

APPENDIX

In this appendix, we attached a sample reading on Integration by Parts and the associated reading quiz from Implementation I. Similarly, we attached a sample reading on Cross Product and the associated reading HW from Implementation II. Additionally, we have 4 charts with captions to expand on our quantitative data.

Calculus-II
Integration by Parts

We have seen so far how to use the substitution rule. It is worth noting that the substitution rule for integrals is the opposite of the chain rule for derivatives:

- Chain Rule: $(f(g(x)))' = f'(g(x))g'(x)$,
- Substitution Rule: $\int f'(g(x))g'(x) dx = f(g(x)) + C$ using $u = g(x)$.

We will find the opposite of the product rule of derivatives. It is called **integration by parts**.

- Product Rule: $(uv)' = u'v + uv'$,
- Integration by Parts: $\int u dv = uv - \int v du$.

To get this formula, we rewrite the product rule as $\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$, then we integrate both sides to get

$$uv = \int v du + \int u dv,$$

which can be written as $\int u dv = uv - \int v du$.

Example 1 To integrate $\int xe^x dx$ we choose $u = x$ and $dv = e^x dx$, thus $du = dx$ and $v = e^x$. Then

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

The other possible choice would be $u = e^x$ and $dv = x dx$, then $du = e^x dx$ and $v = \frac{x^2}{2}$ and thus

$$\int xe^x dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x dx,$$

which is actually more complicated than the one we started with.

Remark 1 The above example shows that when we choose u and dv , we need to keep in mind two things:

1. we have to be able to integrate dv to find v ,
2. the second integral we get should be easier than the one we started with.

Example 2 To integrate $\int x \ln(x) dx$ we choose $u = \ln(x)$, and $dv = x dx$, then $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$.

$$\int x \ln(x) dx = \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2} \cdot \frac{dx}{x} = \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C.$$

These two examples can be generalized as follows. When we have a power of x with e^x , dv is chosen to be $e^x dx$. When we have a power of x with $\ln x$, then dv is chosen to be the power of x .

Reading Quiz:

To find $\int x^2 \ln x dx$ we need to use the formula $\int u dv = uv - \int v du$.

use $u = \ln x$ then $du = \dots\dots\dots dx$

use $dv = x^2 dx$ then integrate to get $v = \dots\dots\dots$

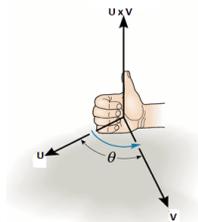
Thus $\int x^2 \ln x dx = (\dots\dots\dots)(\dots\dots\dots) - \int \dots\dots\dots dx =$

Calculus-III

Cross Product

In this section, we introduce a new operation on vectors. We saw previously how the dot product of two vectors gave a scalar. However, the new operation, called cross product, will produce a vector.

Definition 1 The cross product of two vectors \vec{u} and \vec{v} is denoted by $\vec{u} \times \vec{v}$. By definition, it is the vector of magnitude $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ where θ is the angle between the two vectors. The direction



of this new vector is given by the right hand rule

Example 1 Consider the three coordinate vectors $\vec{i}, \vec{j}, \vec{k}$. the angle between any two of these is 90 degrees making $\sin\theta = 1$. Their magnitude is also one. By the right-hand rule we have: $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, and $\vec{k} \times \vec{i} = \vec{j}$.

Definition 2 The cross product has the following properties:

- \vec{u} and \vec{v} are parallel ($\theta = 0, \pi$) if and only if $\vec{u} \times \vec{v} = \vec{0}$.
- If we complete the parallelogram made by \vec{u} and \vec{v} , we can compute its area by $|\vec{u} \times \vec{v}|$.
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
- $(a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v}) = a(\vec{u} \times \vec{v})$.

Example 2 From the first property, we find that $\vec{u} \times \vec{u} = \vec{0}$.

From the third property and example 1, we find that $\vec{j} \times \vec{i} = -\vec{k}$ and $\vec{i} \times \vec{k} = -\vec{j}$.

From the fourth property, we can find $\vec{i} \times (\vec{j} + \vec{k}) = \vec{i} \times \vec{j} + \vec{i} \times \vec{k} = \vec{k} - \vec{j}$.

This last example gives the intuition on finding a formula for $\vec{u} \times \vec{v}$ algebraically. Recall that we can write any vector \vec{u} as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$. Given another vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, we can use the above properties to compute $\vec{u} \times \vec{v}$:

Definition 3 By foiling out $\vec{u} \times \vec{v} = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \times (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$, we get:

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

We will see in class how we can write this formula as a determinant of a matrix, a new idea that you will see in future math classes.

Reading HW:

1. What is the angle between the vector $\vec{u} \times \vec{v}$ and the vectors \vec{u}, \vec{v} ?
2. a. Draw the parallelogram made by the vectors $2\vec{i}$ and $3\vec{j}$. Compute its area using geometrical formulas.
b. Compare this area to $|2\vec{i} \times 3\vec{j}|$.
3. Apply the formula in Definition 3 to compute $\langle 1, 1, 1 \rangle \times \langle -1, 0, 2 \rangle$.

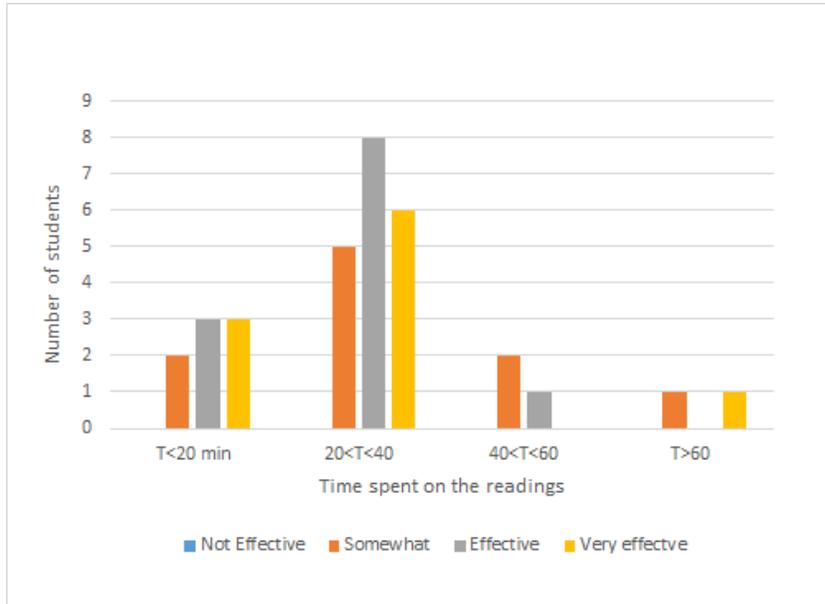


Figure 1. Time spent on readings vs. effectiveness of the readings (Implementation I)

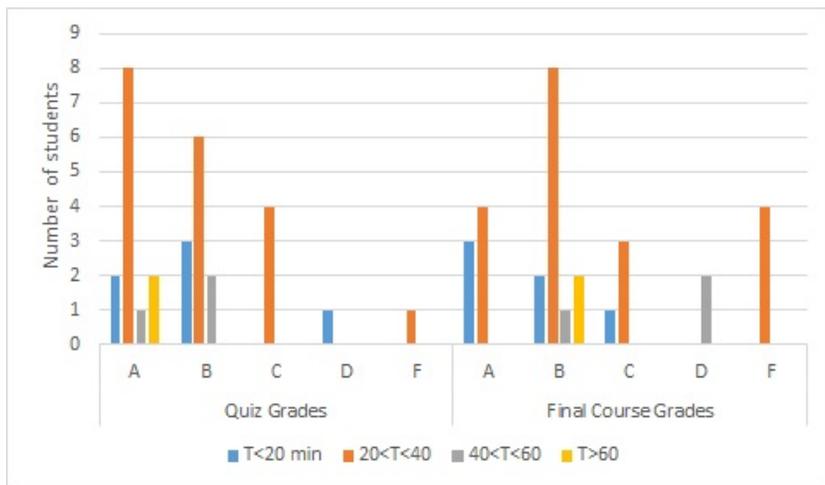


Figure 2. Time spent on readings vs. quiz and final course grades (Implementation I)

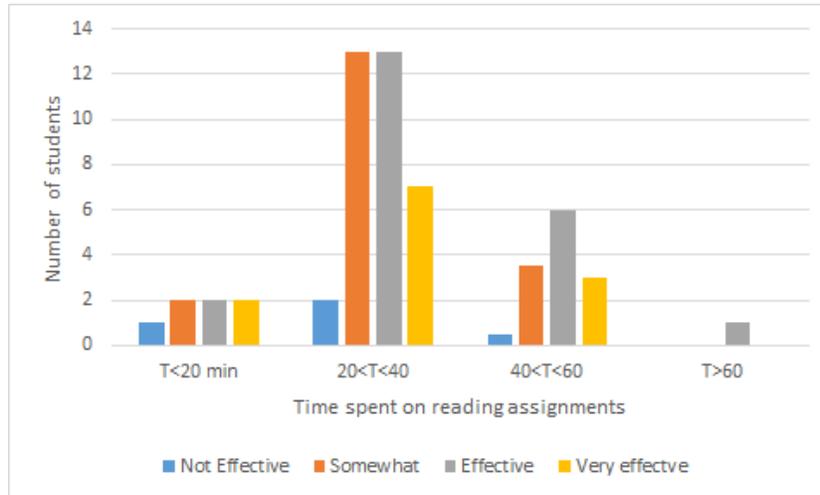


Figure 3. Time spent on readings vs. effectiveness of the readings (Implementation II)

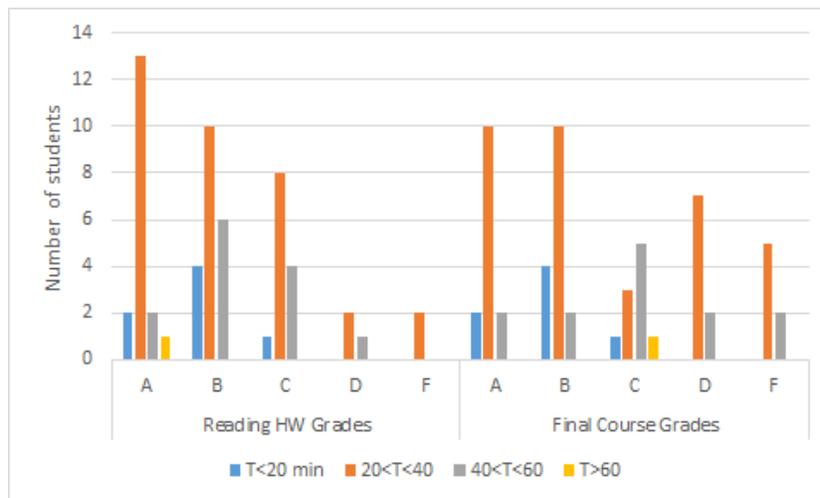


Figure 4. Time spent on readings vs. reading homework and final course grades (Implementation II)