a conclusion to stability condition, where  $\alpha$ ,  $\gamma$ ,  $\beta$ , and  $\delta$  are all positive, and  $n_1$  and  $n_2$  need to be odd.

### 3 NUMERICAL STUDIES OF PARAMETER IDENTIFICATION

This section performs a series of simulation work in a logical manner, to investigate the effects of parameter changes on the hysteresis dynamics. Simulation studies were conducted using the Matlab/Simulink software. The full simulation time for all the cases was set as 4 s, and the external input signal  $d$  was sinusoids with frequency 1 Hz. Thus, the input signals  $d$  and  $\dot{d}$  are given by

$$d = \frac{500}{2\pi} \sin 2\pi t \tag{9}$$

$$\dot{d} = 500 \cos 2\pi t \tag{10}$$

Here, the amplitude of  $d$  is specifically chosen as  $500/2\pi$ , which is meant to let the amplitude of  $\dot{d}$  be 500, allowing for the analysis to be conducted in a convenient manner.

Figure 1 displays the block diagram of Duffing-like model in Simulink, with two inputs  $d$  and  $\dot{d}$ , and an output  $y$ . All the relevant parameters and signal flows related to  $x$  and  $\dot{x}$  are included in the block. The typical input-output curve of the Duffing-like model is shown in Fig. 2, where the complete hysteresis curve is separated into linear and nonlinear regions. The nonlinear region is controlled by the nonlinear terms related to  $\gamma$  and  $\delta$ , and the linear region is associated with the linear coefficients of  $\alpha$  and  $\beta$ . In addition, four transitional points are labelled as  $T_1$ ,  $T_2$ ,  $T_3$ , and  $T_4$ , which represents the transition area of transformation from linear to nonlinear or from nonlinear to linear regions. The hysteresis curve is assumed to run counterclockwise.

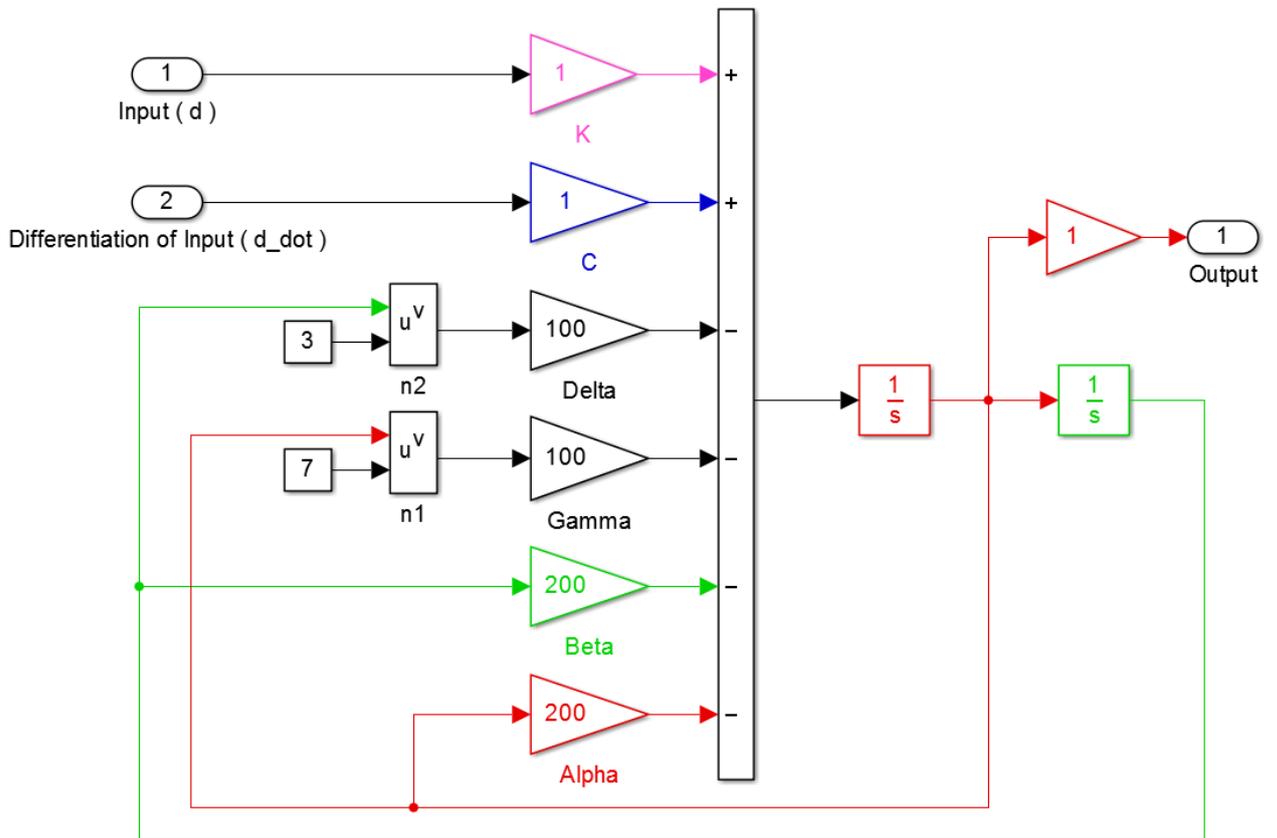


Figure 1 – Block diagram of Duffing-like model in Matlab/Simulink

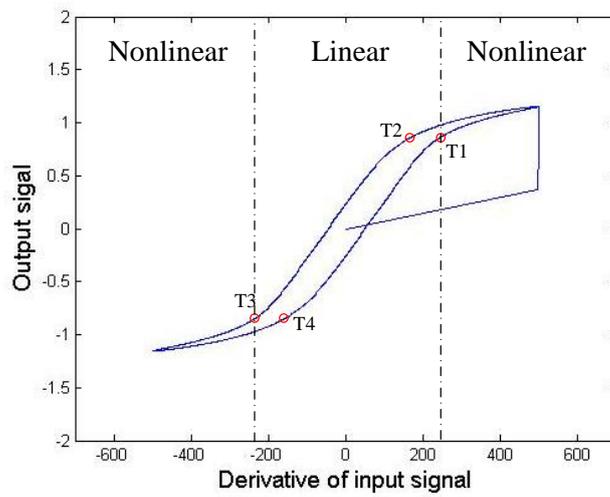


Figure 2 – Hysteresis curve of Duffing-like model

According to Eq. (2), Fig. 1, and Fig. 2, the benchmark parameters of the Duffing-like model are defined as  $\alpha = 200$ ,  $\beta = 300$ ,  $\gamma = 100$ ,  $\delta = 100$ ,  $c = 1$ ,  $k = 1$ ,  $n_1 = 7$ , and  $n_2 = 3$ . Then, the parameters of  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $c$ , and  $k$  are increased or decreased, to explore the physical meaning of the parameters in relation with the hysteresis dynamics. Figs. 3-8 depict the simulation results of the six parameters, and the observation associated with linearity and nonlinearity, and with damping, stiffness, and inertial dynamics, are discussed as follows:

(1) The effect of linear damping coefficient  $\alpha$

As shown in Fig. 3, when  $\alpha$  is increased from 20 to 2000, the magnitude of  $y$  becomes small, and the nonlinear region gradually becomes obscure. In other words, when  $\alpha$  is decreased and the output  $y$  is increased, the transition area becomes obvious. Therefore, the internal dynamics associated with  $\alpha$  behaves like a damper within the hysteresis system, which dissipates the system energy and drops down the output magnitude.

(2) The effect of linear stiffness coefficient  $\beta$

Figure 4 shows that  $\beta$  adjusts the total area of hysteresis curve; as  $\beta$  increases from 200 to 400, the area is reduced. However, the coefficient  $\beta$  does not change the maximum amplitude of  $y$  and the transition point. It is noted that, when  $\beta$  is large enough, the hysteresis area is reduced and the input-output paths are almost overlapped. In other words, a large  $\beta$  provides sufficient potential energy in resistance to  $d$  and  $\dot{d}$ , to overcome the inertia of the hysteresis system. Therefore, as the input velocity  $\dot{d}$  reaches zero, the output  $y$  is simultaneously driven to zero. The dynamics related to  $\beta$  behaves like a spring component, and thus  $\beta$  is considered the linear stiffness in the Duffing-like model.

(3) The effect of nonlinear damping coefficient  $\gamma$

The variations of  $\gamma$  are shown in Fig. 5; as  $\gamma$  increases from 10 to 1000, the magnitude of  $y$  is reduced. In addition, it is observed that  $\gamma$  controls the rate of the curve in the nonlinear region, but has no effects on the rate of the linear curves. Meanwhile, the location of transition area varies with  $\gamma$ ; as  $\gamma$  increases, the transition area is shift closer to the equilibrium point. In other words, the nonlinear region is expanded, and the rates of linear and nonlinear curves are getting distinct. From the above observation,  $\gamma$  is therefore considered in relation to the damping characteristics in the nonlinear region.

(4) The effect of nonlinear stiffness coefficient  $\delta$

The effects of  $\delta$  are compared and plotted in Fig 6; when  $\delta$  is promoted from 10 to 1000, the area of hysteresis loop is reduced, in a similar pattern to  $\beta$ ; thus,  $\delta$  is considered the nonlinear stiffness coefficient in relation with the nonlinear restoring force. However, it is observed that  $\delta$  has an upper limit, so that the area of hysteresis loop reaches a maximum. The physical phenomenon related to  $\delta$  constitutes the authors' ongoing work.

(5) The effect of linear damping coefficient  $c$  and linear stiffness coefficient  $k$

The simulation results of Figs. 7 and 8 as a whole show that the nonlinearity varies in proportional to  $c$  and  $k$ . However,  $c$  and  $k$  have reverse effects as compared to  $\alpha$  and  $\beta$ , respectively.

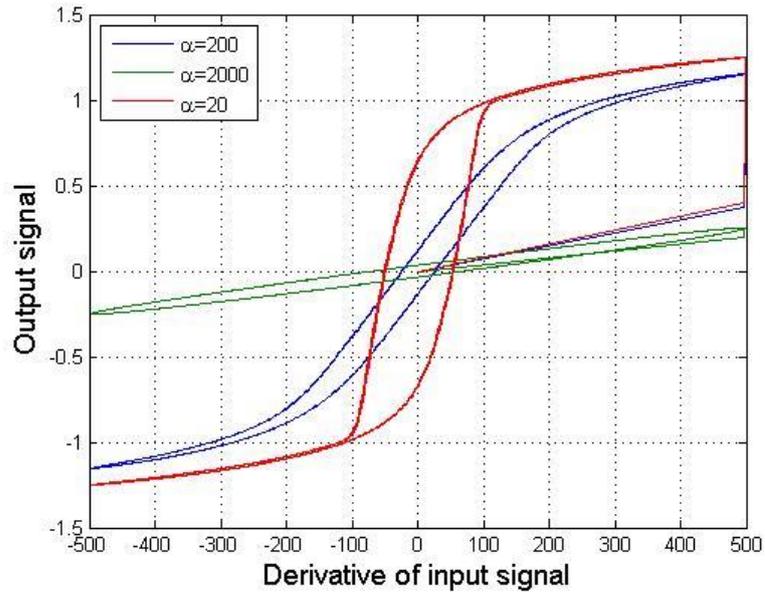


Figure 3 – Input-output relation depending on  $\alpha$

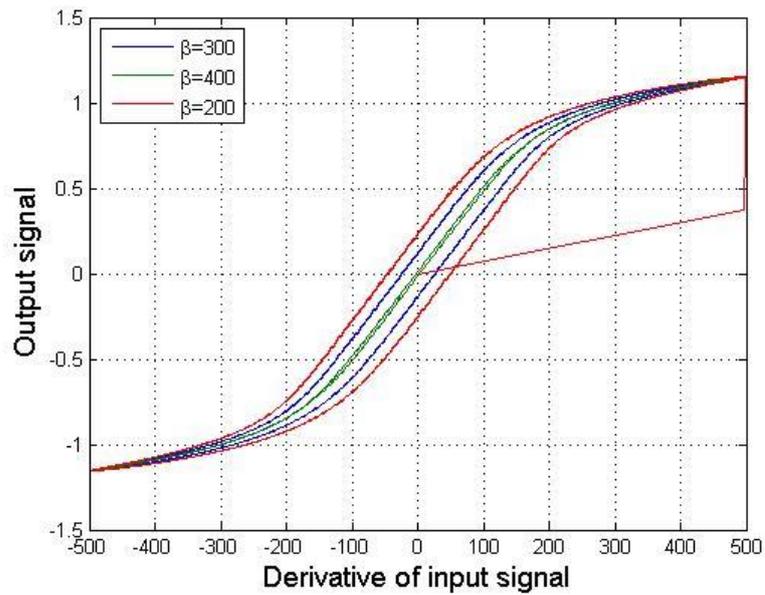


Figure 4 – Input-output relation depending on  $\beta$

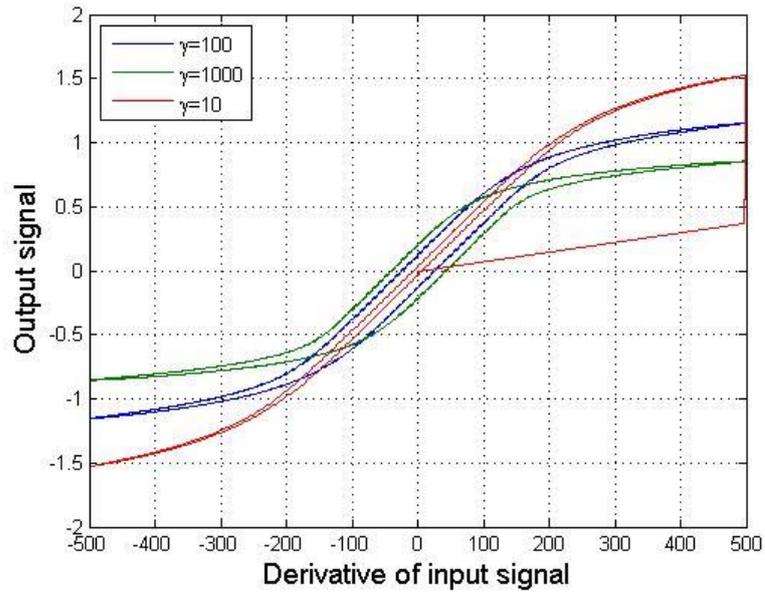


Figure 5 - Input-output relation depending on  $\gamma$

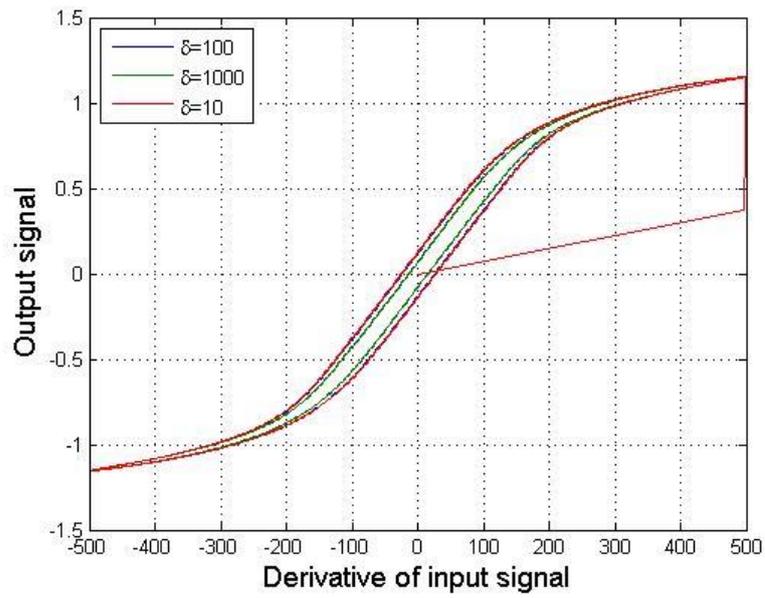


Figure 6 - Input-output relation depending on  $\delta$

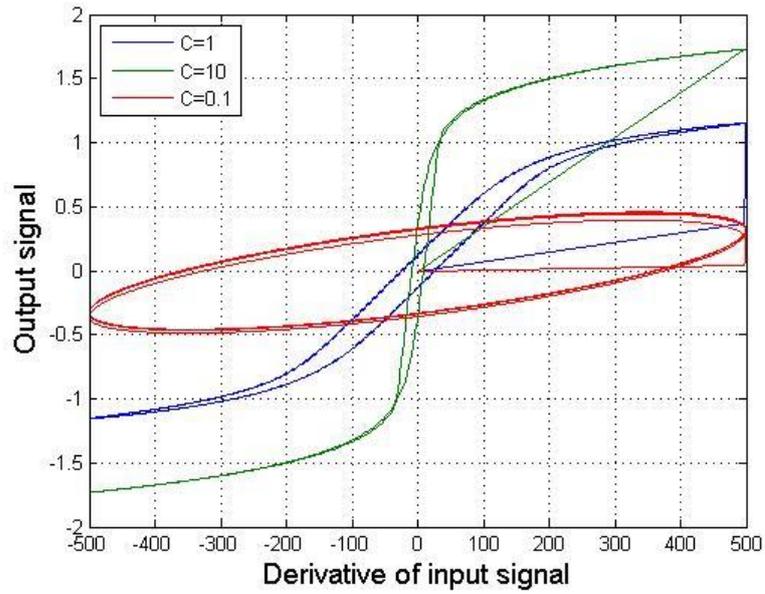


Figure 7 - Input-output relation depending on  $c$

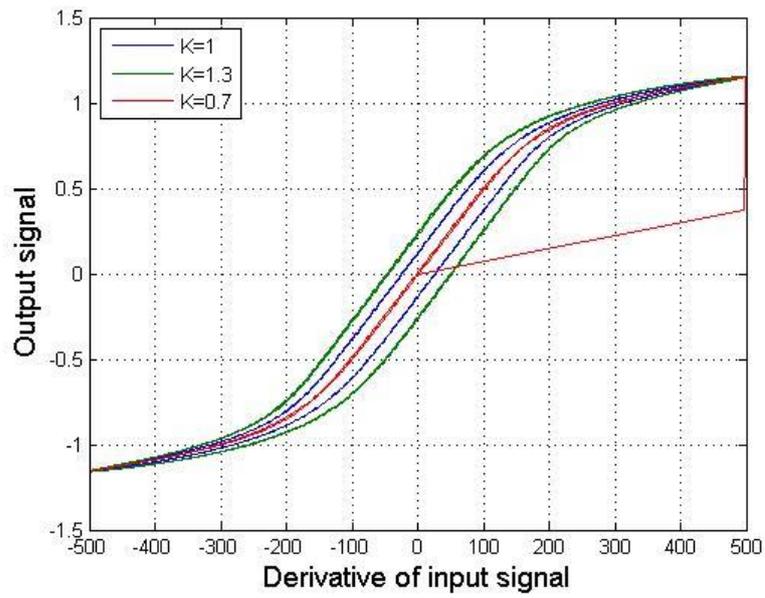


Figure 8 - Input-output relation depending on  $k$

#### 4 NUMERICAL AND EXPERIMENTAL RESULTS

In this section, the Duffing-like model is applied to identify the hysteresis dynamics of a magnetoresistive sensor, and the experimental data of the sensor are shown in Fig. 9 [6]. For the magnetoresistive sensor, the hysteresis behaviour is defines as the input-output relation between the input of magnetic field intensity and the output of sensor resistance. Therefore, the  $x$  axis of Fig. 9 denotes the input of the magnetic field intensity in micro tesla (mT), and the  $y$  axis is the output of sensor resistance in ohm ( $\Omega$ ); the circuit of the sensor is illustrated in Fig. 9. The input magnitude of magnetic field intensity is  $0.5 \times 10^{-3}$  mT, and the excitations are constant cosine waves with a frequency of 39 Hz; the output range of the sensor is between 650  $\Omega$  and 970  $\Omega$ [6].

With reference to Fig. 9, the fitting result of the hysteresis curve using the Duffing-like model is presented in Fig. 10. The parameter tuning procedure begins with  $\alpha$ , and the transition region of Fig. 9 indicates the nonlinear rate for adjusting  $\gamma$  in Fig. 10. Furthermore, the loop area of Fig. 9 gives hints for adjusting  $\beta$  and  $\delta$ . Ultimately, the parameters are tuned as  $\alpha = 4200$ ,  $\beta = 0$ ,  $\gamma = 100$ ,  $\delta = 0$ ,  $n_1 = 11$ ,  $n_2 = 9$ ,  $c = 50000$ , and  $k = 0$ , showing the effectiveness of the Duffing-like model. It is to be noted that, although Fig. 9 has a reverse pattern in comparison with Fig. 2, the tuning procedure is still similar to that addressed in Section 3. The reverse hysteresis dynamics will be discussed in future work.

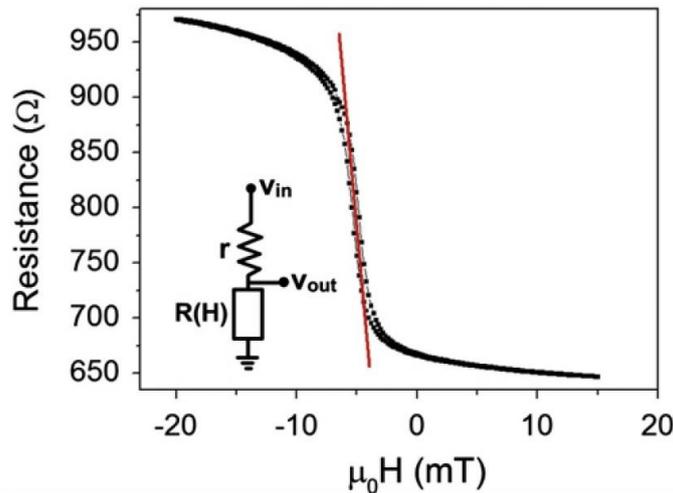


Figure 9 –input-output relation of a magnetorheological sensor [6]

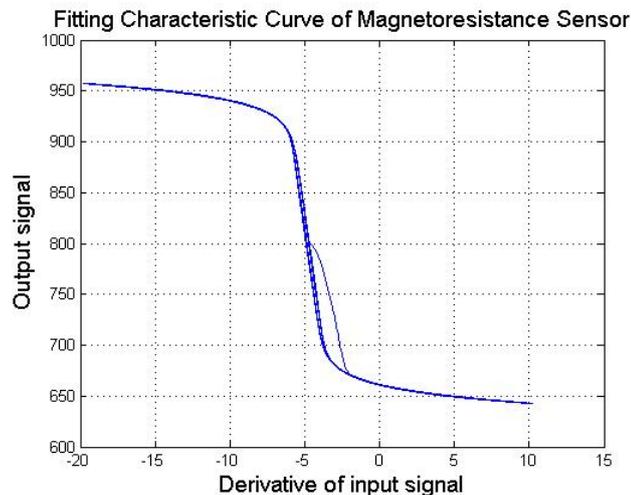


Figure 10 – Identification result of magnetoresistive sensor dynamics, with  $\alpha = 4200$ ,  $\beta = 0$ ,  $\gamma = 100$ ,  $\delta = 0$ ,  $n_1 = 11$ ,  $n_2 = 9$ ,  $c = 50000$ , and  $k = 0$ .

## 5 CONCLUSIONS

Novel Duffing-like model for identifying hysteresis dynamics has been proposed, and this paper gains a deeper insight into the issue of parameter tuning. This research work defines the hysteresis dynamics as the combination of linear and nonlinear regions; the dynamic behaviours of the linear and nonlinear regions are controlled by the linear and nonlinear coefficients, respectively. The parameters adjust the rate of curve, the area of loop, the magnitude of output, and the transition state from linear to nonlinear domains. Compare to other hysteresis models in literature, the parameters of the Duffing-like model are decoupled with explicit physical meaning, and thus, the identification work can be conducted in a more systematic and traceable manner. In future work, stability analysis and control design in relation with the parameters will be explored in more detail. In addition, general and standard application of the Duffing-like model for engineering systems will be continued.

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