

(QA track): This file is extracted from the original derivation file "Lsh2.mw", July 29, 2015. It documents the calculation that yields Eq. (6.1) of the text of the paper "Exploring Riemann's Functional Equation".

> restart;

Here are both sides of the basic equations, left and right

> Gen := Zeta(1, 1-tau)+Zeta(1, tau)\*Pi^(1/2-tau)\*GAMMA((1/2)\*tau)/GAMMA(1/2-(1/2)\*tau) = (ln(2\*Pi)-Psi(tau)+(1/2)\*Pi\*tan((1/2)\*Pi\*tau))\*Zeta(1-tau);

$$Lsh := \text{subs}\left(\tau = \frac{1}{2} + I\rho, \frac{\text{lhs}(\text{Gen})}{\zeta(1, \tau)}\right)$$

Lsh := Zeta(1, 1/2-I\*rho)/Zeta(1, 1/2+I\*rho)+2\*cos((1/2)\*Pi\*(1/2+I\*rho))\*GAMMA(1/2+I\*rho)\*(2\*Pi)^(-1/2-I\*rho);

$$Rsh := \text{subs}\left(\tau = \frac{1}{2} + I\rho, \frac{\text{rhs}(\text{Gen})}{\zeta(1, \tau)}\right)$$

Rsh:=subs(s=1/2+I\*rho, (log(2\*Pi)-Psi(s)+Pi/2\*tan(Pi\*s/2))\*Zeta(1-s)/Zeta(1,s));

check:

evalf(subs(tau=1/2+I\*rho, rho=1.456, Gen));  
 evalf(subs(tau=1/2+I\*rho, rho=1.456, [abs(lhs(Gen)/Zeta(1, tau))^2, abs(rhs(Gen)/Zeta(1, tau))^2]));  
 evalf(subs(tau=1/2+I\*rho, rho=1.456, [abs(Lsh)^2, abs(Rsh)^2]));

$$\text{Gen} := \zeta(1, 1 - \tau) + \frac{\zeta(1, \tau) \pi^{\frac{1}{2} - \tau} \Gamma\left(\frac{1}{2} \tau\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2} \tau\right)} = \left(\ln(2 \pi) - \Psi(\tau) + \frac{1}{2} \pi \tan\left(\frac{1}{2} \pi \tau\right)\right) \zeta(1 - \tau)$$

$$Lsh := \frac{\zeta\left(1, \frac{1}{2} - I\rho\right)}{\zeta\left(1, \frac{1}{2} + I\rho\right)} + 2 \cos\left(\frac{1}{2} \pi \left(\frac{1}{2} + I\rho\right)\right) \Gamma\left(\frac{1}{2} + I\rho\right) (2 \pi)^{-\frac{1}{2} - I\rho}$$

$$Rsh := \frac{\left(\ln(2 \pi) - \Psi\left(\frac{1}{2} + I\rho\right) + \frac{1}{2} \pi \tan\left(\frac{1}{2} \pi \left(\frac{1}{2} + I\rho\right)\right)\right) \zeta\left(\frac{1}{2} - I\rho\right)}{\zeta\left(1, \frac{1}{2} + I\rho\right)}$$

$$0.5126316472 + 0.7583516623 I = 0.5126316466 + 0.7583516625 I$$

$$[3.412337258, 3.412337258]$$

$$[3.412337262, 3.412337258]$$

(1)

This is the modified function, called Q - Eq.(5.3) of the text.

> Rsh0:=Rsh+Zeta(1/2-I\*rho)/Zeta(1/2+I\*rho)-Zeta(1,1/2-I\*rho)/Zeta(1,1/2+I\*rho);;

$$Rsh0 := \frac{\left( \ln(2\pi) - \Psi\left(\frac{1}{2} + I\rho\right) + \frac{1}{2} \pi \tan\left(\frac{1}{2} \pi \left(\frac{1}{2} + I\rho\right)\right) \right) \zeta\left(\frac{1}{2} - I\rho\right)}{\zeta\left(1, \frac{1}{2} + I\rho\right)} \quad (2)$$

$$+ \frac{\zeta\left(\frac{1}{2} - I\rho\right)}{\zeta\left(\frac{1}{2} + I\rho\right)} - \frac{\zeta\left(1, \frac{1}{2} - I\rho\right)}{\zeta\left(1, \frac{1}{2} + I\rho\right)}$$

Calculate the real and imaginary parts of Rsh0 and check  
NOTES:

1. define  $PsR = \Re\left(\Psi\left(\frac{1}{2} + I\rho\right)\right) - \log(2\pi)$
2.  $\Im\left(\Psi\left(\frac{1}{2} + I\rho\right)\right) = \frac{\pi \tanh(\pi\rho)}{2}$  - see NIST 5.4.17
3. Real and imaginary components of  $\zeta\left(\frac{1}{2} + I\rho\right)$  are represented by ZR and ZI;
4. Real and imaginary components of  $\zeta\left(1, \frac{1}{2} + I\rho\right)$  - first derivative - are represented by Z1R and Z1I;
5. Many of the simplification expressions in the following are of the form  $subs(A) = simplify(A)$  where "simplify" can be a combination of "simplify/combine/expand/factor".

The expressions enclosed in brackets were obtained by cut/paste to eliminate the possibility of transcription errors. Each was chosen to coincide with elements of the

Maple "op" internal representation, thus invoking Maple's pattern matching. However, since Maple has an annoying tendency not to yield the same result twice in a row, sometimes writing  $(A-B) = -(-A+B)$ , the pattern matching used here cannot be guaranteed to be reproducible. If that happens, it is recommended that the sub-calculation be restarted, several times if necessary.

```
> PsR=Re (Psi (1/2+I*rho) ) -log (2*Pi) ;
```

$$PsR = \Re\left(\Psi\left(\frac{1}{2} + I\rho\right)\right) - \ln(2\pi) \quad (3)$$

```
> RshC:=subs (Psi (1/2+I*rho)=PsR+log (2*Pi)+I*Pi/2*tanh (Pi*rho) , Zeta (1/2-I*rho)=ZR-I*ZI , Zeta (1, 1/2+I*rho)=Z1R+I*Z1I , Rsh) ;
RshR:=collect (combine (evalc (Re (RshC) ) ) , [cosh , tanh , ln , PsR , sinh] ) ;
RshR:=collect (subs (sinh (Pi*rho)=tanh (Pi*rho) *cosh (Pi*rho) , RshR) , [cosh , tanh , ln , PsR , sinh , Pi] ) ;
RshR:=factor (subs ( (-Z1I*ZR-Z1R*ZI) / (2*Z1I^2+2*Z1R^2) + (Z1I*ZR+Z1R*ZI) / (2*Z1I^2+2*Z1R^2) =simplify ( (-Z1I*ZR-Z1R*ZI) / (2*Z1I^2+2*Z1R^2) + (Z1I*ZR+Z1R*ZI) / (2*Z1I^2+2*Z1R^2) ) , RshR) ) ;
RshI:=collect (combine (evalc (Im (RshC) ) ) , [cosh , tanh , ln , PsR , sinh] ) ;
RshI:=collect (subs (sinh (Pi*rho)=tanh (Pi*rho) *cosh (Pi*rho) , RshI) , [cosh , tanh , ln , PsR , sinh , Pi] ) ;
RshI:=factor (subs ( (Z1I*ZI-Z1R*ZR) / (2*Z1I^2+2*Z1R^2) + (-Z1I*ZI+Z1R*ZR) / (2*Z1I^2+2*Z1R^2) =simplify ( (Z1I*ZI-Z1R*ZR) / (2*Z1I^2+2*Z1R^2) + (-Z1I*ZI+Z1R*ZR) / (2*Z1I^2+2*Z1R^2) ) , RshI) ) ;
```

$$RshC := \frac{\left( -PsR - \frac{1}{2} I\pi \tanh(\pi\rho) + \frac{1}{2} \pi \tan\left(\frac{1}{2} \pi \left(\frac{1}{2} + I\rho\right)\right) \right) (ZR - IZI)}{Z1R + IZ1I}$$

$$\begin{aligned}
RshR &:= \left( \frac{-ZII ZR - ZIR ZI}{2 ZI^2 + 2 ZIR^2} + \frac{ZII ZR + ZIR ZI}{2 ZI^2 + 2 ZIR^2} \right) \pi \tanh(\pi \rho) \\
&+ \frac{(2 ZII ZI - 2 ZIR ZR) PsR}{2 ZI^2 + 2 ZIR^2} + \frac{(-ZII ZI + ZIR ZR) \pi}{(2 ZI^2 + 2 ZIR^2) \cosh(\pi \rho)} \\
RshR &:= \frac{1}{2} \frac{(ZII ZI - ZIR ZR) (2 PsR \cosh(\pi \rho) - \pi)}{\cosh(\pi \rho) (ZI^2 + ZIR^2)} \\
RshI &:= \frac{1}{2} \frac{(ZII ZR + ZIR ZI) (2 PsR \cosh(\pi \rho) - \pi)}{\cosh(\pi \rho) (ZI^2 + ZIR^2)} \tag{4}
\end{aligned}$$

```

> test:=evalf(subs(PsR = Re(Psi(1/2+I*rho))-ln(2*Pi), Z1I=Im(Zeta(1,
1/2+I*rho)), Z1R=Re(Zeta(1,1/2+I*rho)), ZI=Im(Zeta(1/2+I*rho)), ZR=
Re(Zeta(1/2+I*rho)), rho=1.2345, [Rsh, RshR+I*RshI]));
test := [-0.4928239294 + 1.722009030 I, -0.4928239290 + 1.722009030 I] \tag{5}

```

And here are the squares. alpha is the argument of Zeta(1/2+I\*rho); alpha[1] is the argument of Zeta(1, 1/2+I\*rho). In the text, this is called beta

```

> RshR2:=collect(RshR^2, cosh);
RshI2:=collect(RshI^2, cosh);
Rsh2:=collect(RshR2+RshI2, [cosh, PsR, Pi]);
Rsh2:=subs((Z1I*ZI-Z1R*ZR)^2/(Z1I^2+Z1R^2)^2+(Z1I*ZR+Z1R*ZI)^2/
(Z1I^2+Z1R^2)^2=simplify((Z1I*ZI-Z1R*ZR)^2/(Z1I^2+Z1R^2)^2+(Z1I*
ZR+Z1R*ZI)^2/(Z1I^2+Z1R^2)^2),
-(Z1I*ZI-Z1R*ZR)^2/(Z1I^2+Z1R^2)^2-(Z1I*ZR+Z1R*ZI)^2/(Z1I^2+
Z1R^2)^2=simplify(-(Z1I*ZI-Z1R*ZR)^2/(Z1I^2+Z1R^2)^2-(Z1I*ZR+Z1R*
ZI)^2/(Z1I^2+Z1R^2)^2),
(1/4)*(Z1I*ZI-Z1R*ZR)^2/(Z1I^2+Z1R^2)^2+(1/4)*(Z1I*ZR+Z1R*ZI)^2/
(Z1I^2+Z1R^2)^2=simplify((1/4)*(Z1I*ZI-Z1R*ZR)^2/(Z1I^2+Z1R^2)^2+
(1/4)*(Z1I*ZR+Z1R*ZI)^2/(Z1I^2+Z1R^2)^2), Rsh2);
Rsh2:=factor(Rsh2);

Rsh2:=subs(ZI^2+ZR^2=(alpha^2+1)*ZR^2, Z1I^2+Z1R^2=(alpha[1]^2+1)*
Z1R^2, Rsh2);

```

$$\begin{aligned}
RshR2 &:= \frac{(ZII ZI - ZIR ZR)^2 PsR^2}{(ZI^2 + ZIR^2)^2} - \frac{(ZII ZI - ZIR ZR)^2 \pi PsR}{(ZI^2 + ZIR^2)^2 \cosh(\pi \rho)} \\
&+ \frac{1}{4} \frac{(ZII ZI - ZIR ZR)^2 \pi^2}{(ZI^2 + ZIR^2)^2 \cosh(\pi \rho)^2} \\
RshI2 &:= \frac{(ZII ZR + ZIR ZI)^2 PsR^2}{(ZI^2 + ZIR^2)^2} - \frac{(ZII ZR + ZIR ZI)^2 \pi PsR}{(ZI^2 + ZIR^2)^2 \cosh(\pi \rho)} \\
&+ \frac{1}{4} \frac{(ZII ZR + ZIR ZI)^2 \pi^2}{(ZI^2 + ZIR^2)^2 \cosh(\pi \rho)^2} \\
Rsh2 &:= \frac{1}{4} \frac{(ZI^2 + ZR^2) (2 PsR \cosh(\pi \rho) - \pi)^2}{(ZI^2 + ZIR^2) \cosh(\pi \rho)^2} \tag{6}
\end{aligned}$$

$$Rsh2 := \frac{1}{4} \frac{(\alpha^2 + 1) ZR^2 (2 PsR \cosh(\pi \rho) - \pi)^2}{(\alpha_1^2 + 1) ZIR^2 \cosh(\pi \rho)^2} \quad (6)$$

Determine the real and imaginary parts of the additional terms

```
> evalf (subs (rho=11.567, abs (Rsh0) ^2) );
AddT:=Zeta (1/2-I*rho) /Zeta (1/2+I*rho) -Zeta (1, 1/2-I*rho) /Zeta (1,
1/2+I*rho) ;
AddT:=subs (Zeta (1/2-I*rho)=ZR-I*ZI, Zeta (1/2+I*rho)=ZR+I*ZI, Zeta
(1, 1/2-I*rho)=Z1R-I*Z1I, Zeta (1, 1/2+I*rho)=Z1R+I*Z1I, AddT) ;
AddR:=evalc (Re (AddT) ) ;
AddI:=evalc (Im (AddT) ) ;
```

4.0000000008

$$AddT := \frac{\zeta\left(\frac{1}{2} - I\rho\right)}{\zeta\left(\frac{1}{2} + I\rho\right)} - \frac{\zeta\left(1, \frac{1}{2} - I\rho\right)}{\zeta\left(1, \frac{1}{2} + I\rho\right)}$$

$$AddT := \frac{ZR - IZI}{ZR + IZI} - \frac{Z1R - IZ1I}{Z1R + IZ1I}$$

$$AddR := \frac{ZR^2}{ZI^2 + ZR^2} - \frac{ZI^2}{ZI^2 + ZR^2} - \frac{Z1R^2}{Z1I^2 + Z1R^2} + \frac{Z1I^2}{Z1I^2 + Z1R^2}$$

$$AddI := -\frac{2 ZI ZR}{ZI^2 + ZR^2} + \frac{2 Z1I Z1R}{Z1I^2 + Z1R^2}$$

(7)

It is known that  $Rsh0^2=4$ . This is  $\Omega$  of the text. Check and convert to standard notation. Calculate  $Rsh0a=Rsh0^2$  and check

```
> Digits:=25:
Rsh0a:= RshR+AddR+I*(RshI+AddI) ;
evalf (subs (Z1I=Im (Zeta (1, 1/2+I*rho) ) , PsR = Re (Psi (1/2+I*rho) ) -ln
(2*Pi) , Z1I=Im (Zeta (1, 1/2+I*rho) ) , Z1R=Re (Zeta (1, 1/2+I*rho) ) , ZI=Im
(Zeta (1/2+I*rho) ) , ZR=Re (Zeta (1/2+I*rho) ) , rho=11.567, [abs (Rsh0a)
^2, (RshR+AddR) ^2+ (RshI+AddI) ^2] ) ) ;
```

$$Rsh0a := \frac{1}{2} \frac{(Z1I ZI - Z1R ZR) (2 PsR \cosh(\pi \rho) - \pi)}{\cosh(\pi \rho) (Z1I^2 + Z1R^2)} + \frac{ZR^2}{ZI^2 + ZR^2} - \frac{ZI^2}{ZI^2 + ZR^2} - \frac{Z1R^2}{Z1I^2 + Z1R^2} + \frac{Z1I^2}{Z1I^2 + Z1R^2} + I \left( \frac{1}{2} \frac{(Z1I ZR + Z1R ZI) (2 PsR \cosh(\pi \rho) - \pi)}{\cosh(\pi \rho) (Z1I^2 + Z1R^2)} - \frac{2 ZI ZR}{ZI^2 + ZR^2} + \frac{2 Z1I Z1R}{Z1I^2 + Z1R^2} \right)$$

[4.00000000000000000000000000000000000008, 4.00000000000000000000000000000000000007]

(8)

Square by individual components in an attempt to keep things simple(r). Simplify by a sequence of substitutions

```
> Rsh0ASq:= (RshR+AddR) ^2 ;
RshRSq:=RshR2+2*AddR*RshR+AddR^2 ;
RshISq:=RshI2+2*AddI*RshI+AddI^2 ;
Rsh0Sq:=RshR2+2*AddR*RshR+AddR^2+RshI2+2*AddI*RshI+AddI^2 ;
Rsh0Sq:=Rsh2+2*AddR*RshR+AddR^2+2*AddI*RshI+AddI^2 ;
factor (combine (Rsh0Sq) ) ;;
```

```

collect(%, [PsR, cosh(Pi*rho), cosh(2*Pi*rho), ZR, ZI, Z1R, Z1I, Pi]) ;;
> Rsh0Sq:=subs(((2*alpha^2+2)*Z1R^2+(2*alpha^2+2)*Z1I^2)*ZR^4+((2*
alpha^2+2)*Z1R^2+(2*alpha^2+2)*Z1I^2)*ZI^2*ZR^2)*cosh(2*Pi*rho)+
(2*alpha^2+2)*Z1R^2+(2*alpha^2+2)*Z1I^2)*ZR^4+((2*alpha^2+2)*
Z1R^2+(2*alpha^2+2)*Z1I^2)*ZI^2*ZR^2=factor(((2*alpha^2+2)*
Z1R^2+(2*alpha^2+2)*Z1I^2)*ZR^4+((2*alpha^2+2)*Z1R^2+(2*
alpha^2+2)*Z1I^2)*ZI^2*ZR^2)*cosh(2*Pi*rho)+((2*alpha^2+2)*Z1R^2+
(2*alpha^2+2)*Z1I^2)*ZR^4+((2*alpha^2+2)*Z1R^2+(2*alpha^2+2)*
Z1I^2)*ZI^2*ZR^2),
(((2*alpha^2-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2
-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZI^2*ZR^2)*cosh(2*Pi*rho)+((-2*
alpha^2-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2-2)*
Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZI^2*ZR^2)*PsR^2=factor((((2*
alpha^2-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2-2)*
Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZI^2*ZR^2)*cosh(2*Pi*rho)+((-2*
alpha^2-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2-2)*
Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZI^2*ZR^2)*PsR^2),
((-2*alpha^2-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2-2)*
Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZI^2*ZR^2=factor(((2*alpha^2-2)*
Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2-2)*Z1R^2+(-2*
alpha^2-2)*Z1I^2)*ZI^2*ZR^2),
(((4*alpha^2-4)*Pi*Z1R^2+(-4*alpha^2-4)*Pi*Z1I^2)*ZR^4+((-4*
alpha^2-4)*Pi*Z1R^2+(-4*alpha^2-4)*Pi*Z1I^2)*ZI^2*ZR^2)*cosh(Pi*
rho)*PsR/(Z1R^2*(alpha[1]^2+1)*(ZI^2+ZR^2)*(Z1I^2+Z1R^2)*(cosh(2*
Pi*rho)+1))=factor((((4*alpha^2-4)*Pi*Z1R^2+(-4*alpha^2-4)*Pi*
Z1I^2)*ZR^4+((-4*alpha^2-4)*Pi*Z1R^2+(-4*alpha^2-4)*Pi*Z1I^2)*
ZI^2*ZR^2)*cosh(Pi*rho)*PsR/(Z1R^2*(alpha[1]^2+1)*(ZI^2+ZR^2)*
(Z1I^2+Z1R^2)*(cosh(2*Pi*rho)+1))), op(3,%)=factor(op(3,%)), cosh
(2*Pi*rho)+1=2*cosh(Pi*rho)^2, cosh(2*Pi*rho)=2*cosh(Pi*rho)^2-1,
-(2*(alpha^2+1))*(Z1I^2+Z1R^2)*ZR^2*(ZI^2+ZR^2)*(2*cosh(Pi*rho)^2
-1)+((-2*alpha^2-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2
-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZI^2*ZR^2=factor(-(2*(alpha^2+1)
)*(Z1I^2+Z1R^2)*ZR^2*(ZI^2+ZR^2)*(2*cosh(Pi*rho)^2-1)+((-2*alpha^2
-2)*Z1R^2+(-2*alpha^2-2)*Z1I^2)*ZR^4+((-2*alpha^2-2)*Z1R^2+(-2*
alpha^2-2)*Z1I^2)*ZI^2*ZR^2),
((4*alpha^2+4)*Pi*Z1R^2+(4*alpha^2+4)*Pi*Z1I^2)*ZR^4+((4*
alpha^2+4)*Pi*Z1R^2+(4*alpha^2+4)*Pi*Z1I^2)*ZI^2*ZR^2=factor(((4*
alpha^2+4)*Pi*Z1R^2+(4*alpha^2+4)*Pi*Z1I^2)*ZR^4+((4*alpha^2+4)*
Pi*Z1R^2+(4*alpha^2+4)*Pi*Z1I^2)*ZI^2*ZR^2),%);
Rsh0Sq:=collect(%, [PsR, alpha[1], alpha, cosh(Pi*rho), Z1I, Z1R, Pi, ZR,
Z1I]) ;;
Rsh0Sq:=subs(16*Z1I^2*Z1R^2*ZR^2-32*Z1I*Z1R^3*ZI*ZR+16*Z1R^4*
ZI^2=factor(16*Z1I^2*Z1R^2*ZR^2-32*Z1I*Z1R^3*ZI*ZR+16*Z1R^4*
ZI^2), ZI^2*ZR^2+ZR^4=factor(ZI^2*ZR^2+ZR^4),
ZI^2+ZR^2=abs(Zeta)^2,%);
Rsh0Sq:=collect(%, [PsR, abs(Zeta)^2, alpha[1], alpha, cosh(Pi*rho),
Z1I, Z1R, Pi, ZR, Z1I]) ;;
Rsh0Sq:=subs(16*Z1I^2*Z1R^2*ZR^2-32*Z1I*Z1R^3*ZI*ZR+16*Z1R^4*
ZI^2=factor(16*Z1I^2*Z1R^2*ZR^2-32*Z1I*Z1R^3*ZI*ZR+16*Z1R^4*
ZI^2),
(1/4)*((Pi^2*Z1I^2*ZR^2+Pi^2*Z1R^2*ZR^2)*alpha^2+Z1I^2*Pi^2*ZR^2+
Pi^2*ZR^2*Z1R^2)/(Z1R^2*(alpha[1]^2+1)*(Z1I^2+Z1R^2)*cosh(Pi*rho)
^2)=factor((1/4)*((Pi^2*Z1I^2*ZR^2+Pi^2*Z1R^2*ZR^2)*alpha^2+
Z1I^2*Pi^2*ZR^2+Pi^2*ZR^2*Z1R^2)/(Z1R^2*(alpha[1]^2+1)*(Z1I^2+
Z1R^2)*cosh(Pi*rho)^2)),
(1/4)*(16*Z1R^2*(Z1I*ZR-Z1R*ZI)^2*cosh(Pi*rho)^2*alpha[1]^2+16*

```



$$\begin{aligned}
soln &:= \frac{16 \cosh(\pi \rho)^2 (|\zeta|^2 |\zeta I|^2 - ZI^2 ZR^2 + 2 ZII ZIR ZI ZR - ZIR^2 ZI^2)}{|\zeta|^4} \\
Ps &= \frac{16 \cosh(\pi \rho)^2 ((ZI^2 + ZR^2) (ZI^2 + ZIR^2) - ZI^2 ZR^2 + 2 ZII ZIR ZI ZR - ZIR^2 ZI^2)}{|\zeta|^4} \\
(2 PsR \cosh(\pi \rho) - \pi)^2 &= \frac{16 \cosh(\pi \rho)^2 (ZII ZI + ZIR ZR)^2}{|\zeta|^4} \tag{11}
\end{aligned}$$

Taking the square root of both sides gives, up to a +/- sign and check:

```

> Psol := (2*PsR*cosh(Pi*rho) - Pi) = 4*cosh(Pi*rho) * (Z1I*ZI + Z1R*ZR)
/abs(Zeta)^2;
evalf(subs(PsR=Re(Psi(1/2+I*rho)) - log(2*Pi), 1/abs(Zeta)^2=1/
(ZI^2+ZR^2), abs(Zeta)^2=(ZI^2+ZR^2), Z1I=Im(Zeta(1,1/2+I*rho)),
Z1R=Re(Zeta(1,1/2+I*rho)), ZI=Im(Zeta(1/2+I*rho)), ZR=Re(Zeta(1/2+
I*rho)), rho=1.234,
[Psol]));
Result:=abs(Zeta)^2=numer(rhs(Psol))/lhs(Psol);
evalf(subs(PsR=Re(Psi(1/2+I*rho)) - log(2*Pi), 1/abs(Zeta)^2=1/
(ZI^2+ZR^2), abs(Zeta)^2=(ZI^2+ZR^2), Z1I=Im(Zeta(1,1/2+I*rho)),
Z1R=Re(Zeta(1,1/2+I*rho)), ZI=Im(Zeta(1/2+I*rho)), ZR=Re(Zeta(1/2+
I*rho)), rho=1.234, Result));

```

$$Psol := 2 PsR \cosh(\pi \rho) - \pi = \frac{4 \cosh(\pi \rho) (ZII ZI + ZIR ZR)}{|\zeta|^2}$$

[-83.276438115529105082910687599312199215294862197141  
= 83.276438115529105082910687599312199215294862197140]

$$Result := |\zeta|^2 = \frac{4 \cosh(\pi \rho) (ZII ZI + ZIR ZR)}{2 PsR \cosh(\pi \rho) - \pi}$$

0.42935225146093315788510734749383948334681143755536 = (12)  
-0.42935225146093315788510734749383948334681143755536

> Which sign is it?? Remember, by squaring originally, an extraneous sign was introduced.

```

> ResultX := subs(PsR=Re(Psi(1/2+I*rho)) - log(2*Pi), 1/abs(Zeta)^2=1/
(ZI^2+ZR^2), abs(Zeta)^2=(ZI^2+ZR^2), Z1I=Im(Zeta(1,1/2+I*rho)),
Z1R=Re(Zeta(1,1/2+I*rho)), ZI=Im(Zeta(1/2+I*rho)), ZR=Re(Zeta(1/2+
I*rho)), Result);

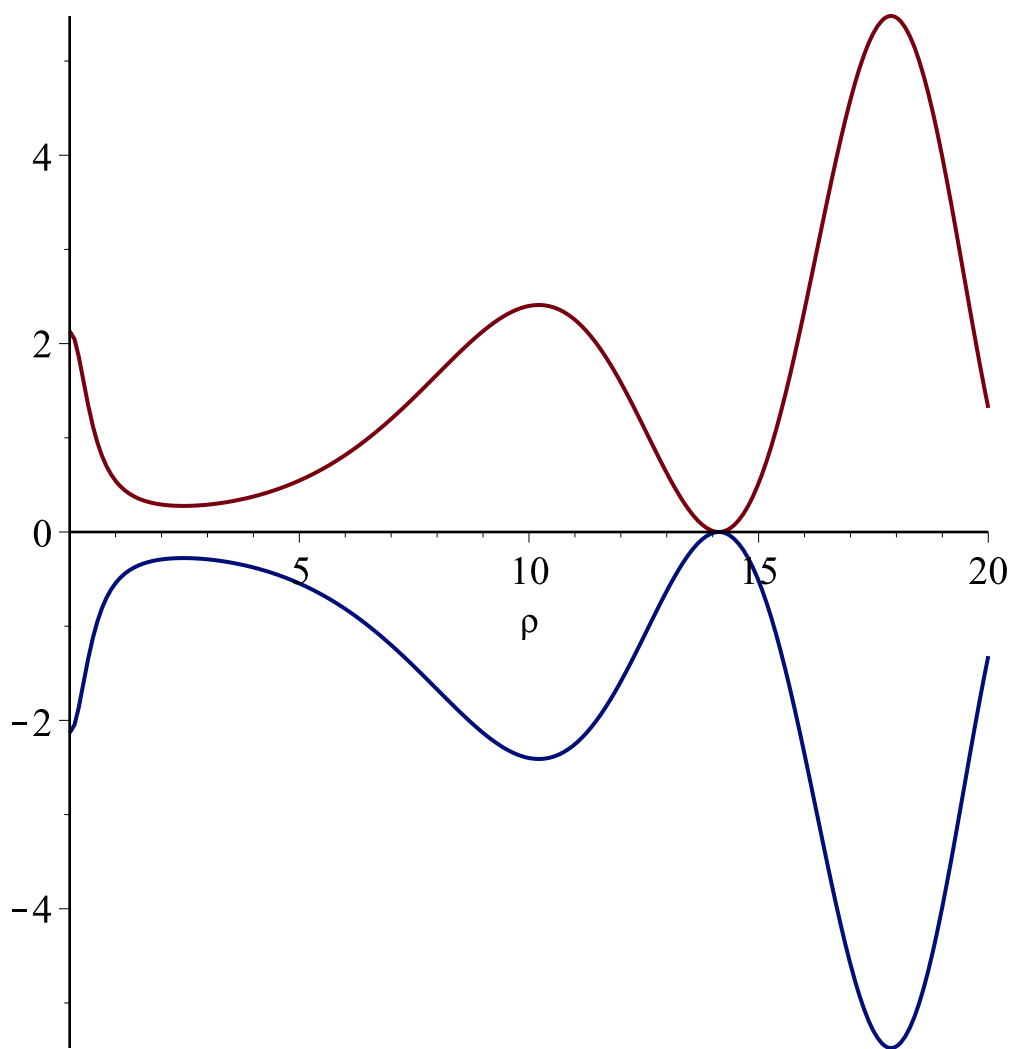
```

$$\begin{aligned}
ResultX &:= \Im\left(\zeta\left(\frac{1}{2} + I\rho\right)\right)^2 + \Re\left(\zeta\left(\frac{1}{2} + I\rho\right)\right)^2 = \left(4 \cosh(\pi \rho) \left(\Im\left(\zeta\left(\frac{1}{2} + I\rho\right)\right) \Im\left(\zeta\left(1, \frac{1}{2} + I\rho\right)\right) + \Re\left(\zeta\left(1, \frac{1}{2} + I\rho\right)\right) \Re\left(\zeta\left(\frac{1}{2} + I\rho\right)\right)\right) / \\
&\left(2 \left(\Re\left(\Psi\left(\frac{1}{2} + I\rho\right)\right) - \ln(2\pi)\right) \cosh(\pi \rho) - \pi\right) \tag{13}
\end{aligned}$$

```

> plot([lhs(ResultX), rhs(ResultX)], rho=0..20);

```



Since the lhs is inherently positive, must choose the minus sign for the rhs. Therefore the final result (Eq. 6.1 of the text) becomes

```
> Result:=abs(Zeta)^2=-numer(rhs(Psol))/lhs(Psol);
Result:=subs(PsR=Re(Psi(1/2+I*rho))-log(2*Pi),%);
```

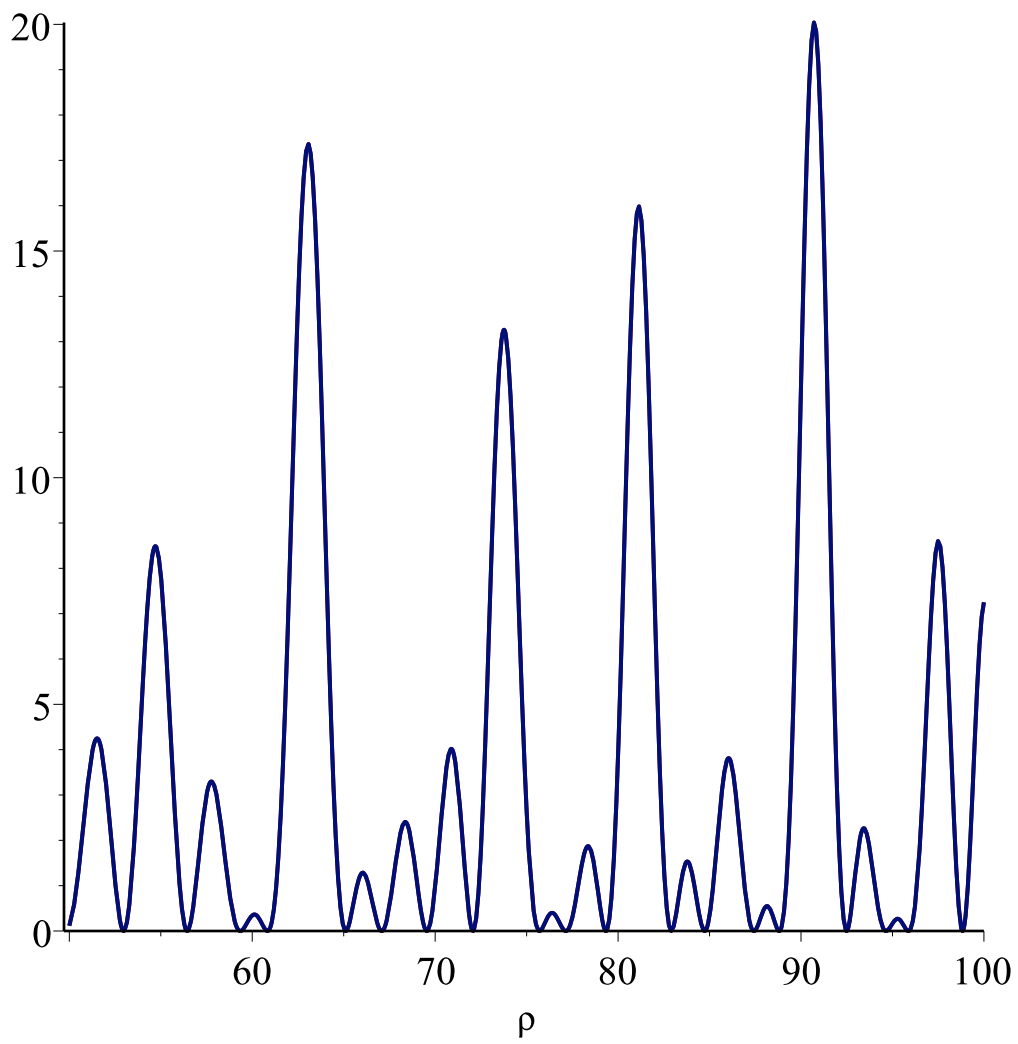
$$Result := |\zeta|^2 = - \frac{4 \cosh(\pi \rho) (ZII ZI + ZIR ZR)}{2 PsR \cosh(\pi \rho) - \pi}$$

$$Result := |\zeta|^2 = - \frac{4 \cosh(\pi \rho) (ZII ZI + ZIR ZR)}{2 \left( \Re \left( \Psi \left( \frac{1}{2} + I \rho \right) \right) - \ln(2 \pi) \right) \cosh(\pi \rho) - \pi}$$

(14)

```
> plot([lhs(ResultX), -rhs(ResultX)], rho=50..100);
```





The two curves are perfect superpositions. Therefore QED.