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Simple Asymptotic BER Expressions for LoRa System over Rice and Rayleigh Channels

V. Savaux and G. Ferré

This paper deals with the bit error rate (BER) of LoRa signal over Rayleigh and Rice channels. Exact analytical expressions of the BER in LoRa systems have been proposed in the literature, but they are not tractable and raise problems of computation precision. We then hereby suggest asymptotic expressions of the BER, which are at the same time tractable and accurate over a wide BER range.

Introduction: Among the digital communication technologies used in the low power wide area network (LPWAN) world, LoRa is the most studied by the scientific community to date. This is mainly due to the open deployment strategy chosen by Semtech. The digital communication technique used in LoRa modulation is the chirp spread spectrum (CSS) initially described by Winkler in [1]. Despite a high sensitivity to poor time/frequency synchronization, this technique offers many advantages such as i) a constant envelope which allows to operate with power amplifiers (PAs) in saturation mode and thus to have excellent amplification efficiency, ii) a sensitivity that can be optimized at the cost of a loss of spectral efficiency, iii) a low complexity demodulation based on fast Fourier transform (FFT).

LoRa or more generally CSS has given rise to numerous studies and analyses in recent years. At the physical layer level, one will retain works proposing to optimize the receivers in order to improve the synchronization steps or to find solutions to destructive collisions [2]. However, in order to be able to judge the effectiveness of the proposed processing, it is necessary to be able to situate the performance in terms of bit error rate (BER) of the developed algorithms based on the theoretical results. Thus [3], [4,5] have proposed approximates and closed expressions of the binary error probabilities of LoRa system over additive white Gaussian noise (AWGN) channels. However, LoRa propagation channels are generally non frequency-selective Rayleigh channels for which they do not exist in the literature closed form expression of binary error probability. In [4], expressions of the BER of LoRa signal in Rayleigh channel have been suggested as well, but they are approximations based on error functions. In [6], the exact expression of the BER for LoRa signal over different channel models has been derived. However, despite their exactness, the BER expressions in [6] are not tractable, and lead to problem of precision. This is mainly due to the fact that the solutions involve sums of binomial coefficients whose full precision computation requires hundreds of bits.

In this paper we derive a very simple and tractable asymptotic expression of the BER of LoRa signal over Rice and Rayleigh channels from that developed in [6], which tends to the exact BER for low noise variance value. It is shown through simulations that that this approximation matches well the exact BER behaviors [6] in large BER range. Furthermore, the suggested solution only involves straightforward functions. Thus, the proposed BER expressions is a good trade-off between accurateness and simplicity, allowing for a fast assessment of the achievable performance of a LoRa signal over Rice and Rayleigh channels. Moreover, we provide a summary of the literature dealing with more sophisticated BER expressions in different channel models, as well as including the effect of interference [7] and channel coding [8].

Next sections are dedicated to: the description of the LoRa modulation and demodulation principles and the introduction of the exact BER expression is introduced; the derivation of a very simple approximation of these closed form expressions; the simulation results, and the conclusion.

Background: LoRa uses several spreading factors (SF) to control the bit rate, improve the range and decrease energy consumption. Initially, the binary information flow generated from the medium access control (MAC) layer is divided into subsequences, each of length $SF \in [7 \dots 12]$. The set of SF consecutive bits constitutes a symbol. The number of possible symbols is hence equal to $M = 2^{SF}$. In the LoRa context, SF also indicates the spreading factor and the relation between the bit rate D_b and the symbol rate D_s that can be written as: $D_s = D_b/SF$. Spread spectrum is obtained through a signal known as chirp that varies continuously and linearly in frequency. When the derivative of the frequency variation is positive, then we deal with an up chirp, conversely it is a down chirp.

When the chirp is up or down over the entire symbol period T_s , which is also called the signaling interval, it is identified as raw chirp. Its complex envelope is mathematically expressed as follows, for $t \in [-\frac{T_s}{2}, \frac{T_s}{2}]$:

$$c(t) = e^{j\theta_c(t)}, \text{ with } \theta_c(t) = \pm\pi \frac{B}{T_s} t^2, \quad (1)$$

where B is the frequency excursion of the transmitted chirp. The '+' and '-' signs stand for up and down raw chirp, respectively. If we note $f_c(t)$ the chirp frequency variation in time, we have:

$$f_c(t) = \frac{1}{2\pi} \frac{d\theta_c(t)}{dt} = \pm \frac{B}{T_s} t. \quad (2)$$

For digital communication systems with no spreading spectrum, the bandwidth used by the transmitted signal is proportional to the symbol rate. The coefficient of proportionality depends on the shaping filter, which is in general a half Nyquist one. In the CSS modulation used in LoRa, the signal bandwidth is fixed by B which has the following relationship with T_s :

$$M = B \times T_s. \quad (3)$$

To distinguish between the M different symbols of the constellation, M orthogonal chirps have to be defined so that each symbol exhibits a specific instantaneous phase trajectory. In the sequel, we denote m_i the transmitted symbol at time iT_s , with $m_i \in \{0, \dots, M-1\}$. The chirp associated to the i^{th} transmitted symbol is then obtained from the raw chirp by applying a delay $\tau_i = \frac{m_i}{B}$ and the raw chirp outside the interval $[-\frac{T_s}{2}, \frac{T_s}{2}]$ is cyclically brought back into $[-\frac{T_s}{2}, -\frac{T_s}{2} + \tau_i]$. As a consequence, the the instantaneous phase modulated chirp (e.g. up chirp) related to the transmission of the symbol m_i breaks down into two parts:

$$\theta_c^i(t) = \pi \left[\frac{B}{T_s} (t - 2\tau_i)t + 2Bt \right] \quad \text{for } t \in \left[-\frac{T_s}{2}, -\frac{T_s}{2} + \frac{m_i}{B} \right)$$

$$\theta_c^i(t) = \frac{\pi B}{T_s} (t - 2\tau_i)t \quad \text{for } t \in \left[-\frac{T_s}{2} + \frac{m_i}{B}, \frac{T_s}{2} \right).$$

Accordingly, the complex envelope of the LoRa symbol transmitting the symbol m_i is $s_i(t) = e^{j\theta_c^i(t-iT_s)}$.

In the following, we consider perfect time and frequency synchronizations and a one tap block fading channel $h \sim \mathcal{N}_C(\mu, \sigma_h^2)$. Thus, $|h|$ obeys a Rice distribution when $\mu \neq 0$, and a Rayleigh distribution if $\mu = 0$. Moreover, we introduce the Rice coefficient $\kappa = \frac{\lambda}{\sigma_h^2}$. Under these assumptions, the received signal sampled at rate T_e and denoted by $y_i(nT_e)$ can be written as

$$y_i(nT_e) = h \times s_i(nT_e) + w(nT_e), \quad (4)$$

where $w(nT_e) \sim \mathcal{N}_C(0, \sigma^2)$ represents the complex noise assumed to be white, Gaussian and circular. Furthermore, n is the sample index with $n \in [-\frac{N}{2}, \frac{N}{2} - 1]$, N being the number of samples of a LoRa symbol. At Nyquist rate, we have $T_e = \frac{1}{B}$, and then $M = N$.

According to the patent EP2449690, the transmitted symbols are detected by multiplying every received signal y_i by the conjugate version of the raw chirp (1) used in the transmitter, i.e. $r_i(nT_e) = y_i(nT_e)e^{-j\theta_c(nT_e)}$. Then, the optimal estimation of m_i can be performed by searching for the maximum of the periodogram of $r_i(nT_e)$, computed as $\mathcal{P}_i[k] = |R_i[k]|^2$, where $R_i[k]$, for $k \in [0, N-1]$, is the discrete Fourier Transform (DFT) of N samples from $r_i(nT_e)$. Thus, the estimation of m_i can be achieved by computing:

$$\hat{m}_i = N - \underset{k \in [0, N-1]}{\operatorname{argmax}} (\mathcal{P}_i[k]). \quad (5)$$

The symbol error probability, denoted by P_s , is defined as the probability that \hat{m}_i is not equal to the actual transmitted symbol m_i , i.e. $P_s = \mathbb{P}[\hat{m}_i \neq m_i]$. The BER is then obtained through:

$$P_b = \frac{2^{SF}-1}{2^{SF}-1} P_s. \quad (6)$$

It has been proved in [6] that, in the absence of prior information, all the symbols can be considered equiprobable so that the symbol error

probability over a Rice channel, denoted by P_s^{Rice} , is expressed as

$$P_s^{Rice} = - \sum_{k=1}^{N-1} \frac{\binom{N-1}{k} (-1)^k \sigma^2}{kN\sigma_h^2 + (k+1)\sigma^2} \exp\left(-\frac{\lambda k N}{kN\sigma_h^2 + (k+1)\sigma^2}\right), \quad (7)$$

where $\lambda = |\mu|^2$. It must be noticed that the error probability over Rayleigh channel (denoted by P_s^{Ray} in the following) is straightforwardly deduced from (7) by setting $\lambda = 0$. Alternatively, the symbol error probability in AWGN channel is obtained by setting $(\lambda = 1, \sigma_h^2 = 0)$. In any case, despite the exactness of the expression (7), it requires the computation of the high-valued binomial coefficient $\binom{N-1}{k}$. This may suffer from precision problems since we remind that, in LoRa we have $N \in \{128, 256, \dots, 2048\}$. It results that (7) is hardly computable in practice nor tractable, therefore limiting its application to theoretical results. This is the reason why we hereby provide simple expressions of P_s^{Rice} and P_s^{Ray} in asymptotic cases where the noise variance value is low.

Asymptotic BER Expression: In this section we provide simple expressions of asymptotic BER value for LoRa system over Rice and Rayleigh channels.

Proposition 1: Let the noise variance value be low enough to consider that $k \geq 1 \gg \frac{1}{\sigma^2}$, and $kN\sigma_h^2 \gg (k+1)\sigma^2$. Then, for any $N = 2^{SF}$, $N \cdot \frac{\sigma_h^2}{\sigma^2} + 1$

we can make the following approximation

$$P_s^{Rice} \approx e^{-\kappa} \cdot \frac{\gamma + \ln(N-1)}{N \cdot \frac{\sigma_h^2}{\sigma^2} + 1}, \quad (8)$$

where γ is the Euler-Mascheroni constant $\gamma \approx 0.577$.

Proof: According to the assumption of low σ^2 value, then, for any $k = 0, 1, \dots, N-1$, the exponential functions in (7) can be approximated as

$$\exp\left(-\frac{\lambda k N}{kN\sigma_h^2 + (k+1)\sigma^2}\right) \approx \exp(-\kappa). \quad (9)$$

The substitution of (9) into (7) then leads to

$$P_s^{Rice} \approx -e^{-\kappa} \sum_{k=1}^{N-1} \frac{\binom{N-1}{k} (-1)^k \sigma^2}{kN\sigma_h^2 + (k+1)\sigma^2}. \quad (10)$$

Then, in conditions where $1 \gg \frac{1}{\sigma^2}$, P_s^{Rice} in (10) can be simplified as

$$P_s^{Rice} \approx -\frac{e^{-\kappa}}{N \cdot \frac{\sigma_h^2}{\sigma^2} + 1} \sum_{k=1}^{N-1} \frac{\binom{N-1}{k} (-1)^k}{k}. \quad (11)$$

Moreover, we consider N large enough to assume that the approximation $\binom{N-1}{k} \underset{N \rightarrow +\infty}{\sim} \frac{(N-1)^k}{k!}$ holds, which yields

$$P_s^{Rice} \approx -\frac{e^{-\kappa}}{N \cdot \frac{\sigma_h^2}{\sigma^2} + 1} \sum_{k=1}^{N-1} \frac{(N-1)^k}{k \cdot k!}. \quad (12)$$

Then, let

$$E_1(z) = \int_z^{+\infty} \frac{e^{-t}}{t} dt,$$

with $z \in \mathbb{C} \setminus \mathbb{R}^-$, be the exponential integral function, defined in [9]. The series representation of E_1 is

$$E_1(z) = -\gamma - \ln(z) - \sum_{k=1}^{+\infty} \frac{(-1)^k z^k}{k \cdot k!}. \quad (13)$$

Since $\lim_{z \rightarrow +\infty} E_1(z) = 0$, and reminding that we assume $N \rightarrow +\infty$, then the substitution of z by $N-1$ in the sum of (13) and then in (12) leads to (8), which concludes the proof. ■

The asymptotic error probability in Rayleigh channel is straightforwardly obtained when $\kappa = 0$, leading to

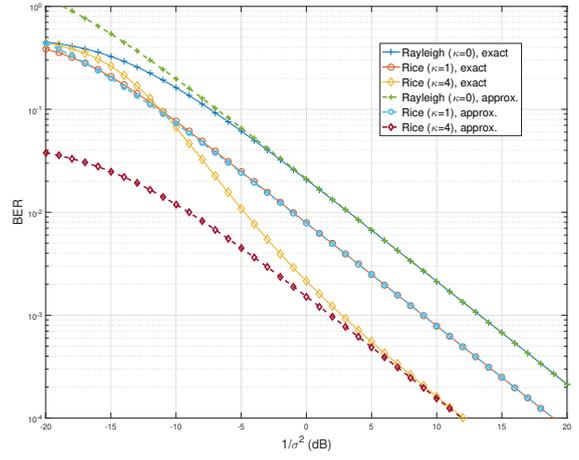


Fig. 1 BER versus $1/\sigma^2$ (dB) for SF7, comparison of the exact BER expression (7), and the suggested approximations (8) and (14) for $\kappa = 0$ (Rayleigh), $\kappa = 1$, and $\kappa = 4$.

$$P_s^{Ray} \approx \frac{\gamma + \ln(N-1)}{N \cdot \frac{\sigma_h^2}{\sigma^2} + 1}. \quad (14)$$

It can be noted that (8) and (14) are more tractable expression than (7) as they only involve usual functions. Furthermore, the suggested approximations do not require the computation of $\binom{N-1}{k}$, which leads to problems of computation precision, especially when N is large (e.g. $N = 1024, 2048$). In the following, we investigate through simulations the precision and the domain of validity of such approximations of the error probability.

Simulations Results: Simulations have been performed to verify the accuracy of the suggested approximations of the error probability in (8) and (14). Theoretical results have been obtained using Python 3.6 with *scipy* and *gmpy2* packages for mathematical computing and precision improvement, respectively¹. It must be emphasized that the theoretical results presented in [6] had been validated through simulations. As a consequence, we do not reproduce the Monte-Carlo simulations to avoid excessive plots.

Fig. 1 shows the BER performance of LoRa system over Rayleigh and Rice channel models versus $1/\sigma^2$ (in dB) for SF7. Other series of simulations have been undertaken for SF8 to SF12, but the results are not shown in this paper, as the BER performance is equivalent to that of SF7, except that the BER curves are shifted on the left (a shift of about -2.5 dB per SF). The performance of the exact expression (7) obtained from [6] (referred as "exact") is compared with that of suggested approximations (8) and (14) for $\kappa = 0$ (Rayleigh), $\kappa = 1$, and $\kappa = 4$ (referred as "approx.").

It can be observed that the larger the Rice coefficient κ , the lower the BER. This is a very well-known result since asymptotically, the case $\kappa = +\infty$ corresponds to the AWGN channel, which is a lower performance bound. Otherwise, we notice that the domain of validity of the suggested approximations depends on the coefficient κ . In fact, we can see that in the Rayleigh case ($\kappa = 0$), the asymptotic expression well matches the exact performance for BER values lower than 10^{-1} , which is relevant as a BER range larger than 0.1 is usually not of interest in practice. For the Rice channel with $\kappa = 1$, we observe that the approximation well matches the exact expression for any BER value, and for $\kappa = 4$, the asymptotic expression of the BER well matches the exact one for BER values lower than 10^{-3} . We deduce that the suggested approximations are more relevant for lower values of the Rice coefficient κ , which is consistent with theory since no asymptotic BER expression can be derived for the AWGN channel corresponding to $\kappa = +\infty$ [5].

The accurateness of the suggested BER expressions compared with the exact one is very interesting in practice, especially since (7) is not tractable, and raises some computing precision problems. In fact, to give an idea of the required precision, the binomial coefficient $\binom{N-1}{k}$ in (7) with full

¹ The code can be found ...

Table 1: Overview of BER analysis of LoRa system according to different models and approaches.

Reference	Channel	Approach
[3–5]	AWGN	exact [5] approximation [4] Monte-Carlo [3]
[8]	AWGN+channel coding	approximation
[7]	AWGN+interferences	approximation
[6]	Rayleigh, Rice, Nakagami	exact

precision needs to be stored over up to 150 bits in SF7, and even up to 4000 bits in SF12 (e.g. $\binom{2047}{200}$ is of order 9.10^{282}), which can be achieved only with the use of the package *gmpy2* in python. This practically shows the relevance of the suggested asymptotic BER expressions, in addition to their accurateness.

To complete the suggested analysis, we provide an overview of the literature dealing with BER derivation in LoRa system according to the channel (AWGN, Rayleigh, etc.) model and the approach (exact or approximated BER expression) is provided in Table 1. Note that we have added [7] and [8] to enlarge the scope of this overview. In fact, even if these paper does not deal with BER of LoRa signal over specific channel model, but their BER analysis consider the effect of the channel coding and the impact of interference on the BER performance, respectively.

Conclusion: In this paper, we have derived simple asymptotic expressions of the BER of LoRa signal over Rice and Rayleigh channel, considering low noise variance value. We have shown through simulations the validity of the suggested approximated expressions compared with the exact one. The advantage of the proposed analysis is to obtain tractable and easily computable solutions, whereas the computation of the exact BER expression may raise problems of precision. Moreover, an overview of the available literature dealing with BER analysis in LoRa system according to different models and approaches has been provided as well.

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