Appendix for "Who is the Key Player? A Network Analysis of Juvenile Delinquency"

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A 2SLS and GMM Estimation

A.1 Asymptotic properties

Under the maintained assumptions, it follows by similar arguments as in Kelejian and Prucha (2010) that

$$\sqrt{n}(\widehat{\theta}_{2sls} - \theta) \xrightarrow{d} N(0, [\text{plim}_{n\to\infty} n^{-1}(\underline{X}'P^*\underline{X})^{-1}\underline{X}'P^*\Sigma P^*\underline{X}(\underline{X}'P^*\underline{X})^{-1}]^{-1}),$$

where $P^* = Z^*(Z^{*\prime}Z^*)^{-1}Z^{*\prime}$.

Furthermore, let

$$\Omega = \begin{bmatrix} Z^{*\prime}\Sigma Z^* & 0 \\ 0 & \text{tr}(\Sigma G^*\Sigma G^*) + \text{tr}(\Sigma G^*\Sigma G^{*\prime}) \end{bmatrix}$$

and

$$D = \begin{bmatrix} Z^{*\prime}G(I_n - \lambda G)^{-1}(\beta_0\iota_n + X\beta_1 + \bar{X}\beta_2) & Z^{*\prime}\iota_n & Z^{*\prime}X & Z^{*\prime}\bar{X} \\ \text{tr}[(G^* + G^{*\prime})G(I_n - \lambda G)^{-1}\Sigma] & 0 & 0 & 0 \end{bmatrix}$$

where $S = I_n - \lambda G$. It follows by similar arguments as in Lin and Lee (2010) that

$$\sqrt{n}(\widehat{\theta}_{gmm} - \theta) \xrightarrow{d} N(0, (\text{plim}_{n\to\infty} n^{-1}D'\Omega^{-1}D)^{-1}).$$

A.2 Monte Carlo experiments

To investigate the finite sample performance of the proposed estimators, we conduct some simulation experiments. In the experiments, the outcome equation is given by

$$y = \lambda Gy + X\beta + u,$$

and the corresponding network g is realized with the probability

$$\Pr(g) = \frac{\exp[\mathcal{Q}(g)]}{\sum_{\widetilde{g}\in\mathcal{G}_n}\exp[\mathcal{Q}(\widetilde{g})]}.$$

The *potential function* $\mathcal{Q}(g)$ takes the form

$$\mathcal{Q}(g) = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}[\delta_0 + W_{ij}\delta_1 + \delta_2 \sum_{k\neq i,j} g_{ik} + \eta_i + \eta_j],$$

where $W_{ij} = 2 - (X_i - X_j)^2$ measures the similarity between agents i and j in exogenous characteristics. We generate X_i independently from the standard normal distribution. We generate v_i and η_i jointly from a bivariate normal distribution

$$\begin{bmatrix} v_i \\ \eta_i \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sigma_{12} \\ \sigma_{12} & 1 \end{bmatrix}\right),$$

and $u_i = \sigma_i v_i$, where σ_i takes the value of $\{1, \sqrt{2}, \sqrt{3}\}$ with equal probability. In the data generating process, we set $\lambda = 0.1$, $\beta = 0.5$, $\delta_0 = 0$, $\delta_1 = 0.5$ and $\delta_2 = -0.5$, and experiment with different values for σ_{12}.

Let

$$\widehat{g}_{ij} = \frac{\exp(\widehat{\delta}_0 + W_{ij}\widehat{\delta}_1)}{1 + \exp(\widehat{\delta}_0 + W_{ij}\widehat{\delta}_1)}$$

where $\widehat{\delta}_0$ and $\widehat{\delta}_1$ are obtained from a logistic regression of g_{ij} on W_{ij}, and let $\widehat{G} = [\widehat{g}_{ij}/\widehat{d}]$, where $\widehat{d} = \max\{\max_i \sum_{j=1}^{n} \widehat{g}_{ij}, \max_j \sum_{i=1}^{n} \widehat{g}_{ij}\}$. The McFadden's pseudo-$R^2$ of the logistic regression is about 0.04. We consider the following estimators: (a) "2SLS-1", the conventional 2SLS estimator with the IV matrix $Q = [X, GX]$; (b) "2SLS-2", the 2SLS estimator defined in Equation (10) with the IV matrix $\widehat{Q} = [X, \widehat{G}X]$; and (c) "GMM", the GMM estimator defined in Equation (11). The estimation results are reported in Table A.1. We use robust measures of central tendency and dispersion, namely, the median, the median of the absolute deviations (Med. AD), the difference between the 0.1 and 0.9 quantile (Dec. Rge) in the empirical distribution of the estimates from 1000 simulation replications. There are two main observations from the experiment. First, the conventional 2SLS estimator, namely "2SLS-1", is biased when the adjacency matrix is endogenous. The "2SLS-1" estimate of λ is upwards biased and the "2SLS-1" estimate of β is downwards biased. As σ_{12} increases, the magnitude of the bias increases. By contrast, "2SLS-2" and "GMM" estimates are essentially unbiased for all cases considered. Second, The "GMM" estimator substantially reduces the dispersion (in terms of Med. AD and Dec. Rge) of the "2SLS-1" and "2SLS-2" estimators.

B DMH Algorithm

We assign the prior distributions of model parameters and unknown variables as follows:

1. Individual latent variable: $\eta_i \sim N(0, \sigma_\eta^2 I_n)$, with $\sigma_\eta^2 \sim \kappa \text{Inv}\chi^2(\alpha)$. The hyper-parameters κ and α are to be specified by the user.

2. Parameters in the link formation: $\delta \sim N(\mu_\delta, \sigma_\delta^2 I)$.

The DMH algorithm takes the following steps:

Step I. Simulate $\widetilde{\eta}_i$ from $p(\widetilde{\eta}_i | g, \delta, \eta_{-i})$ by the DMH algorithm, for $i = 1, \cdots, n$.

I.1. Propose $\widetilde{\eta}_i$ from a random walk proposal density $q_\eta(\widetilde{\eta}_i | \eta_i)$.

I.2. Simulate auxiliary data $\widetilde{g}$ by R runs of Algorithm 1, starting from the observed g.

I.3. Accept $\widetilde{\eta}_i$ according to the acceptance probability

$$\begin{aligned} \alpha_\eta &= \min\left\{1, \frac{\pi(g|\delta,\widetilde{\eta})}{\pi(g|\delta,\eta)} \cdot \frac{p(\widetilde{\eta}_i|\sigma_\eta^2)}{p(\eta_i|\sigma_\eta^2)} \cdot \frac{\pi(\widetilde{g}|\delta,\eta)}{\pi(\widetilde{g}|\delta,\widetilde{\eta})}\right\} \\ &= \min\left\{1, \frac{\exp[\mathcal{Q}(g|\delta,\widetilde{\eta})]}{\exp[\mathcal{Q}(g|\delta,\eta)]} \cdot \frac{p(\widetilde{\eta}_i|\sigma_\eta^2)}{p(\eta_i|\sigma_\eta^2)} \cdot \frac{\exp[\mathcal{Q}(\widetilde{g}|\delta,\eta)]}{\exp[\mathcal{Q}(\widetilde{g}|\delta,\widetilde{\eta})]}\right\}, \end{aligned}$$

where $p(\eta_i|\sigma_\eta^2)$ denotes the density function of $N(0, \sigma_\eta^2 I_n)$.

Step II. Simulate $\widetilde{\sigma}_\eta^2$ from $[\kappa + \sum_{i=1}^n (\widetilde{\eta}_i)^2]\text{Inv}\chi^2(\alpha + n)$ by a standard Gibbs sampler.

Step III. Simulate $\widetilde{\delta}$ from $p(\widetilde{\delta}|g, Y, \widetilde{\eta})$ by the DMH algorithm.

III.1. Propose $\widetilde{\delta}$ from a random walk proposal density $q_\delta(\widetilde{\delta}|\delta)$.

III.2. Simulate auxiliary data $\widetilde{g}$ by R runs of Algorithm 1, starting from the observed g.

III.3. Accept $\widetilde{\delta}$ according to the acceptance probability

$$\begin{aligned}\alpha_\delta &= \min\left\{1, \frac{\pi(g|\widetilde{\delta},\widetilde{\eta})}{\pi(g|\delta,\widetilde{\eta})} \cdot \frac{p(\widetilde{\delta})}{p(\delta)} \cdot \frac{\pi(\widetilde{g}|\delta,\widetilde{\eta})}{\pi(\widetilde{g}|\widetilde{\delta},\widetilde{\eta})}\right\} \\ &= \min\left\{1, \frac{\exp[\mathcal{Q}(g|\widetilde{\delta},\widetilde{\eta})]}{\exp[\mathcal{Q}(g|\delta,\widetilde{\eta})]} \cdot \frac{p(\widetilde{\delta})}{p(\delta)} \cdot \frac{\exp[\mathcal{Q}(\widetilde{g}|\delta,\widetilde{\eta})]}{\exp[\mathcal{Q}(\widetilde{g}|\widetilde{\delta},\widetilde{\eta})]}\right\}.\end{aligned}$$

References

Kelejian, H. H. and Prucha, I. R. (2010). Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances, *Journal of Econometrics* **157**: 53–67.

Lin, X. and Lee, L. F. (2010). GMM estimation of spatial autoregressive models with unknown heteroskedasticity, *Journal of Econometrics* **157**: 34–52.

Table A.1: Monte Carlo Simulation Results

	$n = 200$		$n = 400$	
	$\lambda = 0.1$	$\beta = 0.5$	$\lambda = 0.1$	$\beta = 0.5$
$\sigma_{12} = 0.4$				
2SLS-1	0.113(0.053)[0.337]	0.479(0.113)[0.474]	0.108(0.035)[0.199]	0.483(0.095)[0.387]
2SLS-2	0.101(0.045)[0.271]	0.492(0.109)[0.525]	0.100(0.029)[0.180]	0.504(0.083)[0.402]
GMM	0.101(0.020)[0.074]	0.503(0.076)[0.311]	0.099(0.011)[0.040]	0.504(0.054)[0.204]
$\sigma_{12} = 0.6$				
2SLS-1	0.112(0.046)[0.260]	0.477(0.104)[0.450]	0.110(0.027)[0.171]	0.483(0.078)[0.337]
2SLS-2	0.101(0.029)[0.173]	0.497(0.098)[0.410]	0.100(0.019)[0.113]	0.503(0.073)[0.294]
GMM	0.101(0.013)[0.047]	0.503(0.072)[0.290]	0.100(0.007)[0.026]	0.503(0.052)[0.193]
$\sigma_{12} = 0.8$				
2SLS-1	0.117(0.040)[0.217]	0.482(0.107)[0.429]	0.112(0.025)[0.137]	0.483(0.078)[0.316]
2SLS-2	0.101(0.022)[0.122]	0.496(0.090)[0.349]	0.100(0.015)[0.085]	0.503(0.066)[0.263]
GMM	0.100(0.010)[0.035]	0.503(0.070)[0.278]	0.100(0.005)[0.019]	0.502(0.052)[0.192]

Median (Med. AD) [Dec. Rge]