**APPENDIX: CLASS HANDOUTS AND SAMPLE ASSIGNMENT**

The Appendix consist of of the following materials:

1. Class Activity Handout: Function Iteration.
    An introduction to cobweb diagrams using a parabolic and a linear function.

2. Class Activity Handout: Cobweb Diagram Handout to illustrate variety of behavior.

3. Response Sheet for *Chaos, Fractals, and Dynamics* video.

4. Sample Exploratory Assignment. This assignment is written with the use of Maple in mind, but could easily be adapted to any CAS. Resources for exploring function iteration on other platforms are available; for example, a handy applet that shows the cobweb and the solution sequence side-by-side:
    http://mathinsight.org/applet/function_iteration_cobweb_combined , and a cobweb plotter implemented on GeoGebra: https://www.geogebra.org/m/QJ79IWCL

5. Sample Maple Worksheet. This gives Maple commands for producing cobweb diagrams and plotting solution sequences as well as seeking periodic solutions. This is useful for creating classroom demonstrations and also for student use with the Exploratory Assignment. (PDF is provided. I am happy to share the actual Maple worksheet upon request.)

Function Iteration : Cobweb Diagram for $f(x) = a \cdot x^2$ and Linear Equation

Example $x_{n+1} = a\, x_n^2$ $\quad (a = 0.5)$

To make Cobweb plot, start on the (horizontal) axis with a value $x = x_0$.
Draw vertical line to curve $y = f(x)$. You have actually calculated $x_1 = f(x_0)$.
Draw horizontal line to the diagonal ($y = x$) line. This "transfers" x_1 back to the axis
Draw vertical line to curve $y = f(x)$. This computes $x_2 = f(x_1)$ etc.
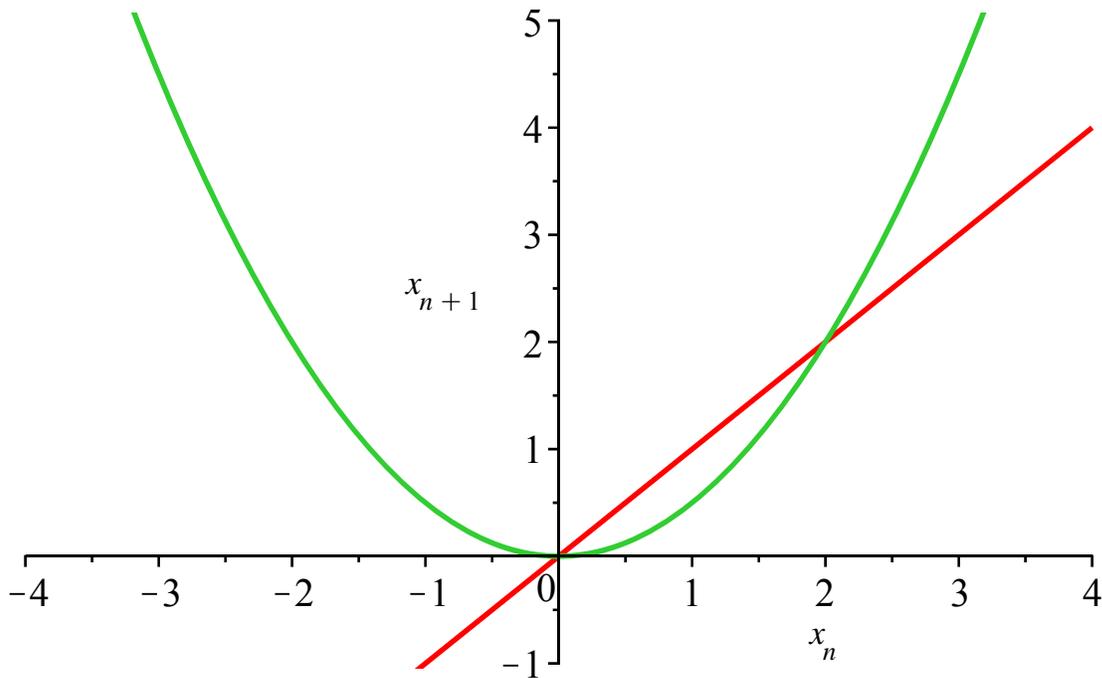
Let $x_0 = 1.5$ -- Do a cobweb plot.

Let $x_0 = 2.5$ -- Do a cobweb plot.

What happens?

Try $x_0 = -1$

$\quad$ $x_0 = -2.5$ (You'll need to extend the diagonal line.)

Are there **Steady States**? $\qquad$ Are they **stable** or **unstable** ?

Example 2: $y_{n+1} = 2 \cdot y_n - 3$
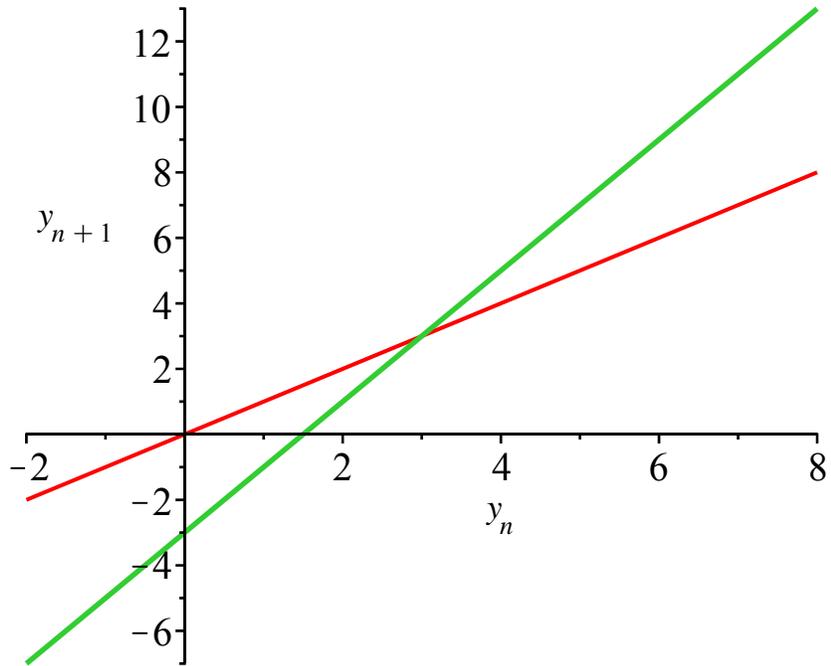

Find the steady state algebraically:

Make a cobweb diagram using two or three different starting values.
Does the s.s. appear to be stable or not?

COBWEB DIAGRAM for DISCRETE LOGISTIC EQUATION

To make Cobweb plot, start on the (horizontal) x-axis with a chosen value x_0.
Draw vertical line to curve $y = f(x)$. This calculates $x_1 = f(x_0)$.
Draw horizontal line to the diagonal ($y = x$) line. This "transfers" x_1 back to the axis
Draw vertical line to curve $y = f(x)$. This computes $x_2 = f(x_1)$ etc.

$$x_{n+1} = a \cdot x_n \cdot (1 - x_n) \quad \text{with various values of the parameter "}a\text{"}$$

$a = 0.8$
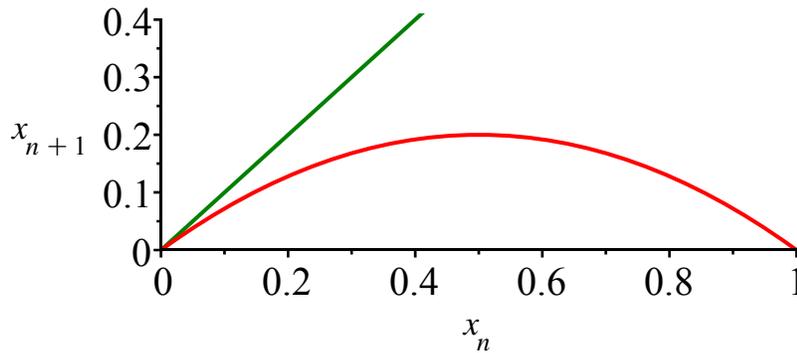
$a = 1.7$
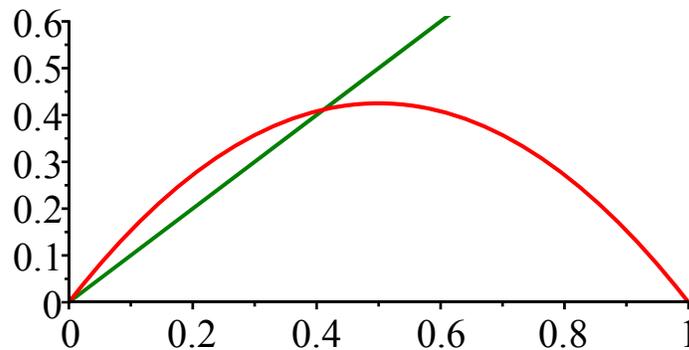
$a = 2.7$
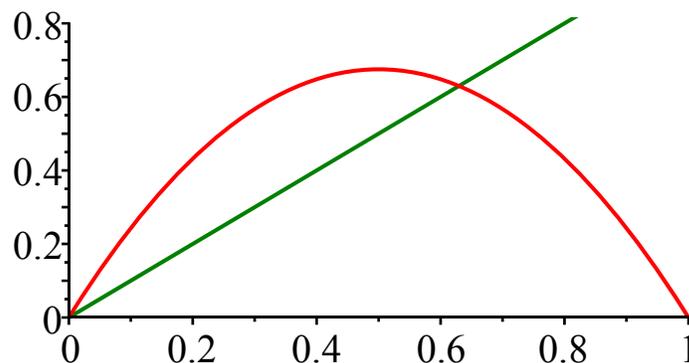
a = 3.2
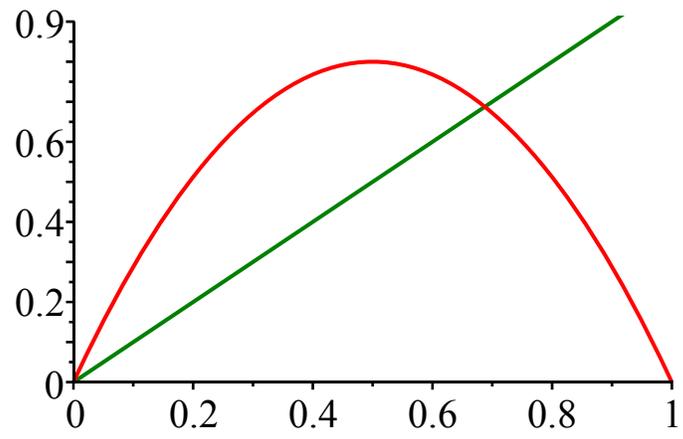
a = 3.6
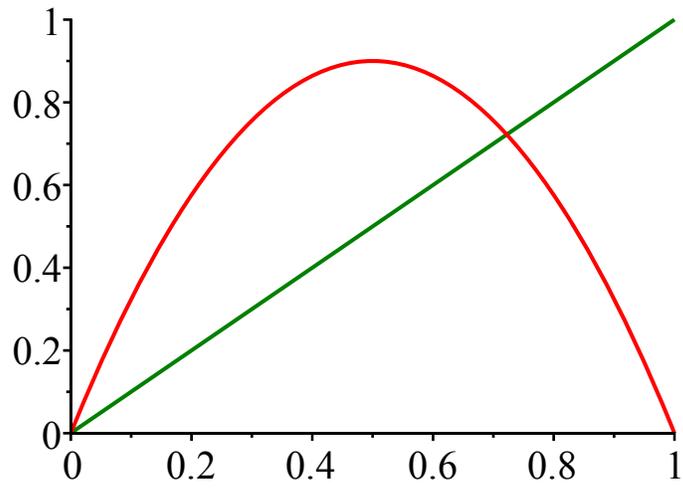
Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics
Robert Devaney; Boston University

Use this handout to take notes. The copy you hand in should be neatly written or typed.
 (This sheet is posted if you want to re-write or type your responses.)

- **Give a short answer to each question – a sentence or two if needed. Be clear and specific**.

1. Give an example of a simple **mathematical** dynamical system that is not predictable.

2. What is meant by the **"orbit"** of a point in a dynamical system or function iteration?

3. What makes an orbit a **"Stable Orbit"?**

4. Why are stable orbits good in mathematical models?

5. What makes an orbit "Chaotic"?

6. What is a Julia Set?

7. How are Julia sets found (in this video)?

8. What do the colored/black regions mean in these pictures?

9. What is one important property of a "fractal"?

Use other side of sheet for these questions:
10. Describe the two most interesting things you learned from this video.
                                    (A couple of sentences each, please.)
11. Was this video useful/interesting to you? Other comments on this presentation?

Homework Assignment: Iterated Functions

If you like, you may team up with one other person and turn in one report together. (No more than 2 people to a "team".)

To do these problems, use the Maple file **Worksheet for Function Iteration HW** which is posted.
Copy the file onto your **own drive/disc** so you can **modify it** to answer the following questions. (Also edit out my comments and put in your own.)

The purpose of these problems is to explore the behavior of the Discrete Logistic Equation and another related equation. When you create various graphs and solve equations for steady states and periodic points, be sure you understand what the graphs and calculations show. One of the challenges of this assignment is **organization.** You will be making lots of pictures, and the challenge is to get the "bigger picture" – to see what is going on as you vary the parameter.

Please SAVE TREES – delete calculations that are not useful.
Make sure you label which problem/part you are doing as you go along; inserting comments is helpful.
You can switch between Math and Text in Maple , so you can and should produce readable documents.
Always USE PRINT PREVIEW before printing._____

Maple will need to compute lots of decimal places – in your written response and the chart below, round to 4 decimal places.

Problems 1, 2 and 3
First you will take another look at the behavior of solutions of the discrete logistic equation:

$$x_{n+1} = \mathbf{a}\, x_n\, (1 - x_n) = f(x_n)$$

1. For a = 3.3
 (a) Find the steady states.
 (b) Find the 2-point periodic solution.
    Start by finding a solution of $f(f(x)) = x$. Then make sure you have both points of the periodic solution.
 (c) Plot (and hand in) a graph showing $y = f(f(x))$ and $y = x$.
    On the graph, label the steady states and the points with period 2 .   (3+4+3=10 points)

2. Also for a = 3.3, decide whether the difference equation has a 4-point periodic solution.
Explain how you came to your conclusion. Include a graph to support your conclusion.
(You will need greater accuracy for this calculation. In Maple, use the command: Digits:=24)
                                                                                        (4 points)

3. Now use a = 3.5
(a) Find a 2-point periodic solution and a 4-point periodic solution.
    Which one appears to be the attractor? Explain how you know. Refer to your Maple work.
(b) Is there a 3-point periodic solution? Support your answer with a Maple graph.
                                                                                        (8+2=10 points)

Problem 4.
As we saw in the Devaney video, functions other than the quadratic (logistic) curve can also exhibit similar types of behavior when iterated – a key feature is that the function must have a single "hump" in the interval we are studying.

Consider $f(x) = a \cos(x)$;   that is: $x_{n+1} = a \cos(x_n)$ on the interval $-4 \leq x \leq 4$

                                         (Note the INTERVAL is different from Prob. 3!)

Modify and use the **Maple worksheet** to study this function.

For this problem, hand in your Maple worksheet – delete all long lists of output – keep important graphs and commands. SHRINK graphs!

Also -- summarize your findings in the table below – fill in the values for SS and periodic points.

                                         (Give 4 decimal places in the chart.)

Part (A) To begin, let **a = 1.** Do the following:
- (i)   Pick an initial value and plot the cobweb diagram
- (ii)   State what the sequence $\{x_n\}$ seems to be doing (approaching SS, approaching periodic solution?)
- (iii)   Find the steady state.
- (iv)   Determine whether there is 2-point periodic solution; if so, find it.
- (v)   Determine whether there is a 4-point periodic solution; if so, find it.

Repeat steps (i) through (v) for
Part (B)   **a** = 1.4,   **Part (C)**   **a** = 1.9;   and   **Part (D)**   **a** = 2.9   (24 points)

	Steady state	2-pt Periodic	4-pt Periodic	Which is stable?
(A) a = 1				
(B) a = 1.4				
(C) a= 1.9				
(D) a = 2.9				

Problem 5.
Use linear stability analysis to determine whether the steady state is stable in Parts (A), (B), (C) above. Does Linear Stability Analysis give the same results as your computer exploration in Problem 4?

                                         (8 points)

Problem 6.
(a) True or False?
    Solutions of the (continuous) logistic differential equation: $dy/dt = r\, y(1 - y)$ with $y \geq 0$; $r > 0$

    and solutions of the discrete logistic equation: $x_{n+1} = a\, x_n (1 - x_n)$; $a > 0$

    display similar behavior for all values of the parameters **a** and **r**.   Explain briefly.

(b) If you are attempting to model a population, does it matter whether you use the continuous or discrete logistic equation? Is one more appropriate than the other? Explain briefly.

                                         (6 points)

MAPLE WORKSHEET FOR ITERATED FUNCTIONS

Cobweb Plot and Solutions for Iterated Functions

This worksheet can be used to study the behavior of iterations of a function:
$$x_{n+1} = f(x_n) \quad \text{which includes the parameter "}a\text{"}.$$
As currently set up, f is a function that maps [0,1] into [0,1]
You will need to modify the functions, and **you may need to modify the interval**, the number of points you want to plot, etc.

This sheet includes commands for
  ◦ Plotting the Cobweb Diagram
  ◦ Plotting x_n vs. n
  ◦ Calculating x_n
  ◦ Finding steady states and periodic points

The easiest way to use this sheet is to make a copy of the whole thing, modify for your function, and DELETE all the parts you don't need.(Delete my comments and put in your own.)

You will need to re-use sections of this worksheet repeatedly. You should **restart** when you change parameter values to avoid Maple errors -- saving each **"a"** value as a separate worksheet would work as well.

(Written by Z. Kadas; last updated 10/21/2016)

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> *restart* :

> *with*(*plots*) :

Define the function f to be iterated *(right now it is the Discrete Logistic Eqn.)*
> $f := x \rightarrow a \cdot x \cdot (1 - x);$

Pick a value for parameter "a"
> $a := 3.15;$

Choose a starting value:

> $x[0] := 0.2;$

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*Plotting the Cobweb
> $cob := [x[0], 0];$

This loop creates a list called "*cob*" consisting of all the points in the cobweb diagram in the order in which they are to be traced.
> **for** i **from** 1 **to** 20 **do**
   $x[i] := f(x[i-1]) :$
   $cob := cob, [x[i-1], x[i]], [x[i], x[i]] :$
   end do:
The following command just prints the list of points -- you can delete it:
> $cob;$

Next we create the plot structures for the curve, cobweb and diagonal, and display them:

> $curve := plot(f(x), x = 0..1, color = blue)$:

> $cobweb := plot([cob], color = red)$:

> $diag := plot(x, x = 0..1, color = black)$:

> $display(\{curve, diag, cobweb\});$

∘ **Plotting** x_n **vs.** n

We may want to look at the sequence of values x_n (the horizontal axis is just n).

> $sol := seq([n, x[n]], n = 0..20);$

> $plot([sol]);$

Note that the "plot" command connects the dots.

∘ **Calculating repeated iterations**

To help see what is going on, we can simply calculate x_j for as many j values as we want. Make a longer list if the trend isn't obvious. (**But beware** -- sometimes even a long list doesn't make things clear! -- and doing hundreds of calculations does produce roundoff error.)

> **for** j **from** 1 **to** 20 **do**
    $x[j] := f(x[j-1])$
    end do;

Please DON'T print out long lists -- you can copy part of the output if needed.

∘ **Steady States and Periodic Points - numerically and graphically**

When seeking steady states and periodic solutions, additional accuracy is sometimes helpful. For 16 digits, use the command:

> $Digits := 16;$

We can find a steady state by saying

> $fsolve(f(x) = x, x);$

Maple solves the equation exactly if it can; otherwise it provides an approximation.

Maple **may** not find all solutions of this equation without some guidance -- you may have to specify an interval:

> $fsolve(f(x) = x, x, x = 0.5..1);$

To decide whether you have found all the steady states, it helps to look at a plot of $y = f(x)$ and $y = x$.

> $plot(\{f(x), x\}, x = 0..1);$

To find the periodic points of period 2 we can do this:

> $fsolve(f(f(x)) = x, x);$

To check graphically whether all solutions have been found:

> $plot(\{f(f(x)), x\}, x = 0..1);$

Sometimes Maple finds all the solutions. If not, (or if you are looking for a specific solution) you can indicate the interval where Maple should seek the solution--based on the graph

$> fsolve(\ f(f(x)) = x, x, 0.3..0.6);$

 To get the whole periodic solution, "follow" the periodic point(s) :

$>\ f(\%);$

$>\ f(\%);$

So you are back to the first point of the two-point periodic solution.
The solution (rounded to 4 decimal places) consists of the pair of values { 0.7840, 0.5335 } .

To find solutions of different periods, adjust the "*fsolve*" command accordingly. (What equation must a point in the 3-point periodic solution solve?)