A Comparison with Horowitz (1998)

It is possible to study Horowitz’s (1998) smoothed QR estimator using the same tools we employ to document the asymptotic behavior of our convolution-type kernel QR estimator. Let \( \tau \in (0, 1) \) and Assumptions X, Q and K hold. Let now \( \mathcal{R}_h^{(j)}(b; \tau) := \mathbb{E}[\tilde{R}_h^{(j)}(b; \tau)] \) for \( j = 0, 1, 2 \) and \( \mathbf{b}_h(\tau) := \text{arg min}_b \mathcal{R}_h(b; \tau) \). The latter corresponds to the unique solution of the first-order condition \( \mathcal{R}_h^{(1)}(\mathbf{b}_h(\tau); \tau) = 0 \) for \( h \) small enough. It turns out that \( \tilde{R}_h^{(2)}(b; \tau) = \frac{1}{n} \sum_{i=1}^n X_i \omega_i \alpha \left( e_i(b)/h \right) \), where \( \alpha(t) := 2k(t) + tk^{(1)}(t) \). Integrating by parts shows that \( \int \psi k^{(1)}(t) dt = -(j + 1) \int \psi k(t) dt \), so that \( \alpha(\cdot) \) is a kernel function with the same order than \( k(\cdot) \). Accordingly, Horowitz’s (1998) smoothed objective function also satisfies Lemma 1.

Along the same lines as in the proof of Theorem 1,

\[
\mathbf{b}_h(\tau) - \beta_h(\tau) = - \left[ \mathcal{R}_h^{(2)}(\beta(\tau); \tau) + o(1) \right]^{-1} \mathcal{R}_h^{(1)}(\beta_h(\tau); \tau)
\]

\[
= \left[ D^{-1}(\tau) + o(1) \right] \mathbb{E} \left[ X \left( \frac{e(\beta_h(\tau))}{h} \right) k \left( \frac{e(\beta_h(\tau))}{h} \right) \right].
\]

Because \( k(\cdot) \) is symmetric and of order \( s + 1 \), Theorem 1 implies that

\[
\mathbb{E} \left[ X \left( \frac{e(\beta_h(\tau))}{h} \right) k \left( \frac{e(\beta_h(\tau))}{h} \right) \right] = h \mathbb{E} \left[ X \int z k(z) f \left( X' \beta_h(\tau) + h z \right) k(z) d z \right] = h^{s+1} \int z^{s+1} k(z) d z \mathbb{E} \left[ Xf(s) \left( X' \beta(\tau) \right) | X \right] + o(h^{s+1})
\]

and that \( \mathbf{b}_h(\tau) = \beta_h(\tau) + (s + 1) h^{s+1} B(\tau) + o(h^{s+1}) = \beta(\tau) + s h^{s+1} B(\tau) + o(h^{s+1}) \). This means that \( \mathbf{b}_h(\tau) - \beta(\tau) = -s(\beta_h(\tau) - \beta(\tau)) + o(h^{s+1}) \), so that Horowitz’s (1998) smoothing approach amplifies the bias by a factor \(-s\) asymptotically.

We next consider the asymptotic covariance matrix of Horowitz’s smoothed QR estimator. Consider \( b_h(\tau) = \beta(\tau) + O(h^2) \) and let \( \Delta_h(\tau) := \tilde{R}_h^{(1)}(b_h(\tau); \tau) - \tilde{R}_h^{(1)}(\beta_h(\tau); \tau) \). We first observe that \( \mathbb{V} \left[ \sqrt{n} \Delta_h(\tau) \right] = O(h) \), whereas using the fact \( y = X'b_h(\tau) - hu \) yields under Assumption Q2 that

\[
n \text{Cov} \left( \tilde{R}_h^{(1)}(b_h(\tau); \tau), \Delta_h(\tau) \right) = \mathbb{E} \left[ \mathbb{E} \left[ X X' \int \left[ \tau - K \left( \frac{e(b_h(\tau))}{h} \right) \right] \frac{e(b_h(\tau))}{h} k \left( \frac{e(b_h(\tau))}{h} \right) f(y|X) d y \right] \right] = h \int [K(u) - \tau] u k(u) d u \mathbb{E} \left[ X X' f \left( X' \beta_h(\tau) \right) | X \right] d y + O(h^2) = h \int_{0}^{\infty} [K(u) - K(-u)] u k(u) d u \mathbb{E} \left[ X X' f \left( X' \beta(\tau) \right) | X \right] d y + O(h^2)
\]

for any symmetric kernel \( k(\cdot) \). Because \( \int_{0}^{\infty} [K(u) - K(-u)] u k(u) d u > 0 \) for second-order and \( \text{bona fide} \) higher-order kernels, there exists a symmetric positive \( M_\tau \) such that

\[
\mathbb{V} \left[ \sqrt{n} \tilde{R}_h^{(1)}(\beta(\tau); \tau) \right] = \mathbb{V} \left[ \sqrt{n} \tilde{R}_h^{(1)}(\beta(\tau); \tau) \right] + h[M_\tau + o(1)].
\]
It then follows from Lemma 1 that $\mathcal{R}_h^{(2)}(b_h(\tau); \tau) = D(\tau) + o(1)$, and hence
\[
\mathbb{V}[\mathcal{R}_h^{(2)}(b_h(\tau); \tau)^{-1} \mathcal{R}_h^{(1)}(b_h(\tau); \tau)] = \mathbb{V}\left[ R_h^{(2)}(b_h(\tau); \tau)^{-1} R_h^{(1)}(b_h(\tau); \tau) \right] + h D^{-1}(\tau) M_r D^{-1}(\tau) + o(h).
\]

Horowitz’s estimator has a Bahadur-Kiefer representation as in Theorem 2, ergo the above equality shows that the asymptotic covariance matrix of Horowitz’s estimator is larger than ours at the second order.

B Technical proofs

Proof of Lemma 1 Under Assumption Q2, a Taylor expansion with integral remainder yields
\[
f(v + hz | x) = \sum_{\ell=0}^{s} f^{(\ell)}(v | x) \frac{(hz)^{\ell}}{\ell!} + \frac{(hz)^{s}}{(s-1)!} \int_{0}^{1} (1-w)^{s-1} \left[ f^{(s)}(v + whz | x) - f^{(s)}(v | x) \right] \, dw.
\]

(i) Assumption K1 ensures that
\[
\mathbb{E}[k_h(v - Y) | x] - f(v | x) = \int k_h(v - y) f(y | x) \, dy - f(v | x)
\]
\[
= \int k(z) \left[ f(v + hz | x) - f(v | x) \right] \, dz
\]
\[
= \int_{0}^{1} (1-w)^{s-1} \int \frac{(hz)^{s}}{(s-1)!} k(z) \left[ f^{(s)}(v + whz | x) - f^{(s)}(v | x) \right] \, dz \, dw \quad (23)
\]
through a change of variables $y = v + hz$. Now, the check function is such that
\[
\int \rho_\tau(v) \, dG(v) = (1 - \tau) \int_{-\infty}^{0} G(v) \, dv + \tau \int_{0}^{\infty} [1 - G(v)] \, dv
\]
for any arbitrary cdf $G$, and hence
\[
R(b; \tau) = \int \left\{ (1 - \tau) \int_{-\infty}^{0} \int_{-\infty}^{t+x'b} f(v | x) \, dv \, dt + \tau \int_{0}^{\infty} f(v | x) \, dv \, dt \right\} dF_X(x),
\]
where $F_X(x)$ is the cdf of $X$. Similarly,
\[
R_h(b; \tau) = \int \left\{ (1 - \tau) \int_{-\infty}^{0} \int_{-\infty}^{t+x'b} \mathbb{E}[k_h(v - Y) | x] \, dv \, dt + \tau \int_{0}^{\infty} \int_{t+x'b}^{\infty} \mathbb{E}[k_h(v - Y) | x] \, dv \, dt \right\} dF_X(x).
\]
It follows from (23) that
\[
L_1 := \left| \int_{-\infty}^{0} \int_{-\infty}^{t+x'b} \left\{ \mathbb{E}[k_h(v - Y) | x] - f(v | x) \right\} \, dv \, dt \right|
\]
\[
= \left| \int_{0}^{1} (1-w)^{s-1} \int \frac{(hz)^{s}}{(s-1)!} k(z) \int_{-\infty}^{0} \int_{-\infty}^{t+x'b} \left[ f^{(s)}(v + whz | x) - f^{(s)}(v | x) \right] \, dv \, dt \, dz \, dw \right|
\]
\[
= \left| \int_{0}^{1} (1-w)^{s-1} \int \frac{(hz)^{s}}{(s-1)!} k(z) \left[ f^{(s-2)}(x'b + whz | x) - f^{(s-2)}(x'b | x) \right] \, dz \, dw \right|
\]
given that $\int |z^{s+1} k(z)| \, dz < \infty$ by Assumption K1 and that $f^{(s-2)}(\cdot | \cdot)$ is Lipschitz. Analogously,
\[
\left| \int_{0}^{\infty} \int_{t+x'b}^{\infty} \mathbb{E}[k_h(v - Y) | x] - f(v | x) \, dv \, dt \right| \leq C h^{s+1},
\]
establishing the result.
(ii) By the definitions of $R(b; \tau)$ and $R_h(b; \tau)$, it follows from the Lebesgue dominated convergence theorem that

\begin{equation}
R^{(1)}(b; \tau) = \mathbb{E} \left[ X \left( F(X'b \mid X) - \tau \right) \right] = \int x \left[ \int_{-\infty}^{x^b} f(y \mid x) \, dy - \tau \right] \, dF_X(x),
\end{equation}

and that

\begin{equation}
R^{(1)}_h(b; \tau) = \mathbb{E} \left\{ X \left[ K \left( \frac{X'b - Y}{h} \right) - \tau \right] \right\} = \int x \left\{ \int_{-\infty}^{x^b} \mathbb{E} \left[ k_h(v - Y) \mid x \right] \, dv - \tau \right\} \, dF_X(x). \tag{24}
\end{equation}

In view that $\int z^s k(z) \, dz = 0$ and $\int |z^{s+1} k(z)| \, dz < \infty$, integrating (23) yields

\begin{equation}
L_2 := \int_{-\infty}^{x^b} \mathbb{E} \left[ k_h(v - Y) \mid x \right] - f(v \mid x) \, dv \begin{align*}
&= \int_0^1 (1 - w)^{s-1} \int \left( \frac{h^s z}{(s-1)!} \right) k(z) \int_{-\infty}^{x^b} \left[ f^{(s)}(v + wz \mid x) - f^{(s)}(v \mid x) \right] \, dv \, dz \, dw \\
&= \int_0^1 (1 - w)^{s-1} \int \left( \frac{h^s z}{(s-1)!} \right) k(z) \left[ f^{(s)}(z' + wz \mid x) - f^{(s-1)}(z' \mid x) \right] \, dz \, dw \\
&= \int_0^1 w(1 - w)^{-1} \int \left( \frac{h^s z}{(s-1)!} \right) k(z) \int_0^1 f^{(s)}(z' + twz \mid x) \, dt \, dz \, dw \leq C h^{s+1}, \tag{25}
\end{align*}
\end{equation}

uniformly given that $f^{(s)}$ is bounded. The result then readily follows from Assumption X.

(iii) Differentiating $R^{(1)}(b; \tau)$ with respect to $b$ results in

\begin{equation}
R^{(2)}(b; \tau) = \mathbb{E} \left[ XX' f(X'b \mid X) \right] = \int xx' f(x'b \mid x) \, dF_X(x)
\end{equation}

and, likewise,

\begin{equation}
R^{(2)}_h(b; \tau) = \mathbb{E} \left[ XX' k_h(X'b - Y) \right] = \int xx' \mathbb{E} \left[ k_h(x'b - Y) \mid x \right] \, dF_X(x).
\end{equation}

Setting $v = x'b$ in (23) then yields

\begin{equation}
\left\| R^{(2)}_h(b; \tau) - R^{(2)}_h(b; \tau) \right\| \leq C \left\| \mathbb{E} \left[ k_h(v - Y) \mid x \right] - f(v \mid x) \right\|
\end{equation}

\begin{equation}
\leq C h^{s} \int |z^s K(z)| \sup_{(x,y) \in \mathbb{R}^{d+1}} \sup_{|t| \leq hz} \left| f^{(s)}(y + t \mid x) - f^{(s)}(y \mid x) \right| \, dz = o(h^s),
\end{equation}

under Assumptions X and Q2 by the Lebesgue dominated convergence theorem, as stated.

(iv) Recall that

\begin{equation}
R^{(2)}_h(b; \tau) = \mathbb{E} \left[ XX' k_h(X'b - Y) \right] = \int k(z) \int xx' f(x'b + hz \mid x) \, dF_X(x) \, dz.
\end{equation}

Under Assumption Q2, it ensues from $f(\cdot \mid \cdot)$ being Lipschitz that

\begin{equation}
\left\| R^{(2)}_h(b + \delta; \tau) - R^{(2)}_h(b; \tau) \right\| \leq C \int |k(z)| \int \|xx'\| \|x'\delta\| \, dF_X(x) \, dz \leq C \|\delta\|,
\end{equation}

uniformly in $(b, h, \delta, \tau)$, completing the proof. \hfill \blacksquare
Proof of Lemma 3  For $\eta > 0$,
\[
\left\{ \sup_{(\tau, h)} \| \hat{b}_h(\tau) - b_h(\tau) \| \geq 2\eta \right\} \subseteq \bigcup_{(\tau, h)} \left\{ \inf_{\{b : \|b - b_h(\tau)\| \geq 2\eta\}} \hat{R}_h(b; \tau) \leq \inf_{\{b : \|b - b_h(\tau)\| \leq 2\eta\}} \hat{R}_h(b; \tau) \right\}
\subseteq \bigcup_{(\tau, h)} \left\{ \inf_{\{b : \|b - b_h(\tau)\| \geq 2\eta\}} \hat{R}_h(b; \tau) \leq \hat{R}_h(b_h(\tau); \tau) \right\}
\subseteq \bigcup_{(\tau, h)} \left\{ \inf_{\{b : \|b - b_h(\tau)\| \geq 2\eta\}} \hat{R}_h(b; \tau) \leq 0 \right\}.
\]
given that $\hat{R}_h(b_h(\tau); \tau) = 0$. Theorem 1 ensures that
\[
\left\{ b : \|b - b_h(\tau)\| \geq 2\eta \right\} \subseteq \left\{ b : \|b - \beta(\tau)\| + \sup_{(\tau, h)} \|\hat{b}_h(\tau) - \beta(\tau)\| \geq 2\eta \right\}
\subseteq \left\{ b : \|b - \beta(\tau)\| + O(\bar{h}^{-1}) \geq 2\eta \right\}
\subseteq \left\{ b : \|b - \beta(\tau)\| \geq \eta \right\}
\]
for all $(\tau, h)$ provided that $n$ is large enough. This means that
\[
\left\{ \sup_{(\tau, h)} \| \hat{b}_h(\tau) - b_h(\tau) \| \geq 2\eta \right\} \subseteq \bigcup_{(\tau, h)} \left\{ \inf_{\{b : \|b - \beta(\tau)\| \geq \eta\}} \hat{R}_h(b; \tau) \leq 0 \right\}.
\]
As $t \mapsto \rho_t(t)$ is 1-Lipschitz, it follows from
\[
\hat{R}_h(b; \tau) = \frac{1}{n h} \sum_{i=1}^{n} \int \rho_t(t) k \left( \frac{t - (Y_i - X_i'b)}{h} \right) dt = \frac{1}{n} \sum_{i=1}^{n} \int \rho_t(Y_i - X_i'b + h z) k(z) dz
\]
that
\[
\left| \hat{R}_h(b; \tau) - \hat{R}(b; \tau) \right| = \frac{1}{n} \sum_{i=1}^{n} \left| \int \rho_t(Y_i - X_i'b + h z) - \rho_t(Y_i - X_i'b) \right| k(z) dz \leq h \int |z k(z)| dz < \infty,
\]
for all $b$, $\tau$ and $h$ by Assumption K1. Theorem 1 and the Lipschitz property of $b \mapsto \hat{R}(b; \tau)$ then ensures that $\hat{R}_h(b; \tau) \geq \hat{R}(b; \tau) - C h$ uniformly in $b$ and $\tau$, so that
\[
\left\{ \sup_{(\tau, h)} \| \hat{b}_h(\tau) - b_h(\tau) \| \geq 2\eta \right\} \subseteq \bigcup_{(\tau, h)} \left\{ \inf_{\{b : \|b - \beta(\tau)\| \geq \eta\}} \hat{R}_h(b; \tau) \leq C h \right\}.
\]
The next step is a convexity argument. We first perform the change of variables $b = \beta(\tau) + \rho u$ with $\|u\| = 1$ and $\rho \geq \eta$. In view that $b \mapsto \hat{R}(b; \tau)$ is convex with $\hat{R}(\beta(\tau); \tau) = 0$,
\[
\frac{\eta}{\rho} \hat{R}(\beta(\tau) + \rho u; \tau) = \frac{\eta}{\rho} \hat{R}(\beta(\tau) + \rho u; \tau) + \left( 1 - \frac{\eta}{\rho} \right) \hat{R}(\beta(\tau); \tau) \geq \hat{R}(\beta(\tau) + \eta u; \tau).
\]
It follows from the above inequality that
\[
\left\{ \inf_{\{b : \|b - \beta(\tau)\| \geq \eta\}} \hat{R}(b; \tau) \leq C h \right\} \subseteq \left\{ \inf_{\{b : \|b - \beta(\tau)\| = \eta\}} \hat{R}(b; \tau) \leq C h \right\},
\]
for all $(\tau, h)$ provided that $n$ is large enough.
and hence
\[
\bigcup_{(\tau, h)} \left\{ \left\| \beta_h(\tau) - \beta_h(\psi) \right\| \geq 2\eta \right\} \subset \bigcup_{\tau} \left\{ \inf_{(b, \|b - \beta(\psi)\| = \eta)} \hat{R}(b; \tau) \leq C \tilde{h}_n \right\}
\]
\[
\subset \left\{ \inf_{\tau} \left\{ \inf_{(b, \|b - \beta(\psi)\| = \eta)} \left[ \hat{R}(b; \tau) - R(b; \tau) \right] \right\} \leq C \tilde{h}_n - \inf_{\tau} \left\{ \inf_{(b, \|b - \beta(\psi)\| = \eta)} R(b; \tau) \right\} \right\}.
\]

We next establish an upper bound for \(C \tilde{h}_n - \inf_{\tau \in [T, \bar{T}]} \inf_{(b, \|b - \beta(\psi)\| = \eta)} R(b; \tau)\) using the fact that the eigenvalues of \(R^{(1)}(b; \tau)\) are bounded away from 0 uniformly in \(b\), for \(\|b - \beta(\psi)\| \leq 1\) and \(\tau \in [T, \bar{T}]\).

Given that \(R^{(1)}(\beta(\psi), \tau) = 0\), a second-order Taylor expansion of \(R(b; \tau) = R(b; \tau) - R(\beta(\psi); \tau)\) gives way to
\[
R(b; \tau) = 0 + (b - \beta(\psi))' \left[ \int_0^1 (1 - t) R^{(2)}(\beta(\psi) + t[b - \beta(\psi)]; \tau) \, dt \right] (b - \beta(\psi)) \geq C \eta^2
\]
for all \(b\) such that \(\|b - \beta(\psi)\| = \eta\). This means that, for any \(\eta_2 = \eta - \epsilon_2 < \eta\) with conformable \(\epsilon_2\) and \(\tilde{h}_n\) small enough,
\[
\bigcup_{(\tau, h)} \left\{ \left\| \beta_h(\tau) - \beta_h(\psi) \right\| \geq 2\eta \right\} \subset \left\{ \sup_{\tau \in [T, \bar{T}]} \sup_{(b, \|b - \beta(\psi)\| = \eta)} \left| \hat{R}(b; \tau) - R(b; \tau) \right| \geq C \eta_2^2 \right\}.
\]

Now, let \(Z_i = (Y_i, X_i')\), \(\theta = (\tau, b')\) and \(g_1(Z_i, \theta) = \rho_\tau(Y_i - X_i'b) - \rho_\tau(Y_i - X_i'\beta(\psi))\), so that
\[
\hat{R}(b; \tau) - R(b; \tau) = \frac{1}{n} \sum_{i=1}^n \left\{ g_1(Z_i, \theta) - \mathbb{E}[g_1(Z_i, \theta)] \right\}.
\]

Under Assumption X, it follows from \(\eta \leq 1\) that, for all \(b\) such that \(\|b - \beta(\psi)\| = \eta\) and \(\tau \in [T, \bar{T}]\),
\[
|g_1(Z_i, \theta)| \leq \|X_i\| \|b - \beta(\psi)\| \leq C,
\]
implying that \(\mathbb{V}(g_1(Z_i, \theta)) \leq \sigma^2 \leq C\). Observe also that pairing Assumption X with the Lipschitz conditions on \(\tau \mapsto \beta(\psi)\) in Assumption Q1 and on \(\tau \mapsto \rho_\tau(u)\) entails, for all admissible \(z\),
\[
|g_1(z, \theta_1 - g_1(z, \theta_2)| \leq C \|\theta_1 - \theta_2\|, \tag{26}
\]
where \(\|\theta\|^2 = \|b\|^2 + |\tau|^2\). Next, for \(\delta > 0\), let \(\Theta_j\), with \(j = 1, \ldots, J(\delta) \leq C \delta^{-(d+1)}\), be such that
\[
\Theta = \left\{ \theta = (b, \tau): \tau \in [T, \bar{T}], \|b - \beta(\psi)\| = \eta_1 \right\} \subset \bigcup_{j=1}^{J(\delta)} \mathcal{B}(\Theta_j, \delta),
\]
where \(\mathcal{B}(\Theta_j, \delta)\) is the \(\|\cdot\|\)-ball with center \(\Theta_j\) and radius \(\delta\). Define \(\bar{g}_{1,j}(\cdot)\) and \(\bar{g}_{1,j}(\cdot)\) respectively as \(\bar{g}_{1,j}(z) := \inf_{\theta \in \mathcal{B}(\Theta_j, \delta)} g_1(z, \theta)\) and \(\bar{g}_{1,j}(z) := \sup_{\theta \in \mathcal{B}(\Theta_j, \delta)} g_1(z, \theta)\), so that \(\{g_1(\cdot, \theta): \theta \in \mathcal{B}(\Theta_j, \delta)\} \subset \{g_{1,j}, \bar{g}_{1,j}\}\). Let \(G_{1,j, \Theta} := \{g_1(\cdot, \theta): \theta \in \Theta\} \subset \bigcup_{j=1}^{J(\delta)} \{g_{1,j}, \bar{g}_{1,j}\}\). It follows from (26) that \(\bar{g}_{1,j}(z) - g_{1,j}(z) \leq C \delta \leq C\) and \(\mathbb{E} \left[ \left| g_{1,j}(Z_i) - \bar{g}_{1,j}(Z_i) \right|^2 \right] \leq C \delta^2\). By conditions (i) and (ii) in Lemma 2, it follows from (18) that setting \(H(\delta) = -(d + 1) \ln \delta + C\) leads to
\[
\Pr \left( \sup_{\delta \in \Theta} \left| \hat{R}(b; \tau) - R(b; \tau) \right| \geq C \frac{1 + \sqrt{\tau} + r/\sqrt{n}}{\sqrt{n}} \right) \leq \exp(-r).
\]
This means that, for \( n \) large enough with respect to \( \eta_2 \),

\[
\Pr \left( \sup_{\mathcal{R}} \sup_{\{ b : \| b - \beta(\tau) \| = \eta_2 \}} \left| \hat{R}(b, \tau) - \mathcal{R}(b, \tau) \right| \geq C \eta_2^2 \right) \leq C \exp \left( -n C \eta_2^2 \right),
\]

and hence

\[
\Pr \left( \sup_{(\tau, h)} \left\| \hat{\beta}_h(\tau) - \beta_h(\tau) \right\| \geq 2\eta \right) \leq C \exp \left( -n C \eta_2^2 \right),
\]

completing the proof.

**Proof of Lemma 4** We start with the first deviation probability. As \( \tilde{R}_h^{(1)}(\beta_h(\tau), \tau) = 0 \),

\[
\sup_{(\tau, h)} \left\| \sqrt{n} \tilde{R}_h^{(1)}[\beta_h(\tau), \tau] \right\| \leq \sup_{(\tau, h)} \sup_{\{ b : \| b - \beta_h(\tau) \| \leq \eta \}} \left\| \sqrt{n} \left( \tilde{R}_h^{(1)}(b, \tau) - \tilde{R}_h^{(1)}(b, \tau) \right) \right\|.
\]

However,

\[
\tilde{R}_h^{(1)}(b, \tau) = \frac{\partial}{\partial b} \left[ \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{Z}} \rho_{\tau}(Y_i - X_i'b + b h z) k(z) \, dz \right] = \frac{1}{n} \sum_{i=1}^{n} X_i \left[ \int \mathbb{I}(Y_i - X_i'b + b h z < 0) k(z) \, dz - \tau \right],
\]

implying that \( \tilde{R}_h^{(1)}(b, \tau) = \sum_{i=1}^{n} g_2(Z_i, \theta) / n \), with

\[
g_2(Z_i, \theta) = X_i \left[ \int \mathbb{I}(Y_i - X_i'b + h z < 0) k(z) \, dz - \tau \right],
\]

for \( Z_i = (Y_i, X_i)' \) and \( \theta \in \Theta := \{ (\tau, h) : (\tau, h) \in [\mathbb{Z}, \tau] \times [b_n, \bar{h}_n], \| b - \beta_h(\tau) \| \leq \eta \} \). We bound each of the entries of \( \tilde{R}_h^{(1)}(b, \tau) \), so that there is no loss of generality in assuming that \( X_i \) is univariate. Note that \( |g_2(Z_i, \theta)| \leq C, \mathbb{V}(g_2(Z_i, \theta)) \leq \sigma^2 \leq C \), and \( |g_2(Z_i, \theta_2) - g_2(Z_i, \theta_1)| \leq C \) for all \( \theta_1 \) and \( \theta_2 \). Let \( \| \theta \| = \| b \|^2 + |h|^2 + |\tau|^2 \) and let \( B(\theta, \delta^2) \) denote the \( \| - \| - \) ball with center \( \theta \) and radius \( \delta^2 \). Assumption X ensures that, for any \( \theta_1 \) and \( \theta_2 \) in \( B(\theta, \delta^2) \),

\[
|g_2(Z_i, \theta_2) - g_2(Z_i, \theta_1)| \leq C \left[ \int \mathbb{I}(Y_i - X_i'b + h z \in [-C \delta^2, C \delta^2]) |k(z)| \, dz + \delta^2 \right]. \tag{27}
\]

Consider a covering of \( \Theta \) with \( J(\delta^2) \leq C \delta^{-2(d+1)} \) balls \( B(\theta_j, \delta^2) \). Letting \( g_{2,j}(z) := \inf_{\theta \in B(\theta_j, \delta^2)} g_2(z, \theta) \) and \( \bar{g}_{2,j}(z) = \sup_{\theta \in B(\theta_j, \delta^2)} g_2(z, \theta) \) implies not only that \( \{ g_{2,j}(\cdot, \theta) : \theta \in B(\theta_j, \delta^2) \} \subset [g_{2,j}, \bar{g}_{2,j}] \), but also that \( G_{2,\delta} := \{ g_2(\cdot, \theta) : \theta \in \Theta \} \subset \bigcup_{j=1}^{J(\delta^2)} [g_{2,j}, \bar{g}_{2,j}] \). Equation (27) ensures that, uniformly in \( j \) and \( \delta^2 \leq \sigma^2 \),

\[
\mathbb{E} \left[ \left| \bar{g}_{2,j}(Z_i) - g_{2,j}(Z_i) \right|^2 \right] \leq C \delta^4 + C \mathbb{E} \left[ \int \mathbb{I}(Y_i - X_i'b + h z \in [-C \delta^2, C \delta^2]) |k(z)| \, dz \right]^2.
\]

Applying the Cauchy-Schwarz inequality under Assumptions K and Q2 then gives way to

\[
L_4 := \mathbb{E} \left[ \int \mathbb{I}(Y_i - X_i'b - h z \in [-C \delta^2, C \delta^2]) k(z) \, dz \right]^2 \leq \mathbb{E} \left[ \int \mathbb{I}(Y_i - X_i'b - h z \in [-C \delta^2, C \delta^2]) |k(z)| \, dz \right] \times \int |k(z)| \, dz \leq \mathbb{E} \left\{ \Pr \left( Y_i - X_i'b - h z \in [-C \delta^2, C \delta^2] \mid X_i \right) \right\} |k(z)| \, dz \times \int |k(z)| \, dz \leq C \delta^2,
\]

The proof is complete.
implying that $\mathbb{E} \left[ \left| \tilde{g}_{2j}(Z_i) - g_{2j}(Z_i) \right|^2 \right] \leq C(\delta^4 + \delta^2) \leq C\delta^2$, uniformly in $j$ and $\delta^2 \leq \sigma^2$. As a result, conditions (i) and (ii) in Lemma 2 hold for $\ln H(\delta) = -2(d + 1) \ln \delta + C$, so that (18) gives

$$\Pr \left( \sup_{\delta \in \Theta} \left\| \frac{1}{\sqrt{n}} \left( \tilde{R}_{h}^{(1)}(b, \tau) - R_{h}^{(1)}(b, \tau) \right) \right\| \geq C \left( \sqrt{\tau + 1 + \tau / \sqrt{n}} \right) \right) \leq 2 \exp(-r).$$

Accordingly, the first bound holds for $n$ large enough. As for the second bound, there is no loss of generality to assume that $X_i$ is unidimensional. Note that $\sqrt{n}h / \ln n \tilde{R}_{h}^{(2)}(b, \tau) = \sum_{i=1}^{n} g_{3}(Z_i, \theta) / \sqrt{n}$, with

$$g_{3}(Z_i, \theta) := \frac{1}{h \ln n} X_{i}^{2} k \left( \frac{X_{i} b - Y_{i}}{h} \right).$$

Assumptions K and X ensure that, uniformly for $\theta \in \Theta$,

$$|g_{3}(Z_i, \theta)| \leq C \frac{1}{h \ln n} \leq C \frac{O(\sqrt{n})}{\ln^{2} n}.$$

It also follows from Assumption Q2 that, uniformly for $\theta \in \Theta$,

$$\mathbb{V}(g_{3}(Z_i, \theta)) \leq C \frac{C}{\ln n} \int \int k \left( \frac{x' b - y}{h} \right) f(y | x) \, dy \, dF_{X}(x)$$

$$= C \frac{1}{\ln n} \int \int k(v) f(x' b + h v | x) \, dv \, dF_{X}(x) \leq C \frac{1}{\ln n} = \sigma^2.$$

Assumption K posits that, for any $\theta_1$ and $\theta_2$ in $\Theta$, $|g_{3}(Z_i, \theta_1) - g_{3}(Z_i, \theta_2)| \leq C n^{-\gamma} \| \theta_1 - \theta_2 \|. $ Consider a covering of $\Theta$ with $J(\delta / n^{\gamma}) \leq C(\delta / n^{\gamma})^{-\gamma(d+1)}$ balls $B(\theta, \delta / n^{\gamma})$ and let $g_{3j}(z) := \inf_{\theta \in B(\theta, \delta)} g_{3}(z, \theta)$ and $\bar{g}_{3j}(z) := \sup_{\theta \in B(\theta, \delta)} g_{3}(z, \theta)$. It then turns out that $\{g_{3j}(z, \theta) : \theta \in B(\theta_j, \delta) \} \subset [g_{3j}, \bar{g}_{3j}]$ and hence $\mathcal{G}_{3,\Theta} = \{g_{3} (\cdot, \theta) : \theta \in \Theta \} \subset \bigcup_{j=1}^{J(\delta / n^{\gamma})} [g_{3j}, \bar{g}_{3j}]$, with $\mathbb{E} \left[ \left| g_{3}(Z_i) - g_{3}(Z_i) \right|^2 \right] \leq C \delta^2$. Conditions (i) and (ii) in Lemma 2 thus hold for $\ln H(\delta) = -2(d + 1) (\ln \delta - C \ln n) + C$, so that (18) results for any $u > 0$ in

$$\Pr \left( \sup_{\delta \in \Theta} \left\| \frac{\sqrt{n}h}{\ln n} \left( \tilde{R}_{h}^{(2)}(b, \tau) - R_{h}^{(2)}(b, \tau) \right) \right\| \geq C \left( 1 + \frac{\sqrt{u}}{\sqrt{\ln n}} + \frac{u}{\ln n} \right) \right) \leq 2 \exp(-u).$$

Setting $u = r \ln n$ then yields the exponential inequality.

Suppose now, without loss of generality, that $\mathcal{B}$ is convex. Recall that

$$\tilde{R}_{h}^{(2)}(b, \tau) - R_{h}^{(2)}(b, \tau) = \frac{1}{n} \sum_{i=1}^{n} X_{i} X_{i}' (b_{i} - b_{0}) \int_{0}^{1} \frac{1}{h^2} k^{(1)} \left( \frac{Y_{i} - X_{i}' [b_{1} + t(b_{1} - b_{0})]}{h} \right) \, dt$$

and that the variance of $h^{-2} k^{(1)} ((Y_{i} - X_{i}' b) / h)$ is of order $h^{-3} = o(n / \ln n)$ under Assumption K. Applying now the same arguments as in the proof of the exponential inequality yields

$$\frac{1}{n} \sum_{i=1}^{n} X_{i} X_{i}' (b_{i} - b_{0}) \frac{1}{h^2} k^{(1)} \left( \frac{Y_{i} - X_{i}' [b_{1} + t(b_{1} - b_{0})]}{h} \right)$$

$$= \mathbb{E} \left[ X X' X (b_{1} - b_{0}) \int_{-\infty}^{\infty} \frac{1}{h} k \left( \frac{y - X [b_{1} + t(b_{1} - b_{0})]}{h} \right) f^{(1)}(y | X) \, dy \right] + O_{P} \left( \sqrt{\ln n / nh^3} \right),$$

uniformly in $(\tau, h, b_{0}, b_{1})$ for $t \in [0, 1]$. The proofs of the remaining results follow similarly.
Proof of Proposition 3  Let
\[ \mathcal{E}_{n}^{3}(\varepsilon) := \left\{ \sup_{(\tau, h)} \left\| \hat{\beta}_{h}(\tau) - \beta_{h}(\tau) \right\| \geq \varepsilon^{1/4} \right\}, \]
which is such that \( \Pr(\mathcal{E}_{n}^{3}(\varepsilon)) \leq C \exp(-C n \varepsilon) \) by Lemma 3. The bounds for \( \Pr(\mathcal{E}_{n}^{1}(r)) \) and \( \Pr(\mathcal{E}_{n}^{2}(r)) \) follow from Lemma 4. In particular, \( \lim_{n \to \infty} \Pr(\mathcal{E}_{n}^{2}(r)) = 0 \), whereas Lemma 1 ensures under Assumption X that \( b \to \hat{R}_{h}(b; \tau) \) is strictly convex for \( b \) in a vicinity of \( \beta_{h}(\tau) \), for all \( \tau \) in \( [\tau, \bar{\tau}] \) with probability at least \( 1 - \Pr(\mathcal{E}_{n}^{1}(r)) - \Pr(\mathcal{E}_{n}^{2}(r)) \). But, by Lemma 3 and Theorem 1, all minimizers of \( \hat{R}_{h}(b; \tau) \) lie in such a vicinity with a probability tending to 1. This means that we can make \( 1 - \Pr(\mathcal{E}_{n}^{1}(r)) - \Pr(\mathcal{E}_{n}^{2}(r)) \) arbitrarily close to 1 by increasing \( r \), and hence \( \hat{\beta}_{h}(\tau) \) is unique with a probability going to 1 as \( n \) increases. It also follows that, in case \( \hat{\mathcal{E}}_{n}^{1}(r), \hat{\mathcal{E}}_{n}^{2}(r) \) and \( \hat{\mathcal{E}}_{n}^{3}(\varepsilon) \) are all true and \( n \) is large enough, \( \hat{\beta}_{h}(\tau) \) satisfies the first-order condition \( \hat{R}_{h}^{(1)}(\hat{\beta}_{h}(\tau); \tau) = 0 \). Recall from the proof of Theorem 1 that \( \hat{R}_{h}^{(2)}(\cdot; \tau) \) has an inverse in the vicinity of \( \beta_{h}(\tau) \) for \( n \) large enough on \( \mathcal{E}^{2}(r) \). Applying the implicit function theorem then yields \( \hat{\beta}_{h}(\tau) \) continuous over the admissible \((\tau, h)\). Accordingly,
\[
-\hat{R}_{h}^{(1)}(\hat{\beta}_{h}(\tau); \tau) = \hat{R}_{h}^{(1)}(\hat{\beta}_{h}(\tau); \tau) - \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau)
= [\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)] \int_{0}^{1} \hat{R}_{h}^{(2)}(\beta_{h}(\tau) + t[\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)]; \tau) \, dt.
\]
Now, if \( \varepsilon \) in \( \mathcal{E}_{n}^{3}(\varepsilon) \) is small enough, the eigenvalues of the above matrix are in \([1/C, C]\) for a large \( C \) provided that \( n \) is large enough, uniformly in \( \tau \) and \( h \). This means that
\[
\hat{\beta}_{h}(\tau) - \beta_{h}(\tau) = -\left[ \int_{0}^{1} \hat{R}_{h}^{(2)}(\beta_{h}(\tau) + u[\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)]; \tau) \, du \right]^{-1} \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau).
\]
Lemma 1(iv) then implies that, for a generic constant \( C \) coming from Bernstein-type inequalities,
\[
P_{2} := \left\| \sqrt{n} (\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)) + \left[ R_{h}^{(2)}(\beta_{h}(\tau); \tau) \right]^{-1} \sqrt{n} \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau) \right\|
\leq C \left\| \int_{0}^{1} \left[ \hat{R}_{h}^{(2)}(\beta_{h}(\tau) + u[\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)]; \tau) - R_{h}^{(2)}(\beta_{h}(\tau) + u[\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)]; \tau) \right] \, du \left\| \sqrt{n} \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau) \right\| \right. \\
+ C \left\| \int_{0}^{1} \left[ \hat{R}_{h}^{(2)}(\beta_{h}(\tau) + u[\hat{\beta}_{h}(\tau) - \beta_{h}(\tau)]; \tau) - R_{h}^{(2)}(\beta_{h}(\tau); \tau) \right] \, du \left\| \sqrt{n} \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau) \right\| \right. \\
\leq C \left\{ \sqrt{\frac{\ln n}{nh}} r^{2} + \left\| \hat{\beta}_{h}(\tau) - \beta_{h}(\tau) \right\| \sqrt{n} \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau) \right\} \\
\leq C \left\{ \sqrt{\frac{\ln n}{nh}} r^{2} + n^{-1/2} \left\| \sqrt{n} \hat{R}_{h}^{(1)}(\beta_{h}(\tau); \tau) \right\|^{2} \right\} \\
\leq C \left\{ \sqrt{\frac{\ln n}{nh}} + \frac{1}{\sqrt{n}} \right\} r^{2}
\]
on \( \mathcal{E}_{n}^{1}(r) \) and \( \mathcal{E}_{n}^{2}(r) \), implying that \( \mathcal{E}_{n}(r) \) holds as long as \( C_{0} \) of the Proposition is large enough. ■

Proof of Lemma 5  Let \( h = h_{n} \) to simplify notation. We first note that \( \mathbb{E}(\sqrt{n} \hat{S}_{h}(\tau)) = 0 \). In addition, for any \( \alpha, \tau \in [\tau, \bar{\tau}] \), it follows that
\[
\mathbb{V}(\sqrt{n} \hat{S}_{h}(\tau), \sqrt{n} \hat{S}_{h}(\tau)) = \mathbb{E} \left\{ XX' \left[ K \left( -\frac{e(\beta_{h}(\tau))}{h} \right) - \tau \right] \left[ K \left( -\frac{e(\beta_{h}(\tau))}{h} \right) - \tau \right] \right\}
\]
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converges to \( \mathbb{E}\{XX'(I[X^\prime \beta(\cdot) \geq Y] - \tau)(I[X^\prime \beta(\cdot) \geq Y] - \tau)\} \) as \( n \to \infty \). A simple computation using iterated expectations then yields the limiting covariance structure in (22).

By the Cramér-Wold device, in order to obtain weak convergence for the \( d \)-dimensional process \( \{\sqrt{n} \hat{S}_h : \tau \in [\bar{\tau}, \bar{\tau}]\} \), it suffices to consider the convergence in distribution of the linear form \( \{\sqrt{n} \lambda' \hat{S}_h : \tau \in [\bar{\tau}, \bar{\tau}]\} \), where \( \lambda \) is an arbitrary (fixed) vector in \( \mathbb{R}^d \). Assume without loss of generality that \( \|X\| \leq 1 \) and \( \|\lambda\| \leq 1 \), and let \( Z = (Y, X) \in \mathbb{R} \times \mathbb{R}^d \) and, similarly, \( Z_i = (Y_i, X_i) \). Define now \( g_{n,\tau} : \mathbb{R} \times \text{supp}X \to \mathbb{R} \) for \( x = (y, x) \) as

\[
g_{n,\tau}(x) := x \lambda \left\{ K \left( \frac{x' \beta_h(\tau) - y}{\hat{h}} \right) - \tau \right\},
\]

where \( x_\lambda = \lambda' x \) and \( X_\lambda = \lambda' X \), and consider the class of functions \( \mathcal{G}_n = \{g_{n,\tau} : \tau \in [\bar{\tau}, \bar{\tau}]\} \). Letting \( \mathbb{P} \) and \( \mathbb{P}_n \) respectively denote the distribution of \( Z \) and the empirical distribution of the sample \( (Z_1, \ldots, Z_n) \) yields

\[
\sqrt{n} \lambda' \hat{S}_h(\tau) = \sqrt{n}(\mathbb{P}_n g_{n,\tau} - \mathbb{P} g_{n,\tau}).
\]

In other words, the process \( \{\sqrt{n} \lambda' \hat{S}_h : \tau \in [\bar{\tau}, \bar{\tau}]\} \) is an empirical process indexed by a (changing) class of functions \( \mathcal{G}_n \). By Theorem 19.28 in van der Vaart (1998), it suffices to establish that

\[
\sup_{|\tau - \varsigma| < \delta(n)} \mathbb{E}|g_{n,\tau}(Z) - g_{n,\varsigma}(Z)|^2 \to 0 \tag{30}
\]

and that, for any \( \delta(n) \downarrow 0 \),

\[
\int_0^{\delta(n)} \sqrt{\ln N_{\|\|}(\epsilon, \mathcal{G}_n, L^2(\mathbb{P}))} \, d\epsilon \to 0 \tag{31}
\]

with \( N_{\|\|}(\epsilon, \mathcal{G}_n, L^2(\mathbb{P})) \) denoting the minimum number of \( \epsilon \)-brackets in \( L^2(\mathbb{P}) \) required to cover \( \mathcal{G}_n \). The remaining requirements of Theorem 19.28 indeed hold trivially in view that the index set \( [\bar{\tau}, \bar{\tau}] \) is a compact—and so, totally bounded—metric space, and that the changing classes \( \mathcal{G}_n \) admit envelope functions \( G_n \equiv 1 \) for all \( n \) that satisfy the Lindeberg condition \( \mathbb{E}_2(G_n^2 \mathbb{E}[G_n > \sqrt{n}\epsilon]) \to 0 \).

Let \( \partial_\tau := \frac{\partial}{\partial \tau} \). By Lemma 1 and Theorem 1, applying twice the implicit function theorem yields

\[
\partial_\tau \beta_h(\tau) = -D_h(\tau)^{-1} \partial_\tau R_h^{(1)}(\beta_h(\tau); \tau) = D_h(\tau)^{-1} \mathbb{E}(X) = (D(\tau) + o(1))^{-1} \mathbb{E}(X) = \partial_\tau \beta(\tau) + o(1)
\]

uniformly for \( (\tau, h) \in [\bar{\tau}, \bar{\tau}] \times [\bar{h}, \bar{h}] \). This implies, by Assumption Q1, that \( \sup \|\partial_\tau \beta_h(\tau)\| \leq C \) for \( n \) large enough, with supremum taken over \( (\tau, h) \in [\bar{\tau}, \bar{\tau}] \times [\bar{h}, \bar{h}] \), and so \( \|\beta_h(\tau) - \beta_h(\varsigma)\| \leq C|\tau - \varsigma| \). It also follows from the inverse function theorem and Assumption Q1 that \( \tau \mapsto x' \beta_h(\tau) \) is strictly increasing in \( \tau \), for any \( x \in \text{supp}X \) and \( n \) large enough. In what follows, we assume that \( n \) is large enough, so that the above holds.

Now, let \( \tau_L \leq \tau_U \leq \bar{\tau} \) and consider two random elements (possibly degenerate) \( \bar{\tau} \) and \( \bar{\varsigma} \) in \( [\tau_L, \tau_U] \). The mean value theorem and Assumption Q2 then ensure that

\[
\Pr\left( x' \beta_h(\bar{\tau} \wedge \bar{\varsigma}) - hu \leq Y \leq x' \beta_h(\bar{\tau} \vee \bar{\varsigma}) - hu \mid X = x \right) \leq C |\tau_U - \tau_L|, \tag{32}
\]

uniformly for \( u \in \mathbb{R} \) and \( x \in \text{supp}X \), given that \( [x' \beta_h(\bar{\tau} \wedge \bar{\varsigma}), x' \beta_h(\bar{\tau} \vee \bar{\varsigma})] \subset [x' \beta_h(\tau_L), x' \beta_h(\tau_U)] \) and \( |x' \beta_h(\tau_U) - x' \beta_h(\tau_L)| \leq C|\tau_U - \tau_L| \). Define \( \Upsilon_u = \{X' \beta_h(\bar{\tau} \wedge \bar{\varsigma}) - Y \leq hu \leq X' \beta_h(\bar{\tau} \vee \bar{\varsigma}) - Y\} \).
It follows from \( |g_{n,\tau}(Z) - g_{n,\zeta}(Z)| \leq \int \mathbb{I}(\mathcal{Y}_u) \, |k(u)| \, du + |\tau - \zeta| \) that

\[
\mathbb{E}|g_{n,\tau}(Z) - g_{n,\zeta}(Z)|^2 \leq \mathbb{E}|\tau - \zeta|^2 + 2\mathbb{E}\left[ |\tau - \zeta| \int \mathbb{I}(\mathcal{Y}_u) \, |k(u)| \, du \right] + \mathbb{E}\left[ \int \mathbb{I}(\mathcal{Y}_u) \, |k(u)| \, du \right]^2 \\
\leq |\tau_U - \tau_L|^2 + 2 |\tau_U - \tau_L| \int \Pr(\mathcal{Y}_u) \, |k(u)| \, du + C \int \Pr(\mathcal{Y}_u) \, |k(u)| \, du \\
\leq C |\tau_U - \tau_L|, \tag{33}
\]

(34) given that the Cauchy-Schwarz inequality implies that \( \int \mathbb{I}(\mathcal{Y}_u) \, |k(u)| \, du \int |k(u)| \, du \) is an upper bound for \( \left( \int \mathbb{I}(\mathcal{Y}_u) \, |k(u)|^{1/2} \, |k(u)|^{1/2} \, du \right)^2 \), \( \int |k(u)| \, du < \infty \) by Assumption K1, and \( \Pr(\mathcal{Y}_u) = \mathbb{E}\left[ \Pr(\mathcal{Y}_u | X) \right] \leq C |\tau_U - \tau_L| \) by iterated expectations and (32). Taking \( \tau \) and \( \zeta \) to be deterministic shows that (30) holds, for all \( \delta(n) \downarrow 0 \).

We now obtain a set of brackets whose bracketing number is of order \( 1/\epsilon \). For \( \epsilon > 0 \) small enough, we cover the interval \([\tau, \tau]\) with \( J(\epsilon) \leq \lfloor (\tau - \tau) / \epsilon + 1 \rfloor \leq 2/\epsilon \) open intervals \( B_i = (\tau_i - \epsilon, \tau_i + \epsilon) \), and let \( \tilde{g}_i^i(z) = \sup_{r \in B_i} g_{n,\tau}(z) \) and \( g_i^i(z) = \inf_{r \in B_i} g_{n,\tau}(z) \). It is straightforward to appreciate that the collection formed by the brackets \([g_i^i, \tilde{g}_i^i]\), with \( i = 1, \ldots, J(\epsilon) \), covers \( \mathcal{G}_n \) and that these suprema and infima are attained in the closure of \( B_i \).\(^6\) In particular, \( \tilde{g}_i^i(Z) = g_{n,\tau_i}(Z) \) and \( g_i^i(Z) = g_{n,\zeta_i}(Z) \), where \( \tau_i \) and \( \zeta_i \) are random elements in \([\tau_i - \epsilon, \tau_i + \epsilon]\). Resorting to (34) once more then gives

\[
\mathbb{E}|\tilde{g}_i^i(Z) - g_i^i(Z)|^2 \leq C\epsilon,
\]

and, as a result, \( N_{\|/(\epsilon, \mathcal{G}_n, L^2(P)) \leq C/\epsilon \). This ensures that (31) holds for all \( \delta(n) \downarrow 0 \).

---

\(^6\) For \( n = 1 \) and \( n = J(\epsilon) \), the intervals are actually \([\tau, \tau_i + \epsilon]\) and \([\tau_i - \epsilon, \tau]\), respectively. For simplicity of exposition, we keep the notation as above.